Ulrich ideals of Gorenstein numerical semigroup rings *

Takahiro Numata †

Graduate School of Integrated Basic Sciences, Nihon University

1 Introduction

Throughout this paper let $\mathbb{N}$ and $\mathbb{Z}$ denote the set of nonnegative integers and integers, respectively. A *numerical semigroup* is a subset of $\mathbb{N}$ which is closed under addition, contains the zero element and whose complement in $\mathbb{N}$ is finite. Every numerical semigroup H is finitely generated and has the unique minimal system of generators $a_1, ..., a_r \in \mathbb{N}$; that is

$$H = \langle a_1, ..., a_r \rangle := \{\lambda_1 a_1 + \cdots + \lambda_r a_r \mid \lambda_1, ..., \lambda_r \in \mathbb{N}\},$$

where $\gcd(a_1, ..., a_r) = 1$. The *Frobenius number* of H, denoted by $\mathrm{F}(H)$, is the maximal integer which is not belonging to H. A numerical semigroup H is *symmetric* if for any integers $x \in \mathbb{Z}$, either $x \in H$ or $\mathrm{F}(H) - x \in H$. Let k be a field and t be an indeterminate over k. The ring

$$k[[H]] := k[[t^{a_1}, ..., t^{a_r}]] \subset k[[t]]$$

is called the *numerical semigroup ring* associated to $H = \langle a_1, ..., a_r \rangle$. A numerical semigroup ring $k[[H]]$ is a one-dimensional Cohen-Macaulay local ring with maximal ideal $\mathfrak{m} = (t^{a_1}, ..., t^{a_r})$. It is well known that $k[[H]]$ is Gorenstein if and only if H is symmetric.

Our purpose in this paper is to investigate Ulrich ideals of Gorenstein numerical semigroup rings which generated by monomials. The notion of Ulrich ideals was introduced recently by S. Goto, K. Ozeki, R. Takahashi, K-.i. Watanabe and K. Yoshida in [GOTWY].

Definition 1.1. [GOTWY] Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim R \geq 0$ and I be an $\mathfrak{m}$-primary ideal of R. Suppose that I contains a parameter ideal $Q = (a_1, ..., a_d)$ of R as a minimal reduction. Then I is called an *Ulrich ideal* of R if the following two conditions hold true:

*This paper is an announcement of our result and the detailed version will be submitted to somewhere.

†E-mail: takahiro.n0913@gmail.com

(1) $I^2 = QI$.

(2) R/I-module I/I^2 is free.

By definition, any parameter ideal of R is Ulrich. For convenience, in this paper, we except parameter ideals whenever we refer to Ulrich ideals. We put $R = k[[H]]$ and let χ_R^g denotes the set of Ulrich ideals of R which generated by monomials in t. When H is a numerical semigroup generated by 2-elements, the set χ_R^g is completely described in [GOTWY]. So, in Section 3, we consider the case where H is generated by 3-elements, that is $H = \langle a, b, c\rangle$. In this case, we completely determine when $\chi_{k[[H]]}^g$ is empty or not so. In section 4, we consider the case where H is a numerical semigroup generated by an *arithmetic sequence*, that is $H = \langle a, a+d, ..., a+nd\rangle$, where $a, d > 0, n \geq 2$ and $\gcd(a, d) = 1$. Then we state that $\chi_{k[[H]]}^g \neq \emptyset$ if and only if $n = 2$.

2 Preliminaries

We start this section by recalling some results on Ulrich ideals from [GOTWY]. The following theorem is very important to achieve our goal.

Theorem 2.1. [GOTWY] *Suppose that $R = k[[H]]$ is a Gorenstein numerical semigroup ring (equivalently, H is a symmetric numerical semigroup) and let I be an ideal of R. Then the following conditions are equivalent.*

(1) $I \in \chi_R^g$.

(2) $I = (t^\alpha, t^\beta)$ $(\alpha, \beta \in H, \alpha < \beta)$ *and if we put $x = \beta - \alpha$, the following conditions hold.*

  (i) $x \notin H$, $2x \in H$.

  (ii) *the numerical semigroup $H_1 = H + \langle x\rangle$ is symmetric, and*

  (iii) $\alpha = \min\{h \in H \mid h + x \in H\}$.

In particular, we note that $\chi_R^g \neq \emptyset$ if and only if there is an integer $x \in \mathbb{Z}$ which satisfies that conditions (i) and (ii) above.

Example 2.2. (1) Let $H = \langle 4, 5\rangle = \{0, 4, 5, 8, 9, 10, 12 \rightarrow\}$. We can find the integers which satisfy the condition (i):

$$x = 2, 6, 7, 11.$$

In these integers, 2 and 6 just satisfy the condition (ii). Therefore we have

$$\chi_{k[[H]]}^g = \{(t^8, t^{10}), (t^4, t^{10})\}$$

by the condition (iii).

(2) If $H = \langle 3, 5\rangle$, then $\chi_{k[[H]]}^g = \emptyset$ since we can check that there are no integers which satisfy the conditions (i) and (ii).

Actually, when H is generated by 2-elements, the set χ_R^g is completely described in [GOTWY]. In particular, the following assertion holds true.

Theorem 2.3. [GOTWY] *Let $H = \langle a, b \rangle$ be a numerical semigroup. Then the following conditions are equivalent.*

(1) $\chi_{k[[H]]}^g \neq \emptyset$.

(2) *a or b is even.*

3 The case of $H = \langle a, b, c \rangle$

In this section, we consider the case where $H = \langle a, b, c \rangle$. There is a characterization for H to be symmetric.

Lemma 3.1. [He], [Wa] *Let $H = \langle a, b, c \rangle$ be a numerical semigroup generated by 3-elements. Then following assertions are equivalent.*

(1) *H is symmetric.*

(2) *Changing order of a, b and c if necessary, we can write $a = a'd, b = b'd$, where $\gcd(a, b) = d > 1$ and $c \in \langle a', b' \rangle$, $c \neq a', b'$.*

If this case occurs, we denote by $H = \langle d \langle a', b' \rangle, c \rangle$.

Example 3.2. (1) $H = \langle 8, 12, 15 \rangle$ is symmetric since we can write as $H = \langle 4 \langle 2, 3 \rangle, 15 \rangle$.

(2) $H = \langle 7, 11, 13 \rangle$ is not symmetric since any two pairs of minimal generators of H are pairwise coprime.

Using Lemma 3.1, we state our main theorem.

Theorem 3.3. *Let $H = \langle a, b, c \rangle$ be a symmetric numerical semigroup and assume that $H = \langle d \langle a', b' \rangle, c \rangle$. We set $R = k[[H]]$, $H_1 = \langle a', b' \rangle$ and $R_1 = k[[H_1]]$. Then the following assertions hold true.*

(1) *If d and c are odd, then*

$$\chi_R^g = \{(t^{d\alpha}, t^{d\beta}) \mid \alpha, \beta \in H_1 \text{ such that } (t^\alpha, t^\beta) \in \chi_{R_1}^g\}.$$

In particular, $\#\chi_R^g = \#\chi_{R_1}^g$.

(2) *If a, b and c are odd, then $\chi_R^g = \emptyset$.*

In the following, we assume that a' and b' are odd.

(3) *If d is odd and c is even, then*

(i) *$\chi_R^g \neq \emptyset$ if and only if $H + \langle c/2 \rangle$ is symmetric.*

(ii) *if* $\chi_R^g \neq \emptyset$, *then*

$$\chi_R^g = \{(t^{\frac{c}{2}l}, t^{\frac{c}{2}d}) \mid \ l \text{ is even with } \ 1 < l < d\}.$$

In particular, $\#\chi_R^g = (d-1)/2$.

(4) *If* d *is even and* c *is odd, then* $\chi_R^g \neq \emptyset$.

We provide some lemmas to prove this.

Definition 3.4. [RG] For two numerical semigroups $H_1 = \langle a_1, ..., a_m \rangle$ and $H_2 = \langle b_1, ..., b_n \rangle$, we define a *gluing* of H_1 and H_2 as follows:

$$\langle d_1 H_1, d_2 H_2 \rangle := \langle d_1 a_1, ..., d_1 a_m, d_2 b_1, ..., d_2 b_n \rangle,$$

where $d_1 \in H_2 \backslash \{b_1, ..., b_n\}, d_2 \in H_1 \backslash \{a_1, ..., a_m\}$ and $\gcd(d_1, d_2) = 1$.

By the constructions of gluings, we have the following result.

Proposition 3.5. *Let* $H = \langle d_1 H_1, d_2 H_2 \rangle$ *be a gluing of two numerical semigroups* H_1 *and* H_2. *Then the ring* $k[[H]]$ *is a* $k[[H_1]]$ *(resp.* $k[[H_2]]$*) - free module of rank* d_1 *(resp.* d_2*).*

We say that a numerical semigroup H is a *complete intersection* if its semigroup ring $k[[H]]$ is a complete intersection.

Theorem 3.6. [De], [RG] *The following assertions hold true.*

(1) *Let* $H = \langle d_1 H_1, d_2 H_2 \rangle$ *be a gluing of two numerical semigroups* H_1 *and* H_2. *Then* H *is symmetric (resp. a complete intersection) if and only if* H_1 *and* H_2 *are symmetric (resp. complete intersections).*

(2) *A numerical semigroup other than* $\mathbb{N}$ *is a complete intersection if and only if it is a gluing of two complete intersection numerical semigroups.*

Remark 3.7. When a numerical semigroup H is generated by 3-elements, H is symmetric if and only if H is a complete intersection. Therefore Lemma 3.1 is a special case of Theorem 3.6 (2) since $H = \langle a, b, c \rangle = \langle d \langle a', b' \rangle, c \rangle$ is a gluing of $\langle a', b' \rangle$ and $\langle 1 \rangle = \mathbb{N}$.

Lemma 3.8. *Let* $H = \langle d_1 H_1, d_2 H_2 \rangle$ *be a gluing of two symmetric numerical semigroups* $H_1 = \langle a_1, ..., a_m \rangle$ *and* $H_2 = \langle b_1, ..., b_m \rangle$. *We put* $R = k[[H]]$ *and* $R_i = k[[H_i]]$ *for* $i = 1, 2$. *The following assertions hold true for* $i = 1, 2$.

(1) *If* $(t^\alpha, t^\beta) \in \chi_{R_1}^g$, *then* $(t^{d_i \alpha}, t^{d_i \beta}) \in \chi_{R_i}^g$.

(2) *If* $(t^\gamma, t^\delta) \in \chi_R^g$ *and* d_i *divides* $x := \delta - \gamma > 0$, *then there exists two integers* $\alpha, \beta \in H_i$ *with* $c/d_i = \beta - \alpha > 0$ *such that* $(t^\alpha, t^\beta) \in \chi_{R_i}^g$.

(3) $\#\chi_{R_i}^g \leq \#\chi_R^g$.

Proof. (1) There is an natural injection from R_1 to R:

$$\begin{array}{cccc} \phi: & R_1 & \hookrightarrow & R \\ & t^{a_j} & \longmapsto & t^{d_1 a_j} \end{array}$$

We note that $R \cong R_1^{\oplus d_1}$ by Proposition 3.5. If $I \subset R_1$ is an Ulrich ideal, then

$$\mathrm{e}(IR) = d_1 e(I) = d_1 \mu_{R_1}(I)\ell_{R_1}(R_1/I) = \mu_R(IR)\ell_R(R/IR),$$

where IR is the extension of I from R_1 to R (see [GOTWY, Lemma 2.3.]).

(2) This easily follows form Theorem 2.1.

(3) This is clear by (1) and (2). □

Example 3.9. (1) Let $H_1 = \langle 4, 5\rangle$ and $H_2 = \mathbb{N}$. We know that

$$\chi_{R_1}^g = \{(t^8, t^{10}), (t^4, t^{10})\}$$

and $\chi_{R_2}^g = \emptyset$ (see Example 2.2). Let $H = \langle 3H_1, 13H_2\rangle = \langle 12, 13, 15\rangle$ be a gluing of H_1 and H_2. By Theorem 2.1, we can check that

$$\chi_R^g = \{(t^{24}, t^{30}), (t^{12}, t^{30})\}.$$

In this case, there is a one-to-one correspondence between the sets $\chi_{R_1}^g$ and χ_R^g. In other word, all Ulrich ideals of R are extensions from those of R_1. This example illustrate Theorem (1) since 3 and 13 are odd.

(2) Let H_1 and H_2 be as above and let $H = \langle 3H_1, 16H_2\rangle$ be a gluing of H_1 and H_2. Then we see that

$$\chi_R^g = \{(t^{24}, t^{30}), (t^{12}, t^{30}), (t^{16}, t^{24}), (t^{16}, t^{30})\}.$$

In this case, the ideals (t^{16}, t^{24}) and (t^{16}, t^{30}) are not extensions from those of R_1.

By using Bresinsky's results in [Br], we have the following.

Lemma 3.10. *Let $H = \langle a, b, c\rangle$ be a symmetric numerical semigroup. If $H + \langle x\rangle$ is symmetric for an integer $x \in \mathbb{Z}$ such that $x \notin H$ and $2x \in H$, then $H + \langle x\rangle$ is a complete intersection.*

Using Lemma 3.10, we can prove the following lemma.

Lemma 3.11. *Let $H = \langle a, b, c\rangle = \langle d\langle a', b'\rangle, c\rangle$ be a symmetric numerical semigroup. Suppose that $S = H + \langle x\rangle$ is symmetric for an integer $x \in \mathbb{Z}$ such that $x \notin H$ and $2x \in H$. We write as $2x = \lambda_1 a + \lambda_2 b + \lambda_3 c$, where $\lambda_1, \lambda_2, \lambda_3 \geq 0$. If $\lambda_3 > 0$, and $\lambda_1 > 0$ or $\lambda_2 > 0$, then the following statements hold.*

(1) $\mathrm{emb}(S) > 2$.

(2) *a or b is even, and c is even.*

Now we give the proof of Theorem 3.9.

Proof of Theorem 3.9. (1) We assume that $H + \langle x \rangle$ is symmetric for an integer x such that $x \notin H$ and $2x \in H$. Then it suffices to prove that d divides x by Lemma 3.8. Since $2x \in H$, we can write as $2x = \lambda_1 a + \lambda_2 b + \lambda_3 c$, where $\lambda_1, \lambda_2, \lambda_3 \geq 0$. If $\lambda_3 = 0$, then d divides x since d is odd, and so we are done. Therefore we assume that $\lambda_3 > 0$. Then it must be $\lambda_1 > 0$ or $\lambda_2 > 0$ since otherwise we see that x is a multiple of c, which is a contradiction. By Lemma 3.11, c is even, which contradict to our assumption. Hence we obtain $\lambda_3 = 0$.

(2) This follows from (1) and Theorem 2.1.

(3) First, we claim that if $H + \langle x \rangle$ is symmetric for an integer $x \in \mathbb{Z}$ such that $x \notin H$ and $2x \in H$, then $x = \lambda c/2$, where λ is an odd positive integer. We write as $2x = \lambda_1 a + \lambda_2 b + \lambda_3 c$, where $\lambda_1, \lambda_2, \lambda_3 \geq 0$. If $\lambda_3 = 0$, then d divides x, which is a contradiction. Hence we have $\lambda_3 > 0$. And if $\lambda_1 > 0$ or $\lambda_2 > 0$, then we see that either a or b is even by Lemma 3.11. This is contradict to our assumption, and hence $\lambda_1 = \lambda_2 = 0$. Therefore we have $x = \lambda_3 c/2$, where λ_3 is odd.

Now we prove the first assertion (i). We assume that $H + \langle c/2 \rangle$ is not symmetric and $\chi^g_{k[[H]]} \neq \emptyset$. Then

$$\frac{c}{2} \notin \langle a', b' \rangle . \tag{3.1}$$

since otherwise $H + \langle c/2 \rangle = \langle d \langle a', b' \rangle, c/2 \rangle$ which is symmetric. By our assumption, there is an integer $x \in \mathbb{Z}$ such that $x \notin H$, $2x \in H$ and $H + \langle x \rangle$ is symmetric. We can write as $x = \lambda c/2$ for some odd integer $\lambda > 0$ by the claim in previous paragraph. If $\lambda = 1$, then $H + \langle x \rangle$ never be symmetric. Hence we have $\lambda \geq 3$. Then it is easily seen that $H + \langle x \rangle = \langle a, b, c, \lambda c/2 \rangle$ is generated by 4-elements, and which is symmetric. Therefore this numerical semigroup is a complete intersection by Lemma 3.10. Hence we can write as

$$H + \langle x \rangle = \left\langle d \langle a', b' \rangle, \frac{c}{2} \langle 2, \lambda \rangle \right\rangle$$

since we know that both $\langle a', b' \rangle$ and $\langle 2, \lambda \rangle$ are complete intersections. This contradict to the condition (3.1). This complete the proof of (i). The second assertion (ii) easily follows from the previous arguments.

(4) By using Lemma 3.1, we can check that $H + \langle da'/2 \rangle$ or $H + \langle db'/2 \rangle$ is symmetric. Hence $\chi^g_{k[[H]]} \neq \emptyset$ by Theorem 2.1. $\square$

4 The case of $H = \langle a, a+d, ..., a+nd \rangle$

We say that a numerical semigroup H is generated by an *arithmetic sequence* if $H = \langle a, a+d, ..., a+nd \rangle$, where $a, d > 0, n \geq 2$ and $\gcd(a, d) = 1$. The following is the main result about this.

Theorem 4.1. [Nu] *Let $H = \langle a, a+d, ..., a+nd \rangle$ be a symmetric numerical semigroup generated by an arithmetic sequence. Then $\chi^g_{k[[H]]} \neq \emptyset$ if and only if $n = 2$.*

When $H = \langle a, a+d, ..., a+nd\rangle$ is a symmetric numerical semigroup generated by an arithmetic sequence, H is a complete intersection if and only if $n = 2$ (for example, see [GSS]). So we had expected that if $k[[H]]$ is a Gorenstein numerical semigroup ring which is not a complete intersection, then $\chi^g_{k[[H]]} = \emptyset$. But, unfortunately, there are counter examples.

Example 4.2. A numerical semigroup $H = \langle 10, 12, 13, 14, 15\rangle$ is symmetric but not a complete intersection. However $H + \langle 5\rangle = \langle 5, 12, 13, 14\rangle$ is symmetric, and hence the ideal $(t^{10}, t^{15}) \in \chi^g_{k[[H]]}$. In general, $H_m = \langle 2m, 2m+2, 2m+3, ..., 3m\rangle$ is symmetric but not a complete intersection if $m \geq 5$. Then we can check that $(t^{2m}, t^{3m}) \in \chi^g_{k[[H_m]]}$.

References

[Br] H. Bresinsky, *Symmetric semigroups of integers generated by 4 elements*, Manuscripta Math. **17** (1975), 205-219.

[De] D. Delorme, *Sous-monoïdes d'intersection complète de* $\mathbb{N}$, Ann. Sci. Ècole Norm. Sup. **9** (1976), 145-154.

[GSS] P. Gimenez, I. Sengupta and H. Srinivasan, *Minimal graded free resolutions for monomial curves defined by arithmetic sequences*, J. Algebra, **388** (2013) 294-310.

[GOTWY] S. Goto, K. Ozeki, R. Takahashi, K-.i. Watanabe, K. Yoshida, *Ulrich ideals and modules*, arXiv: 1206.3197 (2012).

[GW] S. Goto, K.-i. Watanabe, *On graded rings*, J. Math. Soc. Japan **30** (1978), 172-213.

[He] J. Herzog, *Generators and relations of abelian semigroups and semigroup rings*, Manuscripta Math. **3** (1970) 175-193.

[RG] J. C. Rosales and P. A. García-Sánchez, *Numerical semigroups*, Springer Developments in Mathematics, Volume **20** (2009).

[Nu] T. Numata, *Numerical semigroups generated by arithmetic sequences*, to appear in Proceedings of the Institute of Natural Sciences, Nihon University **49** (2014).

[Wa] K.-i. Watanabe, *Some examples of one dimensional Gorenstein domains*, Nagoya Math. J. **49** (1973), 101-109.