The Lefschetz property of coinvariant algebras of complex reflection groups

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Abstract

The cohomology rings of the flag varieties of complex Lie groups have the strong Lefschetz property thanks to the Hard Lefschetz theorem, and these rings are isomorphic to the coinvariant algebras of the corresponding Weyl groups as graded rings. Namely, the coinvariant algebras of Weyl groups are typical and basic examples of graded rings that have the strong Lefschetz property. The coinvariant algebras of the real reflection groups, which contains the Weyl groups, are known to have the strong Lefschetz property. In this note, we first focus on the complex reflection groups, which contain the real reflection groups, and show that the coinvariant algebras of the complex reflection groups except five primitive groups.

Second, we focus on ideals generated by polynomials invariant under the action of complex reflection groups. We study when the ideas are complete intersections, when the quotient of the ideals have the strong Lefschetz property, and other related problems.

1 The Lefschetz property

1.1 The strong Lefschetz property

Definition 1 (The strong Lefschetz property)**.** An artinian graded algebra A

$$A = \bigoplus_{i=0}^{c} A_i \qquad \text{(the graded decomposition. } A_0 \simeq K,\ \dim_K A_c = 1\text{)}$$

over a field K has the *strong Lefschetz property* if there exist $L \in A_1$ such that the multiplication map

$$(\times L^{c-2i}) : A_i \to A_{c-i}$$

is bijective for all $i = 0, 1, \ldots, \lfloor c/2 \rfloor$.

In this case, the Hilbert function of A should be symmetric. Although we can define the strong Lefschetz property for graded algebras whose Hilbert functions are not symmetric, we have defined the strong Lefschetz property as above since we consider graded algebras whose Hilbert functions are symmetric in this note.

1.2 The Hard Lefschetz theorem

The strong Lefschetz property is an abstraction of the property satisfied by the cohomology rings due to the Hard Lefschetz theorem.

Theorem 2 (Hard Lefschetz theorem. See [2], for example)**.** Let X be a d-dimensional complex compact Kähler manifold, and ω the Kähler metric. Define the endomorphism L on the cohomology ring $H^*(X, \mathbb{C})$ to be the multiplication map by the class of ω, which belongs to $H^2(X, \mathbb{C})$. Then the map

$$L^{d-k} : H^k(X, \mathbb{C}) \to H^{2d-k}(X, \mathbb{C})$$

is bijective for $k = 0, 1, \ldots, d-1$.

Example 3. (1) $\mathbb{C}[x]/(x^m)$ has the strong Lefschetz property. This algebra is isomorphic to the cohomology ring of the complex projective space $\mathbb{CP}^{m-1}$.

Table 1: irreducible real reflection groups

type	G	order	fundamental degree	
A_n	S_{n+1}	$(n+1)!$	$2, \ldots, n+1$	$(n \geq 1)$
B_n	$S_n \ltimes (\mathbb{Z}_2)^n$	$2^n n!$	$2, 4, \ldots, 2n$	$(n \geq 2)$
D_n	$S_n \ltimes (\mathbb{Z}_2)^{n-1}$	$2^{n-1} n!$	$2, 4, \ldots, 2(n-1); n$	$(n \geq 4)$
E_6		51840	$2, 5, 6, 8, 9, 12$	
E_7		2903040	$2, 6, 8, 10, 12, 14, 18$	
E_8		696729600	$2, 8, 12, 14, 18, 20, 24, 30$	
F_4		1152	$2, 6, 8, 12$	
G_2		12	$2, 6$	
H_3		120	$2, 6, 10$	
H_4		14400	$2, 12, 20, 30$	
$\mathrm{I}_2(m)$		$2m$	$2, m$	$m = 5$ or $m \geq 7$

(2) $\mathbb{C}[x_1, x_2, \ldots, x_n]/(e_1, e_2, \ldots, e_n)$ has the strong Lefschetz property. This algebra is isomorphic to the cohomology ring of the flag varieties $GL(n;\mathbb{C})/B$, where $B \subset GL(n;\mathbb{C})$ is a Borel subgroup, and e_j denotes the elementary symmetric polynomials of degree j.

2 The coinvariant algebra of complex reflection groups

2.1 The classification of complex reflection groups

Reflection groups are by definition groups generated by reflections. In this note we consider finite reflection groups only.

Definition 4 (Complex reflection groups)**.** A unitary transform σ on $\mathbb{C}^n$ is called a *reflection* if σ is diagonalizable, is not the identity, but the identity on some hyperplane. In particular, reflections of finite order are diagonalizable unitary transforms such that the eigenvalues are $(n-1)$-fold 1 and a root of unity.

A (finite) subgroup of $U(n)$ generated by reflections is called a *complex reflection group*. If a complex reflection group $G \subset U(n)$ is contained in the orthogonal group $O(n)$, then G is called a *real reflection group*. Table 1 is the classification of the irreducible real reflection groups.

Theorem 5 (Chevalley)**.** Let G be a finite subgroup of $U(n)$ acting on $R = \mathbb{C}[x_1, x_2, \ldots, x_n]$. Then the following conditions are equivalent.

(1) G is a complex reflection group.

(2) The invariant subring R^G of R is the polynomial ring generated by n algebraically independent G-invariant polynomials.

When these conditions are satisfied, n algebraically independent G-invariant polynomials are called the *fundamental invariants*. The fundamental invariants are not unique. They can be taken to be homogeneous, and the their degrees are then unique. The degrees are called *fundamental degrees* of G.

It is known that the fundamental invariants form a regular sequence. For example, the sequence $e_1, e_2, \ldots, e_n$ consisting of elementary symmetric polynomials in the polynomial ring in n variables is a regular sequence.

Table 2: $G(r,p,n)$

group	rank	order	fundamental degree
$G(r,p,n)$	n	$r^n n!/p$	$r, 2r, \dots, (n-1)r;\ nr/p$

Theorem 6 (Shephard-Todd [7])**.** Irreducible finite complex reflection groups are classified as follows:

1. the nth symmetric group S_n $(n \geq 2)$,
2. $G(r,p,n)$ $(r \geq 2,\ n \geq 1,\ p \mid r,\ (r,p,n) \neq (2,2,2))$,
3. 34 primitive groups,

where $G(r,p,n)$ is defined as

$$G(r,p,n) = A(r,p,n) \rtimes S_n,$$

$$A(r,p,n) = \left\{ \begin{pmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_n \end{pmatrix} \;\middle|\; \begin{array}{l} (\omega_j)^r = 1 \text{ and} \\ (\omega_1\omega_2\cdots\omega_n)^{r/p} = 1 \end{array} \right\}.$$

We list their data in Table 2 and Table 3.

$G(2,1,n)$, $G(2,2,n)$ and $G(m,m,2)$ are isomorphic to the Weyl group of types B_n, D_n and $\mathrm{I}_2(m)$, respectively.

2.2 The strong Lefschetz property of the coinvariant algebras

Definition 7 (coinvariant algebras)**.** Let $G \subset U(n)$ be a complex reflection group of rank n acting on the polynomial ring $R = \mathbb{C}[x_1, x_2, \dots, x_n]$. Let I be an ideal of R generated by G-invariant polynomials without constant terms. We call the quotient algebra R/I the *coinvariant algebra* of G. Namely, if we denote the fundamental invariants by $f_1, f_2, \dots, f_n \in R^G$, then

$$R/(f_1, f_2, \dots, f_n)$$

is the coinvariant algebra of G.

Since the fundamental invariants form a regular sequence, the coinvariant algebra is a complete intersection. Let us consider the strong Lefschetz property of the coinvariant algebras of complex reflection groups. The coinvariant algebra S_n is already proved to have the strong Lefschetz property. This is the first case in the classification (Theorem 6). The second case in the classification is $G(r,p,n)$ $(p \mid r)$, for which the strong Lefschetz property is also already proved. The proof needs the theory of central simple modules, and we omit the details of the proof.

Lemma 8 ($G(r,p,n)$ [3, Proposition 4.26])**.** For $r \geq 2$, $n \geq 1$, $p \mid r$, the coinvariant algebra of $G(r,p,n)$ has the strong Lefschetz property.

In the classification the remaining groups are 34 primitive groups. Some of them are already proved to have the strong Lefschetz property.

Table 3: primitive irreducible complex reflection groups

No.	rank	order	fundamental degree
4	2	24	4, 6
5	2	72	6, 12
6	2	48	4, 12
7	2	144	12, 12
8	2	96	8, 12
9	2	192	8, 24
10	2	288	12, 24
11	2	576	24, 24
12	2	48	6, 8
13	2	96	8, 12
14	2	144	6, 24
15	2	288	12, 24
16	2	600	20, 30
17	2	1200	20, 60
18	2	1800	30, 60
19	2	3600	60, 60
20	2	360	12, 30
21	2	720	12, 60
22	2	240	12, 20

No.	rank	order	fundamental degree	
23	3	120	2, 6, 10	H_3
24	3	336	4, 6, 14	
25	3	648	6, 9, 12	
26	3	1296	6, 12, 18	
27	3	2160	6, 12, 30	
28	4	1152	2, 6, 8, 12	F_4
29	4	7680	4, 8, 12, 20	
30	4	14400	2, 12, 20, 30	H_4
31	4	$64 \cdot 6!$	8, 12, 20, 24	
32	4	$216 \cdot 6!$	12, 18, 24, 30	
33	5	$72 \cdot 6!$	4, 6, 10, 12, 18	
34	6	$108 \cdot 9!$	6, 12, 18, 24, 30, 42	
35	6	$72 \cdot 6!$	2, 5, 6, 8, 9, 12	E_6
36	7	$8 \cdot 9!$	2, 6, 8, 10, 12, 14, 18	E_7
37	8	$192 \cdot 10!$	2, 8, 12, 14, 18, 20, 24, 30	E_8

Lemma 9 (primitive groups of rank two and real reflection groups)**.** In Table 3 of primitive complex reflection groups the coinvariant algebras of the groups of Nos. $4, 5, \ldots, 22; 23, 28, 30, 35, 36, 37$ have the strong Lefschetz property.

Proof. Since every artinian graded algebra which is a quotient of the polynomial ring of two variables has the strong Lefschetz property, the coinvariant algebras of the primitive groups from No. 4 to No. 22 in Table 3 have the strong Lefschetz property.

As indicated in Table 3, the primitive groups of Nos. 23, 28, 30, 35, 36 and 37 are the real reflection groups of types H_3, F_4, H_4, E_6, E_7 and E_8, respectively. It follows from the Hard Lefschetz theorem that the coinvariant algebras of Weyl groups have the strong Lefschetz property. In addition, for the real reflection groups the coinvariant algebras have the strong Lefschetz property by [9], [6], [3, Theorem 8.13], [4], etc. Hence the coinvariant algebras of primitive complex reflection groups of Nos. 23, 28, 30, 35, 36 and 37 have the strong Lefschetz property. □

There remain nine groups for which we need to prove the strong Lefschetz property. For groups of small rank we can use a computer.

Lemma 10 (primitive groups of rank three)**.** For the primitive groups of No. 24, 25, 26 and 27 in Table 3, the coinvariant algebras have the strong Lefschetz property.

Proof. Once we obtain the fundamental invariants it is easy to check the strong Lefschetz property with a computer. We list the fundamental invariants of the primitive complex reflection groups of Nos. 24, 25, 26 and 27.

No. 24 (See Springer [8, §4.6, p. 98], Miller-Blichfeldt-Dickson [5, §125 (J), p. 254] , e.g.):

$$f_4 = x^3y + y^3z + z^3x,$$
$$f_6 = xy^5 + yz^5 + zx^5 - 5x^2y^2z^2,$$
$$f_{14} = \frac{1}{9}\begin{vmatrix} (f_4)_{xx} & (f_4)_{xy} & (f_4)_{xz} & (f_6)_x \\ (f_4)_{yx} & (f_4)_{yy} & (f_4)_{yz} & (f_6)_y \\ (f_4)_{zx} & (f_4)_{zy} & (f_4)_{zz} & (f_6)_z \\ (f_6)_x & (f_6)_y & (f_6)_z & 0 \end{vmatrix}.$$

No. 25 (See Springer [8, §4.7, p. 101], Miller-Blichfeldt-Dickson [5, §125 (G), p. 253], e.g.):

$$f_6 = x^6 + y^6 + z^6 - 12(x^3y^3 + x^3z^3 + y^3z^3),$$
$$f_9 = (x^3 - y^3)(x^3 - z^3)(y^3 - z^3),$$
$$f_{12} = (x^3 + y^3 + z^3)((x^3 + y^3 + z^3)^3 + 216x^3y^3z^3).$$

No. 26 (See Shephard-Todd [7, §6, p. 286], Miller-Blichfeldt-Dickson [5, §125 (G), p. 253], e.g.): Polynomials f_6, f_9, f_{12} are the same as in No. 25, and the fundamental invariants are f_6, f_{12}, f_{18}, where

$$f_{18} = 432f_9^2 - f_6^3 + 3f_6f_{12}.$$

No. 27 (See Miller-Blichfeldt-Dickson [5, §125 (I), p. 254], e.g.):

$$f_6 = (x^2 + yz)^3 + \lambda x(x^5 + y^5 + z^5 + 5xy^2z^2 - 5x^3yz),$$
$$f_{12} = \begin{vmatrix} (f_6)_{xx} & (f_6)_{xy} & (f_6)_{xz} \\ (f_6)_{yx} & (f_6)_{yy} & (f_6)_{yz} \\ (f_6)_{zx} & (f_6)_{zy} & (f_6)_{zz} \end{vmatrix},$$
$$f_{30} = \begin{vmatrix} (f_6)_{xx} & (f_6)_{xy} & (f_6)_{xz} & (f_{12})_x \\ (f_6)_{yx} & (f_6)_{yy} & (f_6)_{yz} & (f_{12})_y \\ (f_6)_{zx} & (f_6)_{zy} & (f_6)_{zz} & (f_{12})_z \\ (f_{12})_x & (f_{12})_y & (f_{12})_z & 0 \end{vmatrix},$$

where

$$\lambda = \frac{-9 \pm \sqrt{-15}}{20}.$$

□

We conjecture the strong Lefschetz property for the remaining primitive groups of Nos. 29, 31, 32, 33, 34.

Conjecture 11 (The strong Lefschetz property of the coinvariant algebras of complex reflection groups)**.** The coinvariant algebras of complex reflection groups have the strong Lefschetz property.

3 Regular sequences consisting of invariant polynomials

The coinvariant algebra of a complex reflection group is a quotient by the regular sequence consisting of the fundamental invariants. In this section we study regular sequences consisting of (not necessarily fundamental) invariant polynomials. We give a necessarily condition for a sequence consisting of invariant polynomials to be a regular sequence.

This topic is related to the conjecture that complete intersections have the strong Lefschetz property, or that artinian graded Gorenstein algebras which admit the action of S_n have the strong Lefschetz property.

3.1 Results of Conca-Krattenthaler-Watanabe

In Conca-Krattenthaler-Watanabe [1] they study the condition for sequences consisting of power sum symmetric polynomial or complete symmetric polynomials to be regular sequences, and the give necessary conditions and conjectures.

Proposition 12 ([1, Conca-Krattenthaler-Watanabe])**.** Let a, b, c, d be different positive integers.

(1) In the polynomial ring in two variables the sequence p_a, p_b of power sum symmetric polynomials is a regular sequence if and only if at least one of a/g and b/g is even, where $g = \gcd(a, b)$.

(2) In the polynomial ring in two variables the sequence h_a, h_b of complete symmetric polynomials is a regular sequence if and only if $\gcd(a + 1, b + 1) = 1$.

(3) In the polynomial ring in three variables if the sequence p_a, p_b, p_c is a regular sequence, then $6 \mid abc$.

(4) In the polynomial ring in three variables if the sequence h_a, h_b, h_c is a regular sequence, then the following holds:

   (a) $6 \mid abc$

   (b) $\gcd(a + 1, b + 1, c + 1) = 1$

   (c) for any integer $t \geq 3$, there exists $u \in \{a, b, c\}$ such that $u + 2 \not\equiv 0, 1 \pmod{t}$.

(5) In the polynomial ring in four variables if the sequence p_a, p_b, p_c, p_d is a regular sequence, then the following three conditions hold.

   (a) at least two of a, b, c, d are even, at least one of them is a multiple of three, and at least one of them is a multiple of four.

   (b) Let E be the subset of $\{a, b, c, d\}$ consisting of even numbers, and $g = \gcd(E)$. Then the set $\{t/g \mid t \in E\}$ contains an even number.

   (c) $\{a, b, c, d\}$ does not contain any subset of the form $\{t, 2t, 5t\}$.

Remark 13. In [1], they conjecture that the converse of Proposition 12 (3), (4) and (5) hold.

3.2 A necessary condition for sequences of invariant polynomials to be regular sequences

Theorem 14. Let G be a complex reflection group of rank n, and $\delta_1, \delta_2, \ldots, \delta_n$ be its fundamental degrees. Let $f_1, f_2, \ldots, f_n \in R$ be a regular sequence consisting of homogeneous G-invariant polynomials with $d_j := \deg f_j > 0$. Then we have the following.

(1) $\dfrac{(1 - q^{d_1})(1 - q^{d_2}) \cdots (1 - q^{d_n})}{(1 - q^{\delta_1})(1 - q^{\delta_2}) \cdots (1 - q^{\delta_n})}$ is a polynomial with coefficients in nonnegative integers.

(2) For any $k \geq 1$ we have $\#\{j \mid d_j \text{ is a multiple of } k\} \geq \#\{j \mid \delta_j \text{ is a multiple of } k\}$.

(3) $\delta_1\delta_2 \cdots \delta_n \mid d_1 d_2 \cdots d_n$.

Proof. (1) Since $f_1, f_2, \ldots, f_n$ is a regular sequence in R^G, R^G factors as

$$R^G = U \otimes_{\mathbb{C}} \mathbb{C}[f_1, f_2, \ldots, f_n]$$

for some graded vector subspace of R^G. By comparing the Hilbert functions, we have

$$\begin{aligned}\mathrm{Hilb}(U) &= \frac{\mathrm{Hilb}(R^G)}{\mathrm{Hilb}(\mathbb{C}[f_1, f_2, \ldots, f_n])} \\ &= \frac{(\prod_{i=1}^n (1-q^{\delta_i}))^{-1}}{(\prod_{i=1}^n (1-q^{d_i}))^{-1}} \\ &= \frac{\prod_{i=1}^n (1-q^{d_i})}{\prod_{i=1}^n (1-q^{\delta_i})}.\end{aligned}$$

Since this is a Hilbert function of finite dimensional graded vector space U, it should be a polynomial with coefficients in nonnegative integers.

(2) Let $\Phi_k = \Phi_k(q)$ be a cyclotomic polynomial whose roots are the primitive kth roots of unity. Φ_k is an monic irreducible polynomial in $\mathbb{Z}[q]$, and satisfies

$$1 - q^d = \prod_{k|d} \Phi_k.$$

If we write the fraction in (1) in terms of Φ_k, the number of Φ_k in the denominator is greater than or equal to that in the numerator since the fraction is a polynomial. This condition is nothing but the desired condition.

(3) Since the fraction in (1) converges to an integer as q tends to one, the limit $d_1 d_2 \cdots d_n / \delta_1 \delta_2 \cdots \delta_n$ is an integer. □

When $G = S_n$, Theorem 14 (2) and (3) are read as follows.

Corollary 15. Set $R = K[x_1, x_2, \ldots, x_n]$. Let $f_1, f_2, \ldots, f_n \in R$ be a regular sequence consisting of symmetric homogeneous polynomials with $d_j = \deg f_j > 0$. Then we have the following.

(1) For any $k \geq 1$ we have $\#\{j \mid d_j \text{ is a multiple of } k\} \geq \lfloor n/k \rfloor$.

(2) $n! \mid d_1 d_2 \cdots d_n$.

When $G = G(r, 1, n)$, Theorem 14 (2) and (3) are read as follows.

Corollary 16. Let $r \geq 2$, $n \geq 1$, and $R = K[x_1, x_2, \ldots, x_n]$. Let $f_1, f_2, \ldots, f_n \in R$ be a regular sequence consisting of homogeneous symmetric polynomials in $x_1^r, x_2^r, \ldots, x_n^r$ with $\deg f_j = r d_j > 0$. Then we have

(1) For any $k \geq 1$ we have $\#\{j \mid d_j \text{ is a multiple of } k\} \geq \lfloor n/k \rfloor$.

(2) $n! \mid d_1 d_2 \cdots d_n$.

Proof. (1) Since the fundamental degrees are $r, 2r, \ldots, nr$, It follows from Theorem 14 (2) that $\#\{j \mid r d_j \text{ is a multiple of } k\} \geq \#\{j \mid rj \text{ is a multiple of } k\}$ for any $k \geq 1$. When k is a multiple rk' of r, this condition becomes $\#\{j \mid d_j \text{ is a multiple of } k'\} \geq \#\{j \mid j \text{ is a multiple of } k'\}$ for any $k' \geq 1$. Since the right-hand side is equal to $\lfloor n/k' \rfloor$, we have obtained the desired condition.

(2) By Theorem 14 (3) we have $r \cdot 2r \cdots nr \mid r d_1 \cdot r d_2 \cdots r d_n$. This is nothing but the desired condition. □

Example 17. Set $R = K[x_1, x_2, x_3, x_4]$.

(1) Let $f_1, f_2, f_3, f_4 \in R$ be a sequence consisting of non-constant symmetric polynomials with degrees 1, 2, 5, 12, respectively. Since this sequence satisfies the condition of Corollary 15 (2), but does not satisfy the condition in (3), the sequence is never a regular sequence independent of the choice of f_j.

(2) The sequence $p_1, p_3, p_4, p_{12} \in R$ consisting of power sum symmetric polynomials is not a regular sequence, but the degrees 1, 3, 4, 12 satisfy the condition in Corollary 15. Instead of the power sums the sequence $h_1, h_3, h_4, h_{12} \in R$ consisting of complete symmetric polynomials is a regular sequence.

Example 18. Let us compare our results with the results of Conca-Krattenthaler-Watanabe when $G = S_3, S_4$.

(1) When $G = S_3$ the condition in Corollary 15 (2) comes to $6 \mid d_1 d_2 d_3$. In this case there are at least one even number and at least one multiple of three among d_1, d_2, d_3. Then the condition in Corollary 15 (1) is satisfied automatically.

    Thus our result for $G = S_3$ is a generalization of Conca-Krattenthaler-Watanabe's result.

(2) When $G = S_4$ the condition in Corollary 15 (2) is read as there are at least $\lfloor 4/k \rfloor$ multiple of k among d_1, d_2, d_3, d_4 for each $k = 1, 2, 3, 4$. This corresponds to Conca-Krattenthaler-Watanabe's result, Proposition 12 (4) (a).

    Conca-Krattenthaler-Watanabe essentially proved Corollary 15 (2) for any n. Thus Corollary 15 (2) generalizes Conca-Krattenthaler-Watanabe's result from power sums to any symmetric polynomials.

3.3 The Macaulay duals of complete intersections by regular sequences consisting of invariant polynomials

Definition 19 (Macaulay dual)**.** Let $R = K[x_1, \dots, x_n]$ and $Q = K[\partial_1, \dots, \partial_n]$, where ∂_j is the partial differential operator corresponding to x_j. Q acts on R by differentiation. When Q/J is antinian Gorenstein for an ideal J of Q, there exists a unique $F \in R$ such that

$$\mathrm{Ann}_Q(F) = J$$

up to scaling.

When R/I is artinian Gorenstein for an ideal I of R, let J be the ideal of Q corresponding to I under the identification of Q with R ($\partial_j \leftrightarrow x_j$). We call the unique $F \in R$ such such that $\mathrm{Ann}_Q(F) = J$ the **Macaulay dual** of R/I. It is known that all the differentiations of F span R/I.

Theorem 20. Let $G \subset U(n)$ be a complex reflection group of rank n, and $f_1, f_2, \dots, f_n$ be a regular sequence consisting of invariant homogeneous polynomials of positive degree. Then the Macaulay dual of the artinian Gorenstein algebra $R/(f_1, \dots, f_n)$ is skew invariant, where a polynomial F is said to be skew invariant if $g.F = \det(g^{-1})F$ for any $g \in G$.

Proof. Since the fundamental invariants form a regular sequence in R, and $f_1, f_2, \dots, f_n$ form a regular sequence in R^G, we have

$$R \simeq H \otimes_{\mathbb{C}} U \otimes_{\mathbb{C}} \mathbb{C}[f_1, \dots, f_n],$$

where H is a graded subspace of R, and U is a graded subspace of R^G. Let I be the ideal of R generated by the fundamental invariants, and $J = (f_1, f_2, \dots, f_n)$. Then we have the following isomorphisms as graded G-modules.

$$R/I \simeq H, \qquad R/J \simeq H \otimes_{\mathbb{C}} U$$

Since the basis of the highest-degree component of the coinvariant algebra R/I of the complex reflection group G is known to be skew invariant, that of R/J is also skew invariant thanks to the isomorphism above.

Finally since the Macaulay dual of R/J can be a basis of the highest-degree component of the R/J, we conclude that the Macaulay dual of R/J is skew invariant. ☐

Example 21. (1) The Macaulay dual of the coinvariant algebra of S_n is the difference product $\prod_{i<j}(x_i - x_j)$ of $x_1, x_2, \ldots, x_n$, and it is skew symmetric.

(2) The Macaulay dual of the complete intersection $(h_d, h_{d+1}, \ldots, h_{d+n-1}) \subset \mathbb{C}[x_1, x_2, \ldots, x_n]$ by a regular sequence consisting of complete symmetric polynomials of consecutive degrees is

$$(x_1 x_2 \cdots x_n)^{d-1} \prod_{i<j}(x_i - x_j).$$

This is skew symmetric.

(3) The complex reflection group $G(3,1,2)$ of rank two is generated by S_n and the map σ ($\sigma(x_1) = \exp(2\pi/3)x_1$, $\sigma(x_2) = x_2$). The fundamental invariants are power sum symmetric polynomials p_3, p_6. The sequence p_6, p_9 consisting of invariants is a regular sequence, and the Macaulay dual of its quotient is $F = (ab)^2(a^3 - b^3)(2a^6 + 35a^3b^3 + 2b^6)$. Noting that $\sigma F = \exp(-2\pi/3)F$ it is easy to see that F is skew invariant.

References

[1] Aldo Conca, Christian Krattenthaler, and Junzo Watanabe. Regular sequences of symmetric polynomials. *Rend. Semin. Mat. Univ. Padova*, 121:179–199, 2009.

[2] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry.* Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[3] Tadahito Harima, Toshiaki Maeno, Hideaki Morita, Yasuhide Numata, Akihito Wachi, and Junzo Watanabe. *The Lefschetz Properties*, volume 2080 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2013.

[4] Chris McDaniel. The strong Lefschetz property for coinvariant rings of finite reflection groups. *J. Algebra*, 331:68–95, 2011.

[5] G. A. Miller, H. F. Blichfeldt, and L. E. Dickson. *Theory and applications of finite groups.* Dover Publications Inc., New York, 1961.

[6] Yasuhide Numata and Akihito Wachi. The strong Lefschetz property of the coinvariant ring of the Coxeter group of type H_4. *J. Algebra*, 318(2):1032–1038, 2007.

[7] G. C. Shephard and J. A. Todd. Finite unitary reflection groups. *Canadian J. Math.*, 6:274–304, 1954.

[8] T. A. Springer. *Invariant theory.* Lecture Notes in Mathematics, Vol. 585. Springer-Verlag, Berlin, 1977.

[9] Richard P. Stanley. Weyl groups, the hard Lefschetz theorem, and the Sperner property. *SIAM J. Algebraic Discrete Methods*, 1(2):168–184, 1980.