Almost purity theorem and the homological conjectures *

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1 Homological conjectures

The homological conjectures are a string of statements that describe certain phenomena of (local) Noetherian rings and modules over them (see [5] for details and its history). These questions were suggested and studied extensively in the 1970's by many researchers. Those pioneers include Serre, Bass, Peskine, Szpiro, Hochster, Roberts and others. Among them, M. Hochster was leading this area and published many interesting results that stemmed from the conjectures. One of the simplest forms of the homological conjectures is the direct summand conjecture. It is formulated as follows.

Conjecture 1 (Direct Summand Conjecture)**.** *Assume that R is a Noetherian regular domain and $R \to S$ is a torsion free module-finite extension. Then $R \to S$ splits as an R-module.*

Since the splitting question is local, we may localize $R \to S$ at a prime ideal of R so that we have a module-finite map $R \to S$, where R is a regular local ring. Furthermore, we may assume that $R \to S$ is module-finite and S is a domain. The direct summand conjecture is easy to prove, if R contains a field of characteristic zero. We sketch its proof. Let $\mathrm{Tr} : \mathrm{Frac}(S) \to \mathrm{Frac}(R)$ be the trace map in the field theory. Then since R is regular, it induces a map $\mathrm{Tr} : S \to R$. Let $d = [\mathrm{Frac}(S) : \mathrm{Frac}(R)]$. Then $\frac{1}{d}\mathrm{Tr} : S \to R$ gives a splitting to the R-module map $R \to S$. If we analyze this proof carefully, the proof works even when R is an integrally closed domain. In positive characteristic, Hochster gave a proof in [4] using the Frobenius map which is a ring homomorphism sending an element to its p-th power. Peskine and Szpiro also made an effective use of the Frobenius map. In [4], he formulated the monomial conjecture which is equivalent to the direct summand conjecture. The direct summand conjecture is still open when the

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local ring has mixed characteristic and the dimension at least 4. Hochster realized that the homological conjectures follow from the existence of big Cohen-Macaulay modules (algebras).

Definition 2. Let (R, m) be a local Noetherian ring and let B be an R-module (algebra). Then B is a *balanced big Cohen-Macaulay R-module (algebra)*, if every system of parameters of R is a B-regular sequence and $B \neq mB$.

We should emphasize that the point of this definition is to allow B to be non-Noetherian or non-finitely generated over R. When B is taken to be a finitely generated R-module, then B is said to be a *small Cohen-Macaulay R-module*. It looks a bit suspicious that every complete local ring has a small Cohen-Macaulay module. Moreover, it is unknown if every local ring of mixed characteristic has a big Cohen-Macaulay algebra. The study of the mixed characteristic case will be our main theme in this article, and we address some topics and techniques that are related to problems in mixed characteristic case. At the current status, the following theorem is known ([8] for details).

Theorem 3. *Let R be a regular local ring of mixed characteristic $p > 0$ and let S be a torsion free module-finite R-algebra such that the localization $R[\frac{1}{p}] \to S[\frac{1}{p}]$ is finite étale. Then S has a balanced big Cohen-Macaulay algebra.*

The proof of this theorem is based on the almost purity theorem. We have the following corollary.

Corollary 4. *Let R be a Noetherian regular domain and let S be a torsion free module-finite R-algebra. Assume that R is p-torsion free and the localization $R[\frac{1}{p}] \to S[\frac{1}{p}]$ is finite étale for some prime integer $p > 0$. Then $R \hookrightarrow S$ splits as an R-module homomorphism.*

2 Almost ring theory

The idea of almost ring theory first appeared in Faltings' proof of the p-adic comparison theorem between the p-adic étale cohomology and the de Rham cohomology for a smooth proper scheme X over a p-adic field K via Fontaine's functor. More precisely, there is a canonical isomorphism of G_K-representations:

$$\mathbf{C}_K \otimes_{\mathbf{Q}_p} H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \cong \bigoplus_q (\mathbf{C}_K(-q) \otimes_K H^{n-q}(X, \Omega^q_{X/K})).$$

Faltings deduced this isomorphism from the almost purity theorem. At the same time, Heitmann [3] proved the direct summand conjecture in dimension 3 in a similar spirit to the almost purity theorem. To state the almost purity theorem, we introduce notation. Let (V, pV, k) be an unramified complete discrete valuation ring of mixed characteristic with perfect residue field k. Let $R := V[[x_2, \dots, x_d]]$. Then we may form an increasing chain of regular local rings:

$$R_0 := R \to R_1 \to R_2 \to \cdots \to R_n \cdots$$

such that $R_n = V[p^{\frac{1}{p^n}}][[x_2^{\frac{1}{p^n}}, \dots, x_d^{\frac{1}{p^n}}]]$. Now we put

$$R_\infty := \bigcup_{n>0} R_n.$$

From the construction, the Frobenius endomorphism on R_∞/pR_∞ is surjective with kernel equal to $p^{\frac{1}{p}}R_\infty/pR_\infty$, and $R \to R_\infty$ is a faithfully flat integral extension. Finally, R_∞ is a coherent domain. The following is a weak version of the almost purity theorem [1].

Theorem 5 (Faltings)**.** *Assume that B is an integral extension domain of R_∞ such that B is normal and $R_\infty[\frac{1}{p}] \to B[\frac{1}{p}]$ is finite étale. Then the Frobenius endomorphism on B/pB is surjective and $R_\infty \to B$ is an almost flat extension.*

We will later explain almost flat extensions. Let B be an integral domain that comes equipped with a function $v : B \to \mathbf{R} \cup \{\infty\}$ such that

1. $v(ab) = v(a) + v(b)$ for $a, b \in B$.
2. $v(a + b) \geq \max\{v(a), v(b)\}$ for $a, b \in B$.
3. $v(b) \geq 0$ for all $b \in B$.
4. $v(b) = \infty \iff b = 0$.

Definition 6. Let the notation be as above. Then a B-module M is called *almost zero*, if for any $m \in M$ and $\epsilon > 0$, there exists an element $b \in B$ such that $v(b) < \epsilon$ and $b \cdot m = 0$.

Notice that the above definition of almost zero modules is slightly weaker than the one given in the book of Gabber and Ramero [2], in which one starts with a valuation ring (V, m) such that $m^2 = m$ and say that a V-module M is almost zero (in the sense of Gabber-Ramero) if $mM = 0$. There are another possible definitions of almost zero modules and it depends on the situation, but we will employ the above definition for the applications to the homological conjectures. It is easy to prove the following properties.

1. If $0 \to L \to M \to N \to 0$ is a short exact sequence of B-modules, then M is almost zero if and only if both L and N are almost zero.

2. The direct limit of almost zero modules is almost zero.

Definition 7. We say that a B-module M is *almost flat*, if $\operatorname{Tor}_i^B(M, N)$ is almost zero for any B-module N and $i > 0$.

Let (R, m) be a complete local domain. Then we propose the following conjecture.

Conjecture 8. *Let R^+ be the integral closure of R in a fixed algebraic closure of the field of fractions of R. Then the local cohomology module $H_m^i(R^+)$ is an almost zero R^+-module for all $0 \leq i \leq \dim R - 1$.*

Heitmann's proof of the direct summand conjecture in dimension 3 shows that, if Conjecture 8 is true, then the direct summand conjecture follows in mixed characteristic and we refer the reader to [6] for the proof. One might wonder that $H_m^i(R^+) = 0$ for $0 \leq i \leq \dim R - 1$. However, this is not true. If R is a local domain of dimension 3 and contains the field of rationals $\mathbf{Q}$, then it is shown that $H_m^2(R^+) \neq 0$. Indeed, it is easy to construct an example of a 3-dimensional local domain that is normal, but not Cohen-Macaulay. Call such a ring R. Then by the trace map argument, it is shown that the natural inclusion $R \hookrightarrow R^+$ splits. Hence R^+ cannot be a big Cohen-Macaulay R-algebra.

3 Fontaine rings and the purity theorem

We introduce the Fontaine rings and prove the purity theorem in mixed characteristic. This is the easiest version of the almost purity theorem and does not require almost ring theory at all. Its proof is to reduce the problem in mixed characteristic to the problem in positive characteristic via Fontaine rings and then study the Frobenius action on the differential modules for perfect rings. This proof is a simple exercise using the Fontaine rings and the author believes that this is a good place to demonstrate its usefulness to make readers acquainted with the philosophy of almost ring theory.

Definition 9 (Fontaine ring)**.** Let A be a ring and let $p > 0$ be a prime integer. Then we define

$$\mathbf{E}(A) := \varprojlim_{n \in \mathbf{N}} A_n,$$

where $A_n := A/pA$ for all n and the map $A_{n+1} \to A_n$ is the Frobenius map. The ring structure of $\mathbf{E}(A)$ is induced by the ring structure of A.

An element of $\mathbf{E}(A)$ is of the form $a = (a_n \mid a_n \in A/pA)$ with $a_{n+1}^p = a_n$. We denote by $\langle p \rangle$ the element $(p, p^{\frac{1}{p}}, p^{\frac{1}{p^2}}, \ldots) \in \mathbf{E}(A)$. It is easy to see that $\mathbf{E}(A)$ is a perfect ring of characteristic p and there is a natural map:

$$\Phi_A : \mathbf{E}(A) \to A/pA$$

defined by $\Phi_A(a_0, a_1, a_2, \ldots) = a_0$. This is a ring homomorphism. The following lemma is immediate.

Lemma 10. *If the Frobenius endomorphism on A/pA is surjective, then Φ_A is also surjective.*

If d is a derivation on a ring A of characteristic $p > 0$, then we have $dx^p = px^{p-1}dx = 0$. Using this fact, we have the following fact.

Lemma 11. *Assume that $R \to S$ is a ring homomorphism of perfect rings of characteristic $p > 0$. Then we have*

$$\Omega^1_{S/R} = 0.$$

Proof. By assumption, the relative Frobenius map: $S \otimes_R R^{(1)} \to S^{(1)}$ defined by $s \otimes r \mapsto s^p r$ is an isomorphism. Then this induces an isomorphism on differential modules:

$$\Omega^1_{S/R} \otimes_R R^{(1)} \cong \Omega^1_{S^{(1)}/R^{(1)}}$$

Then the relative Frobenius map maps dx to $dx^p = 0$. Hence we have $\Omega^1_{S/R} = 0$, as required. $\square$

We have constructed a big ring R_∞ in the previous section. The Fontaine ring $\mathbf{E}(R_\infty)$ maps onto R_∞/pR_∞, because the Frobenius endomorphism on R_∞/pR_∞ is surjective. Let $R_\infty \to S_\infty$ be a torsion free module-finite extension of normal domains.

Proposition 12. *Assume further that the Frobenius endomorphism on S_∞/pS_∞ is surjective. Then*

$$\Omega^1_{S_\infty/R_\infty} = 0.$$

Proof. We put $\overline{R}_\infty := R_\infty/pR_\infty$ and $\overline{S}_\infty := S_\infty/pS_\infty$. Then there is a commutative diagram whose rows are short exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{E}(R_\infty) & \xrightarrow{\langle p\rangle} & \mathbf{E}(R_\infty) & \xrightarrow{\Phi_{R_\infty}} & R_\infty/pR_\infty & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{E}(S_\infty) & \xrightarrow{\langle p\rangle} & \mathbf{E}(S_\infty) & \xrightarrow{\Phi_{S_\infty}} & S_\infty/pS_\infty & \longrightarrow & 0
\end{array}$$

Indeed, both Φ_{R_∞} and Φ_{S_∞} are surjective by assumption ([7] for the exactness at other spots). By this diagram and Lemma 11, we have $\Omega^1_{\overline{S}_\infty/\overline{R}_\infty} = 0$, because the Fontaine rings are perfect. On the other hand,

$$\Omega^1_{S_\infty/R_\infty} \otimes_{R_\infty} R_\infty/pR_\infty \cong \Omega^1_{\overline{S}_\infty/\overline{R}_\infty}.$$

Since $R_\infty \to S_\infty$ is a torsion free module-finite extension and R_∞ is coherent, it is a finitely presented extension. So $\Omega^1_{S_\infty/R_\infty}$ is a finitely generated R_∞-module. Hence we have $\Omega^1_{S_\infty/R_\infty} = 0$ by Nakayama's lemma. □

Now assume that $R_\infty \to S_\infty$ is a generically finite, torsion free integral extension of normal domains. Then we have $\mathrm{Frac}(S_\infty) = \mathrm{Frac}(R_\infty)[\alpha]$ for some $\alpha \in \mathrm{Frac}(S_\infty)$. Let K_n denote the field of fractions of R_n, let $L_n := K_n[\alpha]$ and let S_n be the integral closure of R in L_n. Then since R_n is a complete local domain, S_n is also a complete local domain that is module-finite over R_n. However, $R_\infty \to S_\infty$ is not necessarily module-finite.

Theorem 13 (Purity theorem)**.** *Let the notation be as above. Then the following statements are equivalent:*

1. *The extension $R_n \to S_n$ is finite étale for $n \gg 0$.*
2. *The extension $R_\infty \to S_\infty$ is finite étale.*
3. *$R_\infty \to S_\infty$ is module-finite and the Frobenius endomorphism on S_∞/pS_∞ is surjective.*

Proof. *1 ⇒ 2* We have that $S_n \otimes_{R_n} R_\infty$ is an integral normal domain for $n \gg 0$ by linearly disjoint property. Hence we have $S_\infty \cong S_n \otimes_{R_n} R_\infty$ and thus, $R_\infty \to S_\infty$ is finite étale by base change.

2 ⇒ 3 This is immediate from the fact that the relative Frobenius map induced on the ring extension $R_\infty \to S_\infty$ is an isomorphism and the Frobenius endomorphism on R_∞/pR_∞ is surjective ([2], Theorem 3.5.13).

3 ⇒ 1 By Proposition 12, we have $\Omega^1_{S_\infty/R_\infty} = 0$. Note that $S_\infty \cong S_n \otimes_{R_n} R_\infty$ for $n \gg 0$. Then since R_∞ is faithfully flat over R_n, it follows that

$$\Omega^1_{S_\infty/R_\infty} \cong \Omega^1_{S_n \otimes_{R_n} R_\infty/R_\infty} \cong \Omega^1_{S_n/R_n} \otimes_{R_n} R_\infty$$

and thus, $\Omega^1_{S_n/R_n} = 0$ for $n \gg 0$. This implies that $R_n \to S_n$ is an unramified local map of local rings. Then since R_n is regular, S_n is also regular, form which the flatness of $R_n \to S_n$ is clear. Therefore, $R_n \to S_n$ is finite étale. □

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