DUAL F-SIGNATURE OF COHEN-MACAULAY MODULES OVER QUOTIENT SURFACE SINGULARITIES

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1. INTRODUCTION

Throughout this paper, we suppose that k is an algebraically closed field of prime characteristic $p>0$. Let R be a Noetherian ring of prime characteristic $p>0$, then we can define the Frobenius morphism $F: R \to R$ $(r \mapsto r^p)$. For $e \in \mathbb{N}$, we also define the e-times iterated Frobenius morphism $F^e: R \to R$ $(r \mapsto r^{p^e})$. For any R-module M, we define the R-module F^e_*M via F^e as follows. That is, F^e_*M is just M as an abelian group, and its R-module structure is defined by $r \cdot m := F^e(r)m = r^{p^e}m$ $(r \in R,\ m \in M)$. We say R is F-finite if F_*R is a finitely generated R-module. For example, if R is an essentially of finite type over a perfect field or complete Noetherian local ring with a perfect residue field k, then R is F-finite. In this article, we only discuss such rings, thus the F-finiteness is always satisfied.

In positive characteristic commutative algebra, we understand the properties of R through the structure of F^e_*M. For this purpose, several numerical invariants are defined. Firstly, we introduce the notion of F-signature defined by C. Huneke and G. Leuschke.

Definition 1.1 ([HL]). *Let $(R,\mathfrak{m},k)$ be a d-dimensional reduced F-finite Noetherian local ring of prime characteristic $p>0$. For each $e \in \mathbb{N}$, we decompose F^e_*R as follows*

$$F^e_*R \cong R^{\oplus a_e} \oplus M_e,$$

where M_e has no free direct summands. We call a_e the e-th F-splitting number of R. Then, the limit $s(R) := \lim\limits_{e\to\infty} \dfrac{a_e}{p^{ed}}$ is called the F-signature of R.

And K. Tucker showed its existence [Tuc]. By Kunz's theorem, R is regular if and only if F^e_*R is a free R-module of rank p^{ed} [Kun]. Thus, roughly speaking, the F-signature $s(R)$ measures the deviation from regularity. The next theorem confirms this intuition.

Theorem 1.2 ([HL], [Yao2], [AL]). *Let $(R,\mathfrak{m},k)$ be a d-dimensional reduced F-finite Noetherian local ring with* $\mathrm{char} R = p > 0$. *Then we have*

(1) *R is regular if and only if $s(R)=1$,*
(2) *R is strongly F-regular if and only if $s(R)>0$.*

This notion is extended for a finitely generated R-module as follows.

Definition 1.3 ([San]). *Let $(R,\mathfrak{m},k)$ be a d-dimensional reduced F-finite Noetherian local ring of prime characteristic $p>0$. For a finitely generated R-module M and $e \in \mathbb{N}$, we set*

$$b_e(M) := \max\{n \mid \exists \varphi : F^e_*M \twoheadrightarrow M^{\oplus n}\},$$

THIS PAPER IS AN ANNOUNCEMENT OF OUR RESULT AND THE DETAILED VERSION WILL BE SUBMITTED TO SOMEWHERE.

and call it the e-th F-surjective number of M. Then we call the limit $s(M) \coloneqq \lim_{e\to\infty} \dfrac{b_e(M)}{p^{ed}}$ *dual F-signature of M if it exists.*

Remark 1.4. Since the morphism $F^e_*R \twoheadrightarrow R^{\oplus b_e(R)}$ splits, if M is isomorphic to the basering R, then the dual F-signature of R in sense of Definition 1.3 coincides with the F-signature of R. Thus, we use the same notation unless it causes confusion.

Remark 1.5. Since the $\mathfrak{m}$-adic completion commutes with $F^e_*(-)$, we can easily reduce the case of complete local ring in Definition 1.1 and 1.3. Thus, we may assume that the Krull-Schmidt condition holds for R.

Just like the F-signature, the dual F-signature also characterizes some singularities.

Theorem 1.6 ([San]). *Let $(R,\mathfrak{m},k)$ be a d-dimensional reduced F-finite Cohen-Macaulay local ring with* $\operatorname{char} R = p > 0$. *Then we have*

(1) *R is F-rational if and only if $s(\omega_R) > 0$,*
(2) *$s(R) \le s(\omega_R)$,*
(3) *$s(R) = s(\omega_R)$ if and only if R is Gorenstein.*

In this way, the value of $s(R)$ and $s(\omega_R)$ characterize some singularities. Now we have some questions. Let M be a finitely generated R-module which may not be R or ω_R. Then

· Does the value of $s(M)$ have some pieces of information about singularities ?
· What does the explicit value of $s(M)$ mean ?
· Is there any connection between $s(M)$ and other numerical invariants ?

However, it is difficult to try these questions for now. Because the value of dual F-signature is not known and we don't have an effective method for determining it except only a few cases. For example, the case of two-dimensional Veronese subrings is studied in [San, Example 3.17]. Thus, in this article, we investigate the dual F-signature for Cohen-Macaulay (=CM) modules over two-dimensional rational double points. Therefore, in the rest of this article, we suppose that G is a finite subgroup of $\mathrm{SL}(2,k)$ and the order of G is coprime to $p = \operatorname{char} k$. We remark that G contains no pseudo-reflections in this situation and it is well known that a finite subgroup of $\mathrm{SL}(2,k)$ is conjugate to one of the type so-called $(A_n), (D_n), (E_6), (E_7)$ or (E_8). We denote the invariant subring of $S \coloneqq k[[x,y]]$ under the action of G by $R \coloneqq S^G$ and the maximal ideal of R by $\mathfrak{m}$. In this situation, the invariant subring R is Gorenstein by [Wat]. We call R (or equivalently $\operatorname{Spec} R$) rational double points (or Du Val singularities, Kleinian singularities, ADE singularities in the literature).

Let $V_0 = k, V_1, \cdots, V_n$ be the full set of non-isomorphic irreducible representations of G. We set $M_t \coloneqq (S \otimes_k V_t)^G$ $(t = 0,1,\cdots,n)$. Under the assumption G contains no pseudo-reflections, we can see that each M_t is an indecomposable maximal Cohen-Macaulay (=MCM) R-module ($\operatorname{rank}_R M_t = \dim_k V_t$) and $M_s \not\cong M_t$ $(s \neq t)$. For more details refer to [HN, Section 2].

In this article, we will investigate the value of $s(M_t)$. In order to determine the dual F-signature, we have to understand the following topics;

(1) The structure of $F^e_*M_t$, namely
  – What kind of MCM appears in $F^e_*M_t$ as a direct summand?
  – The asymptotic behavior of $F^e_*M_t$ on the order of p^{2e}.
(2) How do we construct a surjection $F^e_*M_t \twoheadrightarrow M_t^{\oplus b_e}$?

To show the former one, we need the notion of generalized F-signature. So we review it in Section 2. After that we will use the notion of the Auslander-Reiten quiver to show the latter problem. Thus, we give a brief summary of the Auslander-Reiten theory in Section 3. In Section 4, we actually determine the value of dual F-signature of CM modules. Since the strategy for determining the dual F-signature is almost the same for all the ADE cases, we will give a concrete explanation only for the case of D_5. In Section 5, we give the complete list of the value of dual F-signature for all the ADE cases.

2. Generalized F-signature of invariant subrings

Firstly it is known that R is of finite CM representation type, that is, it has only finitely many non-isomorphic indecomposable MCM modules $\{R, M_1, \cdots, M_n\}$. Since $F_*^e R$ is an MCM R-module, we can describe as

$$F_*^e R \cong R^{\oplus c_{0,e}} \oplus M_1^{\oplus c_{1,e}} \oplus \cdots \oplus M_n^{\oplus c_{n,e}}.$$

Since the Krull-Schmidt condition holds for R, the multiplicities $c_{t,e}$ are determined uniquely. For understanding the asymptotic behavior of the multiplicity $c_{t,e}$, we consider the limit

$$s(R, M_t) := \lim_{e \to \infty} \frac{c_{t,e}}{p^{2e}} \quad (t = 0, 1, \cdots, n).$$

We call it generalized F-signature of M_t. In our situation, this limit exists [SVdB, Yao1]. And the value of this limit is known as follows.

Theorem 2.1. ([HS, Lemma 4.10], *see also* [HN, Theorem 3.4]) *For $t = 0, 1, \cdots, n$, we have*

$$s(R, M_t) = \frac{\operatorname{rank}_R M_t}{|G|} = \frac{\dim_k V_t}{|G|}$$

Remark 2.2. In the case of $t = 0$, we have $s(R, R) = s(R)$ and the above result is also due to [HL, Example 18], [WY, Theorem 4.2].

As a corollary, we also have the next statement .

Corollary 2.3. ([HN, Corollary 3.10]) *Suppose an MCM R-module $F_*^e M_t$ decomposes as follows.*

$$F_*^e M_t \cong R^{\oplus d_{0,e}^t} \oplus M_1^{\oplus d_{1,e}^t} \oplus \cdots \oplus M_n^{\oplus d_{n,e}^t}.$$

Then, for all $s, t = 0, \cdots, n$, we have

$$s(M_t, M_s) := \lim_{e \to \infty} \frac{d_{s,e}^t}{p^{2e}} = (\operatorname{rank}_R M_t) \cdot s(R, M_s) = \frac{(\operatorname{rank}_R M_t) \cdot (\operatorname{rank}_R M_s)}{|G|}.$$

Remark 2.4. As Corollary 2.3 shows, every indecomposable MCM R-modules appear in $F_*^e M_t$ as a direct summand for sufficiently large $e >> 0$. Therefore, the additive closure $\operatorname{add}_R(F_*^e M_t)$ coincides with the category of MCM R-modules $\mathrm{CM}(R)$. So we can apply several results so-called Auslander-Reiten theory to $\operatorname{add}_R(F_*^e M_t)$. We discuss it in the next section.

3. Review of Auslander-Reiten theory

From Nakayama's lemma, when we discuss the surjectivity of $F_*^e M_t \to M_t^{\oplus b}$, we may consider each MCM module as a vector space after tensoring the residue field k. Thus, we want to know a basis of $M_t/\mathfrak{m}M_t$ (i.e. minimal generators of M_t). For some $m \in \mathbb{N}$, we have

$$
\begin{array}{ccc}
M_t & \cong & \mathrm{Hom}_R(R, M_t) \\
\cup & & \cup \\
\mathfrak{m}M_t & \cong & \{R \overset{\text{non split}}{\to} R^{\oplus m} \to M_t\}.
\end{array}
$$

From this observation, we identify a minimal generator of M_t with a morphism from R to M_t which doesn't factor through free modules except the starting point. In order to find such morphisms, we will use the notion of Auslander-Reiten (=AR) quiver. So we review some results of Auslander-Reiten theory in this section. For more details, see some textbooks (e.g. [Yos]).

In order to define the AR quiver, we introduce the notion of irreducible morphism.

Definition 3.1 (Irreducible morphism)**.** *Suppose M and N are MCM R-modules. We decompose M and N into indecomposable modules as $M = \oplus_i M_i$, $N = \oplus_j N_j$ and also decompose $\psi \in \mathrm{Hom}_R(M,N)$ along the above decomposition as $\psi = (\psi_{ij} : M_i \to N_j)_{ij}$. Then we define submodule $\mathrm{rad}_R(M,N) \subset \mathrm{Hom}_R(M,N)$ as*

$$\psi \in \mathrm{rad}_R(M,N) \overset{def}{\Longleftrightarrow} \textit{no } \psi_{ij} \textit{ is an isomorphism.}$$

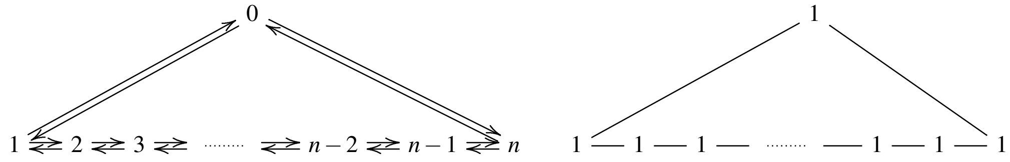
Furthermore, we define submodule $\mathrm{rad}_R^2(M,N) \subset \mathrm{Hom}_R(M,N)$. The submodule $\mathrm{rad}_R^2(M,N)$ consists of morphisms $\psi : M \to N$ such that ψ decomposes as $\psi = fg$, where $f \in \mathrm{rad}_R(M,Z)$, $g \in \mathrm{rad}_R(Z,N)$ and Z is an MCM R-module. We say that a morphism $\psi : M \to N$ is irreducible if $\psi \in \mathrm{rad}_R(M,N) \setminus \mathrm{rad}_R^2(M,N)$. In this setting, we define the k-vector space $\mathrm{Irr}_R(M,N)$ as $\mathrm{Irr}_R(M,N) := \mathrm{rad}_R(M,N) \big/ \mathrm{rad}_R^2(M,N)$.

By using this notion, we define the AR quiver.

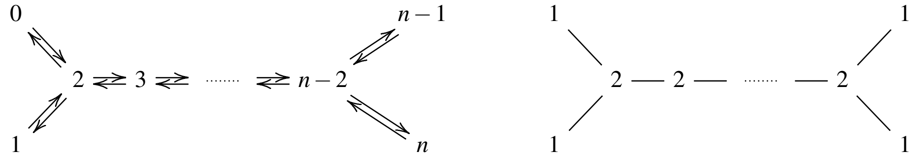
Definition 3.2 (Auslander-Reiten quiver)**.** *The AR quiver of R is an oriented graph whose vertices are indecomposable MCM R-modules $\{R, M_1, \cdots, M_n\}$ and draw $\dim_k \mathrm{Irr}_R(M_s, M_t)$ arrows from M_s to M_t $(s,t = 0,1,\cdots,n)$.*

In our situation, the AR quiver of R coincides with the McKay quiver of G by [Aus1], so we can describe it from representations of G (for the definition of McKay quiver, refer to [Yos, (10.3)]). And more fortunately, the AR quiver of R coincides with the extended Dynkin diagram corresponding to a finite subgroup of $\mathrm{SL}(2,k)$ after replacing each edges " $-$ " by arrows " $\leftrightarrows$ ". This is a kind of McKay correspondence. Therefore the Auslander-Reiten quiver of R is the left hand side of the following,

(A_n)



(D_n)



$$(E_6)\qquad \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \uparrow\downarrow & & & & \\ & & & & 1 & & & & \\ & & & & \uparrow\downarrow & & & & \\ 5 & \rightleftarrows & 3 & \rightleftarrows & 2 & \rightleftarrows & 4 & \rightleftarrows & 6 \end{array} \qquad \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & | & & & & \\ & & & & 2 & & & & \\ & & & & | & & & & \\ 1 & — & 2 & — & 3 & — & 2 & — & 1 \end{array}$$

$$(E_7)\qquad \begin{array}{ccccccccccccc} & & & & & & 7 & & & & & & \\ & & & & & & \uparrow\downarrow & & & & & & \\ 0 & \rightleftarrows & 1 & \rightleftarrows & 2 & \rightleftarrows & 3 & \rightleftarrows & 4 & \rightleftarrows & 5 & \rightleftarrows & 6 \end{array} \qquad \begin{array}{ccccccccccccc} & & & & & & 2 & & & & & & \\ & & & & & & | & & & & & & \\ 1 & — & 2 & — & 3 & — & 4 & — & 3 & — & 2 & — & 1 \end{array}$$

$$(E_8)\qquad \begin{array}{ccccccccccccccc} & & & & & & & & & & 8 & & & & \\ & & & & & & & & & & \uparrow\downarrow & & & & \\ 0 & \rightleftarrows & 1 & \rightleftarrows & 2 & \rightleftarrows & 3 & \rightleftarrows & 4 & \rightleftarrows & 5 & \rightleftarrows & 6 & \rightleftarrows & 7 \end{array} \qquad \begin{array}{ccccccccccccccc} & & & & & & & & & & 3 & & & & \\ & & & & & & & & & & | & & & & \\ 1 & — & 2 & — & 3 & — & 4 & — & 5 & — & 6 & — & 4 & — & 2 \end{array}$$

where a vertex t corresponds the MCM R-module M_t and the right hand side of the figure means $\operatorname{rank}_R M_t$.

4. DUAL F-SIGNATURE OVER RATIONAL DOUBLE POINTS

The strategy for determining the dual F-signature is almost the same for all the ADE cases. So from now on, we will explain the method of determining it by using the following example.

Example 4.1. *The binary dihedral group* $G := \mathscr{D}_3 = \langle \begin{pmatrix} \zeta_6 & 0 \\ 0 & \zeta_6^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{pmatrix} \rangle$ *is the type* D_5 *in the list* [Yos, (10.15)] *and* $|G| = 12$. *For the invariant subring under the action of* G, *the AR quiver takes the form of* D_5,

$$\begin{tikzcd} 0 \arrow[dr, shift left, "a"] & & & 4 \arrow[dl, shift left, "D"] \\ & 2 \arrow[ul, shift left, "A"] \arrow[dl, shift left, "B"] \arrow[r, shift left, "C"] & 3 \arrow[l, shift left, "c"] \arrow[ur, shift left, "d"] \arrow[dr, shift left, "e"] & \\ 1 \arrow[ur, shift left, "b"] & & & 5 \arrow[ul, shift left, "E"] \end{tikzcd}$$

with relations

$$\begin{cases} A\circ a = 0, & C\circ c + D\circ d + E\circ e = 0, \\ B\circ b = 0, & d\circ D = 0, \\ a\circ A + b\circ B + c\circ C = 0, & e\circ E = 0 \end{cases} \tag{4.1}$$

In order to find morphisms from R to M_t which doesn't factor through free modules except the starting point, we define the stable category $\underline{\mathrm{CM}}(R)$ as follows. The objects of $\underline{\mathrm{CM}}(R)$ are same as those of $\mathrm{CM}(R)$ and the morphism set is given by

$$\underline{\mathrm{Hom}}_R(X,Y) := \mathrm{Hom}_R(X,Y)\big/\mathscr{P}(X,Y), \quad X,Y \in \mathrm{CM}(R)$$

where $\mathscr{P}(X,Y)$ is the submodule of $\mathrm{Hom}_R(X,Y)$ consisting of morphisms which factor through a free R-module.

By the property of AR quiver, we can see that morphisms from R to non-free indecomposable $M_t \in \mathrm{CM}(R)$ (on the AR quiver of D_5) always go through the vertex M_2 at the beginning. Thus, the composition of $R \xrightarrow{a} M_2$ and non-zero elements of $\underline{\mathrm{Hom}}_R(M_2, M_t)$ are exactly what we wanted. Therefore we will find non-zero elements of $\underline{\mathrm{Hom}}_R(M_2, M_t)$. For this purpose we rewrite the AR quiver as a repetition of the original one.

$$
\begin{tikzcd}
0 \arrow[dr, "a"] & & 0 \arrow[dr] & & 0 \arrow[dr] & & 0 \arrow[dr] & & 0 \\
1 \arrow[r, "b"] & 2 \arrow[r, "B"] \arrow[ur, "A"] \arrow[dr, "C"] & 1 \arrow[r] & 2 \arrow[r] \arrow[ur] \arrow[dr] & 1 \arrow[r] & 2 \arrow[r] \arrow[ur] \arrow[dr] & 1 \arrow[r] & 2 \arrow[r] \arrow[ur] \arrow[dr] & 1 \\
3 \arrow[r, "d"] \arrow[ur, "c"] \arrow[dr, "e"] & 4 \arrow[r, "D"] & 3 \arrow[r] \arrow[ur] \arrow[dr] & 4 \arrow[r] & 3 \arrow[r] \arrow[ur] \arrow[dr] & 4 \arrow[r] & 3 \arrow[r] \arrow[ur] \arrow[dr] & 4 \arrow[r] & 3 \\
& 5 \arrow[ur, "E"] & & 5 \arrow[ur] & & 5 \arrow[ur] & & 5 \arrow[ur] &
\end{tikzcd}
$$

Since this quiver has relations, it seems to be difficult to extract non-zero morphisms from the above picture. But there is a useful technique so-called counting argument of AR quiver. By using such a technique, we can extract desired morphisms. This method first appeared in the work of Gabriel [Gab] and it is also used for classifying special CM modules over quotient surface singularities [IW]. For the details of this kind of counting argument, see [Gab, Iya, IW].

After applying such a technique, we obtain the following picture. And paths on this quiver represent non-zero morphisms in $\underline{\mathrm{Hom}}_R(M_2,M_t)$.

$$
\begin{tikzcd}
2 \arrow[r] \arrow[dr] & 1 \arrow[r] & 2 \arrow[dr] & & 2 \arrow[r] \arrow[dr] & 1 \arrow[r] & 2 \\
& 3 \arrow[r] \arrow[ur] \arrow[dr] & 4 \arrow[r] & 3^2 \arrow[r] \arrow[ur] \arrow[dr] & 4 \arrow[r] & 3 \arrow[ur] & \\
& & 5 \arrow[ur] & & 5 \arrow[ur] & &
\end{tikzcd}
$$

The following quiver is the composition of $R \xrightarrow{a} M_2$ and non-zero element of $\underline{\mathrm{Hom}}_R(M_2,M_t)$ for $t = 1,2,\cdots,5$ (the exponent of each vertex implies the multiplicity).

$$
\begin{tikzcd}
0 \arrow[dr] & & & & & & & \\
& 2 \arrow[r] \arrow[dr] & 1 \arrow[r] & 2 \arrow[dr] & & 2 \arrow[r] \arrow[dr] & 1 \arrow[r] & 2 \\
& & 3 \arrow[r] \arrow[ur] \arrow[dr] & 4 \arrow[r] & 3^2 \arrow[r] \arrow[ur] \arrow[dr] & 4 \arrow[r] & 3 \arrow[ur] & \\
& & & 5 \arrow[ur] & & 5 \arrow[ur] & &
\end{tikzcd}
\tag{4.2}
$$

Thus, we identify paths on this quiver with minimal generator of each MCM R-module M_t. For example, minimal generators of M_1 are identified with

$$
\begin{tikzcd}
0 \arrow[dr] & & & \\
& 2 \arrow[r] & 1 & 2 \\
& & 3 & 4 \\
& & & 5
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
0 \arrow[dr] & & & & & & & \\
& 2 \arrow[dr] & 1 & 2 & & 2 \arrow[r] & 1 & 2 \\
& & 3 \arrow[r] & 4 \arrow[r] & 3^2 \arrow[ur] & 4 & 3 & \\
& & & 5 & & 5 & &
\end{tikzcd}
\tag{4.3}
$$

Of course, there are several paths from R to M_1 not only the above ones. But they are same up to modulo radical because this quiver has relations. When we consider a surjection, it doesn't matter if we identify them.

By using these results, we try to determine $s(M_1)$ as an example. Firstly, we suppose an MCM R-module $F^e_*M_1$ decomposes as $F^e_*M_1 \cong R^{\oplus d_{0,e}} \oplus M_1^{\oplus d_{1,e}} \oplus \cdots \oplus M_n^{\oplus d_{n,e}}$. From Corollary 2.3, we have the following for all $s = 0,1,\cdots,n$,

$$
s(M_1,M_s) = \lim_{e\to\infty} \frac{d_{s,e}}{p^{2e}} = \frac{(\mathrm{rank}_R M_1)(\mathrm{rank}_R M_s)}{|G|} = \frac{\mathrm{rank}_R M_s}{12}.
$$

Thus, we may consider

$$F_*^e M_1 \approx (R^{\frac{1}{12}} \oplus M_1^{\frac{1}{12}} \oplus M_2^{\frac{2}{12}} \oplus M_3^{\frac{2}{12}} \oplus M_4^{\frac{1}{12}} \oplus M_5^{\frac{1}{12}})^{p^{2e}},$$

on the order of p^{2e} in this case. When we try to compute dual F-signature, the part of $o(p^{2e})$ is harmless. So we identify $F_*^e M_1$ with $\left(R^{\frac{1}{12}} \oplus M_1^{\frac{1}{12}} \oplus M_2^{\frac{2}{12}} \oplus M_3^{\frac{2}{12}} \oplus M_4^{\frac{1}{12}} \oplus M_5^{\frac{1}{12}}\right)$ and consider a surjection

$$R^{\frac{1}{12}} \oplus M_1^{\frac{1}{12}} \oplus M_2^{\frac{2}{12}} \oplus M_3^{\frac{2}{12}} \oplus M_4^{\frac{1}{12}} \oplus M_5^{\frac{1}{12}} \twoheadrightarrow M_1^{\oplus b}.$$

As we showed before, minimal generators of M_1 are identified with paths in (4.3). We denote the left (resp. right) of them by g_1 (resp. g_2). In order to construct a surjection, we pay attention to g_1. We can see that M_2 can generate g_1 through the morphism $M_2 \xrightarrow{B} M_1$.

$$\underset{\text{min. gen. of } M_2}{\begin{tikzcd}[column sep=small, row sep=small] 0 \arrow[dr, "a"] & \\ & 2 \end{tikzcd}} \quad + \quad 2 \xrightarrow{B} 1 \quad = \quad g_1 : \begin{tikzcd}[column sep=small, row sep=small] 0 \arrow[dr, "a"] & & \\ & 2 \arrow[r, "B"] & 1 \end{tikzcd}$$

Similarly, R can generate g_1 through the morphism $R \xrightarrow{B \circ a} M_1$. Moreover, M_1 clearly generate g_1 through the identity map $(M_1 \xrightarrow{1_{M_1}} M_1)$. And we have no other such MCMs. Collectively, MCM modules which generate the minimal generator g_1 are $\{R^{\frac{1}{12}}, M_1^{\frac{1}{12}}, M_2^{\frac{2}{12}}\}$. Thus, the value of $s(M_1)$ can take $s(M_1) \leq \frac{1}{12} + \frac{1}{12} + \frac{2}{12} = \frac{4}{12}$ (In order to show the ratio of $s(M_1)$ to the order of G, we don't reduce the fraction). In this way, we obtain the upper bounds of $s(M_1)$.

Next, we will show that we can actually construct a surjection

$$R^{\frac{1}{12}} \oplus M_1^{\frac{1}{12}} \oplus M_2^{\frac{2}{12}} \oplus M_3^{\frac{2}{12}} \oplus M_4^{\frac{1}{12}} \oplus M_5^{\frac{1}{12}} \twoheadrightarrow M_1^{\frac{4}{12}}.$$

As we showed before, each minimal generator of M_2 is identified with a morphism from "0" to "2" in (4.2). Considering the composition of such morphisms and $2 \xrightarrow{B} 1$, we have g_1 and g_2 at the same time. Namely, we have the surjection $M_2^{\frac{2}{12}} \overset{B}{\twoheadrightarrow} M_1^{\frac{2}{12}}$. Moreover, there is the surjection $M_1^{\frac{1}{12}} \twoheadrightarrow M_1^{\frac{1}{12}}$ clearly. In this way, M_1 and M_2 can generate g_1 and g_2 at the same time. But R can't generate g_1 and g_2 at the same time (only generate either g_1 or g_2). We will use it for generating g_1 and use the remaining MCMs $\{M_3, M_4, M_5\}$ for generating g_2. (From the picture (4.2), we read off that these remaining MCMs generate g_2.) For example, we have $R^{\frac{1}{12}} \oplus M_4^{\frac{1}{12}} \twoheadrightarrow M_1^{\frac{1}{12}}$. Thus, we conclude $s(M_1) = \frac{4}{12}$.

5. Summary of the value of Dual F-signature

By using similar method, we can obtain the dual F-signature for all the ADE cases. Finally, we give the complete list of the dual F-signature.

Theorem 5.1. *The following is the Dynkin diagram Q and corresponding values of dual F-signature (In order to show the ratio of dual F-signature to the order of G clearly, we don't reduce fractions).*

(1) *Type A_n*

- *n is an even number (i.e. $n = 2r$)*

$$A_n: \quad 1 - 2 - \cdots - r - r+1 - \cdots - n-1 - n$$

$$\frac{2}{n+1} - \frac{3}{n+1} - \cdots - \frac{r+1}{n+1} - \frac{r+1}{n+1} - \cdots - \frac{3}{n+1} - \frac{2}{n+1}$$

- *n is an odd number (i.e. $n = 2r-1$)*

$$A_n: \quad 1 - 2 - \cdots - r-1 - r - r+1 - \cdots - n-1 - n$$

$$\frac{2}{n+1} - \frac{3}{n+1} - \cdots - \frac{r}{n+1} - \frac{2r+1}{2(n+1)} - \frac{r}{n+1} - \cdots - \frac{3}{n+1} - \frac{2}{n+1}$$

(2) *Type D_n*

- *n is an even number (i.e. $n = 2r$)*

$$D_n: \quad 1 - 2 - \cdots - m - \cdots - r - r+1 - \cdots - n-2 \begin{array}{l} \diagup \; n-1 \\ \diagdown \; n \end{array}$$

$$\frac{4}{4(n-2)} - \frac{6}{4(n-2)} - \cdots - \frac{4m-2}{4(n-2)} - \cdots - \frac{4r-2}{4(n-2)} - \frac{4r-1}{4(n-2)} - \cdots - \frac{4r-1}{4(n-2)} \begin{array}{l} \diagup \; \frac{2(n-2)}{4(n-2)} \\ \diagdown \; \frac{2(n-2)}{4(n-2)} \end{array}$$

- *n is an odd number (i.e. $n = 2r-1$)*

$$D_n: \quad 1 - 2 - \cdots - m - \cdots - r-1 - r - \cdots - n-2 \begin{array}{l} \diagup \; n-1 \\ \diagdown \; n \end{array}$$

$$\frac{4}{4(n-2)} - \frac{6}{4(n-2)} - \cdots - \frac{4m-2}{4(n-2)} - \cdots - \frac{4r-6}{4(n-2)} - \frac{4r-3}{4(n-2)} - \cdots - \frac{4r-3}{4(n-2)} \begin{array}{l} \diagup \; \frac{2(n-2)}{4(n-2)} \\ \diagdown \; \frac{2(n-2)}{4(n-2)} \end{array}$$

(3) *Type E_6*

$$E_6: \quad \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & | & & & & \\ 5 & - & 3 & - & 2 & - & 4 & - & 6 \end{array} \qquad \begin{array}{ccccccccc} & & & & \frac{6}{24} & & & & \\ & & & & | & & & & \\ \frac{9}{24} & - & \frac{16}{24} & - & \frac{18}{24} & - & \frac{16}{24} & - & \frac{9}{24} \end{array}$$

(4) *Type E_7*

$$
\begin{array}{ccccccccccccc}
 & & & & 7 & & & & & & \\
 & & & & | & & & & & & \\
E_7: & 1 & — & 2 & — 3 & — & 4 & — & 5 & — & 6
\end{array}
\qquad
\begin{array}{ccccccccccc}
 & & & & \frac{24}{48} & & & & & & \\
 & & & & | & & & & & & \\
\frac{6}{48} & — & \frac{18}{48} & — & \frac{38}{48} & — & \frac{36}{48} & — & \frac{27}{48} & — & \frac{16}{48}
\end{array}
$$

(5) *Type E_8*

$$
\begin{array}{cccccccccccccc}
 & & & & & & & & 8 & & & & \\
 & & & & & & & & | & & & & \\
E_8: & 1 & — & 2 & — & 3 & — & 4 & — 5 & — & 6 & — & 7
\end{array}
\qquad
\begin{array}{ccccccccccccc}
 & & & & & & & & \frac{66}{120} & & & & \\
 & & & & & & & & | & & & & \\
\frac{6}{120} & — & \frac{18}{120} & — & \frac{40}{120} & — & \frac{75}{120} & — & \frac{102}{120} & — & \frac{80}{120} & — & \frac{51}{120}
\end{array}
$$

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