On Demazure's construction of finite Abelian coverings of normal graded rings[1]

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Abstract. We will study finite Abelian cover of normal graded singularities in terms of Pinkham-Demazure's construction. We will define the new subgroup $Cl^0(R)$ of the torsion part of the divisor class group Cl(R) of normal graded ring. For a finite subgroup G of Cl(R), we show $G/Cl^0(R) \cap G$ is cyclic. Taking the Kummer cover of Proj(R) by $Cl^0(R) \cap G$, the standard generator of $G/Cl^0(R) \cap G$ gives the Demazure divisor of an Abelian cover of R by G.

Introduction

The index one cover trick is one of most important method in the theory of minimal models and other algebraic geometric studies of varieties. In particular, in the case that the canonical module defines a torsion in the divisor class group of local ring, precise study of cyclic cover of singularity in terms of divisor class group is interesting problem. In 1990 and subsequent 10 years, for the graded singularities, the present author and Prof. Kei-ichi Watanabe made some explicit formula of graded cyclic cover by means of Pinkham-Demazure's construction. It began at the talk in the 9th symposium of these series of Japanese Commutative Ring Theory [3].

In recent years, there are many interests in abelian covers of normal two-dimensional singularities. Inspired by these works, here I will discuss the Ablelian covers of normal graded rings in terms of Pinkham-Demazure's construction. Given a finite subgroup G of divisor class group of normal graded ring R, we can construct a graded G-cover of R (see §1). Author's interest is a quick method to represent this new ring in terms of given datum as in the case of cyclic cover. The results in the below will be regarded as natural generalizations of studies of cyclic covers of normal graded rings.

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[1]This paper is an announcement of our result and the detailed version will be submitted to somewhere.
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Let R be a normal graded ring written as $R = \bigoplus_{n \geq 0} R_n$ with the zero-th part R_0 being algebraically closed field. We always assume that R_0 is an algebraically closed, and will study the singularity at the homogeneous maximal ideal $R_+ = \bigoplus_{n > 0} R_n$.

By Pinkham-Demazure, there is an ample $\mathbf{Q}$-Cartier rational divisor D on $\mathrm{Proj}(R) = X$, where the relation

$$R = \bigoplus_{n \geq 0} \mathrm{H}^0(X, O_X(nD))T^n$$

holds [1]. Here T is a homogeneous element of the quotient field of R with degree one, and we regard T as an indeterminate. D is determined up to the choice of T. We will represent this expression as $R = R(X, D)$, and call the Pinkham-Demazure representation of R.

Now, to recall the representation theory of the divisor class group of R by Prof. Watanabe, we introduce the following:

Let $\mathrm{Irr}^1(X)$ be the set of irreducible closed subvarieties of X with codimension one. We shall write D as;

$$D = \sum_{V \in Irr^1(X)} (p_V/q_V)V, \quad (p_V, q_V \in \mathbb{Z}, (p_V, q_V) = 1, q_V \geq 1)$$

Then, by Kei-ichi Watanabe [4], we call represent the divisor class group $Cl(R)$ as follows;

$$Cl(R) \cong \mathrm{Div(X, D)}/P(X) \oplus \mathbf{Z}D$$

Here $P(X)$ denotes the set of principal diviosr of X, $\mathrm{Div(X, D)}$ (resp. $P(X) \oplus \mathbf{Z}D$) denotes the special module corresponds to the homogeneous divisor group of R (resp. homogeneous principal divisors of R) . Recall that $\mathrm{Div(X, D)}$ is introduced as

$$\mathrm{Div(X, D)} = \left\{ \mathrm{E} \in \mathrm{Div(X, \mathbf{Q})} \,\middle|\, \mathrm{E} = \sum_{V \in Irr^1(X)} \mathrm{r_V V} \ \mathrm{with} \ \mathrm{r_V q_V} \in \mathrm{Z} \right\}.$$

So the torsion part of $Cl(R)$ is represented by the subset of $\mathrm{Div(X, D)}$ as

$$\left\{ E \in \mathrm{Div(X, D)} \,\middle|\, {}^\exists \mathrm{r} \in \mathbf{N} \ \mathrm{and} \ {}^\exists \mathrm{a}' \in \mathbf{Z} \ \mathrm{such \ that} \ \mathrm{rE - a'D} \in \mathrm{P(X)} \right\}.$$

Now let us assume that $E \in \mathrm{Div(X, D)}$ defines a torsion element $cl(E) \in Cl(R)$ of order r. There exist an integer a' and a rational function $\phi \in k(X)$ such that the relation $rE - a'D = \mathrm{div}(\phi)$ holds. In [3], to represent the cyclic cover of $R(X, D)$ by $cl(E)$, we introduced the following decomposition of E.

2

Definition. *The central part* and *The polarization part* of E. [3]

Let $GCD(r, a') = s$, and integers α, β satisfy the equality $\alpha a' + \beta r = s$. Set

$$E^0 = \frac{r}{s}E - \frac{a'}{s}D, \ \text{and} \ E^1 = \alpha E + \beta D$$

These are members of $\mathrm{Div}(X, D)$. Further E^0 determined the cyclic branched cover $\rho : Y \to X$ and E^1 is an ample Q-Cartier divisor on X and so is the pullback $\rho^*(E^1)$ on Y. Our fundamental result of cyclic cover [3] is the representation theorem

Theorem (Theorem 1.3 [3]) The Pinkham-Demazure construction for the cyclic cover of $R(X, D)$ by $cl(E)$ is isomorphic to $R(Y, \rho^*(E^1))$.

Here we will call E^0 *the central part* of E and E^1 *the polarization part* of E.

In this paper, we will introduce the decomposition method of subgroup of the torsion part of $Cl(R)$ which is similar to the above decomposition of E.

We shall introduce more small subset $\mathrm{Div}^0(X, D)$ as in the following way

$$\mathrm{Div}^0(X, D) = \left\{ E \in \mathrm{Div}(X, D) \ \middle| \ ^{\exists} r \in \mathbf{N} \ \text{ such that } \ rE \in \mathrm{P}(X) \right\}$$

and define

$$Cl^0(R) = \mathrm{Div}^0(X, D)/P(X).$$

There are natural inclusion relations

the torsion part of $Cl(X) \subset Cl^0(R) \subset$ the torsion part of $Cl(R)$

In other word, the subgroup $Cl^0(R)$ consists in the elements of the form E^0 for $\mathrm{Div}(X, D)$ which give torsion of $Cl(R)$ ((1) of Lemma 1.2). Using $Cl^0(R)$ we can state the results.

Theorem 1. Let $G \subset Cl(R)$ be a finite subgroup. Then the quotient $G/(Cl^0(R) \cap G)$ is a cyclic group.

Now we can choose a generator $cl(D_G)$ of $G/(Cl^0(R) \cap G)$, where $D_G \in \mathrm{Div}(X, D)$ a representative. We define the polarization part $(D_G)^1$ of D_G. In the case $cl(D_G) = 0$ in $G/(Cl^0(R) \cap G)$, that is when the relation $G \subset Cl^0(R)$ or $Cl^0(R) \cap G = G$, we can choose the polarization part as $(D_G)^1 = D$. Let $G^0 \subset G$ be the subgroup defined as $G^0 = Cl^0(R) \cap G$.

3

Theorem 2. Let $G \subset Cl(R)$ be a finite subgroup and G^0 be as above. Using G^0 we can construct a Kummer covering Y_G of X as $\rho : Y_G = \operatorname{Spec}_X(\bigoplus_{E \in G^0} O_X(E)) \to X$ and consider the pullback $\rho^*((D_G)^1) = \tilde{D} \in \operatorname{Div}(Y_G) \otimes Q$. Then $\rho^*((D_G)^1)$ is an ample Q-Cartier divisor on Y_G. Further we can observe that $R(Y_G, \tilde{D})$ is a G-graded cover of $R = R(X, D)$.

Theorem 3. If $G^0 = 0$, G is a cyclic group (by Theorem 1) and $Y_G = X$ in the notation of Theorem 2. In the construction of Theorem 2, we have the relation $R(Y_G, \tilde{D})^{(|G|)} = R$. That is R is the $|G|$-th Veronease subgroup of $R(Y_G, \tilde{D})$.

Conversely, if there exist a natural number r with $R(Y_G, \tilde{D})^{(r)} = R$, then the relation $G^0 = 0$ holds.

Note 4. When G is a cyclic group, our theorems in the above are concerning the cyclic covers. These are nothing but the theorems proven in [3]. For our theorem 3, the assumption $G^0 = 0$ corresponds the the condition $gcd(r, a') = 1$ in the notation of §1 of [3].

§1. A construction of graded G-cover of $R(X, D)$, and proof of Thereoms.

The purpose of this section is to collect the preliminaries of class group of divisor class groups, and make precise the graded structure of our graded covers. After pure careful setting of the structure, theorems in Introduction follow easily.

Let R be a normal graded ring represented as $R = R(X, D) = \bigoplus_{n \geq 0} \operatorname{H}^0(X, O_X(nD))T^n$ by the Pinkham-Demazure construction. Here T is the homogeneous element of $Q(R)$ with $\deg(T) = 1$. Let G be a finite subgroup of the divisor class goup $Cl(R)$. Then by K.-i. Watanabe [4] , there are Q-divisors $E_1, E_2, \cdots, E_m \in \operatorname{Div}(X, D)$ which generate G as a direct sum as

$$G = < cl(E_1) > \oplus < cl(E_2) > \oplus \cdots \oplus < cl(E_m) > \quad in \quad Cl(R).$$

Each $cl(E_i) \in Cl(R)$ is a torsion element. We set r_i the torsion order of $cl(E_i)$. There are the integer a_i' and a rational function $\phi_i \in k(X)$ with the relation $r_i E_i - a_i' D = div(\phi_i)$ in $Div(X)$ and set $s_i = gcd(r_i, a_i')$ for $i = 1, 2, \cdots, m$.

We introduce graded R-algebra as follows:

$$A(G) = \oplus_{(k_1,k_2,\cdots,k_m)\in\mathbf{Z}^m}R(k_1E_1 + k_2E_2 + \cdots + k_mE_m)T_1^{k_1}T_2^{k_2}\cdots T_m^{k_m}$$

$$\subset Q(R)[T_1,T_1^{-1},T_2,T_2^{-1},\cdots,T_m,T_m^{-1}]$$

Here $k_1E_1 + k_2E_2 + \cdots + k_mE_m \in \mathrm{Div}(X,D)$ and

$$R(k_1E_1 + k_2E_2 + \cdots + k_mE_m) = \oplus_{k\in\mathbf{Z}}H^0(O_X(k_1E_1 + k_2E_2 + \cdots + k_mE_m + kD))T^k$$

as standard notation. Let us set the degree of T_i as $\deg(T_i) = \frac{r_i}{a_i'} \in \mathbf{Q}$. We also have $\deg(T) = 1$ by the assumption. Then $T^{r_i} - \phi_i T^{a_i'}$ is a homogeneous element of degree a_i'.

Now we define our graded G-cover $R(G)$ of R by

$$R(G) = A(G)/(T^{r_1} - \phi_i T^{a_1'}, T^{r_2} - \phi_i T^{a_2'}, \cdots, T^{r_m} - \phi_i T^{a_m'})$$

We can easily see that $R(G)$ is a graded finite R-module.

Lemma 1.1 $R(G)$ is a normal domain.

In fact, $R(G)$ is obtained by the following successive procedure. One of the basic result for the proceeding arguments is the injectivity of the divisor class groups of cyclic cover modulo trivial kernels. For the precise informations, I refere our previous paper [2] (in particular Corollary 2.6 [2]).

Note that, in each step, $R[[i]]$ is a normal graded domain by [2,3], and so is $R[[m]] = R(G)$:

Let $R[[i]]$ be as follows; $R[[0]] = R$, and $R[[1]]$ the cyclic cover of R by $cl(E_1) \in Cl(R)$ with the data $r_1E_1 - a_1'D = div(\phi_1)$ in $Div(X)$. Then we have a natural inclusion map of divisor class groups $Cl(R)/\mathbf{Z}cl(E_1) \to Cl(R[[1]])$ and the corresponding elements $cl(E_i) \in Cl(R[[1]])$ has the same torsion order as in $Cl(R)$ for $i = 2, \cdots, m$. The relations $r_iE_i - a_i'D = div(\phi_i)$ in $Div(X)$ induce the corresponding identification there. So we can continue these process as $R[[i]]$ is the cyclic cover of $R[[i-1]]$ by $cl(E_i) \in Cl(R[[i-1]])$ with the data corresponding $r_iE_i - a_i'D = div(\phi_i)$ in $Div(X)$. We conclude that $R[[m]] = R(G)$.

Note that the grading of $R[[1]]$ by the method is given over $\mathbf{Z}$ or over non-negative integers. However in our process on $R[[i]]$ in the above, the grading are given in the sets contained in $\mathbf{Q}$ by the rule on the degree of T_i. Each homogeneous element in

$$H^0(O_X(k_1E_1 + k_2E_2 + \cdots + k_mE_m + kD))T^kT_1^{k_1}T_2^{k_2}\cdots T_m^{k_m}$$

5

has the degree

$$k + k_1 \frac{a'_1}{r_1} + k_2 \frac{a'_2}{r_2} + \cdots + k_m \frac{a'_m}{r_m}.$$

So the grading is given over

$$I = \left\{ k + k_1 \frac{a'_1}{r_1} + k_2 \frac{a'_2}{r_2} + \cdots + k_m \frac{a'_m}{r_m} \;\;\Big|\;\; k, k_1, \cdots, k_m \in \mathbf{Z} \right\} \subset \mathbf{Q}$$

For the group $Cl^0(R)$, we can show the following.

Lemma 1.2 Let R, $Cl(R)$ and $Cl^0(R)$ be as in Introduction.

(1) If $E \in \mathrm{Div}(X, D)$ defines a torsion element of $Cl(R)$, the following relation holds.

$$\mathbf{Z}cl(E) \cap Cl^0(R) = \mathbf{Z}cl(E^0) \quad \text{in} \quad Cl(R)$$

(2) If $E_1, E_2 \in \mathrm{Div}(X, D)$ define torsion elements of $Cl(R)$ and $\mathbf{Z}cl(E_1) \cap \mathbf{Z}cl(E_2) = \{0\}$ in $Cl(R)$, then

$$\mathbf{Z}cl(E_1^0) \cap \mathbf{Z}cl(E_2^0) = \{0\} \quad \text{in} \quad Cl^0(R) = \mathrm{Div}^0(X, D)/P(X).$$

(3) Using the decomposition $G = <cl(E_1)> \oplus <cl(E_2)> \oplus \cdots \oplus <cl(E_m)>$ in $Cl(R)$., we have the relation

$$G \cap Cl^0(R) = <cl(E_1^0)> \oplus <cl(E_2^0)> \oplus \cdots \oplus <cl(E_m^0)> \quad \text{in} \quad Cl^0(R).$$

(4) By (3), we have

$$G/(G \cap Cl^0(R)) \cong \frac{<cl(E_1)>}{<cl(E_1^0)>} \oplus \frac{<cl(E_2)>}{<cl(E_2^0)>} \oplus \cdots \oplus \frac{<cl(E_m)>}{<cl(E_m^0)>}$$

(5) For each E_i, we have the following relation

$$<cl(E_i)>/<cl(E_i^0)> = (<cl(E_i^1)> + <cl(E_i^0)>)/<cl(E_i^0)> \quad \text{in} \quad Cl(R)/Cl^0(R)$$

Inparticular the order of this group is $\dfrac{r_i}{s_i}$.

(6) $\dfrac{r_i}{s_i}$ and $\dfrac{r_j}{s_j}$ are relatively prime for $i \neq j$. Therefore the group of (4) is a cyclic group.

Note on a proof of (6) of Lemma 1.2

6

By the definition,

$$\frac{r_i}{s_i}E_i^1 = \frac{r_i}{s_i}\alpha_i E_i + \frac{r_i}{s_i}\beta_i D = \frac{r_i}{s_i}\alpha_i E_i + \frac{s_i - \alpha_i a_i'}{s_i}D = \alpha_i\left(\frac{r_i}{s_i}E_i - \frac{a_i'}{s_i}D\right) + D$$

Let $d = GCD(\frac{r_i}{s_i}, \frac{r_j}{s_j})$. The divisor

$$\frac{1}{d}\frac{r_i}{s_i}E_i^1 - \frac{1}{d}\frac{r_j}{s_j}E_j^1$$

is an integral combination of E_i^1 and E_j^1. Hence this is an element of $\mathrm{Div}(X, D)$. Now we have

$$d\left(\frac{1}{d}\frac{r_i}{s_i}E_i^1 - \frac{1}{d}\frac{r_j}{s_j}E_j^1\right) = \alpha_i\left(\frac{r_i}{s_i}E_i - \frac{a_i'}{s_i}D\right) - \alpha_j\left(\frac{r_j}{s_j}E_j - \frac{a_j'}{s_j}D\right)$$

This belongs to $\mathrm{Div}^0(X, D)$. By definition, $\frac{1}{d}\frac{r_i}{s_i}E_i^1 - \frac{1}{d}\frac{r_j}{s_j}E_j^1$ also belong to $\mathrm{Div}^0(X, D)$. Hence $cl(\frac{1}{d}\frac{r_i}{s_i}E_i^1) = cl(\frac{1}{d}\frac{r_j}{s_j}E_j^1)$ is a non-zero element of $Cl(R)/Cl^0(R)$ if $d > 1$. This is impossible. So it have to $d = 1$.

Theorem 1 is nothing but the assertions of (6) of Lemma 1.2.

Thereom 2 is a straightforward calculation of Proj using Lemma1.1 and Lemma1.2. I omit the proof here. But I want to give remark on the choice of the Demazure divisor in the following special case. If G is contained in $Cl^0(R)$, we have the conditions $\frac{r_i}{s_i} = 1$ for $i = 1, 2, \cdots, m$. Then $r_i = s_i(= GCD(r_i, a_i'))$ and r_i divides a_i' for $i = 1, 2, \cdots, m$. Hence the grading group I is exactly $\mathbf{Z}$ in the usual grading. The Demazure divisor is defined by the homogeneous element of the degree 1. We can also choose the element as T. So our divisor is determined by D.

§2 Some example and remarks.

Our results are formula for computation of graded rings. Most important things should be several *interesting* examples. But I am sorry for I only talk about very little here.

Example 2.1 Rational double points are given by graded rings. The divisor class groups of these classes are ,in many case, cyclic groups. Here we calculate an abelian cover of D_{2m} case. Then the class group is not cyclic.

Let $R = R(P^1, D)$ be the singularity with $D = \frac{1}{2}P_1 + \frac{1}{2}P_2 - \frac{2m-3}{2m-2}P_3$ where $P_1, P_2, P_3 \in$

P^1. Let $E_1 = \frac{1}{2}P_1$ and $E_2 = \frac{1}{2}P_2$. We have

$$\mathrm{Div}(X, D) = \mathbf{Z}E_1 + \mathbf{Z}E_2 + \mathbf{Z}D + \mathrm{Div}(P^1)$$

and $Cl(R) \cong \mathbf{Z}cl(E_1) \oplus \mathbf{Z}cl(E_2)$

Let us take $G = Cl(R)$, and consider G-cover of R. We can see easily that

$$Cl^0(R) \cong \mathbf{Z}\left(E_1 - (m-1)D\right) \oplus \mathbf{Z}\left(E_2 - (m-1)D\right).$$

So this is the case that $G \subset Cl^0(R)$.

We assume that m is even. (The arguments for case of odd m are similar. So I will omit it.) We have

$$E_1 - (m-1)D \sim -\frac{1}{2}P_2 + \frac{1}{2}P_3, \quad E_2 - (m-1)D \sim -\frac{1}{2}P_1 + \frac{1}{2}P_3$$

Let $\pi_1 : X_1 \to X = P^1$ be the cyclic cover defined by $E_2 - (m-1)D$, and $\pi_2 : Y \to X_1$ the cyclic cover of X_1 by the pullback $\pi_1^*(E_1 - (m-1)D)$. The composition of pi_1 and p_2 defines $\rho : Y \to X = P^1$ and is the Kummer cover of $P^1 = X$ defined by $Cl^0(R)$. One can easily see that $Y \cong P^1$ and $\rho^*(D) = 2P_0 - \frac{m-2}{m-1}P_{3,1}^* - \frac{m-2}{m-1}P_{3,2}^*$. Hence the type of the singularity of $R(Y, \rho^*(D))$ is the rational doube point of A_{2m-3} type.

On universal Abelian Cover of singularities. In the example above , the singularity type is known from the different point of view as the universal abelian covering of singularity.

My first motivation to study the present theme was to understand the phenomenon funded by W. Neumann [6] in the following theorem.

Theorem [6]. Let (V, p) be a quasi homogeneous complex surface singularity. Suppose that $V/\mathbf{C}^*$ is a rational curve. Then the universal abelian cover of (V, p) is a complete intersection of Brieskorn polynomial singularities.

Universal abelian covering is the normal complex filling of the homology cover of the link. This is determined canonically and is unique. Here the assumption implies the the finiteness of the divisor class group of the graded ring. By Okuma's study [7], this is also obtain by covering construction from the resolution manifold by using full divisor class group. Their arguments are algebraic geometric but not purely algebraic (for me). So I hoped to understand their studies in more ring theoretic situation and by more ring theoretic arguments. But I could not yet.

8

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