GOOD IDEALS ON 2-DIMENSIONAL NORMAL SINGULARITIES

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1. Introduction

In a 2-dimensional rational singularity, Lipman showed in [Li] that that every integrally closed ideal is "stable" in the sense that $I^2 = IQ$ holds for every minimal reduction Q of I. This fact plays very important role to study ideal theory on a 2-dimensional rational singularity.

On the other hand, as far as we know, almost nothing was done concerning ideal theory of non-rational singularities. Here we study "good ideals" of 2-dimensional normal singularities and discuss existence and characterization of such ideals.

Definition 1.1. Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring and I be an $\mathfrak{m}$ primary ideal of A.

(1) We say I is *stable* if $I^2 = IQ$ for every minimal reduction Q of I.
(2) (Goto-Iai, [GIW]) We say I is a good ideal if it satisfies the following conditions.
  (a) I is stable and
  (b) $Q : I = I$.

Note that if I is stable, then $I \subset Q : I$ and by the characterization of core of ideals, under the condition (1), the condition (2)(b) is equivalent to the following conditions. Recall that core(I) is the intersection of all minimal reductions of I.

(2′) [[CPU], Example 3.1] core$(I) = I^2$.

If A is Gorenstein, then I is good if and only if I is stable and

(2″) $2\ell_A(A/I) = e(I)$,

where $e(I) = \ell_A(A/Q)$ denotes the multiplicity of I (note that this equivalence does not hold id A is not Gorenstein).

We will discuss about the existence of good ideals for any 2-dimensional normal singularities. Also, we will discuss about the non-existence of Ulrich ideals ([GOTWY]) for certain 2-dimensional Gorenstein rings which are not complete intersections.

Our argument is rather "geometric" and we discuss properties of cycles on a resolution of singularities of Spec (A). We always assume the existence of a resolution of singularities of Spec (A) and that the residue field of A is algebraically closed.

This paper is an announcement of our results and the detailed version [OWY] will be submitted to somewhere else.

This work was partially supported by JSPS Grant-in-Aid for Scientific Research (C) 23540068, 23540059, 25400050. The second and third named authors were partially supported by Individual Research Expense of College of Humanity and Sciences, Nihon University.

2. p_g-CYCLES

In the following, we always assume that A is a two-dimensional normal local ring and $f : X \to \operatorname{Spec} A$ is a resolution of singularities with $E := f^{-1}(\mathfrak{m})$ the exceptional divisor on X. Let $E = \cup_{i=1}^r E_i$ be the decomposition into irreducible components of E. We say that E_i is a (-1) *curve* if $E_i \cong \mathbb{P}^1$ and $E_i^2 = -1$. We say X is a *minimal resolution* of Spec (A) if X contains no (-1) curve.

An $\mathfrak{m}$-primary ideal I is said to be *represented on* X if the sheaf $I\mathcal{O}_X$ is invertible and $I = H^0(X, I\mathcal{O}_X)$. If an ideal I is represented on some resolution $X \to \operatorname{Spec} A$, then it is integrally closed. Conversely, any integrally closed ideal can be represented on some (may not be minimal) resolution X of the singularity.

A cycle on X is a formal sum (or a divisor on X supported on E) $Z = \sum n_i E_i$. For cycles $Z = \sum n_i E_i$ and $Z' = \sum n_i' E_i$, we denote $Z \geq Z'$ if $n_i \geq n_i'$ for every i. In particular, we say that Z is *effective* if $Z \geq 0$. If I is am $\mathfrak{m}$ primary ideal of A and if I is represented on X, then $I\mathcal{O}_X = \mathcal{O}_X(-Z)$ for some effective cycle Z. In this case, we denote $I = I_Z$.

Definition 2.1. Let X be a resolution of Spec (A) as above.

(1) $p_g(A) = \ell_A(H^1(X, \mathcal{O}_X))$ is called the *geometric genus* of A. A is a rational singularity if and only if $p_g(A) = 0$. Also, we denote $h^1(\mathcal{O}_X(-Z)) = \ell_A(H^1(X, \mathcal{O}_X(-Z))$.

(2) An effective cycle $Z = \sum n_i E_i$ is called an *anti-nef cycle* if $Z.E_i \leq 0$ for every irreducible component E_i of E. Note that if I is represented on X and $I = I_Z$, then Z is an anti-nef cycle.

We know the following upper bound of $h^1(X, \mathcal{O}_X(-Z))$ for anti-nef cycles on X.

Proposition 2.2 ([Mo],[OWY]). *If Z is an anti-nef cycle on some resolution X of* Spec (A)*, then $h^1(\mathcal{O}_X(-Z)) \leq p_g(A)$. Moreover, if equality holds, then $\mathcal{O}_X(-Z)$ is generated by global sections.*

An anti-nef cycle Z with the property $h^1(X, \mathcal{O}_X(-Z))$ has many nice properties.

Definition 2.3. Let Z be an anti-nef cycle on X. We call Z a p_g cycle if $h^1(\mathcal{O}_X(-Z)) = p_g(A) = h^1(\mathcal{O}_X)$. in this case, we call I_Z a p_g ideal. If A is a rational singularity, an $\mathfrak{m}$ primary ideal I is a p_g ideal if and only if I is integrally closed.

Theorem 2.4. *(1) If Z is a p_g cycle and if $I = I_Z = H^0(X, \mathcal{O}_X(-Z))$, then $\overline{I^2} = QI$ for any minimal reduction Q of I, where $\overline{I^2}$ means the integral closure of I^2.*

(2) If Z, Z' are both p_g cycles, so is $Z + Z'$. That is, if I, I' are p_g ideals, then II' is also a p_g ideal.

(3) Conversely, if Z, Z' are anti-nef cycles on X, Z and $Z + Z'$ are p_g cycles, then so is Z'.

We can also show the existence of p_g ideals.

Theorem 2.5. *For any 2-dimensional normal ring A, we can construct a resolution X and a p_g cycle Z on X. Moreover, if A is Gorenstein, then we can construct X, Z so that $K_X Z = 0$, where K_X is the canonical divisor on X.*

3. Existence of Good ideals

When A is Gorenstein, the existence of good ideals follows from the Riemann-Roch theorem.

For invertible sheaf $\mathcal{L}$ on X, we define $\chi(\mathcal{L})$ by

$$\chi(\mathcal{L}) = \ell_A\left(H^0(X \setminus E, \mathcal{L})/H^0(X, \mathcal{L})\right) + h^1(\mathcal{L}).$$

Note that $\chi(\mathcal{O}_X) = p_g(A)$ since A is normal.

Theorem 3.1 (**Kato's Riemann-Roch formula**). *([K], [WY]) For a cycle Z on X, we have*

$$\chi(\mathcal{O}_X(Z)) - \chi(\mathcal{O}_X) = -(Z^2 - ZK_X)/2.$$

In particular, if $I = I_Z$, then we have

$$\ell_A(A/I_Z) = -(Z^2 + ZK_X)/2 + (p_g(A) - h^1(\mathcal{O}_X(-Z))).$$

In general, we have the following equality for good ideals.

Lemma 3.2. *Let I be an $\mathfrak{m}$ primary ideal and assume that I_Z is the integral closure of I. We assume I is a good ideal.*

1. *If I_Z is stable, then $I = I_Z$.*
2. *If A is Gorenstein, then $K_X Z = 2(\ell_A(I_Z/I) + (p_g(A) - h^1(\mathcal{O}_X(-Z)))$.*
3. *In particular, if A is Gorenstein and I_Z is a p_g ideal, then I is a good ideal if and only if $I = I_Z$ and $K_X Z = 0$.*

Proof. (1) Let Q be a minimal reduction of I, which is also a minimal reduction of I_Z. Since $I_Z \subset Q : I_Z$, we have

$$I_Z \subset Q : I_Z \subset Q : I = I.$$

(2) The statement follows from Theorem 3.1 and the fact that $e(I) = -Z^2 = 2\ell_A(A/I)$ and (3) follows from (1) and (2). $\square$

As a corollary of Theorem 2.5 we get the existence theorem for good ideals.

Theorem 3.3. *Let $(A, \mathfrak{m})$ be a 2-dimensional Gorenstein normal local ring. then A has a good p_g ideal.*

Example 3.4. *Let A be a 2-dimensional normal local ring.*

1. *([GIW]) If A is a regular local ring, then A has no good ideals. If A is a rational Gorenstein singularity which is not regular, then I is a good ideal if and only if I is integrally closed and represented on the minimal resolution. As we will show later, this part is also true for non- Gorenstein case.*
2. *It is easy to see that $\mathfrak{m}$ is a good ideal if and only if it is stable. Hence if A is Gorenstein, then $\mathfrak{m}$ is a good ideal if and only $e(A) = 2$. Note that if $e(A) \geq 3$, then $\mathfrak{m}$ is not a p_g ideal, too.*
3. *If A is Gorenstein with $p_g(A) = 1, e(A) = 4$, then there is a good ideal with $\bar{I} = \mathfrak{m}$ and $\ell_A(A/I) = 2$ by Lemma 3.2 (2).*

4. Non-Gorenstein case.

When A is not Gorestein, we do not have numerical criterion for good ideals like Lemma 3.2 (2). So we are not successful yet to get criterion for good ideals. But we have some necessary conditions with respect to (-1) curves on X.

Recall that if X contains a (-1) curves C (i.e. $C \cong \mathbb{P}^1$ and $C^2 = -1$), then $f : X \to \operatorname{Spec} A$ is decomposed as $f = \pi \circ g$ such that $\pi : X \to X'$ is a contraction of C and $g : X' \to \operatorname{Spec} A$ is a morphism of schemes with X' regular. In particular, every resolution factors through the minimal resolution.

Lemma 4.1. *Let C be a (-1) curve on X and Z be a p_g cycle on X. Put $Z' = Z - C$ and Q be a minimal reduction of I_Z. If $ZC < 0$ and $I_Z \subsetneq I_{Z'}$, then $I_{Z'} \subset Q : I_Z$ and hence I_Z is not a good ideal.*

If A is a rational singularity, this gives a satisfactory necessary condition for a good ideal.

Theorem 4.2. *Let $(A, \mathfrak{m})$ be a two-dimensional rational singularity and let I be an $\mathfrak{m}$-primary ideal in A. Then I is good if and only if I is represented by some cycle on a* minimal *resolution.*

Proof. Here we show the "only if" part. Let $I = I_Z$ be a good ideal represented on X. We assume that I is not represented on the minimal resolution and show that I is not a good ideal. Then we may assume that X contains a (-1) curve E with $ZE < 0$. Then $Z' = Z - E$ is anti-nef. Since there is one to one correspondence between anti-nef cycles on X and the integrally closed ideals represented on X ([Gi]), we get $I_Z \subsetneq I_{Z'}$. Hence I is not a good ideal by Lemma 4.1. $\square$

Now we investigate what happens if $I = I_Z$ is not a good ideal for a p_g cycle Z. Assume that Q is a minimal reduction of I and take $h \in Q : I$, $h \notin I$. Now, let $Z_h = \sum v_{E_i}(h)E_i$ be the cycle defined by h. Here the condition $h \in I_Z$ is equivalent to $Z \leq Z_h$. Since I is stable, $I \subset I : Q$ and if this is not the case, adding a general element of I_Z to h, we may assume $Z_h = Z - D$ for some positive cycle D.

Lemma 4.3. *Let $D > 0$ defined as above. Then $h^1(\mathcal{O}_X(D)) = p_g(A)$ and we have $D^2 = K_X D$.*

Proof. Look at the map $h : \mathcal{O}_X \to \mathcal{O}_X(-Z_h)$ induced by the multiplication of h. Here, by the definition of Z_h, if we put $\mathcal{C} = \operatorname{Coker}(h : \mathcal{O}_X \to \mathcal{O}_X(-Z_h))$, then $\operatorname{Supp}(\mathcal{C})$ is finite over Spec (A) and hence is affine.

Next, we put $Q = (f, g)$ and consider the short exact sequence

$$(4.3.1) \quad 0 \to \mathcal{O}_X(Z) \to \mathcal{O}_X^{\oplus 2} \to \mathcal{O}_X(-Z) \to 0,$$

where $\mathcal{O}_X^{\oplus 2} \to \mathcal{O}_X(-Z)$ is induced by multiplication of (f, g). Taking the cohomology ling exact sequence we get a short exact sequence

$$0 \to I/Q \to H^1(\mathcal{O}_X(Z)) \to M \to 0,$$

where we put $M = \operatorname{Ker}\,(H^1(\mathcal{O}_X^{\oplus 2}) \to H^1(\mathcal{O}_X(-Z))$. Since Z is a p_g cycle, we have $\ell_A(M) = p_g(A)$.

Tensoring $\mathcal{O}_X(-Z_h)$ to (4.3.1), we get a short exact sequence

(4.3.2) $\quad 0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(-Z_h)^{\oplus 2} \to \mathcal{O}_X(-Z - Z_h) \to 0.$

Now from the exact sequence $0 \to \mathcal{O}_X(Z) \xrightarrow{h} \mathcal{O}_X(D) \to \mathcal{C} \otimes \mathcal{O}_X(Z) \to 0$, we get that $h : H^1(\mathcal{O}_X(Z)) \to H^1(\mathcal{O}_X(D))$ is surjective and since $I_Z/Q \subset H^1(\mathcal{O}_X(Z))$ is annihilated by h, $h^1(\mathcal{O}_X(D)) \leq \ell_A(M) = p_g$.

The converse inequality $h^1(\mathcal{O}_X(D)) \geq p_g$ is deduced from $0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0$ and the vanishing $H^0(\mathcal{O}_D(D)) = 0$ (cf. [Wah]). Then from Theorem 3.1 for $\chi(\mathcal{O}_X(D))$, we get $D(D - K_X) = 0$. $\square$

Now we finish the "if" part of Theorem 4.2. It suffices to show that there does not exist a positive cycle D on the minimal resolution, which satisfies the condition of Lemma 4.3. Actually, since $K_X E_i \geq 0$ for every irreducible compoment on X and hence $K_X D \geq 0$. But since $D^2 < 0$, we cannot have $D^2 = K_X D$.

We are now trying to find the condition for an integrally closed ideal $I = I_Z$ to be a good ideal, where Z is a p_g. We hope to be able to prove the existence of good ideals for any 2-dimensional normal local ring in a near future.

Example 4.4. *Let $A = k[[X^r, X^{r-1}Y, \ldots, Y^r]]$ be the so called "r-th Veronese subring". Then the exceptional set of the minimal resolution of* Spec (A) *consists of a single rational curve E with $E^2 = -r$. Hence a good ideal on A is $\mathfrak{m}^n$ for some n.*

In general, by Theorem 4.2, the set of good ideals on A is countable.

5. Ulrich ideals on Gorenstein singularities with $p_g(A) = 1$.

In a d-dimensional Cohen-Macaulay local ring, and $\mathfrak{m}$ primary ideal I is an Ulrich ideal if I is stable and I/I^2 is a free A/I module.

See [GOTWY] for general property of Ulrich ideals and [GOTWY2] for classification of Ulrich ideals on rational singularities.

Here, we show that two dimensional normal Gorenstein singularity with high multiplicity has no Ulrich ideals.

Theorem 5.1. *Let A be a two-dimensional normal local ring with $p_g(A) \geq 5$. Then A has no Ulrich ideals.*

Proof. It is known that a good ideal I is an Ulrich ideal if and only if $\mu(I) = 3$, where $\mu(I)$ denotes the number of minimal generating system of I (cf. [GOTWY]). The proof splits into several Lemmas. $\square$

Lemma 5.2. *Let A be a normal Gorenstein ring with $p_g(A) = 1$ and $e(A) \geq 3$.*

(1) *If I is an integrally closed $\mathfrak{m}$ primary ideal, then $\mu(I) \geq e(A) = \mu(\mathfrak{m})$.*
(2) *If I is an Ulrich ideal, then $\ell_A(\bar{I}/I) \leq 1$, where $\bar{I}$ is the integral closure of I.*

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