Ulrich ideals on hypersurfaces [1]

Shiro Goto (Meiji University)
Kazuho Ozeki (Yamaguchi University)
Ryo Takahashi (Nagoya University)
Kei-ichi Watanabe (Nihon University)
Ken-ichi Yoshida (Nihon University)

The main aim of this talk is to classify all Ulrich ideals and Ulrich modules for simple singularities. In order to do that, we investigate the relationship between Ulrich ideals and Ulrich modules for hypersurface local domain with the multiplicity $e_{\mathfrak{m}}^0(A) = 2$.

1. Ulrich ideals on hypersurfaces

Throughout this talk, let $(A, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring of dimension d with infinite residue field, and let I be an $\mathfrak{m}$-primary ideal which is not a parameter ideal. Let M be a finitely generated A-module. For such an A-module M, $\mu_A(M)$ (resp.$\ell_A(M)$, $e_I^0(M)$) denotes the minimal number of generators (resp. the length, the multiplicity with respect to I) of M.

First of all, we recall the definition of Ulrich ideals.

Definition 1.1 ([GIW, GOTWY1] etc.)**.** An ideal $I \subset A$ is called *stable* if $I^2 = QI$ holds true (note that this condition does not depend on the choice of the minimal reduction Q of I). An ideal I is called *Ulrich* (resp. *good*) if I is stable and I/I^2 is A/I-free (resp. $I = Q : I$).

The following proposition gives a relationship between Ulrich ideals and good ideals.

Proposition 1.2 ([GOTWY1, Corollary 2.6])**.** *Let A, I be as above.*

(1) *Any Ulrich ideal is a good ideal.*
(2) *Suppose that A is Gorenstein. Then I is an Ulrich ideal if and only if it is a good ideal with $\mu_A(I) = d + 1$ (or A/I is Gorenstein).*

There are many examples of good ideals but not Ulrich ideals. For instance, in dimension 2, all powers of any good ideals are also good ideals. Moreover, if A is a regular local ring of $d = \dim A \geq 3$, then A has no Ulrich ideals but A admits many good ideals.

Let us recall the definition of Ulrich modules.

Definition 1.3 ([GOTWY1])**.** Let M be a finitely generated A-module. Then M is said to be an *Ulrich A-module with respect to I* if the following conditions holds true:

(a) M is a maximal Cohen-Macaulay A-module.
(b) $e_I^0(M) = \ell_A(M/IM)$ (i.e. $IM = QM$ for some minimal reduction Q of I).
(c) M/IM is A/I-free (i.e. $\ell_A(M/IM) = \mu_A(M) \cdot \ell_A(A/I)$).

Remark 1.4. A maximal Cohen-Macaulay A-module which satisfies $\mu_A(M) = e_{\mathfrak{m}}^0(M)$ is called an *MGMCM module* ([BHU]). Such an module is also said to be an *Ulrich A-module*, which means an Ulrich A-module with respect to $\mathfrak{m}$ in our sense.

[1]This paper is announcement of our result and the detailed version will be submitted to somewhere.

Remark 1.5 (See [HKuh]). Let A be a hypersurface local ring of $\mathrm{e}^0_{\mathfrak{m}}(A) = 2$. It is known that any maximal Cohen-Macaulay A-module can be written as a direct sum of a free module and an Ulrich A-module. In particular, $\mathrm{Syz}^1_A(M)$ is an Ulrich A-module for every maximal Cohen-Macaulay A-module M.

We can construct Ulrich modules with respect to I from a given Ulrich ideal I.

Proposition 1.6 (See [GOTWY1, Theorem 4.1]). *Let I be an Ulrich ideal of A. Then $\mathrm{Syz}^i_A(A/I)$ is an Ulrich A-module with respect to I for every $i \geq d$.*

If $I \subset A$ is an Ulrich ideal, then it is a good ideal and there exists an Ulrich A-module M with respect to I. When A is a hypersurface local domain, this characterizes Ulrich ideals.

Theorem 1.7 ([GOTWY1, Corollary 3.5]). *Suppose that A is a hypersurface local domain. Then the following conditions are equivalent*:

(1) *I is an Ulrich ideal.*
(2) *I is a good ideal and there exists an Ulrich A-module M with respect to I.*

Furthermore, if $e^0_{\mathfrak{m}}(A) = 2$, then we can give one more equivalent condition.

Corollary 1.8. *Suppose that A is a hypersurface local domain of $\mathrm{e}^0_{\mathfrak{m}}(A) = 2$. Then the following conditions are equivalent*:

(1) *I is an Ulrich ideal.*
(2) *I is a good ideal and there exists an Ulrich A-module M with respect to I.*
(3) *I is a stable ideal and there exists an Ulrich A-module M with respect to I.*

Proof. It is enough to show (3) $\Longrightarrow$ (2). Since A is not a regular local ring, an Ulrich A-module M with respect to I has no free summands. As $\mathrm{e}^0_{\mathfrak{m}}(A) = 2$, M is an Ulrich A-module in the classical sense. Namely,

$$\mu_A(M) = e^0_{\mathfrak{m}}(M) = e^0_{\mathfrak{m}}(A) \cdot \mathrm{rank}_A M = 2 \cdot \mathrm{rank}_A M.$$

It follows from the freeness of M/IM as an A/I-module that

$$\ell_A(M/IM) = \mu_A(M) \cdot \ell_A(A/I) = 2 \cdot \mathrm{rank}_A M \cdot \ell_A(A/I).$$

On the other hand, by assumption, we have

$$\ell_A(M/IM) = e^0_I(M) = e^0_I(A) \cdot \mathrm{rank}_A M.$$

Hence $e^0_I(A) = 2 \cdot \ell_A(A/I)$, that is, $I = Q \colon I$ for some minimal reduction Q of I. Therefore I is a good ideal. $\square$

In the corollary above, the stablity of I is needed as the next example shows.

Example 1.9. Let $(A, \mathfrak{m})$ be a Gorenstein local domain of dimension 1 and $e = e^0_{\mathfrak{m}}(A) \geq 2$. Then $\mathfrak{m}^{e-1}$ is an Ulrich A-module with respect to $\mathfrak{m}$. But if $e \geq 3$, then $\mathfrak{m}$ is *not* stable.

2. Ulrich ideal and $c(f)$

The main purpose of this section is to classify all Ulrich ideals of simple singularities. In order to do that, we discuss about the relationship between the set of Ulrich ideals and the set $c(f)$ (see the definition below).

In what follows, let S be a formal power series ring over an algebraically closed field k of characteristic 0, and let f be a nonzero element of $\mathfrak{m}_S^2$, where $\mathfrak{m}_S$ denotes the unique maximal ideal of S.

Definition 2.1 (see e.g. [Yos]). Put $c(f) = \{J \mid J \text{ is a proper ideal of } S \text{ with } f \in J^2\}$. The ring $A = S/(f)$ is called a *simple singularity* if $\sharp c(f) < \infty$.

In the last symposium, we classified all Ulrich ideals and Ulrich modules for some ideal for any rational double point ([GOTWY2]). In dimension 2, A is a simple singularity if and only if A is a rational double point, which follows from the following lemma.

Lemma 2.2 (See e.g. [Yos]). *Assume that $A = S/(f)$ is a simple singularity. Then f is one of the following equations*:

$$\begin{array}{lll} (A_n) & x^2 + y^{n+1} + \underline{z}^2 & (n \geq 1) \\ (D_n) & x^2y + y^{n-1} + \underline{z}^2 & (n \geq 4) \\ (E_6) & x^3 + y^4 + \underline{z}^2 & \\ (E_7) & x^3 + xy^3 + \underline{z}^2 & \\ (E_8) & x^3 + y^5 + \underline{z}^2, & \end{array}$$

where $\underline{z}^2$ denotes $z_2^2 + \cdots + z_d^2$.

Therefore if we classify all Ulrich ideals of simple singularities of dimension $d \geq 2$, then it generalizes the main theorem ([GOTWY2, Theorem 1.4]) to higher dimensional case. In order to accomplish it, we investigate the relationship between χ_A and $c(f)$. The following proposition is useful in order to find all Ulrich ideals.

Proposition 2.3. *Let $A = S/(f)$ be as above. Put*

$$\chi_A^* = \{J \subset S \mid J/(f) \text{ is an Ulrich ideal of } A\}.$$

Then we have $\chi_A^* \subset c(f)$.

As an application, we give the following.

Corollary 2.4. *Let $A = S/(f)$ be as above. Put $e = e_{\mathfrak{m}}^0(A) \geq 2$. If $I = J/(f) \subset A$ is an Ulrich ideal, then $J \not\subset \mathfrak{m}_S^{\lceil \frac{e+1}{2} \rceil}$.*

In particular, if A is a hypersurface local ring of $e_{\mathfrak{m}}^0(A) \leq 3$ and I is an Ulrich ideal, then $I \not\subset \mathfrak{m}^2$.

Proof. Suppose that $J \subset \mathfrak{m}_S^{\lceil \frac{e+1}{2} \rceil}$. Then we have

$$f \in J^2 \subset (\mathfrak{m}_S^{\lceil \frac{e+1}{2} \rceil})^2 \subset \mathfrak{m}_S^{e+1}.$$

But this contradicts the assumption that $e = \operatorname{ord}(f)$.

Now suppose that $e = \operatorname{ord}(f) \leq 3$ and $I \subset \mathfrak{m}^2$. Then $J \subset \mathfrak{m}_S^2 + (f)$. Hence $f \in J^2 \subset (\mathfrak{m}_S^2, f)^2$. Then $f = af + b$ for some $a \in \mathfrak{m}_S^2$ and $b \in \mathfrak{m}_S^4$. As $1 - a$ is unit in S, we get $f \in \mathfrak{m}_S^4$, which is a contradiction. $\square$

We need the following lemma in order to prove Proposition 2.3.

Lemma 2.5 ([GOTWY1, Theorem 7.6])**.** *Let A be a Cohen-Macaulay local ring and $I \subset A$ an Ulrich ideal. Let $i \geq d$ be an integer. Put $M = \mathrm{Syz}_A^i(A/I)$. Let*

$$F_1 \xrightarrow{\partial} F_0 \to M \to 0 \quad (ex)$$

be a finite presentation of M over A, where F_i are free A-modules and $\mathrm{rank}_A F_0 = \mu_A(M)$. *Then $I_1(\partial) = I$.*

Proof of Proposition 2.3. Let $I = J/(f)$ be an Ulrich ideal. Consider $M = \mathrm{Syz}_A^d(A/I)$. Since M is a maximal Cohen-Macaulay A-module over a hypersurface $A = S/(f)$, we can find a matrix factorization (φ, ψ) of f as follows:

$$\varphi \circ \psi = \psi \circ \varphi = f \cdot \mathrm{id}_{S^{\oplus n}}, \qquad 0 \to S^{\oplus n} \xrightarrow{\varphi} S^{\oplus n} \to M \to 0 \text{ (ex)}.$$

This gives a minimal free resolution of M over A, which is a periodic free resolution with periodicity 2 (see [Yos]) as follows:

$$\cdots\cdots \to A^{\oplus n} \xrightarrow{\overline{\varphi}} A^{\oplus n} \xrightarrow{\overline{\psi}} A^{\oplus n} \xrightarrow{\overline{\varphi}} A^{\oplus n} \to M \to 0 \text{ (ex)}.$$

Applying Lemma 2.5 implies that $I_1(\overline{\varphi}) = I_1(\overline{\psi}) = I$, that is, $I_1(\varphi) = I_1(\psi) = J$. Then $f \in I_1(\varphi)I_1(\psi) = J^2$. Hence $J \in c(f)$. $\square$

The following main theorem, which generalizes such a result in higher dimensional case. The key idea is to caluculate $c(f)$ in each case.

Theorem 2.6. *Assume that $A = S/(f)$ is a simple singularity of dimension $d \geq 1$. Then $c(f)$ consists of the following ideals:*

$$\begin{array}{cl}
(A_n) & \underline{\{(x, y^k, \underline{z}) \mid k = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor\}}, \\
(D_{2m}) & \{(x, y^k, \underline{z}) \mid k = 1, \ldots, m-1\} \cup \underline{\{(x \pm \sqrt{-1}y^{m-1}, y^m, \underline{z})\}} \cup \underline{\{(x^2, y, \underline{z})\}}, \\
(D_{2m+1}) & \{(x, y^k, \underline{z}) \mid k = 1, \ldots, m-1\} \cup \underline{\{(x, y^m, \underline{z})\}} \cup \underline{\{(x^2, y, \underline{z})\}}, \\
(E_6) & \{(x, y, \underline{z}), \underline{(x, y^2, \underline{z})}\}, \\
(E_7) & \{(x, y, \underline{z}), (x, y^2, \underline{z}), \underline{(x, y^3, \underline{z})}\}, \\
(E_8) & \{(x, y, \underline{z}), (x, y^2, \underline{z})\},
\end{array}$$

where $\underline{z} = z_2, \ldots, z_d$.

Moreover, if $d \geq 2$, then χ_A^* equals to $c(f)$. Note that if $d = 1$, then χ_A^* consists of all ideals of underlined sets.

If A is a simple singularity, then it is of finite Cohen-Macaulay representation type. If A is Gorenstein (and the homomorphic image of a regular local ring), then the converse is also true. On the other hand, if A is of finite Cohen-Macaulay representation type, then the set of Ulrich ideals χ_A is a finite set.

Question 2.7. Let A be a hypersurface local domain. If χ_A is a finite set, then is A a simple singularity?

Example 2.8. Let k be an algebrically closed field of characteristic 0.

(1) Let $A = k[[x,y,z]]/(x^3+y^6+z^2)$. Then

$$\begin{aligned}\chi_A^* = c(f) &= \big\{(x,y,z),(x,y^2,z),(x,y^3,z)\big\} \cup \big\{(x-\varepsilon y^2, y^3, z) \,\big|\, \varepsilon \in k^\times\big\} \\ &\quad \cup\big\{(x-\omega y^2, y^4, z) \,\big|\, \omega^3 = 1\big\}.\end{aligned}$$

(2) Let $A = k[[x,y,z]]/(x^a+y^b+z^2)$, where $a \geq 3$, $b \geq 2a+1$. If $\varepsilon \in k^\times$, then $I_\varepsilon = (x+\epsilon y^2, y^a, z)$ is an Ulrich ideal. In particular, χ_A^* and hence χ_A is an infinite set.

If both a and b are odd integers (≥ 3), then it is known that the set of Ulrich ideals of monomial type (say, χ_A^g) is empty; see [GOTWY1, Theorem 4.7]. But as a corollary of Example 2.8, we have the following.

Proposition 2.9. *Let a,b be integers with $a \geq 3$ and $b \geq 2a+1$. Let $A = k[[t^a, t^b]]$ be a numerical semigroup ring. Then the set of Ulrich ideals χ_A is an infinite set.*

In fact, for each $\varepsilon \in k^\times$, we can show that $I_\varepsilon = (t^b + \varepsilon t^{2a}, t^{a^2})$ is an Ulrich ideal.

3. Ulrich modules over simple singularities

We can determine all Ulrich modules with respect to some stable ideal over simple singularities. The key points are the following theorem and Knörrer's periodicity. In fact, any simple singularity A of dimension $d \geq 2$ is a hypersurface local domain of $e_{\mathfrak{m}}^0(A) = 2$. If I is an $\mathfrak{m}$-primary stable ideal and there exists an Ulrich module M with respect to I, then I is a good ideal (and thus an Ulrich ideal). Such Ulrich ideals are completely classified in Section 2.

Theorem 3.1. *Let $A = S/(f)$ be a hypersurface local domain of $e_{\mathfrak{m}}^0(A) = 2$, where S is a formal power series ring over an algebraically closed field k of characteristic 0. Assume that $I = J/(f)$ is an Ulrich ideal. Let M be a maximal Cohen-Macaulay A-module without free summands. Then the following conditions are equivalent:*

(1) *M is an Ulrich A-module with respect to I.*
(2) *$I(M) \subset J$, where (φ, ψ) is a matrix factorization of f corresponding to M (see the previous section) and $I(M) = I_1(\varphi) + I_1(\psi)$.*

Proof. (1) $\Longrightarrow$ (2) : As $M \cong \mathrm{Syz}_A^2(M)$, we may assume that $M \subset F = A^{\oplus n}$, where $n = \mu_A(M)$. Write

$$M = \left\langle \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix}, \cdots, \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix} \right\rangle \quad \text{and put} \quad \overline{\psi} = (a_{ij})\colon F \to F.$$

Then since $IM = QM$ for some minimal reduction Q of I, we have $Ia_{ij} \subset Q$. Hence $a_{ij} \in Q \colon I = I$ for all i,j. Namely, $I_1(\psi) \subset J$.

As A/I is Gorenstein by Proposition 1.2, $\mathrm{Syz}_A^1(M)$ is also an Ulrich A-module with respect to I;see [GOTWY1, Theorem 4.1]. Hence it follows from the argument as above that $I_1(\overline{\varphi}) \subset I$. Therefore $I(M) = I_1(\varphi) + I_1(\psi) \subset J$, as required.

$(2) \Longrightarrow (1)$: Let $F_1 \xrightarrow{\overline{\varphi}} F_0 \to M \to 0$ (ex) be a minimal free representation of M over A. Since $I_1(\overline{\varphi}) \subset I$, we have $F_0/IF_0 \cong M/IM$. In particular, $\ell_A(M/IM) = \ell_A(F_0/IF_0) = \mu_A(M) \cdot \ell_A(A/I)$.

On the other hand, $e_I^0(M) = e_I^0(A) \cdot \operatorname{rank}_A M = 2 \cdot \ell_A(A/I) \cdot \operatorname{rank}_A M$, where the last equality follows from the goodness of I. Since $e_{\mathfrak{m}}^0(A) = 2$, we have $2 \cdot \operatorname{rank}_A M = e_{\mathfrak{m}}^0(M) = \mu_A(M)$. Therefore $e_I^0(M) = \mu_A(M) \cdot \ell_A(A/I) = \ell_A(M/IM)$. It follows that M is an Ulrich module with respect to I. $\square$

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