A VARIANT OF HORROCKS CRITERION FOR VECTOR BUNDLES ON MULTIPROJECTIVE SPACE

CHIKASHI MIYAZAKI

The purpose of this note, which is based on my talk at the conference of commutative algebra 2013, Kyoto University, RIMS, describes an attempt to give cohomological criteria for vector bundles on multiprojective space. The details will be published elsewhere. Throughout this paper k is an algebraically closed field.

A famous Auslander-Buchsbaum theorem says that a finitely generated maximal Cohen-Macaulay module over a regular local ring is free. Considering a vector bundle counterpart, you may immediately think of a Horrocks criterion [6] that for a vector bundle E on a projective space $\mathbb{P}^n_k$, E is ACM, that is, $\mathrm{H}^i_*(E) = 0$, $1 \leq i \leq n-1$, if and only if E is a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^n_k}(\ell)$. One of the attempts to generalize these facts have been devoted to a generalization of Cohen-Macaulay modules. Goto [4] has shown that a finitely generated Buchsbaum module over a regular local ring A with a maximal ideal $\mathfrak{m}$ is a direct sum of some of the syzygy modules $\mathrm{Syz}^i_A(A/\mathfrak{m})$. A vector bundle E on $\mathbb{P}^n_k$ is called Buchsbaum if $\mathfrak{m}\mathrm{H}^i_*(E|_L) = 0$ for any r-plane of $\mathbb{P}^n_k$, $1 \leq i < r \leq n$. In order to show Hartshorne's conjecture for a Buchsbaum vector bundle, Chang [2] has shown that a Buchsbaum vector bundle E on $\mathbb{P}^n_k$ is a direct sum of $\Omega^p_{\mathbb{P}^n_k}(\ell)$'s.

When does cohomology-type determines vector bundles (or modules)? We are thinking about this problem by generalizing a projective space to a multiprojective space. Cohomological criteria for the splitting of vector bundles on multiprojective space have been studied, for exmaple, in [1, 3]. In particular, Ballico-Malaspina has proved that for a vector bundle E on $X = \mathbb{P}^{n_1}_k \times \mathbb{P}^{n_2}_k$ is a direct sum of line bundles of the form $\mathcal{O}_X(t,t)$ if and only if $\mathrm{H}^i(E(j_1+\ell, j_2+\ell)) = 0$ for all integers i, j_1, j_2 and ℓ satisfying that $1 \leq i \leq n_1+n_2-1$, $j_1+j_2 = -i$, $-n_1 \leq j_1 \leq 0$ and $-n_2 \leq j_2 \leq 0$. They also described variations of

2010 *Mathematics Subject Classification.* 14F05, 14J60.

Key words and phrases. Cohomological Criterion, Castelnuovo-Mumford regularity.

Partially supported by Grant-in-Aid for Scientific Research (C) (21540044) Japan Society for the Promotion of Science.

such results as E is a direct sum of the form $\mathcal{O}_X(t+1,t)$, $\mathcal{O}_X(t,t+1)$, $p_1^*\Omega^p_{\mathbb{P}^{n_1}}(p+1)\otimes\mathcal{O}_X(t,t)$, $p_2^*\Omega^p_{\mathbb{P}^{n_2}}(p+1)\otimes\mathcal{O}_X(t,t)$.

Let us begin with describing the Horrocks theorem through the Castelnuovo-Mumford regularity according to [1].

Definition 1. A coherent sheaf F on $\mathbb{P}^n_k$ is said to be m-regular if $\mathrm{H}^i(F(m-i))=0$ for all integers $i\geq 1$.

Proposition 2. *If a coherent sheaf F on $\mathbb{P}^n_k$ is m-regular, then F is $(m+1)$-regular. Further, $F(m)$ is globally generated.*

Remark 3. By using Proposition 2, we present another proof of the Horrocks criterion. Let E be an ACM vector bundle on $\mathbb{P}^n_k$. Assume that E is m-regular but not $(m-1)$-regular. Thus we have a surjective map $\varphi:\mathcal{O}^{\oplus}_{\mathbb{P}^n_k}\to E(m)$. On the other hand, since E is ACM, we have $\mathrm{H}^n(E(m-1-n))\neq 0$, and by Serre duality $\mathrm{H}^0(E^\vee(-m))\neq 0$. Accordingly, there are a nonzero map $\psi:E(m)\to\mathcal{O}_{\mathbb{P}^n_k}$. Since $\psi\circ\varphi$ is nonzero, it splits. Hence $\mathcal{O}_{\mathbb{P}^n_k}$ is a direct summand of $E(m)$. By applying an inductive hypothesis on the rank of E, we obtain the assertion.

Now let us introduce a multigraded regularity, which will play an important role to get a cohomological criteria for the splitting of vector bundles on multiprojective space. Hoffman-Wang [5] has defined the regularity for multiprojective space, also see [7]. In order to show a Horrocks-type theorem, Ballico-Malaspina have introduced new definition of a multigraded regularity as below.

Definition 4. Let $X=\mathbb{P}^{n_1}_k\times\mathbb{P}^{n_2}_k$, where $n_1\geq 0$ and $n_2\geq 0$. A coherent sheaf F on X is said to be 0-regular if $\mathrm{H}^i(F(j_1,j_2))=0$ for all integers i, j_1 and j_2 such that $i\geq 1$, $j_1+j_2=-i$, $-n_1\leq j_1\leq 0$ and $-n_2\leq j_2\leq 0$.

Further, a coherent sheaf F on X is said to be (m_1,m_2)-regular if $F(m_1,m_2)$ is 0-regular.

Remark 5. Let F be a coherent sheaf on $X=\mathbb{P}^{n_1}_k\times\mathbb{P}^{n_2}_k$. Assume that F is 0-regular. For a generic hyperplane H_1 of $\mathbb{P}^{n_1}_k$, $F|_{L_1}$ is 0-regular on $L_1=H_1\times\mathbb{P}^{n_2}_k$.

Proposition 6. *Let E be a vector bundle on $X=\mathbb{P}^{n_1}_k\times\mathbb{P}^{n_2}_k$. Assume that E is 0-regular. Then $E(m_1,m_2)$ is 0-regular for $m_1\geq 0$, $m_2\geq 0$. Further, E is globally generated.*

The definition of a new multigraded regularity depends on the effectiveness of Proposition 6. So we give a sketch of the proof here although described in [1, (2.2)].

Proof. We need to show that if a vector bundle E on X is regular, then $E(1,0)$ is regular. Considering the exact sequence $\to \mathrm{H}^i(E(j_1, j_2)) \to \mathrm{H}^i(E(j_1+1, j_2)) \to \mathrm{H}^i(E|_{L_1}(j_1+1, j_2)) \to$ for a generic hyperplane L_1 of $\mathbb{P}^{n_1}_k$, we see the vanishing of the first and the third groups implies the second one. What is important here is to prove $\mathrm{H}^i(E|_{L_1}(j_1+1, j_2)) = 0$. Since $E|_{L_1}(1,0)$ is regular from the inductive hypothesis, we have $\mathrm{H}^i(E|_{L_1}(j_1+1, j_2)) = 0$ for $j_1 + j_2 = -i$, $-n_1 + 1 \le j_1 \le 0$, $-n_2 \le j_2 \le 0$, not for $j_1 = -n_1$. Also the regularity property of $E|_{L_1}(0,1)$ implies $\mathrm{H}^i(E|_{L_1}(j_1+1, j_2)) = 0$ for $j_1 = -n_1$, $j_2 = n_1 - i$, $-n_2 \le j_2 \le 0$, where we use the Grothendieck vanishing. Thus we have $\mathrm{H}^i(E|_{L_1}(j_1+1, j_2)) = 0$ for $j_1 + j_2 = -i$, $-n_1 \le j_1 \le 0$, $-n_2 \le j_2 \le 0$.

□

Now we give a sketch of the proof of the following result, which is extracted from the viewpoint of [8].

Theorem 7. *Let E be a vector bundle on $X = \mathbb{P}^{n_1}_k \times \mathbb{P}^{n_2}_k$, where $n_1 \ge 1$ and $n_2 \ge 1$. Let r_1 and r_2 be integers such that $0 \le r_i \le n_j$ for i and j. The vector bundle E is a direct sum of line bundles of the form $\mathcal{O}_X(\ell_1, \ell_2)$ with $-r_1 \le \ell_1 - \ell_2 \le r_2$ if and only if*

$$\mathrm{H}^i(E(j_1 + \ell, j_2 + \ell)) = 0$$

for all integers i, j_1, j_2 and ℓ satisfying that $1 \le i \le n_1 + n_2 - 1$, $-n_1 \le j_1 \le 0$ and $-n_2 \le j_2 \le 0$ except for either $i = n_1$ and $j_2 \ge j_1 + n_1 - r_1 + 1$, or $i = n_2$ and $j_2 \le j_1 - n_2 + r_2 - 1$.

Now we will give the summary of the proof.

Proof. The "only if" part is easy and left to the readers. In order to show the "if" part we take the minimal integer t such that $E(t,t)$ is 0-regular. In case $\mathrm{H}^{n_1+n_2}(E(-n_1+t-1, -n_2+t-1) \ne 0$, we proceed as in Remark 3.

Now we assume that $\mathrm{H}^{n_1+n_2}(E(-n_1+t-1, -n_2+t-1)) = 0$. Since $E(t-1, t-1)$ is not 0-regular. Then we may assume n_1th cohomologies do not vanish. Thus we consider the set of pairs (j_1, j_2) with $\mathrm{H}^{n_1}(E(j_1 + t, j_2 + t)) \ne 0$. Let us put $\mathfrak{S} = \{(j_1, j_2) | j_1 \ge -n_1 - 1, j_1 + n_1 - r_1 + 1 \le j_2 \le -j_1 - n_1 - 1\}$. Then we may assume there are some $(j_1, j_2) \in \mathfrak{S}$ such that $\mathrm{H}^{n_1}(E(j_1+t, j_2+t)) \ne 0$ and $\mathrm{H}^{n_1}(E(j_1+t+1, j_2+t)) = \mathrm{H}^{n_1}(E(j_1+t, j_2+t-1)) = 0$. Let us take $F = E(j_1+t, j_2+t)$ and a nonzero element $s \in \mathrm{H}^{n_1}(F)$.

By using Koszul complex, we can construct a surjective map $\varphi : \mathrm{H}^0(F(n_1+1, 0)) \to \mathrm{H}^{n_1}(F)$. Hence there is a nonzero element $g \in \mathrm{H}^0(F(n_1+1, 0))$ such that $\varphi(g) = s(\ne 0) \in$

$\mathrm{H}^{n_1}(F)$. By the dual argument we also construct a surjective map $\psi : \mathrm{H}^0(F^\vee(-n_1-1,0)) \to \mathrm{H}^{n_2}(F^\vee(-n_1-1,-n_2-1))$. By taking an element $s^* \in \mathrm{H}^{n_2}(F^\vee(-n_1-1,-n_2-1))$ corresponding to $s \in \mathrm{H}^{n_1}(F)$, accordingly, there is a nonzero element $f \in \mathrm{H}^0(F^\vee(-n_1-1,0))$ such that $\varphi(f) = s^*(\neq 0) \in \mathrm{H}^{n_2}(F^\vee(-n_1-1,-n_2-1))$.

The elements g and f is regarded as elements of $\mathrm{Hom}(\mathcal{O}_X, F(n_1+1,0))$ and $\mathrm{Hom}(F(n_1+1,0),\mathcal{O}_X)$ respectively. The natural map $\mathrm{H}^0(F(n_1+1,0)) \otimes \mathrm{H}^0(F^\vee(-n_1-1,0)) \to \mathrm{H}^0(\mathcal{O}_X)$ gives that $f \circ g$ is an isomorphism, which implies $\mathcal{O}_X$ is a direct summand of $F(n_1+1,0) = E(j_1+t+n_1+1, j_2+t)$. □

References

[1] E. Ballico and F. Malaspina, Regularity and cohomological splitting conditions for vector bundles on multiprojective spaces, J. Algebra 345 (2011), 137 – 149.

[2] M. C. Chang, Characterization of arithmetically Buchsbaum subschemes of codimension 2 in $\mathbb{P}^n$, J. Differential Geom. 31 (1990), 323–341.

[3] L. Costa and R.M. Miro-Roig, Cohomological characterization of vector bundles on multiprojective spaces, J. Algebra 294 (2005), 73 – 96, Corrigendum: J. Algebra 319 (2008) 1336 – 1338.

[4] S. Goto, Maximal Buchsbaum modules over regular local rings a nd a structure theorem for generalized Cohen-Macaulay modules, ASPM 11(1987), 39–64.

[5] J. W. Hoffman and H. H. Wang, Castelnuovo-Mumford regularity in biprojective spaces, Adv. Geom. 4 (2004), 513 – 536.

[6] G. Horrocks, Vector bundles on the punctual spectrum of a ring, Proc. London Math. Soc. 14 (1964), 689 – 713.

[7] D. Maclagan and G. Smith, Multigraded Castelnuovo-Mumford regularity, J. Reine. Angew. Math. 571 (2004), 179 – 212.

[8] C. Miyazaki, A cohomological criterion for splitting of vector bundles on multiprojective space, Proc. Amer. Math. Soc., to appear.

[9] D. Mumford, Lectures on curves on an algebraic surface, Annals of Math. Studies 59 (1966), Princeton UP.

Department of Mathematics, Saga University, Honjo-machi 1, Saga 840-8502, Japan

E-mail address: miyazaki@ms.saga-u.ac.jp