Annihilation of Ext modules and generation of derived categories

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1. Introduction

This article is based on joint work with Srikanth Iyengar [16].

Uniform annihilation of Ext modules has been studied in commutative algebra since the 1980s. Let R be a d-dimensional Cohen–Macaulay complete local ring with maximal ideal $\mathfrak{m}$ and perfect coefficient field. Dieterich [13] and Yoshino [29] independently studied the Brauer–Thrall conjectures for maximal Cohen–Macaulay modules, and it was shown that if R is an isolated singularity, then the ideal

$$\mathsf{ann}_R \mathsf{Ext}^1_R(\mathsf{CM}(R), \mathsf{mod}\, R) = \bigcap_{\substack{M \in \mathsf{CM}(R) \\ N \in \mathsf{mod}\, R}} \mathsf{ann}_R \mathsf{Ext}^1_R(M, N)$$

is $\mathfrak{m}$-primary, where $\mathsf{mod}\, R$ denotes the category of finitely generated R-modules and $\mathsf{CM}(R)$ denotes the category of maximal Cohen–Macaulay R-modules. In the 1990s, Popescu and Roczen [23] further studied the Brauer–Thrall conjectures, and proved that the above ideal defines the singular locus of R, namely, they removed the assumption of an isolated singularity from the result of Dieterich and Yoshino. Later on, Wang [28] investigated uniform annihilation in more general cases; he proved that the above ideal contains the Jacobian ideal of R. Recently, Buchweitz and Flenner [7] explored uniform annihilation over noncommutative rings; in several cases they determined the annihilator $\mathsf{ann}_{\mathsf{Z}(\Lambda)} \underline{\mathsf{CM}}(\Lambda)$ of the stable category of maximal Cohen–Macaulay modules over a Gorenstein order Λ over a regular local ring, which is equal to $\mathsf{ann}_{\mathsf{Z}(\Lambda)} \mathsf{Ext}^1_\Lambda(\mathsf{CM}(\Lambda), \mathsf{CM}(\Lambda))$.

Strong finite generation of triangulated categories has been studied widely and deeply so far since around the end of last century. Bondal and Van den Bergh [6] introduced this notion, and investigated representability of functors; they proved that the bounded derived category of coherent sheaves on a smooth proper variety is strongly finitely generated, and that every cohomological functor of finite type on it is representable. Rouquier [24, 25] defined the dimension of a triangulated category which refines strong finite generation. He proved that the bounded derived category of coherent sheaves on a separated scheme of finite type over a field is strongly finitely generated, or equivalently, has finite dimension. Using the dimension of a derived category, he constructed an example of an artin algebra of representation dimension more than 3 for the first time. Lower and upper bounds for the dimensions of derived categories were explored by many authors; see [1, 2, 3, 4, 5, 11, 19] and references therein. Some analogues of the notion of dimension for abelian categories were also given in [10, 12].

The purpose of the present article is to analyze uniform annihilation of Ext modules and investigate strong finite generation of derived categories. To state our main results we give here the precise definition of uniform annihilation. A noetherian ring Λ is said to have the *uniform annihilator property* (*UAP* for short) if there exist a nonnegative integer n and a nonzerodivisor of the center $\mathsf{Z}(\Lambda)$ annihilating $\mathsf{Ext}^n_\Lambda(\mathsf{mod}\, \Lambda, \mathsf{mod}\, \Lambda)$. A trivial example of a ring having the UAP is a ring of finite global dimension.

First, we make fundations of the UAP of a commutative ring. We observe that a commutative ring having the UAP is always reduced, and obtain that:

Theorem 1.1. *A reduced noetherian ring has the UAP if and only if so do the residue rings by minimal primes.*

[♣] This article is an announcement of our results and the detailed version [16] will be submitted in another journal

As a consequence, all Stanley–Reisner rings have the UAP. Also, this theorem says that to consider the UAP of a commutative ring one may assume that it is a domain. We relate the UAP to separability of extensions of the quotient fields by using Wang's results, and show that all reduced complete local rings with perfect coefficient field have the UAP.

Secondly, we consider descent of the UAP from a noncommutative ring to a commutaive ring. Our main result in this direction implies that every positively dimensional Cohen–Macaulay local ring of finite CM-representation type has the UAP. Moreover, it yields the following theorem.

Theorem 1.2. *Let R be a commutative noetherian ring. Let Λ be a module-finite R-algebra. Suppose that the image of R in Λ is contained in $\mathsf{Z}(\Lambda)$ and that Λ has positive rank as an R-module. If Λ has the UAP, then so does R.*

Consequently, a domain admitting a noncommutative resolution in the sense of [9] has the UAP. In particular, any 1-dimensional reduced Nagata (e.g. excellent) ring has the UAP, which extends a result of Wang [28]. It also follows that every quotient singularity has the UAP. On the other hand, we investigate ascent of the UAP, and show that the UAP ascends from a commutative ring to the endomorphism ring of a torsionfree module having positive rank.

Thirdly, we explore the UAP of a ring that is essentially of finite type over a field. We shall get the following theorem.

Theorem 1.3. *A reduced ring essentially of finite type over a field has the UAP.*

The case where the base field is perfect is handled by using work of Dao and Takahashi [10] on the rank of a certain subcategory of modules. The general case is shown by establishing a module category version of a result of Bondal, Keller and Van den Bergh [6] on compact objects in triangulated categories and utilizing the techniques of Keller and Van den Bergh [18] on strong finite generation of derived categories.

Fourthly, we investigate the relationship between the UAP and strong finite generation of derived categories. Our main result in this direction is a structure theorem of the bounded derived category of an abelian category. Applying it to the abelian category $\mathsf{mod}\,\Lambda$ yields the following theorem.

Theorem 1.4. *The following are equivalent for a noetherian ring Λ.*

(1) *Λ has the UAP.*
(2) *There exist a nonzerodivisor $r \in \mathsf{Z}(\Lambda)$ and $n \geq 0$ with $\mathsf{D}^{\mathrm{b}}(\Lambda) = \mathcal{T}_r \diamond \langle \Lambda \rangle_n$.*
(3) *There exist a nonzerodivisor $r \in \mathsf{Z}(\Lambda)$ and $n \geq 0$ with $\mathsf{D}^{\mathrm{b}}(\Lambda) = \langle \Lambda \rangle_n \diamond \mathcal{T}_r$.*
(4) *There exist a nonzerodivisor $r \in \mathsf{Z}(\Lambda)$ and $n \geq 0$ such that r kills the n-ghosts.*

Here, $\mathsf{D}^{\mathrm{b}}(\Lambda)$ is the bounded derived category of $\mathsf{mod}\,\Lambda$ and $\mathcal{T}_r$ is the full subcategory of $\mathsf{D}^{\mathrm{b}}(\Lambda)$ consisting of objects on which the multiplication morphism by r vanishes. As an application of this theorem, it is observed that for a commutative noetherian ring R of finite Krull dimension $\mathsf{D}^{\mathrm{b}}(R)$ is strongly finitely generated if $R/\mathfrak{p}$ has the UAP for all nonmaximal prime ideals $\mathfrak{p}$. This result recovers both a theorem of Keller, Rouquier and Van den Bergh [18, 25] and a theorem of Aihara and Takahashi [2], asserting that $\mathsf{D}^{\mathrm{b}}(R)$ is strongly finitely generated if R is either a ring essentially of finite type over a field, or a complete local ring with perfect coefficient field.

Convention. Throughout this article, we assume that all rings are right noetherian rings with identity. Unless otherwise specified, Λ is a ring, R is a commutative ring, k is a field and a Λ-module means a right Λ-module. Let $\mathsf{Z}(\Lambda)$, $\mathsf{Mod}\,\Lambda$ and $\mathsf{mod}\,\Lambda$ denote the center of Λ, the category of all Λ-modules and the category of finitely generated Λ-modules, respectively. We say that Λ is an *R-algebra* if there is a ring homomorphism $R \to \Lambda$ whose image is contained in $\mathsf{Z}(\Lambda)$. We denote the total quotient ring of R by $\mathsf{Q}(R)$. For a finitely generated Λ-module M and an integer $n \geq 0$, we denote by $\Omega^n_\Lambda M$ the n-th syzygy of M. All subcategories are assumed to be full and strict (i.e. closed under isomorphism). For an additive category $\mathcal{C}$ and a subcategory $\mathcal{X}$ of $\mathcal{C}$, denote by $\mathsf{add}_{\mathcal{C}}\,\mathcal{X}$ the additive closure of $\mathcal{X}$, that is, the smallest subcategory of $\mathcal{C}$ containing $\mathcal{X}$ and closed under finite direct sums and direct summands. When $\mathcal{X}$ consists of a single object G, we simply write $\mathsf{add}_{\mathcal{C}}\,G$. For $\mathcal{C} = \mathsf{mod}\,\Lambda$ we use the notations $\mathsf{add}_\Lambda\,\mathcal{X}$ and $\mathsf{add}_\Lambda\,G$. Subscripts and superscripts are often omitted if there is no danger of confusion.

2. The uniform annihilator property

In this section, we make several fundamental observations on the uniform annihilator property. Let us begin with giving its definition.

Definition 2.1. We say that Λ has the *uniform annihilator property*, *UAP* for short, if there exists an integer $n \geq 0$ and a nonzerodivisor $a \in \mathsf{Z}(\Lambda)$ such that

$$a \operatorname{Ext}^n_\Lambda(M, N) = 0 \tag{2.1.1}$$

for all finitely generated Λ-modules M and N. Such an element a is called a *uniform annihilator* for Λ.

The condition (2.1.1) means that the element a annihilates the functor $\operatorname{Ext}^n_\Lambda(-,-)$ defined on the category $(\operatorname{mod}\Lambda)^{\mathsf{op}} \times \operatorname{mod}\Lambda$. We often write $a \operatorname{Ext}^n_\Lambda(\operatorname{mod}\Lambda, \operatorname{mod}\Lambda) = 0$.

Remark 2.2. (1) Let $a \in \mathsf{Z}(\Lambda)$ and $n \geq 0$ be such that $a \operatorname{Ext}^n_\Lambda(\operatorname{mod}\Lambda, \operatorname{mod}\Lambda) = 0$. Then it holds that $a \operatorname{Ext}^{\geq n}_\Lambda(\operatorname{mod}\Lambda, \operatorname{mod}\Lambda) = 0$.
(2) All rings of finite global dimension have the UAP.

Every local ring of depth zero that is not a field does not have the UAP. Even a local ring of positive depth does not necessarily have the UAP:

Example 2.3. Let $R = k[[x, y]]/(x^2)$. Then the R-module $R/(x)$ has a free resolution

$$\cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \to R/(x) \to 0.$$

It is easy to see from this that $\operatorname{Ext}^i_R(R/(x), R/(x)) \cong R/(x)$ for all $i \geq 0$. Therefore, every element $a \in R$ with $a \operatorname{Ext}^n_R(\operatorname{mod} R, \operatorname{mod} R) = 0$ for some $n \geq 0$ belongs to the ideal (x). Hence $a^2 = 0$, and a cannot be a nonzerodivisor. Thus R does not have the UAP.

In fact, for a commutative ring the UAP forces it to be reduced:

Proposition 2.4. *Let Λ be a module-finite R-algebra having positive rank as an R-module. Assume that for some nonzerodivisor x of R and some integer $n \geq 0$ one has either $x \operatorname{Ext}^n_R(\operatorname{mod}\Lambda, \operatorname{mod} R) = 0$ or $x \operatorname{Ext}^n_\Lambda(\operatorname{mod}\Lambda, \operatorname{mod}\Lambda) = 0$. Then R is a reduced ring. In particular, every commutative ring having the UAP is reduced.*

Remark 2.5. Our definition requires a uniform annihilator to be a nonzerodivisor. If we replace "nonzerodivisor" with "nonzero element", nonreduced rings can possess such an element. In fact, let $(R, \mathfrak{m})$ be an artinian local ring of Loewy length t. Then $\mathfrak{m}^{t-1}$ is a nonzero ideal of R which annihilates $\operatorname{Ext}^1_R(\operatorname{mod} R, \operatorname{mod} R)$.

Remark 2.6. We can also establish a relative version of the uniform annihilator property. Let us say that a subcategory $\mathcal{C}$ of $\operatorname{mod} R$ has the uniform annihilator property (UAP) if there exists a nonzerodivisor $a \in R$ and an integer $n \geq 0$ such that $a \operatorname{Ext}^n_R(\mathcal{C}, \mathcal{C}) = 0$. Then we may wonder if for a Cohen–Macaulay local ring R the subcategory $\mathsf{CM}(R)$ of $\operatorname{mod} R$ has the UAP, even when $\operatorname{mod} R$ does not. However, the following example says that this is not true in general:

Consider the ring $R = k[[x, y]]/(x^2)$. Then it follows from [30, Example (6.5)] that

$$\mathsf{CM}(R) = \mathsf{add}_{0 \leq e \leq \infty}\{I_e = (x, y^e)\},$$

where $I_0 = R$ and $I_\infty = (x)$. It is observed that $\operatorname{Ext}^i_R(I_e, I_e) \cong (R/I_e)^{\oplus 2}$ for all $0 < e < \infty$ and all $i > 0$. Hence if there are a nonzerodivisor $a \in R$ and an integer $n \geq 0$ such that $a \operatorname{Ext}^n_R(\mathsf{CM}(R), \mathsf{CM}(R)) = 0$, then a is in $\bigcap_{0<e<\infty} I_e = (x)$, and $a^2 = 0$, which is a contradiction.

More generally than the above example, one has the following: Let R be a Gorenstein local ring of dimension d. Let $a \in R$ and $n > d$. If $a \operatorname{Ext}^n_R(\mathsf{CM}(R), \mathsf{CM}(R)) = 0$, then $a \operatorname{Ext}^n_R(\operatorname{mod} R, \operatorname{mod} R) = 0$. In particular, $\mathsf{CM}(R)$ has the UAP if and only if so does $\operatorname{mod} R$.

3. Reduction to the domain case

In the previous section we saw that to see whether a given commutative ring has the UAP we may assume that it is a reduced ring. In this section we consider further reduction to the case where it is a domain. For this purpose, we establish several lemmas. The first one is a generalization of [15, Lemma 2.2] and [27, Lemma 2.1].

Lemma 3.1. *Let Λ be an R-algebra.*

(1) *Let M be a finitely generated Λ-module. If $x \in R$ annihilates $\mathsf{Ext}^1_\Lambda(M, \Omega_\Lambda M)$, then there is an exact sequence of Λ-modules*

$$0 \to (0 :_M x) \to M \oplus \Omega_\Lambda M \to \Omega_\Lambda(M/xM) \to 0.$$

(2) *Assume that Λ is module-finite over R. Let $a, b \in R$ and $n > 0$. Suppose $a \,\mathsf{Ext}^n_\Lambda(\mathsf{mod}\,\Lambda, \mathsf{mod}\,\Lambda) = 0$ and $b \,\mathsf{Ext}^i_R(\Lambda, \mathsf{mod}\,R) = 0$ for each $1 \le i \le n$. Then $a^2 b^n \,\mathsf{Ext}^n_R(\mathsf{mod}\,\Lambda, \mathsf{mod}\,R) = 0$.*

Lemma 3.2. (1) *Let $K \xrightarrow{\theta} X \xrightarrow{\phi} Y \xrightarrow{\pi} C$ be an exact sequence of R-modules. Let a, b be elements of R, n an integer and E an R-module. If $aK = 0 = aC$ and $b \,\mathsf{Ext}^n_R(Y, E) = 0$, then $a^2 b \,\mathsf{Ext}^n_R(X, E) = 0$.*

(2) *Let I be an ideal of R. Let $n > 0$ and $a, b \in R$. If $a \,\mathsf{Ext}^n_R(\mathsf{mod}\,R, \mathsf{mod}\,R) = 0$ and $b \,\mathsf{Ext}^i_R(R/I, \mathsf{mod}\,R) = 0$ for every $1 \le i \le n-1$, then $ab^{n-1} \,\mathsf{Ext}^n_{R/I}(\mathsf{mod}\,R/I, \mathsf{mod}\,R/I) = 0$.*

For an R-module M we denote by $\mathsf{NF}(M)$ the *nonfree locus* of M, that is, the set of prime ideals $\mathfrak{p}$ of R such that the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is nonfree. It is well known that $\mathsf{NF}(M)$ is a closed subset of $\mathsf{Spec}\,R$ in the Zariski topology whenever M is finitely generated.

Lemma 3.3. *Let R be a reduced ring.*

(1) *For any finitely generated R-module M there is a nonzerodivisor x of R with $x \,\mathsf{Ext}^{>0}_R(M, \mathsf{mod}\,R) = 0$.*
(2) *Let I be an ideal of R that strictly contains some $\mathfrak{p} \in \mathsf{Ass}\,R$. Then I contains a nonzerodivisor of R.*
(3) *Let $a \in R$ and $n \in \mathbb{Z}$ with $a \,\mathsf{Ext}^n_R(\mathsf{mod}\,R/\mathfrak{p}, \mathsf{mod}\,R) = 0$ for all $\mathfrak{p} \in \mathsf{Ass}\,R$. Then there exists a nonzerodivisor $b \in R$ such that $ab \,\mathsf{Ext}^n_R(\mathsf{mod}\,R, \mathsf{mod}\,R) = 0$.*

Combining the above lemmas, we can prove the main result of this section.

Theorem 3.4. *Let R be a reduced ring and let $n > 0$. The following are equivalent.*

(1) *There exists a nonzerodivisor $a \in R$ such that $a \,\mathsf{Ext}^n_R(\mathsf{mod}\,R, \mathsf{mod}\,R) = 0$.*
(2) *There exists a nonzerodivisor $a \in R$ such that $a \,\mathsf{Ext}^n_{R/\mathfrak{p}}(\mathsf{mod}\,R/\mathfrak{p}, \mathsf{mod}\,R/\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \mathsf{Ass}\,R$.*
(3) *For each $\mathfrak{p} \in \mathsf{Ass}\,R$ there exists $a \in R \setminus \mathfrak{p}$ such that $a \,\mathsf{Ext}^n_{R/\mathfrak{p}}(\mathsf{mod}\,R/\mathfrak{p}, \mathsf{mod}\,R/\mathfrak{p}) = 0$.*

The following result is an immediate consequence of Theorem 3.4. Thus, to check the UAP of a reduced ring one may assume that it is a domain.

Corollary 3.5. *Let R be a reduced ring. Then R has the UAP if and only if $R/\mathfrak{p}$ has the UAP for each $\mathfrak{p} \in \mathsf{Min}\,R$. In particular, if R has finite Krull dimension and $R/\mathfrak{p}$ is regular for all $\mathfrak{p} \in \mathsf{Min}\,R$, then R has the UAP.*

Remark 3.6. (1) The assumption that R is reduced in Corollary 3.5 cannot be removed. More strongly, a commutative ring R does not necessarily have the UAP even if $R/\mathfrak{p}$ has the UAP for all $\mathfrak{p} \in \mathsf{Spec}\,R$.
(2) All Stanley–Reisner rings have the UAP by the second assertion of Corollary 3.5.

4. Rings over perfect fields

We observed in the previous section that for the UAP of a commutative ring one can assume that it is a domain. In this section, we consider the UAP of a domain whose quotient field is a separable extension of the quotient field of some Noether normalization. We start by stating the following lemma.

Lemma 4.1. *Let R be a regular ring of dimension d and Λ a module-finite R-algebra. Let N be an R-module, and let a be an element of R such that $a \,\mathsf{Ext}^i_R(\Lambda, N) = 0$ for all $i > 0$. Then $a^d \,\mathsf{Ext}^1_R(\Omega^d_\Lambda M, N) = 0$ for all finitely generated Λ-modules M.*

Let A be a commutative ring and R a commutative A-algebra. Define a map $\mu : R \otimes_A R \to R$ by $\mu(r \otimes r') = rr'$. This is a ring homomorphism, and the ideal

$$\mathfrak{N}^R_A := \mu(\mathsf{ann}_{R\otimes_A R}\,\mathsf{Ker}\,\mu)$$

of R is called the *Noether different* of R over A. We obtain the following result by taking advantage of a result of Wang [28].

Lemma 4.2. *Let A be a regular ring of dimension d. Let R be a module-finite A-algebra. Then there exists a nonzerodivisor $a \in A$ such that $a\mathfrak{N}^R_A \,\mathsf{Ext}^{d+1}_R(\mathsf{mod}\,R, \mathsf{mod}\,R) = 0$.*

According to the above result, a domain R turns out to have the UAP once a Noether normalization A with $\mathfrak{N}^R_A \neq 0$ is found. This is closely related to separability of the extension of the quotient fields:

Lemma 4.3. *Let $A \subseteq R$ be a module-finite extension of domains. If $\mathsf{Q}(R)$ is separable over $\mathsf{Q}(A)$, then $\mathfrak{N}_A^R \neq 0$.*

The main result of this section is the following theorem, which is immediate from Lemmas 4.2 and 4.3.

Theorem 4.4. *Let R be a domain. Assume that there is a d-dimensional regular subring A of R such that R is module-finite over A and that $\mathsf{Q}(R)$ is separable over $\mathsf{Q}(A)$. Then R has the UAP.*

The assumption of the above theorem is satisfied for a complete local ring with perfect coefficient field:

Corollary 4.5. *Let R be a complete equicharacteristic local ring with perfect residue field. Then R has the UAP if and only if R is reduced.*

5. Ascent and descent

In this section we study ascent and descent of the UAP. To prove our main result on descent, we make a lemma.

Lemma 5.1. *Let Λ be a module-finite R-algebra.*
(1) *If Λ has positive rank as an R-module, then the map $R \to \Lambda$ is injective.*
(2) *Suppose that $(R, \mathfrak{m}, k)$ is a local ring of depth zero and that Λ is a free R-module. If Λ has the UAP, then R is a field.*

The following theorem shows descent of the UAP, which is proved by using Lemmas 3.1, 3.2, 3.3, 5.1 and Proposition 2.4.

Theorem 5.2. *Let Λ be a module-finite R-algebra having positive rank as an R-module. Consider the following four statements.*
(1) *Λ has the UAP.*
(2) *$x \operatorname{\mathsf{Ext}}_\Lambda^n(\operatorname{\mathsf{mod}} \Lambda, \operatorname{\mathsf{mod}} \Lambda) = 0$ for some integer $n > 0$ and nonzerodivisor $x \in R$.*
(3) *$x \operatorname{\mathsf{Ext}}_R^n(\operatorname{\mathsf{mod}} \Lambda, \operatorname{\mathsf{mod}} R) = 0$ for some integer $n > 0$ and nonzerodivisor $x \in R$.*
(4) *R has the UAP.*

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) hold.

Remark 5.3. (1) The assumption in Theorem 5.2 that Λ is module-finite over R is necessary. In fact, let R be a non-reduced ring possessing a prime ideal $\mathfrak{p}$ such that $R_\mathfrak{p}$ is a field (e.g. $R = k[[x, y]]/(x^2y)$). Then $\Lambda := \mathsf{Q}(R)$ has the UAP, but R does not by Proposition 2.4.
(2) The assumption in Theorem 5.2 that Λ has positive rank over R is necessary. Indeed, if R is non-reduced and $\Lambda = R/\mathfrak{m}$ where $\mathfrak{m}$ is a maximal ideal of R, then Λ has the UAP but R does not by Proposition 2.4.

Let G be a finitely generated R-module. We put $\mathsf{thick}_R^1(G) = \mathsf{add}_R(G)$. For $n \geq 2$ we denote by $\mathsf{thick}_R^n(G)$ the subcategory of $\operatorname{\mathsf{mod}} R$ consisting of modules M admitting an exact sequence

$$0 \to X \to Y \to Z \to 0$$

such that two of X, Y, Z are in $\mathsf{thick}_R^{n-1}(G)$ and M is a direct summand of the third. Now we can state a corollary of the theorem.

Corollary 5.4. *Let R be a reduced ring. Let Λ be a module-finite R-algebra having positive rank as an R-module. Assume that there exist $G \in \operatorname{\mathsf{mod}} R$ and $n > 0$ such that $M \in \mathsf{thick}_R^n(G)$ for all $M \in \operatorname{\mathsf{mod}} \Lambda$. Then R has the UAP.*

Recall that a Cohen–Macaulay local ring R is called *of finite CM-representation type* if there exist only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay R-modules. Corollary 5.4 implies that such a ring has the UAP:

Corollary 5.5. *Let R be a d-dimensional Cohen–Macaulay local ring with $d > 0$. If R is of finite CM-representation type, then R has the UAP.*

Following [9], we say that R admits a *noncommutative resolution* if there exists a finitely generated faithful R-module M such that $\mathsf{End}_R(M)$ has finite global dimension. As another application of Corollary 5.4, we have the following result.

Corollary 5.6. *Let R be a reduced ring. If there exists a module-finite R-algebra of finite global dimension and of positive rank as an R-module, then R has the UAP. In particular, every domain admitting a noncommutative resolution has the UAP.*

Remark 5.7. (1) A result of Wang [28, Proposition 2.1] shows that 1-dimensional reduced complete local rings have the UAP. Our Corollary 5.6 generalizes this: it implies that arbitrary 1-dimensional reduced Nagata (e.g. excellent) rings have the UAP, because the integral closure of such a ring is module-finite and regular. Furthermore, Corollary 5.6 also says that all quotient singularities (i.e., invariant subrings of a polynomial ring or a power series ring over a field) have the UAP.
(2) It follows from a theorem of Leuschke [20, Theorem 6] that a Cohen–Macaulay local ring of finite CM-representation type admits a noncommutative resolution. Combining this with Corollary 5.6 recovers Corollary 5.5 in the case where R is a domain.

Next, we consider ascent of the UAP. For this, we prepare a lemma.

Lemma 5.8. *Let M be a finitely generated R-module, and let $\Lambda = \mathsf{End}_R(M)$. Then every second syzygy of a finitely generated Λ-module is isomorphic to $\mathsf{Hom}_R(M, N)$ for some finitely generated R-module N (up to projective Λ-summands).*

Now we can prove the following theorem, which says that the UAP ascends to endomorphism rings. It is obtained by using Lemmas 3.1, 3.3, 5.8 and Proposition 2.4.

Theorem 5.9. *Let M be a finitely generated torsionfree R-module having positive rank. If R has the UAP, then so does $\mathsf{End}_R(M)$.*

Combining Theorems 5.2 and 5.9, one observes that the UAP of a commutative ring is equivalent to the UAP of its integral closure:

Corollary 5.10. *Suppose that the integral closure $\overline{R}$ of R is a finitely generated R-module. Then R has the UAP if and only if so does $\overline{R}$.*

Note that any Nagata ring (hence any excellent ring) satisfies the assumption of the above corollary; see [26, page 235].

6. Rings essentially of finite type

In this section, we consider the UAP of a ring which is essentially of finite type over a field. For this, we introduce the following subcategories.

Definition 6.1. (1) Let $G \in \mathsf{mod}\, R$. We set $|G|_0 = 0$ and $|G|_1 = |G| = \mathsf{add}\, G$. For $n \geq 2$ we denote by $|G|_n$ the subcategory of $\mathsf{mod}\, R$ consisting of modules M admitting an exact sequence

$$0 \to X \to M \oplus N \to Y \to 0$$

in $\mathsf{mod}\, R$ with $X \in |G|_{n-1}$ and $Y \in |G|$.
(2) Let $G \in \mathsf{Mod}\, R$. We set $\overline{|G|}_0 = 0$ and $\overline{|G|}_1 = \overline{|G|} = \overline{\mathsf{add}}\, G$, where $\overline{\mathsf{add}}\, G$ denotes the subcategory of $\mathsf{Mod}\, R$ consisting of direct summands of arbitrary direct sums of copies of G. For $n \geq 2$ we denote by $\overline{|G|}_n$ the subcategory of $\mathsf{Mod}\, R$ consisting of modules M admitting an exact sequence

$$0 \to X \to M \oplus N \to Y \to 0$$

in $\mathsf{Mod}\, R$ with $X \in \overline{|G|}_{n-1}$ and $Y \in \overline{|G|}$.

We establish a module category version of [6, Proposition 2.2.4]. This lemma will play an essential role in the proof of the main result of this section.

Lemma 6.2. *One has $\overline{|G|}_n \cap \mathsf{mod}\, R = |G|_n$ for all $G \in \mathsf{mod}\, R$ and $n \geq 0$.*

For an integer $n \geq 0$ we denote by $\Omega^n(\mathsf{mod}\, R)$ the subcategory of $\mathsf{mod}\, R$ consisting of n-th syzygies of finitely generated R-modules. Dao and Takahashi [10, Theorem 5.7] show that if R is a d-dimensional complete local ring with perfect coefficient field, then $\Omega^d(\mathsf{mod}\, R)$ is contained in $|G|_n$ for some finitely generated R-module G and some integer $n > 0$. Our next result asserts that the same conclusion holds true also in the case where R is an affine algebra over an arbitrary field.

Theorem 6.3. *Let R be a d-dimensional affine k-algebra. Then there exist a finitely generated R-module G and an integer $n > 0$ such that $\Omega^d(\mathsf{mod}\, R) \subseteq |G|_n$.*

The case where k is perfect is shown by using the proof of [10, Theorem 5.7]. The general case is proved by making a module category version of the proof of [18, Proposition 5.1.2].

Proposition 6.4. *Let W be a multiplicatively closed subset of R. If R has the UAP, then so does the localization R_W.*

The main purpose of this section is achieved by Lemma 3.3, Theorem 6.3 and Proposition 6.4:

Theorem 6.5. *A reduced ring essentially of finite type over a field has the UAP.*

7. Strong finite generation

In this section, we apply our results obtained in the preceding sections to investigate strong finite generation of derived categories, whose notion has been introduced by Bondal and Van den Bergh [6] for general triangulated categories. Let us recall the definition.

Definition 7.1. Let $\mathcal{T}$ be a triangulated category, and $\mathcal{C}, \mathcal{X}, \mathcal{Y}$ subcategories of $\mathcal{T}$.
(1) Denote by $\langle \mathcal{C} \rangle$ the smallest subcategory of $\mathcal{T}$ containing $\mathcal{C}$ and closed under finite direct sums, direct summands and shifts.
(2) Denote by $\mathcal{X} * \mathcal{Y}$ the subcategory of $\mathcal{T}$ consisting of objects $E \in \mathcal{T}$ admitting an exact triangle $X \to E \to Y \rightsquigarrow$ in $\mathcal{T}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Put $\mathcal{X} \diamond \mathcal{Y} = \langle \langle \mathcal{X} \rangle * \langle \mathcal{Y} \rangle \rangle$.
(3) Set $\langle \mathcal{C} \rangle_0 = 0$, $\langle \mathcal{C} \rangle_1 = \langle \mathcal{C} \rangle$ and $\langle \mathcal{C} \rangle_n = \langle \mathcal{C} \rangle_{n-1} \diamond \langle \mathcal{C} \rangle$ for $n \geq 2$. When $\mathcal{C}$ consists of a single object G, we simply denote it by $\langle G \rangle_n$.
(4) We say that $\mathcal{T}$ is *strongly finitely generated* if there exist an object $G \in \mathcal{T}$ and an integer $n \geq 0$ such that $\mathcal{T} = \langle G \rangle_n$.

Remark 7.2. (1) Let $\mathcal{T}$ be a triangulated category, and $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{C}$ subcategories of $\mathcal{T}$.
(a) An object $M \in \mathcal{T}$ belongs to $\mathcal{X} \diamond \mathcal{Y}$ if and only if there is an exact triangle $X \to E \to Y \rightsquigarrow$ with $X \in \langle \mathcal{X} \rangle$ and $Y \in \langle \mathcal{Y} \rangle$ such that M is a direct summand of E.
(b) One has $(\mathcal{X} * \mathcal{Y}) * \mathcal{Z} = \mathcal{X} * (\mathcal{Y} * \mathcal{Z})$, $(\mathcal{X} \diamond \mathcal{Y}) \diamond \mathcal{Z} = \mathcal{X} \diamond (\mathcal{Y} \diamond \mathcal{Z})$ and $\langle \mathcal{C} \rangle_a \diamond \langle \mathcal{C} \rangle_b = \langle \mathcal{C} \rangle_{a+b}$ for all $a, b \geq 0$.
(2) Rouquier [25] introduces the notion of dimension for triangulated categories. A triangulated category is strongly finitely generated if and only if its dimension is finite.

Throughout the rest of this section, let $\mathcal{A}$ be an R-linear abelian category with enough projective objects. We denote by $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ the bounded derived category of $\mathcal{A}$.

Definition 7.3. (1) For $r \in R$, let $\mathcal{T}_r$ be the subcategory of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ consisting of objects $X \in \mathsf{D}^{\mathrm{b}}(\mathcal{A})$ such that the multiplication morphism $X \xrightarrow{r} X$ is zero in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$.
(2) Let $\mathcal{G}$ be the ideal of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ consisting of *ghosts*, that is, morphisms whose homology vanishes. Then the ideal $\mathcal{G}^n$ consists of *n-ghosts*, namely, n-fold compositions of ghosts.

The main purpose of this section is to prove the following theorem.

Theorem 7.4. *The following are equivalent.*
(1) *There exist a nonzerodivisor $r \in R$ and an integer $n \geq 0$ with $r \operatorname{\mathsf{Ext}}^n_{\mathcal{A}}(\mathcal{A}, \mathcal{A}) = 0$.*
(2) *There exist a nonzerodivisor $r \in R$ and an integer $n \geq 0$ with $\mathsf{D}^{\mathrm{b}}(\mathcal{A}) = \mathcal{T}_r \diamond \langle \mathsf{proj}\, \mathcal{A} \rangle_n$.*
(3) *There exist a nonzerodivisor $r \in R$ and an integer $n \geq 0$ with $\mathsf{D}^{\mathrm{b}}(\mathcal{A}) = \langle \mathsf{proj}\, \mathcal{A} \rangle_n \diamond \mathcal{T}_r$.*
(4) *There exist a nonzerodivisor $r \in R$ and an integer $n \geq 0$ with $r\mathcal{G}^n = 0$.*

To show Theorem 7.4, it is convenient to introduce the following conditions.

Definition 7.5. For an element $r \in R$ and an integer $n \geq 0$ we define the conditions:

(E$_{r,n}$) $r \operatorname{\mathsf{Ext}}^n_{\mathcal{A}}(\mathcal{A}, \mathcal{A}) = 0$, (D$_{r,n}$) $\mathsf{D}^{\mathrm{b}}(\mathcal{A}) = \mathcal{T}_r \diamond \langle \mathsf{proj}\, \mathcal{A} \rangle_n$,

(G$_{r,n}$) $r\mathcal{G}^n = 0$, (D$'_{r,n}$) $\mathsf{D}^{\mathrm{b}}(\mathcal{A}) = \langle \mathsf{proj}\, \mathcal{A} \rangle_n \diamond \mathcal{T}_r$.

We prepare the following three lemmas.

Lemma 7.6. *Let r, s be elements of R.*

(1) *For an object* $X \in \mathsf{D}^{\mathrm{b}}(\mathcal{A})$ *one has* $\mathsf{cone}(X \xrightarrow{r} X) \in \mathcal{T}_r$.

(2) *For an exact triangle* $X \xrightarrow{f} Y \xrightarrow{g} Z \rightsquigarrow$ *in* $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$, *if* $X \in \mathcal{T}_r$ *and* $Z \in \mathcal{T}_s$, *then* $Y \in \mathcal{T}_{rs}$.

Lemma 7.7. *One has* $\mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathcal{A})}(P, g) = 0$ *for all* $P \in \langle \mathsf{proj}\,\mathcal{A}\rangle_n$ *and* $g \in \mathcal{G}^n$.

Lemma 7.8. *Let* M *be an object of* $\mathcal{A}$, *and let* $n \geq 0$ *be an integer. Then there exists an exact triangle* $P \to M \to \Sigma^n \Omega^n M \rightsquigarrow$ *in* $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$.

Now we can clarify the relationships among the conditions introduced in Definition 7.5.

Proposition 7.9. *The following implications hold.*

(1) $(\mathrm{D}_{r,n}) \Rightarrow (\mathrm{D}'_{r^2,n})$ *and* $(\mathrm{D}'_{r,n}) \Rightarrow (\mathrm{D}_{r^2,n})$.

(2) $(\mathrm{E}_{r,n}) \Rightarrow (\mathrm{D}_{r^3,2n})$, $(\mathrm{D}'_{r,n}) \Rightarrow (\mathrm{G}_{r,n})$ *and* $(\mathrm{G}_{r,n}) \Rightarrow (\mathrm{E}_{r,n})$.

Proof of Theorem 7.4. The theorem immediately follows from Proposition 7.9. ■

From now on, we give several applications of the theorem. Since our ring Λ is noetherian, the category $\mathsf{mod}\,\Lambda$ is abelian and has enough projective objects. We simply denote by $\mathsf{D}^{\mathrm{b}}(\Lambda)$ the bounded derived category of $\mathsf{mod}\,\Lambda$. Applying Theorem 7.4 to $R = \mathsf{Z}(\Lambda)$ and $\mathcal{A} = \mathsf{mod}\,\Lambda$, we obtain the following result.

Corollary 7.10. *The following are equivalent.*

(1) Λ *has the UAP.*

(2) *There exist a nonzerodivisor* $r \in \mathsf{Z}(\Lambda)$ *and an integer* $n \geq 0$ *with* $\mathsf{D}^{\mathrm{b}}(\Lambda) = \mathcal{T}_r \diamond \langle\Lambda\rangle_n$.

(3) *There exist a nonzerodivisor* $r \in \mathsf{Z}(\Lambda)$ *and an integer* $n \geq 0$ *with* $\mathsf{D}^{\mathrm{b}}(\Lambda) = \langle\Lambda\rangle_n \diamond \mathcal{T}_r$.

(4) *There exist a nonzerodivisor* $r \in \mathsf{Z}(\Lambda)$ *and an integer* $n \geq 0$ *with* $r\mathcal{G}^n = 0$.

For a nonzerodivisor $x \in R$ each object of the subcategory $\mathcal{T}_x$ of $\mathsf{D}^{\mathrm{b}}(R)$ is essentially represented by an object of $\mathsf{D}^{\mathrm{b}}(R/(x))$:

Lemma 7.11. *Let* $x \in R$ *be a nonunit. Then the natural triangle functor* $\mathsf{D}^{\mathrm{b}}(R/(x)) \to \mathsf{D}^{\mathrm{b}}(R)$ *factors through the inclusion functor* $\mathcal{T}_x \to \mathsf{D}^{\mathrm{b}}(R)$. *If* x *is a nonzerodivisor, then the functor* $\mathsf{D}^{\mathrm{b}}(R/(x)) \to \mathcal{T}_x$ *is dense up to direct summands.*

Now we can observe the relationship between the UAP and strong finite generation of a derived category.

Corollary 7.12. *Let* R *be of finite Krull dimension. If* $R/\mathfrak{p}$ *has the UAP for all nonmaximal prime ideals* $\mathfrak{p}$, *then* $\mathsf{D}^{\mathrm{b}}(R)$ *is strongly finitely generated.*

Corollary 7.12 simultaneously recovers a theorem of Rouquier and Keller–Van den Bergh and a theorem of Aihara–Takahashi.

Corollary 7.13 (Rouquier, Keller–Van den Bergh, Aihara–Takahashi)**.** *The derived category* $\mathsf{D}^{\mathrm{b}}(R)$ *is strongly finitely generated if* R *is either*

(1) *a ring essentially of finite type over an arbitrary field, or*

(2) *a complete equicharacteristic local ring with perfect residue field.*

Remark 7.14. Corollary 4.5 guarantees the existence of domains that do not have the UAP. This is verified as follows. First, we remark that strong finite generation of $\mathsf{D}^{\mathrm{b}}(R)$ implies that the singular locus $\mathsf{Sing}\,R$ of R is a closed subset of $\mathsf{Spec}\,R$ in the Zariski topology. Indeed, if $\mathsf{D}^{\mathrm{b}}(R)$ is strongly finitely generated, then $\mathsf{D}^{\mathrm{b}}(R) = \langle G\rangle_n$ for some $G \in \mathsf{D}^{\mathrm{b}}(R)$ and $n \geq 0$. Then we may assume that G is an R-module, and it is easy to see that $\mathsf{Sing}\,R$ coincides with the set of prime ideals $\mathfrak{p}$ of R such that the $R_{\mathfrak{p}}$-module $G_{\mathfrak{p}}$ has infinite projective dimension. Thus $\mathsf{Sing}\,R$ is closed. Ferrand and Raynaud [14] construct a 3-dimensional local domain R whose singular locus is not closed. Thus Corollary 4.5 implies that $R/\mathfrak{p}$ does not have the UAP for some prime ideal $\mathfrak{p}$.

We close this section by stating a result on strong finite generation of $\mathsf{D}^{\mathrm{b}}(R)$ when the Krull dimension of R is small.

Corollary 7.15. *Let* R *be a Nagata (e.g. excellent) ring. Then* $\mathsf{D}^{\mathrm{b}}(R)$ *is strongly finitely generated if either*

(1) $\mathsf{dim}\,R \leq 1$, *or*

(2) $\mathsf{dim}\,R = 2$ *and* R *has the UAP.*

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