How much is known for the conjecture: "All complete intersections have the strong Lefschetz property"?

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It seems natural to conjecture that all (Artinian) complete intersections have the strong Lefschetz property over a field of characteristic zero. I report some basic theorems on Lefschetz properties which suggest that this conjecture should indeed be true. Some new observations are contained. [1]

1 Definitions

Definition 1 Let $A = \bigoplus_{i=0}^{c} A_i$ be an Artinian K-algebra over a field $A_0 = K$ of characteristic zero. We say that A has the strong Lefschetz property (SLP) if there exists a linear form $l \in A_1$ which satisfies

$$\times l^k : A_i \to A_{i+k} \text{ has full rank for all } i = 0, 1, \dots, c-k.$$

In such a case we sometimes say that the pair (A, l) has the strong Lefschetz property. We say that l is a (strong) Lefschetz element.

Definition 2 Let $A = \bigoplus A_i$ be a graded K algebra, where $K = A_0$. The Hilbert function of A is the map

$$i \mapsto h_i := \dim_K A_i.$$

If A is Artinian then $h_i = 0$ for all $i \gg 0$. Hence we may write $A = \bigoplus_{i=0}^{c} A_i$. In this case the Hilbert function can be denoted as a vector $(h_0, h_1, \dots, h_c)$. We say that a Hilbert function $(h_0, h_1, \dots, h_c)$ is unimodal if there exists j such that

$$h_0 \le h_1 \le \cdots \le h_j \ge h_{j+1} \ge \cdots \ge h_c.$$

We say that a Hilbert function is symmetric if

$$h_j = h_{c-j}, \quad j = 0, 1, 2, \dots, [c/2].$$

If an Artinian ring has a symmetric Hilbert function then the strong Lefschetz property can be redefined as follows:

Definition and Proposition 3 (SLP in the narrow sense) Let $A = \bigoplus_{i=0}^{c} A_i$ be an Artinian K-algebra with $A_0 = K$. Suppose that A has a symmetric Hilbert function. Then A has the SLP if there exists a linear form $l \in A_1$ such that

$$\times l : A_i \to A_{c-i} \text{ is bijective for all } i = 0, 1, \dots, [c/2].$$

[1] Detailed proof will appear elsewhere.

Thus we have

$$\text{SLP in the narrow sense} \Leftrightarrow \text{SLP} + \text{symmetric Hilbert function.}$$

In all cases in this article we use the term "strong Lefschetz property" in the narrow sense since we apply it mostly to Gorenstein algebras.

2 The SLP and tensor products

The SLP in the narrow sense is characterized in term of the Lie algebra $\mathfrak{sl}_2$. Using this characterization we can prove that the SLP is closed under taking tensor products. Let $\{L, D, H\}$ be the $\mathfrak{sl}_2$-triple; so we have the relations

$$[L, D] = H, [H, L] = 2L, [H, D] = -2D.$$

Theorem 4 *Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded K-algebra with a symmetric Hilbert function. (We assume that the characteristic of K is zero.) Then the following conditions are equivalent.*

1. *(A, l) has the SLP.*
2. *$\exists \Phi : \mathfrak{sl}_2 \to \mathrm{End}(A)$, a representation of the Lie algebra, such that $\Phi(L) = \times l$ and A_i are the eigenspaces of $H \in \mathfrak{sl}_2$ for the eigenvalues $(2i - c)$, for $i = 0, 1, 2, \ldots$.*

This characterization has many important consequences. We will always assume that K has characteristic zero. The grading, however, is not necessarily standard. It is important to consider algebras with non-standard grading to have a better understanding of algebras with standard grading. Next is a direct consequence of Theorem 4.

Theorem 5 *The SLP is closed under taking tensor products. Namely, if (A, l) and (B, m) have the SLP, then the algebra $(A \otimes_K B, l \otimes 1 + 1 \otimes m)$ also has the SLP.*

Corollary 6 *The monomial complete intersection*

$$K[x_1, x_2, \ldots, x_n]/(x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n})$$

has the SLP with $x_1 + x_2 + \cdots + x_n$ as a Lefschetz element.

Remark 7 Compare the following three assertions:

(1) $K[x_1, x_2, \ldots, x_n]/(x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n})$ has the SLP.

(2) $K[x_1, x_2, \ldots, x_n]/(x_1^2, x_2^2, \ldots, x_n^2)$ has the SLP.

(3) $K[x_1, x_2]/(x_1^{a_1}, x_2^{a_2})$ has the SLP.

Obviously we have the chain of implication: (Theorem 4) $\Rightarrow$ (Theorem 5) $\Rightarrow$ (1) $\Rightarrow$ (2), (3). It seems noteworthy that (2) can be used to prove (3) and (3) can be used to prove Theorem 5 or even Theorem 4. For details see [7]. Below we indicate the implications (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) via Ikeda's lemma.

Lemma 8 (Ikeda) *Suppose that (A, l) has the SLP. Then $(A[z]/(z^2), l + z)$ has the SLP. Hence $K[x_1, x_2, \ldots, x_n]/(x_1^2, x_2^2, \ldots, x_n^2)$ has the SLP with $x_1 + \cdots + x_n$ as a Lefschetz element.*

Using this lemma we want to show that $K[x,y]/(x^{a+1}, y^{b+1})$ has the SLP. Consider the algebra homomorphism

$$\phi : K[X,Y] \to A,$$

where

$$A := K[x_1, x_2, \ldots, x_n]/(x_1^2, x_2^2, \ldots, x_n^2),$$

$$X \mapsto x_1 + x_2 + \cdots + x_a,$$

$$Y \mapsto x_{a+1} + x_{a+2} + \cdots + x_{a+b},$$

$$n = a + b.$$

One sees immediately

$$\operatorname{Ker} \phi = (X^{a+1}, Y^{b+1})$$

$$B := \operatorname{Im} \phi = K[X,Y]/(X^{a+1}, Y^{b+1}) \subset A$$

$$A \text{ and } B \text{ has the same socle degree.}$$

Since B is a subring of A, we have the injections piecewise:

$$\begin{tikzcd}
A_0 \arrow[r, "\times l"] & A_1 \arrow[r, "\times l"] & A_2 \arrow[r, "\times l"] & A_3 \arrow[r, "\times l"] & \cdots \arrow[r, "\times l"] & A_{c-1} \arrow[r, "\times l"] & A_{c-1} \arrow[r, "\times l"] & A_c \\
B_0 \arrow[r, "\times l"] \arrow[u, hook] & B_1 \arrow[r, "\times l"] \arrow[u, hook] & B_2 \arrow[r, "\times l"] \arrow[u, hook] & B_3 \arrow[r, "\times l"] \arrow[u, hook] & \cdots \arrow[r, "\times l"] \arrow[u, phantom, "\cdots"] & B_{c-2} \arrow[r, "\times l"] \arrow[u, hook] & B_{c-1} \arrow[r, "\times l"] \arrow[u, hook] & B_c \arrow[u, hook]
\end{tikzcd}$$

So bijectivity of $A_i \overset{\times l^{c-2i}}{\to} A_{c-i}$ implies that $\times l^{c-2i} : B_i \to B_{c-i}$ is injective. Since $\dim_K B_i = \dim_K B_{c-i}$, it is bijective.

Note that $B := \operatorname{Im} \phi$ is the ring of invariants of A by the Young subgroup $S_a \times S_b \subset S_n$. Also note that the ring of invariants of A by the Young subgroup

$$S_{a_1} \times S_{a_2} \times \cdots \times S_{a_r} \subset S_n$$

is isomorphic to

$$K[X_1, X_2, \ldots, X_r]/(X_1^{a_1+1}, X_2^{a_2+1}, \ldots, X_r^{a_r+1}).$$

The same reasoning can be used to prove that this ring has the SLP. Thus the statement of (3) in Remark 7 is the essential part of Theorem 5 (or even Theorem 4).

The above argument can be extended as follows:

Theorem 9 *Suppose that $A = \bigoplus_{i=0}^{c} A_i$ has the SLP and $B = \bigoplus_{i=0}^{c} B_i$ is a graded subring of A with a symmetric Hilbert function. If $B_c = A_c$ and there exists an element $l \in B_1$ which is an SL element of A, then (B, l) has the SLP.*

It seems to be an interesting problem to ask under what conditions complete intersections can appear as subrings of algebras with the SLP. In this direction we have

Theorem 10 (Goto) *Set*

$$A = K[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_n).$$

(In this article we assume A is an Artinian complete intersection.) Suppose that a finite group G acts on R by linear transformation of the variables. If G is generated by pseudo-reflections then the ring R^G of invariants is a complete intersection.

Suppose we are trying to prove that *all* complete intersections have the SLP. Then "SLP" can be replaced by a weaker condition called "WLP (weak Lefschetz property)." See Theorem 12 below.

Definition 11 (weak Lefschetz property (WLP)) Let $A = \bigoplus_{i=0}^{c} A_i$ be an Artinian algebra with $A_0 = K$. We say that A has the weak Lefschetz property if there exists a linear form $l \in A_1$ such that

$$\times l : A_i \to A_{i+1} \text{ has full rank for all } i = 0, 1, \dots, c-1.$$

We call l a (weak) Lefschetz element. We say that the pair (A, l) has the weak Lefschetz property.

Let K be a field of characteristic zero. Let $\mathcal{C}$ be the family of all graded complete intersections over K. We conjecture that all members of $\mathcal{C}$ have the SLP. It is interesting to note the following

Theorem 12 *TFAE*

1. *All members of $\mathcal{C}$ have the SLP.*
2. *All members of $\mathcal{C}$ have the WLP.*

Concerning the WLP we know the following.

Theorem 13 (J. Migliore – U. Nagel) *Let $A = \bigoplus_{i=0}^{c} A_i$ be an Artinian K-algebra with a symmetric Hilbert function. Then A has the WLP if*
$$\begin{cases} \times l : A_{(c-1)/2} \to A_{(c+1)/2} \text{ is bijective for } \exists l, \text{ (for odd } c), \\ \times l^2 : A_{(c/2)-1} \to A_{(c/2)+1} \text{ is bijective for } \exists l, \text{ (for even } c). \end{cases}$$

3 Flat Extension Theorem

The following is a generalization of Theorem 5.

Theorem 14 (Flat Extension Theorem) *Suppose that $B \to A$ is a finite free extension of Artinian rings, where $B_1 \subset A_1$. Let C be the fiber: $C = A/(B_+)A$. If B and C have the SLP, then A has the SLP.* (Problem: Prove the converse.)

Corollary 15 *A simple extension of B with SLP has the SLP. Namely, if B has the SLP then the algebra $A := B[z]/(z^k + b_1 z^{k-1} + b_2 z^{k-2} + \cdots + b_k)$ has the SLP, where $b_i \in B_i$.*

Proof. A is flat over B and the fiber is isomorphic to $K[z]/(z^k)$. So the flat extension theorem applies.

Corollary 16 *Suppose $A = K[x_1, x_2, \dots, x_n]/(f_1, \dots, f_n)$ is a complete intersection, and suppose that a power of a linear form l can replace one of $f_1, \dots, f_n$ as a generating set for the ideal. Assume that $A/(l)$ has the SLP. Then so does A.*

Proof. Suppose that l^k can replace a member in the minimal generating set for the ideal. Then A is flat over $K[Z]/(Z^k)$, where $Z \mapsto l$, and the fiber is A/lA. Thus the flat extension theorem applies.

Here is an application of the flat extension theorem.

Example 17 *Set $R = K[x_1, x_2, \dots, x_n]$. Let $e_1, e_2, \dots, e_n$ be the elementary symmetric functions:*

$$\begin{aligned} e_1 &= x_1 + x_2 + \cdots + x_n \\ e_2 &= x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n \\ &\vdots \\ e_n &= x_1x_2\cdots x_n \end{aligned}$$

Then R/I, where $I = (e_1, e_2, \dots, e_n)$, has the SLP.

Proof. The generator e_n can be replaced by x_n^n as a member in a minimal set of generators for I. On the other hand the algebra $R/(I + x_nR)$ has embedding dimension $n-1$ and the ideal $(I + x_nR)/x_nR$ is generated by the elementary symmetric functions in $n-1$ variables. Thus the assertion is proved by Corollary 16.

4 A generalization of a theorem of Wiebe

A. Wiebe proved the following

Theorem 18 *Let $A = R/I$ be a graded Artinian K-algebra as a quotient of the polynomial ring R. Let $In(I)$ be the ideal generated by the initial monomials of I with respect to the graded reverse lex order. Then if $R/In(I)$ has the SLP, then so does A.*

This can be generalized as follows:

Theorem 19 *Let $A = \bigoplus_{i=0}^{c} A_i$ be an Artinian graded algebra. For a linear element $z \in A_1$, we set*

$$\mathrm{Gr}_z(A) := A/(z) \oplus (z)/(z^2) \oplus (z^2)/(z^3) \oplus \cdots \oplus (z^r)/(z^{r+1}),$$

with r such that $z^r \neq 0, z^{r+1} = 0$. If there exists a linear form z such that $\mathrm{Gr}_z(A)$ has the SLP, then so does A.

Proof. Let $l \in A_1$ and compare the two linear maps obtained by the multiplication

$$L : A \longrightarrow A$$

and

$$L' : \mathrm{Gr}_z(A) \longrightarrow \mathrm{Gr}_z(A),$$

where L is defined by $L(x) = l \times x$ for $x \in A$ and L' by

$$L'(\overline{x}) = (l \bmod z + z \bmod z^2) \times \overline{x},$$

for $\overline{x} \in \mathrm{Gr}_z(A)$. By the definition of multiplication in $\mathrm{Gr}_z(A)$, this means that

$$\begin{aligned} L'(x \bmod z^{k+1}) &= (l \bmod z + z \bmod z^2) \times \overline{x} &(1)\\ &= (lx \bmod z^{k+1}) + (zx \bmod z^{k+2}) \in \mathrm{Gr}_z(A) &(2) \end{aligned}$$

for $x \in (z^k) \setminus (z^{k+1})$. It is not difficult to see that rank $L \geq$ rank L'. This means that A has at least the weak Lefschetz property. Let x_0 be a new variable. Then to prove that A has the SLP it suffices to prove that $A[x_0]/(x_0^\alpha)$ has the WLP for all positive integers $\alpha > 0$. Since $\mathrm{Gr}_z(B) \cong \mathrm{Gr}_z(A)[x_0]/(x_0^\alpha)$, the SLP of A follows from the WLP of B for all $\alpha > 0$ by the same argument. (See [5].)

5 A sufficient condition for $\mathrm{Gr}_z(A)$ to have SLP

Let A be a graded Artinian (Gorenstein) algebra over K and let $z \in A$ be a linear element. Consider the exact sequence

$$0 \to A/(0:z) \overset{\times z}{\to} A \to A/(z) \to 0.$$

It is sometimes the case that both $A/(0:z)$ and $A/(z)$ are Gorenstein algebras. Even in this situation it is not easy to deduce the SLP of A from the assumption that both $A/(0:z)$ and $A/(z)$ have the SLP. It can be proved, however, that if z is an SL element for $A/(0:z)$, then A has the SLP. In fact this is a sufficient condition for $\mathrm{Gr}_z(A)$ to have the SLP. Below we indicate an outline of proof.

We consider the regular representation of A:

$$\times : A \to \mathrm{End}(A),$$
$$a \mapsto (x \mapsto ax).$$

If we fix a K-basis for A, elements of $\mathrm{End}(A)$ are matrices of size $\dim_K A$. We will be thinking that $\times f$ is a matrix (even if we do not specify a basis). Notice that $\times f$ has only one eigenvalue for any element $f \in A$. If f is a non-unit, then it is nilpotent and all eigenvalues of $\times f$ are zero. The Jordan block decomposition of $z \in A_1$ can be denoted by their sizes of Jordan blocks. In other words this is denoted by a partition of the positive integer $\dim_K A$ or a Young diagram of that size. We use the terms "Young diagram of size N" and "partition of the integer N" interchangeably. As is customary they are regarded as decreasing sequence of integers:

$$n_1 \geq n_2 \geq \cdots \geq n_r, \ \sum n_i = \dim_K A. \tag{3}$$

Let

$$f_1 > f_2 > \cdots > f_s > 0$$

be the finest subsequence of the descending chain of the integers

$$n_1 \geq n_2 \geq \cdots .$$

Then we can describe the Young diagram

$$n_1 \geq n_2 \geq \cdots \geq n_r$$

as

$$\underbrace{f_1, \ldots, f_1}_{m_1}, \underbrace{f_2, \ldots, f_2}_{m_2}, \ldots, \underbrace{f_s, \ldots, f_s}_{m_s}.$$

With these integers $f_1, \ldots, f_s$ we put

$$U_i := \frac{0 : z^{f_i} + (z)}{0 : z^{f_i - 1} + (z)}, \ i = 1, \ldots, s.$$

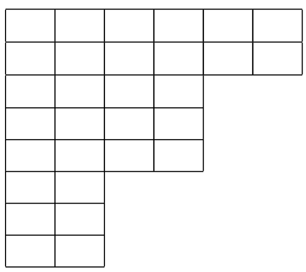
and

$$W_i := \frac{(z^{f_i - 1}) \cap (0 : z)}{(z^{f_i}) \cap (0 : z)}, \ i = 1, \ldots, s.$$

For example if we have the descending sequence of integers (6 6 4 4 4 2 2 2), this is denoted

$$\underbrace{6, \ldots, 6}_{2}, \underbrace{4, \ldots, 4}_{3}, \underbrace{2, \ldots, 2}_{3}.$$

So $s = 3$ and $f_1 = 6, f_2 = 4, f_3 = 2$. The Young diagram for it is:



There are $6 \times 2 + 4 \times 3 + 2 \times 3 = 42$ boxes. This describes a Jordan decomposition of a nilpotent matrix $J = (\times z) \in \mathrm{End}_K(A)$, $A = K^{42}$. If we identify the Young diagram with a Jordan basis for J, we have the following correspondence.

$$\begin{array}{rcl}
\text{Young diagram} & \leftrightarrow & \text{Jordan basis} \\
\text{a box} & \leftrightarrow & \text{basis element} \\
\text{a box } b' \text{ is next to and on the right of } b & \leftrightarrow & J \text{ sends } b \text{ to } b' \\
\text{the box at end of a row} & \leftrightarrow & b \in \ker J \\
\text{the boxes at the end of rows} & \leftrightarrow & \ker J \\
\text{a row} & \leftrightarrow & \text{a Jordan block} \\
\text{the first column} & \leftrightarrow & A/J(A) \\
\text{the } i\text{th column} & \leftrightarrow & J^{i-1}(A)/J^i(A)
\end{array}$$

U_1 consists of the 1st and 2nd boxes of the first column, U_2 consists of the 3rd, 4th and 5th boxes of the first column and U_3 the 6th and 7th of the first column. Similarly W_1 consists of the two boxes at the end of the first and second rows. W_2 are the rightmost boxes of the 3rd, 4th and 5th rows and W_3 the rightmost boxes of the last two rows.

For $z \in A_1$, the sequence of ideals

$$A = (z^0) \supset (z) \supset (z^2) \supset \cdots \supset (z^{N-1}) \supset (z^N) = 0,$$

induces a filtration of $0 : z$ by restriction:

$$(0 : z) \cap (z^0) \supset (0 : z) \cap (z) \supset (0 : z) \cap (z^2) \supset \cdots \supset (0 : z) \cap (z^{N-1}) \supset (z^N) = 0.$$

The non-zero terms of the successive quotients $\frac{(0:z)\cap(z^i)}{(0:z)\cap(z^{i+1})}$ are the modules

$$W_1, W_2, \ldots, W_s.$$

Likewise the sequence of ideals

$$A = 0 : (z^N) \supset 0 : z^{N-1} \supset 0 : z^{N-2} \supset \cdots \supset 0 : z \supset 0$$

induces a co-filtration on the algebra $A/(z)$. Namely a sequence of surjections:

$$A/(z) \twoheadrightarrow A/((z) + (0 : z)) \twoheadrightarrow A/((z) + (0 : z^2)) \twoheadrightarrow \cdots \twoheadrightarrow A/((z) + (0 : z^N)) = 0.$$

Among the surjections $\twoheadrightarrow$, those with a non-trivial cokernel give us the modules

$$U_i := \frac{0 : z^{f_i} + (z)}{0 : z^{f_i - 1} + (z)}, \ i = 1, \ldots, s.$$

Recall that we have chosen $\{f_i\}$ in the decreasing order. So one sees that U_i and W_i are isomorphic as graded $A/(z)$-modules with the shift of degree by $f_i - 1$. So

$$U_i[-f_i + 1] \cong W_i, \ \ U_i \ni b \mapsto bz^{f_i - 1} \in W_i.$$

Observe the following

Proposition 20 *With the same notation as above, suppose that (A, l) is a pair of a graded Artinian ring and a linear element and*

$$\underbrace{f_1, \cdots, f_1}_{m_1}, \underbrace{f_2, \cdots, f_2}_{m_2}, \cdots, \underbrace{f_s, \cdots, f_s}_{m_s}.$$

is the Jordan type of $\times l \in \mathrm{End}_K(A)$.

1. *U_s coincides with W_s if and only if $f_s = 1$. (Recall that U_i and W_i are isomorphic by the map $b \mapsto bz^{f_i-1}$.) In this case $U_s = ((0:l)+(l))/(l) = (0:l)/((0:l)\cap(l)) = W_s$.*

2. *$f_s > 1$ if and only if $(0:l) \subset (l)$.*

3. *The Jordan type of $\times\bar{l} \in \mathrm{End}_K(A/(0:l))$ is given by*

$$\underbrace{f_1-1,\cdots,f_1-1}_{m_1},\underbrace{f_2-1,\cdots,f_2-1}_{m_2},\cdots,\underbrace{f_s-1,\cdots,f_s-1}_{m_s}.$$

(If $f_s = 1$, the last block should be dropped.)

4. *The central irreducible modules (as we call them) $W_1, W_2, \ldots, W_s$ for (A, l) and these modules $W'_1, W'_2, \ldots, W'_{s'}$ for $(A/(0:l), \bar{l})$ may be regarded as the same if $f_s > 1$. (So $s' = s$.)*

5. *If $f_s = 1$, then $W_1, W_2, \ldots, W_{s-1}$ for (A, l) may be regarded as these modules $W'_1, W'_2, \ldots, W'_{s'}$ for $(A/(0:l), \bar{l})$. (So $s' = s-1$.)*

Let Z be a variable, and denote by $K[Z]$ the one dimensional polynomial ring. If $V = \bigoplus_{i=a}^b V_i$ is a finite graded $K[Z]$-module, we say that V has the SLP if the homomorphism

$$\times Z^{b-a-2i} : V_{a+i} \to V_{b-i}$$

is bijective for all $i \le [(b-a)/2]$. Note that the Hilbert function of $V = \bigoplus_{i=a}^b V_i$ is symmetric if $\dim_K V_{a+i} = \dim_K V_{b-i}$ for $i = 0, 1, 2, \ldots, [(a+b)/2]$, provided that $V_a \neq 0, V_b \neq 0$. Let $V = \bigoplus_{i=a}^b V_i$ be a graded vector space over K. Then V has a $K[Z]$-module structure if a degree one homomorphism $L : V \to V$ is given. For a K-vector space V, we write $V[Z]/(Z^f) := V \otimes_K K[Z]/(Z^f)$. If $V = \bigoplus V_{i=a}^b$ is a finite graded vector space with a degree one homomorphism $L : V \to V$, then $\tilde{V} := V[Z]/(Z^f)$ is made into a graded $K[Z]$-module via the degree one homomorphism

$$L + Z : V[Z]/(Z^f) \to V[Z]/(Z^f)$$

defined by

$$\sum v_i Z^{d-i} \mapsto (\sum v_i Z^{d-i})(L+Z) = \sum L v_i Z^{d-i} + \sum v_i Z^{d-i+1}.$$

We use these notation and terminology in the following proposition in this sense. (So we denote by $\tilde{U}_i$ the module $\tilde{U}_i := U_i[Z]/(Z^{f_i}) := U_i \otimes_K K[Z]/(Z^{f_i})$.) We can prove the following

Proposition 21 *Suppose that $A = \bigoplus A_i$ is a graded Gorenstein algebra and let $l \in A_1$ be any linear form. Let U_i be the central irreducible modules for (A, l) as defined before. We have*

1. *U_i has a symmetric Hilbert function for all i.*

2. *$\tilde{U}_i$ has a symmetric Hilbert function for all i.*

3. *If U_i has the SLP as an $A/(l)$-module, then $\tilde{U}_i$ has the SLP as an $(A/(l))[Z]$-module.*

4. *If all U_i have the SLP, then $\mathrm{Gr}_l(A)$ has the SLP.*

Proof. It is easy to see the equivalence of (1) and (2). Denote by $\mathrm{Hilb}(V)$ the Hilbert series

$$\mathrm{Hilb}(V) = \sum (\dim_K V_i) T^i.$$

One sees immediately that

$$\sum_{j=1}^{s} \mathrm{Hilb}(U_j) = \mathrm{Hilb}(A) - \mathrm{Hilb}(A/(0:l)),$$

and

$$\sum_{j=1}^{s} \mathrm{Hilb}(\tilde{U}_j) = \mathrm{Hilb}(A),$$

and moreover the modules $W_1, W_2, \ldots, W_{s-1}$ for (A, l) coincide with those modules for $(A/(0:l), \bar{l})$. This proves (1) and (2) at the same time by induction. The assertion (3) follows from Theorem 4. To see (4) let l be a linear form in A. Then A may be viewed as a $K[Z]$-module by $Z \mapsto l$. The element l induces homomorphisms on U_i and we may view $\tilde{U}_i$ as $K[Z]$-modules and likewise we may view $\mathrm{Gr}_l(A)$ as $K[Z]$-modules by $Z \mapsto (l \bmod (l) + z \bmod (l^2))$. As $K[Z]$-modules we have the isomorphism

$$\bigoplus_{i=1}^{s} \tilde{U}_i \cong \mathrm{Gr}_l(A).$$

This proves (4) (for more details see [2]).

Note that the Hilbert function of the first column (i.e., Hilbert function of $A/(l)$) determines the Hilbert function of A and that W_i are U_i are graded vector spaces and $U_i \cong W_i$ with a shift of degrees by $f_i - 1$. As was remarked earlier, it is possible to define the SLP for graded vector spaces with degree one maps.

With this extended definition of SLP we can state our result as follows:

Theorem 22 *Let A be a graded K-algebra and $l \in A_1$ a linear form. Then the following are equivalent.*

1. $\mathrm{Gr}_l(A) = A/(l) \oplus (l)/(l^2) \oplus \cdots (l^r)/(l^{r+1})$ *has the SLP.*
2. *All of $U_1, \cdots, U_s$ have the SLP as graded $A/(l)$-modules.*

Here are some examples of computation for $U_1, \ldots, U_s$.

Example 23 Let $A = K[x, y, z]/(x^2, y^2, z^2)$. For a basis for A we take the set of square-free monomials of x, y, z. The modules $U_1, U_2, \ldots, W_1, \ldots$ etc. depend on the choice of l.

1. Consider $l = z$. Then the Jordan decomposition of $\times l \in \mathrm{End}(A)$ is given by $8 = 2 + 2 + 2 + 2$. A Jordan basis for $\times l$ is:

1	l
x	xl
y	yl
xy	xyl

$U_1 = \langle 1, x, y, xy \rangle$, $W_1 = \langle l, xl, yl, xyl \rangle$.

2. Consider $l = x+y$. Then the Jordan decomposition of $\times l \in \mathrm{End}(A)$ is $8 = 3+3+1+1$. For simplicity put $f := x - y$. Then a basis for $\times l$ is:

1	l	l^2
z	lz	zl^2
f		
fz		

   $U_1 = \langle 1, z \rangle$, $W_1 = \langle l^2, zl^2 \rangle$, $U_2 = W_2 = \langle f, fz \rangle$.

3. Consider $l = x+y+z$. Then the Jordan decomposition of $\times l \in \mathrm{End}(A)$ is $8 = 4+2+2$. For simplicity put $g := x - y, h = x - z$. Then a basis for $\times l$ is:

1	l	l^2	l^3
g	gl		
h	hl		

   $U_1 = \langle 1 \rangle$, $W_1 = \langle l^3 \rangle$, $U_2 = \langle g, h \rangle$, $W_2 = \langle gl, hl \rangle$.

We exhibit some more examples of complete intersections for which the modules $U_1, \ldots, U_s$ can be computed (for some l) and the SLP can be proved via Theorem 22. In the following examples $e_d(x_1, \ldots, x_n)$ denotes the elementary symmetric functions of degree d. In these examples it is possible to compute the modules $U_1, \ldots, U_s$ by choosing the linear form z as one of the variables. For proof we refer to [2].

Example 24 Let
$$R = K[x_1, \ldots, x_n]$$
$$f_i = e_i(x_1^r, x_2^r, \ldots, x_n^r),\ i = 1, \ldots, n-1,$$
$$f_n = e_n^s = (x_1 \cdots x_n)^s.$$
$$I = (f_1, f_2, \ldots, f_{n-1}, f_n).$$
Then $A = R/I$ has the SLP.

Example 25 Let
$$R = K[x_1, \ldots, x_n]$$
$$f_i = e_i(x_1^r, x_2^r, \ldots, x_n^r),\ i = 1, \ldots, n,$$
$$I = (f_1, f_2, \ldots, f_n).$$
Then $A = R/I$ has the SLP.

6 Inductive argument to compute $U_1, \ldots, U_s$

The Artinian algebras in Examples 24 and 25 are complete intersections and in proving that they have the SLP, it is the key that one of the generators of the ideal is divisible by a linear element. So it seems worth noticing the following

Lemma 26 *1. Let $I = (a_1, \ldots, a_n) \subset R$ be a complete intersection ideal in a regular local ring R, and suppose that $a_n = ab$ is a product of two elements. Then $I' = (I : b) = (a_1, \ldots, a_{n-1}, a)$.*

2. *Suppose that $I \subset R$ is a homogeneous complete intersection ideal in a polynomial ring R and $a_n = al$ with l a linear form. Let*

$$\underbrace{f_1, \cdots, f_1}_{m_1}, \underbrace{f_2, \cdots, f_2}_{m_2}, \cdots, \underbrace{f_s, \cdots, f_s}_{m_s}.$$

   be the Jordan type of $\times l \in \mathrm{End}_K(A)$. Then $f_s > 1$ if and only if $((I : l) + (l))/(l)$ is a complete intersection ideal (i.e., it is generated by $n - 1$ elements).

3. *$f_s = 1$ if and only if $((I : l) + (l))/(l)$ is an almost complete intersection ideal.*

4. *If $((I : l) + (l))/(l)$ is an almost complete intersection ideal, then $U_s = W_s = ((a) + (l))/(l) \cong (A/(l))/(l : a)$.*

Example 27 Let

$$R = K[x_1, \ldots, x_n]$$
$$I = (L_1, L_2, \ldots, L_{n-1}, L_n),$$

where L_i are products of linear forms which are "general enough among themselves." Then $A = R/I$ has the SLP.

Proof. Let l be one of the factors of L_n and consider the exact sequence:

$$0 \to A/(0 : l) \to A \to A/(l) \to 0.$$

The algebras $A/(0 : l)$ and $A/(l)$ are complete intersections by an ideal generated by products of linear forms. So we can use the induction. By the definition $\bar{l}$ is an SL element of $A/(0 : l)$. Consider the central irreducible modules

$$U_1, U_2, \ldots, U_{s-1}, U_s,$$

for (A, l). We want to prove that all of them have the SLP as $A/(l)$-modules. By Proposition 20, the modules

$$U_1, U_2, \ldots, U_{s-1}$$

are central irreducible modules for $(A/(0 : l), \bar{l})$. Since l is general enough, it is an SL element and we get that $U_1, U_2, \ldots, U_{s-1}$ are graded vector spaces with homogeneous components concentrated at one degree. Thus they all have the SLP as $A/(l)$-modules. Consider the module U_s. It is a principal ideal of $A/(l)$ generated by the element $L' := L_s/l$. Thus it is isomorphic to $A/(l : L')$. By the induction hypothesis $A/(l)$ has the SLP. Since L' is a product of general enough linear forms we can conclude that $U_s \cong A/(l : L')$ also has the SLP by Proposition 20.

Remark 28 In the above example we did not define the phrase "general enough linear forms." It is hoped that the reader may understand the meaning of this terminology as the proof reveals it. Generally speaking it is not easy to check if a given linear form is general enough for some purpose in a particular situation. If the ring is a Gorenstein algebra with the SLP, then a general element should be thought of as an SL element. We treat more examples of this kind in the next section, where these modules are computable.

Remark 29 In Example 27, one of L_i can be replaced by $\boxed{\text{any}}$ homogeneous form. This is because we can start the induction with $n = 2$, in which case the SLP for complete intersections are proved.

7 The algebras with the action of the symmetric group

In Section 9 of [7] we indicated how the Schur–Weyl duality can be applied to the monomial complete intersection with generators of equal degree

$$A := K[x_1, x_2, \dots, x_n]/(x_1^d, x_2^d, \dots, x_n^d).$$

Put $B = K[\epsilon]/(\epsilon^2)$. Let B^* be the multiplicative group of B (so it is $\{a + b\epsilon \mid a, b \in K, a \neq 0\}$), and similarly A^* the multiplicative group of A. Define the group homomorphism

$$\Phi : B^* \to A^*$$

by $a + b\epsilon \mapsto a \exp((b/a)(x_1 + x_2 + \cdots + x_n))$. In this case the only eigenvalue of $\Phi(a + b\epsilon)$ is a^{dn} and the Jordan type of it is given by the dual partition to the Hilbert function of A. (See the remark below.)

Remark 30 Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded Artinian algebra. The Hilbert function of A may be regarded as a partition of the integer $\dim_K A$, since

$$\mathrm{Hilb}(A, T) = \sum_{i=0}^{c} (\dim_K A_i) T^i.$$

and

$$\dim_K A = \sum_{i=0}^{c} (\dim_K A_i). \tag{4}$$

(If one prefers, one could re-order it in the decreasing order.) Let $\dim_K A = p_0 + p_1 + p_2 + \cdots + p_r$ be the dual partition of (4). (Assume this is in the decreasing order.) If (4) is unimodal, then the dual partition is given by

$$(p_0, p_1, \dots, p_r) = \tag{5}$$

$$\left(\underbrace{c+1}_{h_0}, \underbrace{c-1, \cdots, c-1}_{h_1 - h_0}, \underbrace{c-3, \cdots, c-3}_{h_2 - h_1}, \underbrace{c-5, \cdots, c-5}_{h_3 - h_2} \cdots \right).$$

It seems natural to expect that the same is true with other Artinian algebras with the action of the symmetric group S_n. So we consider the following problem.

A new problem

Problem 31 *Let $n, d > 0$ be positive integers. Let $R = K[x_1, \dots, x_n]$ be the polynomial ring over a field K and let S_n, the symmetric group, act on R as permutation of the variables. Suppose $I = (g_1, \dots, g_n)$ is a set of homogeneous polynomials permuted among themselves by the action of S_n. Then the group S_n naturally acts on $A := R/I$ as automorphisms of the algebra. Prove that A has the SLP with $x_1 + x_2 + \cdots + x_n$ as a strong Lefschetz element.*

For simplicity we assume that S_n permutes the polynomials

$$g_1, g_2, \cdots, g_n$$

in the same way as S_n permutes the variables.

We can think of two or three different ways of choosing generators.

1. Let F be a symmetric homogeneous polynomial of degree d and put

$$g_1 := x_1^d + F, g_2 := x_2^d + F, \dots, g_n := x_n^d + F. \tag{6}$$

2. Let $(a_1, \ldots, a_n) \in K^n$ be an arbitrary constant vector. Let $e = x_1 + x_2 + \cdots + x_n$.

$$g_i = (e - a_1 x_i)(e - a_2 x_i) \cdots (e - a_d x_i), \quad i = 1, \ldots, n \tag{7}$$

3. Let F be a homogeneous polynomial of degree d and put

$$g_1 := x_1^d - x_2^d, g_2 := x_2^d - x_3^d, \cdots, g_{n-1} := x_{n-2}^d - x_n^d, g_n := F. \tag{8}$$

One notices easily that both (6) and (7) are two different sets generators of the same ideal as described in Problem 31. Since the stabilizer of g_1 should be S_{n-1} the generators in either (6) or (7) can be deconstructed as in (8). Note that in either case (6) or (7), the algebra A is similar to that of Example 27, except that we do not know if the linear forms involved are general enough.

In Section 1 we showed that the algebra $K[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ contains the algebra $K[x_1, \cdots, x_r]/(x_1^d, \ldots, x_r^d)$ as a subring. So, to approach Problem 31, it seems good enough to assume $d = 2$ at least to start with. For the case $d = 2$ we show how Problem 31 can be settled in the next section.

8 Complete intersections generated by quadrics

We work in the polynomial ring $R = K[x_1, \ldots, x_n]$, and we fix the following notation:

$$\begin{array}{rcl} a, p, q & \in & K \\ e_1 & = & x_1 + x_2 + \cdots + x_n \\ e_2 & = & x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n \\ g_1 & = & x_1^2 - x_n^2 \\ g_2 & = & x_2^2 - x_n^2 \\ & \vdots & \\ g_{n-1} & = & x_{n-1}^2 - x_n^2 \\ g_n & = & pe_1^2 + qe_2 \\ I' & = & (g_1, g_2, \ldots, g_{n-1}, g_n) \\ I & = & (g_1, g_2, \ldots, g_{n-1}, e_2 + ax_n^2) \\ J & = & (x_1^2, x_2^2, \ldots, x_{n-1}^2, e_2) \end{array}$$

It is easy to see the following

Proposition 32 (i) *I' and I are the same ideal with $a = np$.*

(ii) *R/J is isomorphic to $R/(x_1^2, \ldots, x_{n-1}^2, x_n^2)$ by the homomorphism*

$$x_i \mapsto \begin{cases} x_i \text{ if } i < n, \\ x_1 + x_2 + + \cdots + x_n \text{ if } i = n. \end{cases}$$

Hence R/J has the SLP with x_n as a general element.

(iii) $\mathrm{In}(I) = \mathrm{In}(J)$ *with respect to the graded reverse lexicographic order.*

(iv) *Put $z := x_n$. Then $\mathrm{Gr}_z(R/I) = \mathrm{Gr}_z(R/J)$. Hence R/J has the SLP with x_n a general element. (In fact it is an SL element.)*

Proof. (We are assuming that p, q, a are chosen so that these ideals are complete intersections.) It is easy to see (i) and (ii) are true and the implication (iii) $\Rightarrow$ (iv). We need a little trick to verify (iii).

Remark 33 1. Let I be the ideal defined in (6). It is possible to compute the central irreducible modules for $(A := R/I, z)$, where z is one of the factors of g_n.

2. Let I be the ideal defined in (7). It is possible to compute the central irreducible modules for $(A := R/I, z)$, where z is one of the factors of g_n.

9 The homomorphism $\mathrm{SL}(2) \to \mathrm{GL}(A)$

Theorem 34 *Let $R = K[x_1, x_2, \ldots, x_n]$ be the polynomial ring over a field K of characteristic zero. Let I be a complete intersection ideal generated by quadrics on which the symmetric group S_n acts by permutation of the variables. Then $A := R/I$ has the SLP with $x_1 + \cdots + x_n$ as an SL element. In particular there is a homomorphism*

$$\Phi : \mathrm{SL}(2) \to \mathrm{GL}(A)$$

such that

$$d\Phi\left(\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}\right) = \times k(x_1 + x_2 + \cdots + x_n),$$

where

$$d\Phi : \mathfrak{sl}_2 \to \mathrm{End}(A)$$

is the induced map of Φ on the Lie algebras. The Jordan type of $\Phi\left(\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\right)$ is the dual to the partition of 2^n given by the coefficients of $(1+T)^n$.

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