# COMMUTATIVE ALGEBRA

Karuizawa, Japan

# The 4-th Symposium on Commutative Algebra in Japan

November 3-6, 1982

Karuizawa

Edited by Shiro Goto

This symposium was partially supported by Grant-in-Aid for Cooperative Research.

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# PREFACE

The 4-th Symposium on Commutative Algebra in Japan was held at the Karuizawa Training Institute of Nihon University during the period 3-6 November 1982, with the financial support from Professor M. Nagata of Kyoto University by the Grant-in-Aid for Cooperative Research. There were 51 participants including two from foreign countries.

This volume consists of the proceedings of almost all of the talks at the Symposium. The papers are arranged in alphabetical order of authors' names. The academic program itself was built with the principle that participants should be and can be speakers and as a logical result, the schedule was so hard that it might be painful to attend all the lectures. However in spite of possible disadvantages, we would like to keep this principle in future too, because it is extremely favorable to younger participants.

In the Problem Session, participants at the symposium were invited to submit open problems on Commutative Algebra or in their own fields of researches, and this volume includes the problems posed in the Session. I am profoundly grateful to the contributors for their cooperation.

It is a great pleasure to record my gratitude and that of my co-organizers Y. Aoyama, S. Itoh and K. Yoshida, to N. Suzuki of Shizuoka College of Pharmacy and T. Kambe of Nihon University who contributed so much to the smooth-running arragements and friendly atmosphere of the symposium. I am also grateful to Mrs. T. Oshitani and Miss S. Rachi for their kind assistance during the preparation for the symposium.

Finally, I would like to express my hearty gratitude to the late Professor M. Fukawa of Tôkai University for his contribution of a lecture at the symposium. He was gone on 14. January, 1983 and his article appearing in this volume is his final paper, though it ended with "to be continued".

March, 1983
S. Goto

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# On the Endomorphism Ring of a Canonical Module

By Ehime Univ. Yoichi Aoyama
Nihon Univ. Shiro Goto

A ring will always mean a commutative noetherian ring with unit. Let $R$ be a ring, $M$ a finitely generated $R$-module and $N$ a submodule of $M$. We denote by $\mathrm{Min}_R(M)$ the set of minimal elements in $\mathrm{Supp}_R(M)$ and put $U_M(N) = \bigcap Q$ where $Q$ runs through all the primary components of $N$ in $M$ such that $\dim M/Q = \dim M/N$. Let $T$ be an $R$-module and $\underline{a}$ an ideal of $R$. $E_R(T)$ denotes an injective envelope of $T$ and $H^i_{\underline{a}}(T)$ is the $i$-th local cohomology module of $T$ with respect to $\underline{a}$. We denote by $\hat{}$ the Jacobson radical adic completion over a semi-local ring. For a ring $R$, $Q(R)$ denotes the total quotient ring of $R$.

<u>Definition</u> ([4, Definition 5.6]). Let $R$ be a local ring of dimension $n$ and with maximal ideal $\underline{p}$. An $R$-module $K_R$ is called a canonical module of $R$ if $K_R \otimes_R \hat{R} \cong \mathrm{Hom}_R(H^n_{\underline{p}}(R), E_R(R/\underline{p}))$. (For elementary properties of canonical modules, we refer the reader to [4, 5 Vortrag und 6 Vortrag] and [2, §1].)

Throughout this note $A$ denotes a $d$-dimensional local ring with maximal ideal $\underline{m}$ and canonical module $K$. We note that $K_{\underline{p}}$ is a canonical module of $A_{\underline{p}}$ for every $\underline{p}$ in $\mathrm{Supp}_A(K)$ by [2, Corollary 4.3]. We put $H = \mathrm{Hom}_A(K,K)$ and $h$ is the natural map from $A$ to $H$. At the 3rd conference on commutative algebra held at Rokko, November 4-7, 1981, we showed the following properties of $H$ :

(1.1) $H$ is a finite $(S_2)$ over-ring of $A/U_A(0)$ contained in $Q(A/U_A(0))$ . ([2, Theorem 3.2])

(1.2) $\dim_A \operatorname{Coker}(h) \le d-2$ . ([2, Proof of Theorem 4.2])
(We define $\dim 0$ to be $-\infty$ .)

In this note we show that $H$ is characterized by the above properties, that is,

Theorem 2. Let $R$ be a ring which satisfies the following conditions:

( i ) $R$ is a finite $(S_2)$ over-ring of $A/U_A(0)$ ,

(ii) For every maximal ideal $\underline{n}$ of $R$ , $\dim R_{\underline{n}} = d$ , and

(iii) $\dim_A \operatorname{Coker}(A \to R) \le d-2$ .

Then $R \cong H$ as $A$-algebras. If $R \subseteq Q(A/U_A(0))$ , the condition (ii) holds (cf. [2, Proof of Theorem 3.2]).

Before proving Theorem 2, we note the following

Proposition 3. The following are equivalent:

(a) The map $\vec{h}$ is an isomorphism.

(b) $\hat{A}$ is $(S_2)$.

(c) $A$ is $(S_2)$.

(Proof) (a) $\Leftrightarrow$ (b) is due to [1, Proposition 2] and (b) $\Rightarrow$ (c) is well known. (c) $\Rightarrow$ (a) was proved by Ogoma [6, Proposition 4.2]. In [3] the writers give a proof, using [2, Corollary 4.3]. (q.e.d.)

Corollary 4. Assume that $\dim A/\underline{p} = d$ for every $\underline{p}$ in $\operatorname{Min}(A)$ . Then the $(S_2)$-locus $\{\underline{p} \in \operatorname{Spec}(A) \mid A_{\underline{p}} \text{ is } (S_2)\}$ is open in $\operatorname{Spec}(A)$ .

(Proof of Theorem 2) We may assume $U_A(0) = 0$ because $K$ is a canonical module of $A/U_A(0)$ ([2,(1.8)]) and $H = \operatorname{Hom}_{A/U_A(0)}(K,K)$ . $L = \operatorname{Hom}_A(R,K)$ is a canonical module of $R$ , that is, $L_{\underline{n}}$ is a canonical module of $R_{\underline{n}}$ for every maximal ideal $\underline{n}$ of $R$ by [4, Satz 5.12]. Since $\dim_A R/A \le d-2$ , $\operatorname{Hom}_A(R/A,K) = 0$ and $\operatorname{Ext}_A^1(R/A,K) = 0$

by [2,(1.10)]. Hence we have an isomorphism $L = \mathrm{Hom}_A(R,K) \xrightarrow{\sim} \mathrm{Hom}_A(A, K) \cong K$ from the exact sequence $0 \longrightarrow A \longrightarrow R \longrightarrow R/A \longrightarrow 0$ . From this isomorphism, we obtain an $A$-algebra isomorphism from $H$ to $\mathrm{Hom}_A(L, L)$ . Because $H$ is commutative, so is $\mathrm{Hom}_A(L,L)$ and $\mathrm{Hom}_A(L,L) = \mathrm{Hom}_R(L,L)$ . Since $R$ is $(S_2)$, $R \cong \mathrm{Hom}_R(L,L)$ . (q.e.d.)

The following proposition is an essential part of the proof of [2, Theorem 4.2].

Proposition 5. Let $B$ be a local ring of dimension $n$ and assume that there is a ring $R$ satisfying the following conditions:

(i) $R$ is a finite $(S_2)$ over-ring of $B$ ,

(ii) For every maximal ideal $\underline{p}$ of $R$ , $\dim R_{\underline{p}} = n$ ,

(iii) $R$ has a canonical module $L$ , i.e., $L_{\underline{p}}$ is a canonical module of $R_{\underline{p}}$ for every maximal ideal $\underline{p}$ of $R$ , and

(iv) $\dim_B R/B \leq n-2$ .

Then $L$ , as a $B$-module, is a canonical module of $B$ , $U_B(0) = 0$ and $R \cong \mathrm{Hom}_B(L,L)$ as $B$-algebras.

The following proposition is rather obvious, but it is worth stating.

Proposition 6. Let $\underline{n}_1, \dots, \underline{n}_r$ be the maximal ideals of $H$ . Then $\hat{K}$ has a decomposition $\hat{K} = \bigoplus_{i=1}^{r} K_i$ by indecomposable $\hat{A}$-modules $K_1, \dots, K_r$ such that $\widehat{H_{\underline{n}_i}} \cong \mathrm{Hom}_{\hat{A}}(K_i,K_i)$ for $i = 1, \dots, r$ and $\mathrm{Hom}_{\hat{A}}(K_i, K_j) = 0$ for $i \neq j$ .

Proposition 7. $H$ is a Cohen-Macaulay ring if and only if $K$ is a Cohen-Macaulay module.

(Proof) Since $H = \mathrm{Hom}_A(K,K)$ and $K \cong \mathrm{Hom}_A(H,K)$ , the assertion follows from the following Lemma 8. (q.e.d.)

For a finitely generated $A$-module $M$ of dimension $d$ , we put

$K_M = \mathrm{Hom}_A(M,K)$ . (Note that $K_M \otimes_A \hat{A} \cong \mathrm{Hom}_A(H^d_{\underline{m}}(M), E_A(A/\underline{m}))$ .) By the same argument as in [1, Lemma 1], we have the following

Lemma 8. Let $M$ be a finitely generated $A$-module of dimension $d$ and depth $t$ .

(1) If $M$ is a Cohen-Macaulay module, then $K_M$ is also a Cohen-Macaulay module.

(2) Assume that $M$ is not a Cohen-Macaulay module and put $s = \max \{ i \mid i < d \text{ and } H^i_{\underline{m}}(M) \neq 0 \}$ .

(i) If $\mathrm{depth}_{\hat{A}} \mathrm{Hom}_A(H^s_{\underline{m}}(M), E_A(A/\underline{m})) = 0$ , then

$$\mathrm{depth}\, K_M = \begin{cases} d-s+1 & \text{if } s > 0 , \\ d & \text{if } s = 0 . \end{cases}$$

(ii) If $s = t$ and $\mathrm{depth}_{\hat{A}} \mathrm{Hom}_A(H^t_{\underline{m}}(M), E_A(A/\underline{m})) = u$ , then

$$\mathrm{depth}\, K_M = \begin{cases} d-t+u+1 & \text{if } u < t , \\ d & \text{if } u = t . \end{cases}$$

Corollary 9 (Schenzel). $A$ is a Cohen-Macaulay ring if and only if $A$ is $(S_2)$ and $K$ is a Cohen-Macaulay module.

Next we consider a relation between $H$ and ideal transforms. In the following we assume that $d \geq 2$ and $U_A(0) = 0$ . Let $\underline{c} = A :_A H$ . Since $K_{\underline{p}}$ is a canonical module of $A_{\underline{p}}$ for every prime ideal $\underline{p}$ of $A$ , $A_{\underline{p}}$ is $(S_2)$ if and only if $\underline{p} \not\supseteq \underline{c}$ by Proposition 3. Let $T$ be the $\underline{c}$-transform of $A$ , i.e., $T = \{ x \in Q(A) \mid \underline{c}^t x \subseteq A$ for some $t \}$ . Then $T$ possesses the following properties:

(10.1) $T$ is finitely generated as an $A$-module. (cf. [5,(2.7.2)])

(10.2) $\dim_A T/A \leq d-2$ .

(10.3) $T$ is $(S_2)$.

Hence, from Theorem 2, we obtain the following

Proposition 11. $T \cong H$ as $A$-algebras.

We denote by $A^g$ the global transform of $A$, i.e., $A^g = \{ x \in Q(A) \mid \underline{m}^t x \subseteq A$ for some $t \}$. By [5,(2.3.2)], $A^g$ is finitely generated as an $A$-module.

Corollary 12. $A^g \cong H$ as $A$-algebras if and only if depth $A_{\underline{p}} \geq \min\{2, \dim A_{\underline{p}}\}$ for every $\underline{p}$ in $\mathrm{Spec}(A) \setminus \{\underline{m}\}$. In particular, if $H^i_{\underline{m}}(A)$ is of finite length for $i < d$, then $A^g \cong H$ as $A$-algebras.

Remark 13. The following are equivalent:

(a) $H^1_{\underline{m}}(A)$ is of finite length (resp. $A$ is a Buchsbaum ring) and $H^i_{\underline{m}}(A) = 0$ for $i \neq 1, d$.

(b) There is a Cohen-Macaulay intermediate ring $B$ between $A$ and $Q(A)$ such that $B$ is finitely generated as an $A$-module and $B/A$ is of finite length (resp. $\underline{m}B \subseteq A$).

In this case, $B$ is uniquely determined, i.e., $B = A^g$, and $H^1_{\underline{m}}(A) \cong B/A$. (cf. the second writer's paper : On the Cohen-Macaulayfication of certain Buchsbaum rings, Nagoya Math. J. 80 (1980) 107 - 116, Theorem(1.1).)

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# GALOIS THEORIES
# FOR PURELY INSEPARABLE MODULAR EXTENSIONS

Kiyoshi Baba (Oita Univ.)

This report gives a summary of Galois theories for purely inseparable modular extensions by using higher derivations, which have been developed mainly by Davis, Heerema, Deveney, and Mordeson from the late 1960's to the early 1980's. We restrict our topics to the case of finite purely inseparable modular extensions and modify their results in part.

Let $L$ be a field of characteristic $p > 0$ and $K$ a subfield of $L$ with $[L : K] < \infty$. A higher derivation $d = \{d_j;\ 0 \leq j \leq m\}$ on $L$ of rank $m$ is a collection of $m$ additive homomorphisms satisfying the following property:

(i) $d_0(a) = a$ for all $a$ in $L$.

(ii) $d_n(ab) = \sum_{i=0}^{n} d_i(a)d_{n-i}(b)$

for all $a, b \in L$ and $0 \leq n \leq m$. This is equivalent to saying that the mapping $d: A \longrightarrow A[t : m] = A[T]/T^{m+1}$ defined by $d(a) = \sum_{j=0}^{m} d_j(a)t^j$ is a ring homomorphism and $d_0 = \mathrm{id}$. The set $H^m(L)$ of all higher derivations on $L$ of rank $m$ is a group with respect to the composition $d \circ e = f = (f_j)$ where $f_j = \sum_{i=0}^{j} d_i e_{j-i}$ for $d = (d_j)$ and $e = (e_j)$. (We abbreviate $d = \{d_j;\ 0 \leq j \leq m\}$ to $d = (d_j)$.) Let $G$ be a subset of $H^m(L)$ and set $L^G = \{a \in L;\ d_j(a) = 0$ for $1 \leq j \leq m$ and $(d_j) \in G\}$. Let $H^m(L/K)$ denote the subgroup of $H^m(L)$ consisting of those $d$ whose fields of constants $L^d$ contain $K$.

## § 1. R. L. Davis ([1], [2])

His approach is to examine the upper central series.

Definition 1.1. Let $G$ be a subgroup of $H^{p^n}(L)$. Then we set

$$G_1 = G,$$
$$G_i = \{ d = (d_j) \in G;\ d_1 = \cdots = d_{i-1} = 0 \}$$

for $2 \leq i \leq p^n + 1$. Then $G_1 \supset G_2 \supset \cdots \supset G_{p^n+1}$ and each $G_i$ is a normal subgroup of $G$. Hence we set

$$D(G_i) = G_i / G_{i+1}$$

for $1 \leq i \leq p^n$. We define a mapping of $D(G_i)$ into $Der(L)$ by $dG_{i+1} \longmapsto d_i$ where $d = (d_j)$. This mapping is injective, therefore we identify $D(G_i)$ with its image in $Der(L)$.

We shall define $ad \in H^{p^n}(L)$ to be $(a^j d_j)$ for $a \in L$ and $d = (d_j) \in H^{p^n}(L)$.

Proposition 1.2. Let $G$ be a subgroup of $H^{p^n}(L)$ and $F$ a subfield of $L$ with the property that the restriction of $G$ to $F$ is a subgroup of $H^{p^n}(F)$. Assume that the restriction of $D(G_1)$ to $F$ is non-zero. Furthermore suppose the following:

(1) $G$ is closed under scalar multiplication by elements of $F$.

(2) The restriction mapping of each $D(G_i)$ into $Der(F)$ is injective for $1 \leq i \leq p^n$.

(3) $d_{hp}(E) \subset E$ for $1 \leq h \leq p^{n-1}$ and $d = (d_j) \in G$, where $E$ is the kernel of $D(G_{p^n})$.

Then the mapping of $G$ into $H^{p^{n-1}}(E)$ given by $d = (d_j) \longmapsto (d_{0|E}, d_{p|E}, d_{2p|E}, \ldots)$ is an injective group homomorphism where $d_{j|E}$ is the restriction of $d_j$ to $E$. Its homomorphic image of $G$ is denoted by $CG$.

Definition 1.3. Let $G$ satisfy the hypothesis of Proposition 1.2. We say that $G$ is structured relative to $F$ if the following conditions are satisfied:

(1) Each $D(G_i)$ is closed under Lie products and the taking of p-th powers for $1 \leq i \leq p^n$.

(2) $D(G_{p^{i}+1}) = D(G_{p^{i+1}})$ for $0 \leq i \leq n-1$.

Definition 1.4. Let $G$ be a subgroup of $H^{p^n}(L)$. Then we denote by $S$ the subgroup of $H^{p^{n-1}}(L)$ obtained by the deletion of the last $p^n - p^{n-1}$ mappings from all elements of $G$. We set

$\mathcal{G}_{n+1} = \{G$; $G$ is a subgroup of $H^{p^n}(L)$ such that:

(i) $C^iG$ $(= C(C^{i-1}G))$ is structured relative to $L^{p^i}$ for $0 \leq i \leq n$,

(ii) $D(G_{p^n})$ is finite dimensional over $L$,

(iii) $S \in \mathcal{G}_n$, i.e., $S$ satisfies the exponent $n$ Galois theory$\}$,

and

$\mathcal{K}_{n+1} = \{K$; $K$ is a subfield of $L$ such that:

(i) $L$ is a purely inseparable modular extension of $K$,

(ii) the exponent of $L$ over $K$ is $n+1\}$.

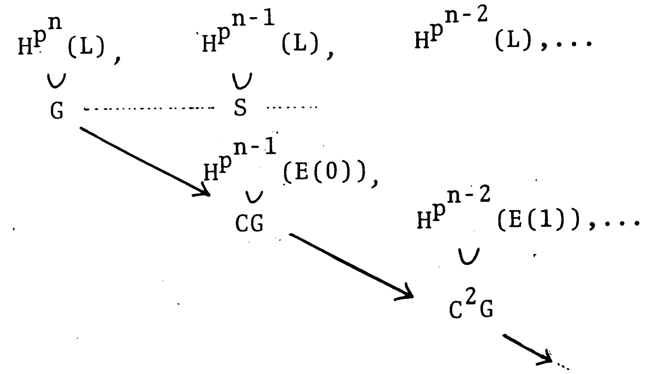
Theorem 1.5. (1) If $G$ is in $\mathcal{G}_{n+1}$, then there exists $K$ in $\mathcal{K}_{n+1}$ such that $G = H^{p^n}(L/K)$.

(2) If $H = H^{p^n}(L/K)$ with $D(H_1) \neq \{0\}$, then $H \in \mathcal{G}_{n+1}$.

We define mappings $\varphi_{n+1}: \mathcal{G}_{n+1} \longrightarrow \mathcal{K}_{n+1}$ and $\psi_{n+1}: \mathcal{K}_{n+1} \longrightarrow \mathcal{G}_{n+1}$ by $\varphi_{n+1}(G) = L^G$ and $\psi_{n+1}(K) = H^{p^n}(L/K)$.

Theorem 1.6. $\psi_{n+1}\varphi_{n+1} = \text{id}$ and $\varphi_{n+1}\psi_{n+1} = \text{id}$.

Remark 1.7. (1) Set $E(i)$ = the kernel of $D(C^iG_{p^{n-i}})$, then $E(i) = K(L^{p^{i+1}})$ for $0 \leq i \leq n$ and $E(0) = E$.




(2) The Galois theory between $\mathcal{G}_1$ and $\mathcal{K}_1$ is Jacobson's.

§ 2. N. Heerema and J. K. Deveney ([3], [8])

"A standard set of generators" consisting of a finite set of higher derivations on L of rank m plays a central role in this Galois theory.

Definition 2.1. A subset $\{x_1, \ldots, x_r\}$ of L is called a subbase for L/K if the following conditions are satisfied:

(1) $\{x_1, \ldots, x_r\} \subset L - K$.

(2) $L = K(x_1) \otimes_K \cdots \otimes_K K(x_r)$.

Definition 2.2. Let $N = N_1 \cup \cdots \cup N_{n+1}$ be a subbase for L/K such that each element of $N_i$ has exponent $i$ over $K$ for $1 \leq i \leq n+1$. Set $N_i = \{x_{i1}, \ldots, x_{ij_i}\}$. Let $\mathcal{F} = \{d^{ih};\ 1 \leq i \leq n+1$ and $1 \leq h \leq j_i\}$ be a subset of $H^m(L/K)$ defined by the following way:

$$d_\alpha^{ih}(x_{rs}) = \begin{cases} \delta_{(i,h)(r,s)} & \text{if } \alpha = [m/p^i] + 1, \\ 0 & \text{if } \alpha \neq [m/p^i] + 1 \end{cases}$$

for $1 \leq i,\ r \leq n+1$, $1 \leq h \leq j_i$ and $1 \leq s \leq j_r$ where

$\delta_{(i,h)(r,s)} = 1$ if $(i, h) = (r,s)$; $\delta_{(i,h)(r,s)} = 0$ if $(i, h) \neq (r, s)$. Then $\mathcal{F}$ is called a standard set of generators for $H^m(L/K)$, and $N$ is called a dual base for $\mathcal{F}$.

Definition 2.3. Let $d$ be a higher derivation on $L$ of rank $m$ defined by $d(a) = d_0(a) + d_\mu(a)t^\mu$ for $a \in L$. Then we define a higher derivation $v(d)$ on $L$ of rank $m$ by $v(d)(a) = d_0(a) + d_\mu(a)t^{\mu+1}$ for $a \in L$, and $cd$ by $cd(a) = d_0(a) + cd_\mu(a)t^\mu$ for $a \in L$ and $c \in L$.

Definition 2.4. Let $\mathcal{F}$ be a standard set of generators for $H^m(L/K)$. We shall set $\bar{v}(\mathcal{F}) = \{v^i(d);\ d \in \mathcal{F},\ i \geq 0\}$ where $v^0 = id$ and $v^i(d) = v(v^{i-1}(d))$. We denote by $\langle \bar{v}(\mathcal{F}) \rangle$ the subgroup of $H^m(L)$ generated by $\{ce;\ c \in L \text{ and } e \in \bar{v}(\mathcal{F})\}$.

Definition 2.5. Let $n$ be a non-negative integer such that $p^n \leq m < p^{n+1}$. We define:

$\mathcal{G} = \{G;\ G$ is a subgroup of $H^m(L)$ and $G = \langle \bar{v}(\mathcal{F}) \rangle$ where $\mathcal{F}$ is a standard set of generators for $H^m(L/L^G)\}$,

and

$\mathcal{K} = \{K;\ K$ is a subfield of $L$, $L$ is a purely inseparable modular extension of $K$ and $L^{p^{n+1}} \subset K\}$.

Definition 2.6. Let $G$ be a subgroup of $H^m(L)$. Then $G$ is called a Galois subgroup of $H^m(L)$ if $G = H^m(L/L^G)$.

Theorem 2.7. $G$ is Galois if and only if $G$ is in $\mathcal{G}$.

We define mappings $\varphi : \mathcal{G} \longrightarrow \mathcal{K}$ and $\psi : \mathcal{K} \longrightarrow \mathcal{G}$ by $\varphi(G) = L^G$ and $\psi(K) = H^m(L/K)$.

Theorem 2.8. $\psi\varphi = id$ and $\varphi\psi = id$.

Remark 2.9. (1) In [8], a finite abelian normal independent iterative subset of $H^m(L)$ is used instead of a standard set of generators for $H^m(L/L^G)$.

(2) If $m = p^n$, then the following holds by Theorem 1.5 and 2.7:

(a) $\mathcal{G}_{n+1} \subset \mathcal{G}$.

(b) If $G$ is in $\mathcal{G}$ with $D(G_1) \neq \{0\}$, then $G \in \mathcal{G}_{n+1}$.

§ 3. N. Heerema, J. K. Deveney and J. N. Mordeson ([4],[5],[7])

In this section we exhibit a Galois theory using pencils of higher derivations. Let $H(L/K)$ be a set of all higher derivations $d$ on $L$ such that the field of d-constants $L^d$ contains $K$ and the rank of $d$ is some power of $p$.

Definition 3.1. Let $d = (d_j)$ be an element of $H(L/K)$ and let $V(d) = e = (e_j)$ be a higher derivation on $L$ whose rank equals $p$ times the rank of $d$ defined by $e_j = d_{j/p}$ if $p$ divides $j$; $e_j = 0$ if $p$ does not divide $j$. Let $d$ and $d'$ be elements of $H(L/K)$. We say that $d'$ is equivalent to $d$ if there exists a non-negative integer $i$ such that $d' = V^i(d)$ or $d = V^i(d')$. The equivalence class of $d$ is denoted by $\bar{d}$ and is called the pencil of $d$. Set $\bar{H}(L/K) = \{\bar{d};\ d \in H(L/K)\}$. We give $\bar{H}(L/K)$ a group structure by defining $\bar{d}\bar{e}$ to be the pencil of $d'e'$ where $d'$ is in $\bar{d}$, $e'$ is in $\bar{e}$ and the rank of $d'$ = the rank of $e'$.

Definition 3.2. A subgroup $G$ of $\bar{H}(L/K)$ is Galois if $G = \bar{H}(L/L^G)$.

In our case a characterization of Galois subgroups is essentially the same as in §2. We define:

$\mathcal{G} = \{ G;\ G$ is a Galois subgroup of $\overline{H}(L/K) \}$,

and

$\mathcal{K} = \{ F;\ L \supset F \supset K$ and $L$ is a purely inseparable modular extension of $F \}$.

Then in the similar way as the preceding sections a Galois theory is established.

Addendum. In [6], M. Gerstenhaber and A. Zaromp has developed a Galois theory by using Artin-Hasse exponentials.

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# Problems on canonical rings of algebraic varieties

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In this report we make a review of problems concerning the following

Guess: The canonical ring of any algebraic manifold is a finitely generated algebra.

Here, manifold means a non-singular irreducible complete scheme defined over an algebraically closed field $k$.

Definition and Notation. Given a line bundle $L$ on an algebraic variety (= irreducible reduced complete k-scheme) $V$, let $G(V, L)$ be the graded $k$-algebra $\oplus_{t \geq 0} H^0(V, L^{\otimes t})$. For the canonical bundle $K$ of a manifold $M$, $G(M, K)$ is called the canonical ring of $M$.

When $|tL| = \emptyset$ for any $t > 0$, we define the L-dimension $\kappa(L, V)$ to be $-\infty$. Otherwise we define $\kappa(L, V)$ to be the maximum of the dimension of the image of $V$ by the rational mapping defined by the linear system $|tL|$, $t$ running through all the positive integers. $\kappa(K, M)$ is called the Kodaira dimension of the manifold $M$ and is denoted by $\kappa(M)$. This is a birational invariant of $M$.

Now we present positive and negative partial results.

Theorem (cf. [Z] & [F2]). $G(V, L)$ is finitely generated if $V$ is normal and if $\kappa(L, V) = 1$.

Theorem (cf. [Z]). $G(M, K)$ is finitely generated if $\dim M = 2$.

Fact (Zariski). $G(V, L)$ is not always finitely generated even if $V$ is a smooth surface.

Fact (Wilson). There exists a locally Gorenstein threefold $V$ whose canonical ring $G(V, \omega_V)$ is not finitely generated.

Question. Is $G(V, \omega_V)$ finitely generated when $V$ is a locally Gorenstein surface ?

Thus the problem is of subtle nature and is related to the geometry of singularities. For example, among Gorenstein singularities, we suspect, there are good ones and bad ones.

Let us recall the proof of the Guess in case $\dim M = 2$. We may assume $\chi(M) = 2$. Then we have a birational morphism $f: M \longrightarrow M'$ onto a manifold $M'$ which contains no exceptional curve (= a smooth rational curve $E$ with $E^2 = -1$). $M'$ turns out to be determined uniquely by $M$ and is called the minimal model of $M$. The canonical bundle $K'$ of $M'$ turns out to be numerically semipositive, i. e., $K'C \geqq 0$ for any curve $C$ in $M'$. Moreover, if $K'C = 0$, we can show that $C \cong \mathbb{P}^1$ and $C^2 = -2$. Using algebraic index theorem we infer that each connected component of the union of such (-2)-curves has a configuration corresponding to one of the Dynkin diagrams $A_n$, $D_n$, $E_6$, $E_7$, $E_8$. So they can be contracted to rational double points. Let $g: M' \longrightarrow M''$ be the contraction morphism. Then $K'$ is the pull-back of the dualizing sheaf $\omega''$ of $M''$. By Nakai's criterion we infer that $\omega''$ is ample. Therefore a positive multiple of it is spanned by global sections and hence so is $K'$. It follows that $G(M, K) = G(M', K')$ is finitely generated. $M'' = \mathrm{Proj}(G(M, K))$ is called the canonical model of $M$.

We used two main tools in the above proof ; the theory of minimal models and the theory of rational double points. We will try to generalize both theories in higher dimension.

One might say a manifold $M$ to be relatively minimal if $M$ is not a blowing-up of another manifold with non-singular center. Then, obviously, for any manifold $M$, we have a birational morphism $f: M \longrightarrow M'$ onto a relatively minimal manifold $M'$. However, such a model is not unique and does not have good properties in general.

For example, $K'$ is not always numerically semipositive. Among experts it is now realized that minimal models should be allowed to have mild singularities in order to play the role of $M'$ in the preceding proof. We would be happy if we can find a locally Gorenstein variety (hopefully with only rational singularities) which is birationally equivalent to the given manifold $M$ and whose canonical sheaf is numerically semipositive. As a matter of fact, this hope is still too optimistic. At least we should consider certain quotient singularities too.

We propose here a different approach. Our "minimal model" is not a variety itself, but a pair of a variety and a $\mathbb{Q}$-divisor on it (modulo certain birational equivalence among such pairs).

Definition. A $\mathbb{Q}$-divisor on a manifold $M$ is a linear combination of prime divisors on $M$ with coefficients being rational numbers. It is said to be effective if the coefficients are non-negative.

A $\mathbb{Q}$-line bundle is an element of $\mathrm{Pic}(M) \otimes \mathbb{Q}$. A $\mathbb{Q}$-line bundle $H$ is said to be (numerically) semipositive if $HC \geq 0$ for any curve $C$ in $M$ (The intersection number $HC \in \mathbb{Q}$ is defined in the obvious way). A $\mathbb{Q}$-line bundle $L$ is said to be pseudo-effective if $\chi(tL + A, M) \geq 0$ for any $t \geq 0$ and any ample line bundle $A$. $\chi(L, M)$ is defined to be $\chi(mL, M)$ for a positive integer $m$ such that $mL$ is a usual line bundle.

Conjecture (Generalized Zariski decomposition). For any pseudo-effective $\mathbb{Q}$-line bundle $L$ on a manifold $M$, there are a birational morphism $f: M^* \longrightarrow M$ and an effective $\mathbb{Q}$-divisor $N$ on $M^*$ having the following properties:

1) $H = f^*L - N$ is (numerically) semipositive.

2) For any surjective morphism $g: W \longrightarrow M^*$ and any effective $\mathbb{Q}$-divisor $E$ on $W$ such that $g^*f^*L - E$ is semipositive, $E - g^*N$ is an effective $\mathbb{Q}$-divisor on $W$.

Remark. Roughly speaking, $N$ is universally the smallest effective

$\mathbb{Q}$-divisor with the property 1). So such a pair $(M^*, N)$ is unique up to a birational equivalence in the following sense: Let $(M^\#, N^\#)$ be another such pair. Then, on any manifold dominating $M^* \times_M M^\#$ birationally, the pull-backs of $N$ and $N^\#$ are the same $\mathbb{Q}$-divisor. Hence we call $N$ (resp. $H$) the negative (resp. semipositive) part of $L$. We have $H^0(M, tL) \cong H^0(M^*, tH)$ for any $t \geq 0$ as long as both $tL$ and $tH$ are usual line bundles.

Remark. When $\dim M = 2$, the classical Zariski decomposition (cf. [Z] & [Fl]) has the above desired property. In particular, we can take $M^*$ to be $M$ itself. If $L$ is the canonical bundle of $M$, $H$ is just the pull-back of the canonical bundle of the minimal model of $M$ (unless $M$ is ruled, in which case $K$ is not pseudo-effective).

Suppose that we have a manifold $M$ whose canonical bundle $K$ admits a Zariski decomposition as above. Since $G(M, L)$ is not always finitely generated even if $L$ admits a Zariski decomposition, the Zariski decomposition of $K$ should enjoy better properties than general line bundles. So we would like to ask:

Question. Does there exist a birational morphism $h: M^* \longrightarrow V$ onto a locally Macaulay variety $V$ such that $\mathcal{L} = \omega_V{}^m$ (= the torsion free part of $\omega_V^{\otimes m}$) is an invertible sheaf on $V$ with $h^*\mathcal{L} = mH$ ?

Conjecturally $V$ is expected to have only certain mild singularities ( called terminal singularities by M. Reid). When $\dim M = 3$, terminal singularities are suspected to be isolated quotient singularities. Any way, if such a variety $V$ exists, it may be called <u>a</u> minimal model of $M$.

Canonical Conjecture. Let $M$ be a manifold whose canonical bundle $K$ is pseudo-effective. Then $K$ admits a generalized Zariski decomposition and its semipositive part $H$ is semiample, that means, $|mH|$ has no base points for some $m > 0$.

Our Guess would follow from this conjecture. In fact, this conjecture is true when $M$ is a surface. As for threefolds we obtained recently the following results. We assume $\mathrm{char}(k) = 0$ in both.

Theorem (Fujita). The canonical conjecture is true if $\kappa(M) = 1$ or 2.

Theorem (Kawamata). The canonical conjecture is true if $K$ is numerically semipositive and if $\kappa(M) = 3$.

The methods used in the proofs are different from the classical one for surfaces. We do not use the existence of a minimal model (generalized Zariski decomposition is enough). We do not construct (a candidate of) the canonical model of $M$ before we show the finitely generatedness of $G(M, K)$. Thus, the following problem remains still wide open.

Problem. Assume $G(M, K)$ is finitely generated. Then, what kind of singularity can the canonical model have ?

When $K$ is semiample and $\kappa(M) = 3$, the canonical model has only rational Gorenstein singularities. Moreover, except at finitely many points, it has only compound Du Val singularities (this means, one gets a rational double point by cutting a general hyperplane passing the singular point). In general, one should expect to encounter quotient singularities of them too.

When $\kappa(M) = 2$, the canonical model (a surface in this case) seems to have only rational singularities, but they are not necessarily rational double points.

Problem. Study locally Macaulay singularities such that the torsion free part (or the reflexive hull) of $\omega^m$ is invertible for some positive integer $m$. Are these singularities quotient of Gorenstein singularities ?

The case of rational singularities are especially important. Note that all the rational surface singularities are quotient singularities. Thus, I think, there is an interesting class of local rings between

Gorenstein ones and Cohen Macaulay ones. Besides this problem, it will be important to study the property of positive multiples of canonical modules.

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## Theory of generalized valuations 1

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Here is an attempt to see what are the common features to Krull valuations and absolute valuations.

### 1. Value systems

**1.1 Definitions.**

$\Gamma$ is a value system (abbreviated by vs.)

$\Leftrightarrow$
- $\Gamma$ is a linearly ordered set.
- $\Gamma$ is a commutative monoid wrt addition. (unity=0)
- $\Gamma$ is a commutative monoid wrt multiplication. (unity=1)
- $0=\mathrm{Min}\,\Gamma$.
- Every non-0-element is multiplicatively invertible.
- $\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma$.
- $\alpha\leq\beta \Rightarrow \alpha+\gamma\leq\beta+\gamma,\ \alpha\gamma\leq\beta\gamma$.
- $\Gamma$ has at least three elements.

Let $\Gamma$ be a value system.

$\Gamma^*$=the multiplicative group consisting of all non-0-elements of $\Gamma$.

$\Gamma$ is Archimedean $\Leftrightarrow$ $\forall\gamma\in\Gamma, \exists n\in\mathbb{N}\ \ \gamma\leq n$. (n means 1+...+1, n times.)

$\Downarrow$

$\Gamma$ is strict $\Longleftrightarrow$ $(\beta\neq 0 \rightarrow \alpha+\beta>\alpha)$

$\Downarrow$

$\Gamma$ is of type A $\Leftrightarrow$ $1<2$ $\Longleftrightarrow$ $1<2<3<4<\ldots$

$\Updownarrow$

$\Gamma$ is of type B $\Leftrightarrow$ $1=2$ $\Longleftrightarrow$ $1=2=3=4=\ldots$

The above $\Downarrow$'s are proper implications, shown by the following examples 3 and 4.

1.2 Examples of value systems.

1. Let $K$ be a subfield of $\mathbb{R}$. Then $K_+ = \{\xi \in K \mid \xi \geq 0\}$ is a vs. (Archimedean.)

2. Let $A$ be a linearly ordered Abelian group multiplicatively represented. Define $\alpha + \beta = \mathrm{Max}(\alpha, \beta)$. Then $\Gamma = \{0\} \cup A$ is a vs. (of type B.) Every vs of type B is of this form.

3. Let $\Gamma_1, \Gamma_2$ be vs's. Then

$\Gamma^* = \Gamma_1^* \times \Gamma_2^*$

order being lexicographic

addition and multiplication componentwise

defines a vs $\Gamma$. (denoted by $\Gamma_1 \times \Gamma_2$)

$\Gamma_1 \times \Gamma_2$ is Archimedean iff $\Gamma_1$ is Archimedean. $\Gamma_1 \times \Gamma_2$ is strict iff $\Gamma_1$ or $\Gamma_2$ is strict. $\Gamma_1 \times \Gamma_2$ is of type A iff $\Gamma_1$ or $\Gamma_2$ is of type A. (We get examples of Archimedean vs of rank $\geqslant 2$, and non-Archimedean strict vs.)

4. Let $\Gamma_1$ be a vs of type B and $\Gamma_2$ be any vs. Then

$\Gamma^* = \Gamma_1^* \times \Gamma_2^*$

order being lexicographic

multiplication being componentwise

$$(\alpha_1, \alpha_2) + (\beta_1, \beta_2) = \begin{cases} (\alpha_1, \alpha_2) & \text{if } \alpha_1 > \beta_1 \\ (\beta_1, \beta_2) & \text{if } \alpha_1 < \beta_1 \\ (\alpha_1, \alpha_2 + \beta_2) & \text{if } \alpha_1 = \beta_1 \end{cases}$$

defines a vs $\Gamma$. $\Gamma$ is necessarily non-strict. $\Gamma$ is of type A iff $\Gamma_2$ is of type A. (We get examples of non-strict vs of type A.)

5. Let $\Gamma_1$ be a vs and $X$ be a variable. We have a vs $\Gamma = \Gamma_1(X)$, the order being induced by

$\sum_{i=0}^{\infty} \alpha_i X^i < \sum_{i=0}^{\infty} \beta_i X^i$ if $\alpha_i = \beta_i$ for all $i > n$ and $\alpha_n < \beta_n$.

$\Gamma_1(X)$ is strict iff $\Gamma_1$ is strict. $\Gamma_1(X)$ is of type A iff $\Gamma_1$ is of type A.

1.3 A subgroup $I$ of $\Gamma^*$ is called an isolated subgroup if it is an interval. Subststems, factor systems modulo isolated subgroups and homomorphisms of vs's are defined. Noether's isomorphism theorem holds.

Isolated subgroups form a linearly ordered family. An isolated subgroup is called principal if it is of the form

$$\langle \gamma \rangle = \{ \xi \in \Gamma \mid \exists m \in \mathbb{Z}, \exists n \in \mathbb{Z}, \ \gamma^m \leq \xi \leq \gamma^n \}.$$

Prop. A principal isolated subgroup $I$ not equal to $\{1\}$ has a maximal properly contained isolated subgroup. (We shall denote it by $I^b$.)

Def. $\operatorname{rank}\Gamma$=ordinal type of the family of all principal isolated subgroups.

Structure theorem. Let $\Gamma$ be of type A. Then there exists unique isolated subgroup $I_1$ such that $\Gamma/I_1$ is of type B and $\Gamma'=\{0\}\cup I_1$ is Archimedean. $I_1$ is actually equal to $\langle 2 \rangle$.

1.4 Definitions.

$A$ is a half line $\Leftrightarrow \phi \neq A \subseteq \Gamma \wedge \forall \alpha \in A, \forall \beta \in \Gamma, (\alpha \leq \beta \Rightarrow \beta \in A)$.

$H(\Gamma)$=the set of all half lines.

$$\begin{array}{l} H(\Gamma) \\ \cup \\ H_0(\Gamma) = \{ A \in H(\Gamma) \mid A \text{ has Min. or } A^c \text{ does not have Max.}\} \ni [\gamma, \rightarrow) \\ \cup \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \text{ identify} \\ \Gamma \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \ni \gamma \end{array}$$

$A \leq B \Leftrightarrow A \supseteq B$.

$A+B = \{ \xi \in \Gamma \mid \exists \alpha \in A, \exists \beta \in B, \alpha+\beta \leq \xi \}$.

$$AB = \{ \xi \in \Gamma \mid \exists \alpha \in A, \exists \beta \in B, \alpha\beta \leq \xi \}$$
$$= \{ \xi \in \Gamma \mid \exists \alpha \in A, \exists \beta \in B, \alpha\beta = \xi \}.$$

Prop. If $\Gamma^* = \langle \omega \rangle$, $\omega < 1$, and $A$ is a half line $\neq 0, \neq \Gamma^*$, then

(1) $\omega A < A$. (2) $\exists \alpha \in A, \omega\alpha < A \leq \alpha$.

1.5 Let $\Gamma$ be a vs. Put

$$\mathcal{V}_\varepsilon = \{(\alpha, \beta) \in \Gamma \times \Gamma \mid \alpha \leq \beta + \varepsilon,\ \beta \leq \alpha + \varepsilon\} \quad (\varepsilon > 0).$$

$\{\mathcal{V}_\varepsilon \mid \varepsilon > 0\}$ gives a uniform structure on $\Gamma$. Addition, multiplication, and inverse forming are continuous.

Prop. (condition of separatedness)

$T_0 \Leftrightarrow T_1 \Leftrightarrow T_2 \Leftrightarrow$ Distinct points have distinct neighbourhood systems $\Leftrightarrow (\alpha < \beta \Rightarrow \exists \varepsilon > 0,\ \alpha + \varepsilon < \beta)$.

Prop. Any vs $\Gamma$ has a universal homomorphism $\varphi : \Gamma \longrightarrow \Gamma^s$ to a separated vs. $\varphi$ is actually a canonical homomorphism $\Gamma \longrightarrow \Gamma/I_0$ where

$$I_0 = \begin{cases} \{1\} & \text{if } \Gamma \text{ is of type B} \\ \langle 2 \rangle^b = \{\xi \in \Gamma \mid (r \in Q_+,\ s \in Q_+,\ r < 1 < s) \Rightarrow r < \xi < s\} & \text{if } \Gamma \text{ is of type A} \end{cases}$$

Prop. Let $\Gamma$ be a separated vs. Then

(1) $\lim \alpha_\lambda = \alpha$, $\lim \beta_\lambda = \beta$, $\forall \lambda\ \alpha_\lambda \leq \beta_\lambda \Rightarrow \alpha \leq \beta$.

(2) $\forall \lambda\ \alpha_\lambda \leq \beta_\lambda \leq \gamma_\lambda$, $\lim \alpha_\lambda = \lim \gamma_\lambda = \beta \Rightarrow \lim \beta_\lambda = \beta$.

(the indexing set $\Lambda$ being a directed set.)

Theorem. There exists a vs $\hat{\Gamma}$ which is a separated completion of $\Gamma$.

Examples.

1. The completion of $K_+$ is $\mathbb{R}_+$.
2. The vs of type B is separated and complete.
3. If $\Gamma_1$ is of type B and $\Gamma_2$ is separated, then $\Gamma_1 \times \Gamma_2$ is separated and complete.
4. If $\Gamma_1 \times \Gamma_2$ is separated, then both $\Gamma_1$ and $\Gamma_2$ are separated, and $\Gamma_1$ is non-strict.

## 2. $\Gamma$-valued rings, normed modules

2.1 Let $\Gamma$ be a separated, complete vs.

A ring $R$ is $\Gamma$-valued if a function $|\ |:R\longrightarrow\Gamma$ is given such that

$$|0|=0,\quad |1|\neq 0,\quad |x+y|\leq|x|+|y|,\quad |xy|=|x|\,|y|$$

holds. A $\Gamma$-valued ring is a topological ring in the usual manner.

Prop. (condition of separatedness of a $\Gamma$-valued ring)

$T_0\Longleftrightarrow T_1\Longleftrightarrow T_2\Longleftrightarrow(|x|=0\Rightarrow x=0)\Longleftrightarrow\{0\}$ is closed.

A $\Gamma$-valued field is necessarily a separated topological field.

We have separated completions of $\Gamma$-valued rings and $\Gamma$-valued fields.

2.2 Let $R$ be a $\Gamma$-valued ring. An $R$-module $M$ is normed if a function $\|\ \|:M\rightarrow H_0(\Gamma)$ is given such that

$$\|0\|=0,\quad \|x+y\|\leq\|x\|+\|y\|,\quad \|ax\|=|a|\,\|x\|\quad (x\in M,\ y\in M,\ a\in R)$$

holds. A normed R-module is a topological R-module in the usual manner.

Prop. (condition of separatedness of a normed R-module)

$T_0\Longleftrightarrow T_1\Longleftrightarrow T_2\Longleftrightarrow(\|x\|=0\Rightarrow x=0)\Longleftrightarrow\{0\}$ is closed.

We have a separated completion of a normed R-module.

2.3 Examples of normed R-modules.

1. $M=R^n$. $\|x\|_1=\sum_{i=1}^{n}|x_i|\in\Gamma$, where $x=(x_1,\dots,x_n)$
2. $M=R^n$. $\|x\|_\infty=\mathrm{Max}\,|x_i|\in\Gamma$.

   $\|\ \|_1$-topology$=\|\ \|_\infty$-topology$=$product topology on $R^n$.
3. $M=\{x=(x_i)\in R^I\mid \Sigma x_i$ is absolutely convergent in $R\}$.

   $\|x\|_1=\Sigma|x_i|\in R$.

   $R^{(I)}$ is dense in $M$.

   $M$ is complete if $R$ is complete.

4. $M=\{x=(x_i)\in R^I \mid \exists \gamma\in\Gamma\ \forall i\in I\ |x_i|\leq\gamma\}$.

$\|x\|_\infty = \bigcap_{i\in I} |x_i| \in H_0(\Gamma)$.

$R^{(I)}$ is not dense in M.

M is complete if R is complete.

$\|\ \|_1$-topology $\supsetneq$ $\|\ \|_\infty$-topology $\supsetneq$ product topology on $R^{(I)}$.

2.4 Let $u:M\longrightarrow M'$ be a linear map of normed R-modules. $\gamma\in\Gamma$ is a bound of u if $\forall x\in M\ \|u(x)\|\leq\gamma\|x\|$ holds. u is bounded if it has bounds. The set of all bounds of u is denoted by $\|u\|$.

Prop. If u is bounded, $\|u\|\in H_0(\Gamma)$.

Prop. If u is bounded, u is continuous. Converse is true if $\Gamma^*$ is principal and the function $|\ |$ is surjective.

Def. $Hom^{bd}(M, M')$=the R-module consisting of all bounded linear maps:$M\longrightarrow M'$.

$GL^{bd}(M)=\{u\in GL(M) \mid$ both u and $u^{-1}$ are bounded$\}$.

Theorem. $GL^{bd}(M)$ is a topological group in case

(a) $\forall x\in M\ \|x\|\in\Gamma$ or (b) $\Gamma^*$ is principal.

We have no counterexamples when conditions (a) and (b) fail.

## 3. The theorem of Hahn-Banach

3.1 Let M be a normed R-module.

Def. M is strongly complete $\Longleftrightarrow$ If families $(t_i)_{i\in I}$, $(A_i)_{i\in I}$ where $t_i\in M$, $A_i\in H_0(\Gamma)$ satisfy $\forall i\in I\ \forall j\in I\ \|t_i-t_j\|\leq A_i+A_j$, then there exists $a\in M$ such that $\forall i\in I\ \|t_i-a\|\leq A_i$ holds.

Strongly complete modules are necessarily complete.

Theorem (generalized Hahn-Banach theorem). Let K be a $\Gamma$-valued field, M and M' be normed K-modules, and N be a K-submodule of M. If M' is strongly complete, any bounded linear map $v:N\longrightarrow M'$ has a linear extension $u:M\longrightarrow M'$ with $\|u\|=\|v\|$.

Cor. If $K$ is strongly complete, then the canonical pairing: $M \times \mathrm{Hom}^{bd}(M, K) \longrightarrow K$ is non-degenerate.

3.2 Examples of strongly complete modules.

1. $\mathbb{R}$ is strongly complete wrt usual absolute value.
2. $\mathbb{C}$ is not strongly complete wrt usual absolute value, whereas the generalized Hahn-Banach theorem holds for $K=M'=\mathbb{C}$.
3. If $K$ is a discretely valued field and $M$ is a complete normed $K$-module, $M$ is strongly complete.
4. If $R$ is strongly complete, then the module $M$ in the example 4 in 2.3 is strongly complete.

continued

A note on quasi-Buchsbaum rings

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1. Introduction.

The purpose of my lecture is to establish the ubiquity of quasi-Buchsbaum rings that are not Buchsbaum and my result is contained in the following

Theorem (1.1). Let $d \geq 3$ and $h_1, h_2, \cdots, h_{d-1} \geqq 0$ be integers. Suppose that at least two of $h_i$'s are positive. Then there exists a quasi-Buchsbaum local domain $A$ which satisfies the following conditions:

(1) $A$ is not a Buchsbaum ring;

(2) $\dim A = d$ ;

(3) $l_A(H_m^i(A)) = h_i$ for all $1 \leqq i \leqq d-1$ .

Moreover if $h_1 = 0$ , the ring $A$ can be taken to be normal.

Now let me briefly recall the definition of Buchsbaum (resp. quasi-Buchsbaum) rings, or more generally that of Buchsbaum (resp. quasi-Buchsbaum) modules. Let $A$ be a Noetherian local ring with maximal ideal $m$ . Then a finitely generated A-module $M$ of dimension $d$ is said to be Buchsbaum if the difference

$$I(M) = l_A(M/qM) - e_M(q)$$

is an invariant of $M$ not depending on the particular choice of a parameter ideal $q$ of $M$ , where $l_A(M/qM)$ and $e_M(q)$ denote respectively the length of the A-module $M/qM$ and the multiplicity of $M$ relative to $q$ .*) In this case the local cohomology modules $H_m^i(M)$ ( $i \neq d$ ) of $M$ relative to the maximal ideal $m$ are vector spaces, that is $m.H_m^i(M) = (0)$ and one has the equality

$$I(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} . l_A(H_m^i(M))$$

(|7|), where $l_A(H_m^i(M))$ denotes the length of $H_m^i(M)$ for each $i \neq d$ . After this fact I would like to say a given finitely generated A-module $M$ to be quasi-Buchsbaum if

$$m.H_m^i(M) = (0)$$

for all $i \neq \dim_A M$ .**) A Noetherian local ring is called a Buchsbaum (resp. quasi-Buchsbaum) ring if it is a Buchsbaum (resp. quasi-Buchsbaum) module over itself. The theory of Buchsbaum or quasi-Buchsbaum rings and modules is now developing. Note that there is given in |4| and |5| a very

*) See |3| and |5| as general references on Buchsbaum modules.

**) c.f. |6|.

powerful criterion of Buchsbaum modules in terms of local cohomology which plays a certain rôle in my lecture (c.f. Lemma (2.3)).

Buchsbaum rings are of course quasi-Buchsbaum rings and provided $H_{\underline{m}}^{i}(A) = (0)$ for all $i \neq t, d$ where $t = \text{depth } A$ and $d = \dim A$, a quasi-Buchsbaum ring $A$ is always Büchsbaum (|5|). Nevertheless without this extra assumption, quasi-Buchsbaum rings are not necessarily Buchsbaum: the first counterexample is of dimension 2 and was given by J. Stückrad (|8|). Expanding his example, one can easily guarantee that for a given integer $d \geqq 2$, there exists a non-Buchsbaum but quasi-Buchsbaum local ring $A$ of dimension $d$ with $H_{\underline{m}}^{i}(A) = (0)$ for all $i \neq 0, 1, d$ (|3|). However even in the latter examples, the rings $A$ are still of depth 0 and almost all the local cohomology modules $H_{\underline{m}}^{i}(A)$ vanish. On the contrast according to Theorem (1.1), one can handle numerous non-Buchsbaum but quasi-Buchsbaum normal rings with arbitrary local cohomology. In this sense my theorem (1.1) may have some interest.

The method of construction of examples is essentially the same as in |2|, which established the ubiquity of Buchsbaum rings. However for the present purpose one needs a few preliminaries on quasi-Buchsbaum modules which I will summarize in the next section. The proof of Theorem (1.1) itself is simple and shall be given in Section 3.

## 2. The ubiquity of quasi-Buchsbaum modules.

In this section let $S = k[X_1, X_2, \ldots, X_n]$ ($n \geqq 3$) be a polynomial ring with $n$ variables over a field $k$ and $\underline{n} = S_+$, the irrelevant maximal ideal of $S$.

Let $M$ be a graded S-module and $p$ an integer. We regard the $p^{\text{th}}$ local cohomology module $H_{\underline{n}}^{p}(M)$ of $M$ relative to $\underline{n}$ as a graded S-module, whose homogeneous component of degree $q$ shall be denoted by $|H_{\underline{n}}^{p}(M)|_q$ ($q \in \mathbb{Z}$). We denote by $M(p)$ the graded S-module which coincides with $M$ as underlying S-modules and whose graduation is defined by $|M(p)|_q = M_{p+q}$ for all $q \in \mathbb{Z}$.

A finitely generated graded S-module $M$ is simply called Buchsbaum (resp. quasi-Buchsbaum) if the $S_{\underline{n}}$-module $M_{\underline{n}}$ is Buchsbaum (resp. quasi-Buchsbaum).

Let

$$(F) \quad 0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0 = S \longrightarrow \underline{k} = S/\underline{n} \longrightarrow 0$$

be a graded minimal free resolution of the graded S-module $\underline{k} = S/\underline{n}$. Recall that the complex (F) can be identified with the Koszul complex of $S$ generated by $X_1, X_2, \ldots, X_n$. For $0 \leqq i \leqq n-1$ let

$$M_i = \underline{k} \qquad (i = 0),$$

$$= \underline{n} \qquad (i = 1),$$

$$= \operatorname{Ker} (F_{i-1} \xrightarrow{f_{i-1}} F_{i-2}) \qquad (n - 1 \geqq i \geqq 2)$$

and we clearly have the following

<u>Lemma</u> (2.1). <u>Let</u> $1 \leqq i \leqq n - 1$ <u>be an integer. Then</u>

(1) $\dim_S M_i = n$ .

(2) $H^p_{\underline{n}}(M_i) = (0) \qquad (p \neq i, n),$

$\qquad\qquad = \underline{k} \qquad (p = i).$

(3) $l_S(M_i/\underline{n}M_i) = \binom{n}{i}$ <u>and</u> $\operatorname{rank}_S M_i = \binom{n-1}{i-1}$ .

Let $e_1, e_2, \dots, e_n$ be an S-free basis of $F_1$ such that each $e_i$ is homogeneous of degree 1 . Let $2 \leqq t \leqq n - 1$ be an integer. For each subset J of $\{1, 2, \dots, n\}$ with $\#J = t$ , we put

$$e_J = e_{j_1} \wedge \cdots \wedge e_{j_t}$$

in $F_t = \bigwedge^t F_1$ where $J = \{j_1, j_2, \dots, j_t\}$ with $j_1 < j_2 < \cdots < j_t$ . Consider the exact sequence

$$F_{t+1} \xrightarrow{f_{t+1}} F_t \xrightarrow{g} M_t \longrightarrow 0$$

of graded S-modules and put

$$L_t = \underline{n}M_t + \sum_J Sg(e_J) ,$$

where J runs through the subsets J of $\{1, 2, \dots, n\}$ with $\#J = t$ such that $J \neq \{1, 2, \dots, t\}$.

<u>Lemma</u> (2.2). (1) $\dim_S L_t = n$ .

(2) $H^p_{\underline{n}}(L_t) = (0) \qquad (p \neq 1, t, n),$

$\qquad\qquad = \underline{k} \qquad (p = t),$

$\qquad\qquad = \underline{k}(-t) \qquad (p = 1).$

(3) $L_t$ <u>is not a Buchsbaum</u> S-<u>module</u>.

<u>Proof</u>. Assertions (1) and (2) follow from (2.1) and the exact sequence

$$0 \longrightarrow L_t \longrightarrow M_t \longrightarrow k(-t) \longrightarrow 0$$

of graded S-modules. Consider assertion (3) and assume that $L_t$ is a Buchsbaum S-module. Then we have the equality

$$I(L_t) = l_S(L_t/\underline{n}L_t) - e_{L_t}(n) .$$

Hence

$$l_S(L_t/\underline{n}L_t) = \left[\binom{n-1}{1} + \binom{n-1}{t}\right] + \binom{n-1}{t-1}$$

$$= \binom{n}{t} + n - 1 ,$$

because $e_{L_t}(\underline{n}) = \binom{n-1}{t-1}$ by (2.1) (3) and

$$I(L_t) = \sum_{i=0}^{n-1} \binom{n-1}{i} \cdot l_S(H^i_{\underline{n}}(L_t))$$

$$= \binom{n-1}{1} + \binom{n-1}{t}$$

by assertion (2) (c.f. |7|). Therefore $L_t$ is minimally generated by $\binom{n}{t}$ + n - 1 elements. On the other hand it is clear that the graded S-module $L_t$ is generated by the $\binom{n}{t} - 1$ elements

$$\left\{ g(e_J) \mid J \subset \{1, 2, \ldots, n\} \text{ such that } J \neq I \text{ and } \#J = t \right\}$$

together with the n elements

$$\{ X_i g(e_I) \}_{1 \leqq i \leqq n} ,$$

where $I = \{1, 2, \ldots, t\}$. Accordingly these elements must form a minimal system of generators for $L_t$ —— this is of course not true, since

$$X_{t+1} g(e_I) \in \sum_{\substack{1 \leqq i \leqq t, \\ J \subset \{1, 2, \ldots, n\} \\ \text{such that } J \neq I \text{ and} \\ \#J = t}} S X_i g(e_J) .$$

Thus $L_t$ is not a Buchsbaum S-module.

Let me recall one lemma.

Lemma (2.3) (|4| and |5|). *Let A be a Noetherian local ring with maximal ideal m and M a finitely generated A-module. If the canonical maps*

$$\mathrm{Ext}^i_A(A/m, M) \longrightarrow H^i_m(M)$$

*are surjective for all $i \neq \dim_A M$, then M is Buchsbaum. In case A is a regular local ring, the converse is also true.*

Let $1 \leqq s < t \leqq n - 1$ be integers. We put $L_{s,t} = L_t$ for $s = 1$. In case $s > 1$, we put $L_{s,t} = \mathrm{Ker}\,(G_{s-2} \longrightarrow G_{s-3})$ where

$$G_{s-2} \longrightarrow \ldots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow L_{t-s+1} \longrightarrow 0$$

is a part of a graded minimal free resolution of $L_{t-s+1}$. Notice that there exists an exact sequence

$$(\#) \quad 0 \longrightarrow L_{s,t} \longrightarrow G_{s-2} \longrightarrow L_{s-1,t-1} \longrightarrow 0$$

of graded S-modules.

Lemma (2.4). (1) $\dim_S L_{s,t} = n$ .

(2) $$\begin{aligned} H^p_{\underline{n}}(L_{s,t}) &= (0) && (p \neq s, t, n), \\ &= \underline{k}(s - t - 1) && (p = s), \\ &= \underline{k} && (p = t). \end{aligned}$$

(3) $L_{s,t}$ is not a Buchsbaum S-module.

Proof. The proof of assertions (1) and (2) is routine. Consider assertion (3). First of all apply functors $\mathrm{Ext}^i_S(S/\underline{n},*)$ and $H^i_{\underline{n}}(*)$ to the exact sequence (#) and we get a commutative square

$$\begin{array}{ccc} \mathrm{Ext}^{i-1}_S(S/\underline{n}, L_{s-1,t-1}) & \cong & \mathrm{Ext}^i_S(S/\underline{n}, L_{s,t}) \\ \downarrow h^{i-1}_{L_{s-1,t-1}} & & \downarrow h^i_{L_{s,t}} \\ H^{i-1}_{\underline{n}}(L_{s-1,t-1}) & \cong & H^i_{\underline{n}}(L_{s,t}) \end{array}$$

for each $i \leqq n - 1$ , where $h^{i-1}_{L_{s-1,t-1}}$ and $h^i_{L_{s,t}}$ are the canonical maps. Hence as $H^{n-1}_{\underline{n}}(L_{s-1,t-1}) = (0)$ by (2), the required assertion (3) follows, by induction on $s$ , from (2.2) (3) and (2.3).

Now let $1 \leqq s < t \leqq n - 1$ be integers. Let $h_0, h_1, \ldots, h_{n-1} \geqq 0$ be integers such that $h_s$ and $h_t$ are positive. We put

$$\begin{aligned} h &= \min\{h_s, h_t\}, \\ u &= s \quad \text{if } h_s \geqq h_t, \\ &= t \quad \text{if } h_s < h_t \end{aligned}$$

and

$$E = \sum_{\substack{0 \leqq i \leqq n-1 \\ \text{such that} \\ i \neq s,\, t}} M_i^{h_i} \oplus M_u^{h_u - h} \oplus L_{s,t}^{h},$$

where $M^r$ denotes, for a given S-module $M$ and an integer $r \geqq 0$ , the direct sum of $r$ copies of $M$ . Then we get by (2.4) the following

Theorem (2.5). (1) $\dim_S E = n$ .

(2) Let $0 \leqq p < n$ be an integer. Then

$$\begin{aligned} H^p_{\underline{n}}(E) &= \underline{k}^{h_p} && (p \neq s), \\ &= \underline{k}^{h_s - h} \oplus \underline{k}(s - t - 1)^h && (p = s). \end{aligned}$$

(3) $E$ is a quasi-Buchsbaum S-module but not Buchsbaum.

3. Proof of Theorem (1.1).

Let $d \geqq 3$ and $h_1, h_2, \ldots, h_{d-1} \geqq 0$ be integers. Assume that $h_s$ and $h_t$ are positive for some $s$ and $t$, $1 \leqq s < t \leqq d-1$. We put $n = d+2$ and

$$\begin{aligned} h'_i &= 0 \qquad ( i = 0, 1, d+1 ), \\ &= h_{i-1} \qquad ( d \geqq i \geqq 2 ). \end{aligned}$$

Let $S = k[X_1, X_2, \ldots, X_n]$ be a polynomial ring with $n$ variables over an infinite field $k$ and consider the graded S-module $E$ obtained by (2.5) for the above integers $h'_i$ ( $0 \leqq i \leqq n-1$ ), $s+1$ and $t+1$. Then as $E_{\underline{p}}$ is a free $S_{\underline{p}}$-module for any prime ideal $\underline{p}$ of $S$ such that $\underline{p} \neq \underline{n}$ and as $\operatorname{depth}_S E \geqq 2$, we have by virtue of |1| a short exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow P(r) \longrightarrow 0$$

of graded S-modules, where $P$ is a graded prime ideal of $S$ with $\operatorname{ht}_S P = 2$, $F$ is a graded free S-module and $r$ an integer. We put $A = S_{\underline{n}}/PS_{\underline{n}}$. Then the local ring $A$ satisfies all the requirements in Theorem (1.1). This is my proof of Theorem (1.1).

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On ASL domains with #Ind(A) $\leq$ 2

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Introduction. The concept of ASL (algebras with straightening laws) is an axiomatization of the "straightening formula" appearing in the invariant theory. This axiomatization, which is lucid and charming, associates commutative algebra with combinatorics through partially ordered sets (abbreviated poset), and moreover with topology through simplicial complexes.

On the other hand, ASL are flat deformations of the Stanley-Reisner rings $R[X_1,\ldots,X_n]/I$, where I is an ideal generated by square-free monomials. In [2] the concept of Hodge algebras is introduced, which is obtained by paying attention to the fact that flat deformations preserve Cohen-Macaulayness and Gorensteinness etc.

The purpose of this note is to determine the structure of certain ASL domains which have relatively simple relations.

## §1. Definitions.

All rings and algebras to be considered here will be commutative and have unit elements.

Suppose R is a ring, A is a R-algebra, and H, a subset of A, is partially ordered set called a poset. A monomial is a

a finite

product of the form $\alpha_1\alpha_2\cdots\alpha_k$ where $\alpha_i \in H$. A monomial $\alpha_1\alpha_2\cdots\alpha_k$ is called <u>standard</u> if $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$.

Then A is called an <u>algebra with straightening law</u> on H over R if the following conditions are satisfied:

(ASL-1) The algebra A is a free R-module whose basis is the set of standard monomials.

(ASL-2) If $\alpha$ and $\beta$ are incomparable (written $\alpha \not\sim \beta$), and

$$(*) \qquad \alpha\beta = \sum r_i \gamma_{i1}\gamma_{i2}\cdots\gamma_{ik_i},$$

where $0 \neq r_i \in R$, and $\gamma_{i1} \leq \gamma_{i2} \leq \cdots$ is the unique expression for $\alpha\beta$ in A as a linear combination of standard monomials, then $\gamma_{i1} \lneq \alpha, \beta$ for every i.

Note that the right-hand side of the relation in (ASL-2) is allowed to be empty sum ( = 0 ), but we require $k_i > 0$ in each term that appears. The relations (*) are called the <u>straightening relations</u> for A. We denote by Ind(A) the subset of H which consists of all the elements $\gamma_{ij}$ that occur on the right-hand side of the straightening relations.

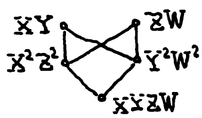
Let H be a finite poset. If we put the right-hand sides of all the straightening relations to be 0, then we can construct the "simplest" ASL on H called the discrete ASL, written R[H], namely

$$R[H] = R[X_\alpha \mid \alpha \in H]/(X_\alpha X_\beta \mid \alpha \not\sim \beta).$$

Example.

$$R[XY, ZW, X^2Z^2, Y^2W^2, XYZW] \subset R[X, Y, Z, W]$$

is an ASL on the poset




More generally, we can show that every subring of a polynomial ring which is generated by a finite number of monomials is a Hodge algebra defined in [2] in some way.

In general, if A is an ASL on a poset H over R, then $\mathrm{Ind}(A) = \emptyset$ if and only if A is discrete, and Ind(A) is a measure of the difference between A and the discrete ASL R[H].

## §2. ASL domains with #Ind(A) = 1.

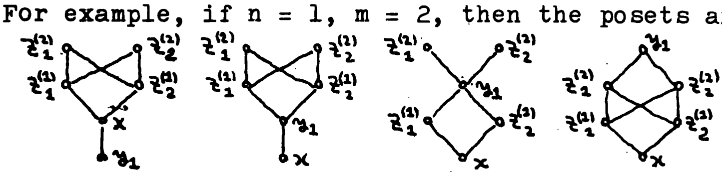
As is remarked in the preceding section, the indiscrete part Ind(A) is a measure of the difference between A and the discrete ASL R[H]. Therefore we may well believe that A has a rather simple structure when #Ind(A) = 1 or 2. And our purpose of this section is to describe the ASL domains with #Ind(A) = 1.

Throughout this section, the base ring R is a domain and A is an ASL domain on a poset H over R which satisfies #Ind(A) = 1. We put $\mathrm{Ind}(A) = \{x\}$. Firstly it is easy to see that x is comparable with any other element of H, and the set $\{\alpha \in H \mid \alpha \leq x\}$ is a totally ordered subset in H.

Secondly, suppose that for any element $\alpha \in H$ there exist $\beta, \gamma \in H$ which are incomparable with $\alpha$, and that $\alpha\beta = f(x)$, $\alpha\gamma = g(x)$ are the straightening relations of (ASL-2), where $f(x), g(x) \in R[X]$ are non-zero polynomials without constant terms. Then we have $\gamma f(x) - \beta g(x) = 0$ which contradicts the linear independence among the standard monomials in (ASL-1). From this, for any element $\alpha \in H$, there exists at most one element which is incomparable with $\alpha$.

Accordingly, if the pairs of incomparable elements of H are $z_1^{(1)}, z_2^{(1)}; z_1^{(2)}, z_2^{(2)}; \dots; z_1^{(m)}, z_2^{(m)}$, then these elements are bigger than x, and if the remaining elements are $y_1, y_2, \dots y_n$, then they are comparable with any other element of H.

For example, if $n = 1$, $m = 2$, then the posets are



Now if the straightening relations in (ASL-2) are

$$z_1^{(i)} z_2^{(i)} = f_i(x) \qquad (1 \leq i \leq m),$$

then we have the natural surjection of R-algebras from

$$R^{[n,m]} = \frac{R[X, Y_1, \dots, Y_n, Z_1^{(1)}, Z_2^{(1)}, \dots, Z_1^{(m)}, Z_2^{(m)}]}{(Z_1^{(1)} Z_2^{(1)} - f_1(X), \dots, Z_1^{(m)} Z_2^{(m)} - f_m(X))}$$

to A. If we show that

$$R^{[n,m]} = R[x, y_1, \dots, y_n, z_1^{(1)}, z_2^{(1)}, \dots, z_1^{(m)}, z_2^{(m)}]$$

is an ASL domain on the same poset H as that of A, then the above surjection turns out to be an isomorphism, and we can conclude that every ASL domain with #Ind(A) = 1 is of the type $R^{[n,m]}$.

In the following we prove that $R^{[n,m]}$ is an ASL domain. We must check the ASL axioms for $R^{[n,m]}$. It is obvious that $R^{[n,m]}$ satisfies (ASL-2). Concerning (ASL-1), we must note first that

$$R^{[n,m]} = R^{[n,m-1]}[Z_1^{(m)}, Z_2^{(m)}]/(Z_1^{(m)} Z_2^{(m)} - f_m(x)).$$

Now by induction on m, we can show that $R^{[n,m]}$ is a domain and satisfies (ASL-1) using the following Lemma 1.

Lemma 1. Let A be a domain, and

$$B = A[x,y] = A[X,Y]/(XY - a),$$

where $0 \neq a \in A$. Then

1) B is a free A-module whose basis is $1, x, x^2, \ldots, y, y^2, \ldots$.

2) B is a domain.

It is easy to see that $R^{[n,m]}$ is a Cohen-Macaulay (resp. Gorenstein) ring if R is so. And moreover if R is a noetherian normal domain, then so is $R^{[n,m]}$. It is an immediate consequence of the following Lemma 2.

Lemma 2. Let A be a noetherian normal domain, and

$$B = A[x,y] = A[X,Y]/(XY - a),$$

where $0 \neq a \in A$. Then B is normal.

Proof. Since $B_x \simeq A[X]_X$, $B_y \simeq A[Y]_Y$ are normal, a prime ideal $\mathfrak{p}$ of B contains x and y if $B_{\mathfrak{p}}$ is not normal. We note that x, y is a regular sequence on B since

$$B/(x) = (A[X,Y]/(XY - a))/((X, XY - a)/(XY - a))$$
$$\simeq A[X,Y]/(X,a) \simeq (A/(a))[Y].$$

Consequently, depth $B_{\mathfrak{p}} \geq 2$ if B is not normal. Now it is well-known that

$$B = \bigcap_{\text{depth } B_{\mathfrak{p}} = 1} B_{\mathfrak{p}}$$

in the quotient field of B. Therefore B is normal. Q.E.D.

## §3. ASL domains with #Ind(A) = 2.

In this section A is an ASL domain with #Ind(A) = 2 on a poset H over a domain R. We put Ind(A) = {x,y}. Since A is a domain, x and y are comparable, say $x < y$.

Now it is obvious that any element of H is comparable with x, and moreover the following facts can be easily shown:

1) There exists at most one element (say z, if exists) which is incomparable with y.
2) Any element $\alpha$ ($\neq x,y,z$) of H is comparable with x,y, and moreover $\alpha$ is comparable with z.
3) For any element $\alpha$ ($\neq x,y,z$) of H, there exists at most one element which is incomparable with $\alpha$.

For example, we show 1) in the following way. Suppose there exist $\alpha \neq \beta \in H$ which are incomparable with y, and that

$$(*) \quad \alpha y = yf(x,y) + g(x), \quad \beta y = yp(x,y) + q(x)$$

are the straightening relations in (ASL-2). From (*) we have

$$\beta(yf(x,y) + g(x)) = \alpha(yp(x,y) + q(x)).$$

And by using (*) again, we have

$$(yp(x,y) + q(x))f(x,y) + \beta g(x)$$
$$= (yf(x,y) + g(x))p(x,y) + \alpha q(x).$$

In this relation, $\beta$ (resp. $\alpha$) appears only in the left-hand (resp. right-hand) side. From this we have $g(x) = 0$ ( resp. $q(x) = 0$) by (ASL-1). And in this case, the relation of (*) turns out to be

$$\alpha y = yf(x,y) \qquad (\text{resp. } \beta y = yp(x,y))$$

but this is a contradiction.

As a result of these facts, it is easy to see that the posets on which there exist ASL domains with #Ind(A) = 2 are the same as those on which there exist ASL domains with #Ind(A) = 1.

Now we determine the structure of ASL domains which satisfy #Ind(A) = 2. We put Ind(A) = $\{x,y\}$, $x < y$. And if there exists an element which is incomparable with y, we denote such an element by z. As in the case of #Ind(A) = 1, the pairs of incomparable elements of H are denoted by $a_1^{(1)}, a_2^{(1)}; \dots; a_1^{(m)}, a_2^{(m)}$, and the remaining elements $b_1, \dots, b_n$ are comparable with any other element of H.

Let

$$yz = f(x,y) \quad \text{and}$$

$$a_1^{(i)} a_2^{(i)} = g_i(x,y) \qquad (1 \leq i \leq m)$$

be the straightening relations in (ASL-2). Here $f(X,Y)$, $g(X,Y) \in R[X,Y]$ are non-zero polynomials without constant terms, and $X$ divides $f(X,Y)$ while $Y$ does not divide $f(X,Y)$, and moreover $X$ must divide $g_i(X,Y)$ if $a_1^{(i)}, a_2^{(i)} < y$.

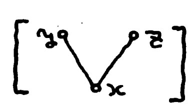
Then we show that

$$R[x,y,z,a_1^{(1)},a_2^{(1)},\dots,a_1^{(m)},a_2^{(m)},b_1,\dots,b_n]$$
$$= \frac{R[X,Y,Z,A_1^{(1)},A_2^{(1)},\dots,A_1^{(m)},A_2^{(m)},B_1,\dots,B_n]}{(YZ - f(X,Y), A_1^{(1)}A_2^{(1)} - g_1(X,Y),\dots,A_1^{(m)}A_2^{(m)} - g_m(X,Y))}$$

is an ASL domain on the poset H. Since it is obvious that this ring satisfies (ASL-2), seeing Lemma 1, we have only to show that

$$R[x,y,z] = R[X,Y,Z]/(YZ - f(X,Y))$$

is an ASL domain on the poset




If we consider $f(X,Y)$ as a polynomial in $Y$ over $R[X]$, $f(X,Y)$ has a non-zero constant term since $Y$ does not divide $f(X,Y)$.

Now we can prove that $R[x,y,z]$ is an ASL domain by using the following Lemma 3.

Lemma 3. Let $A$ be a domain, and

$$B = A[y,z] = A[Y,Z]/(YZ - f(Y)),$$

where $f(Y) \in A[Y]$ has a non-zero constant term. Then

1) B is a free A- module whose basis is $1, y, y^2, \ldots, z, z^2, \ldots$.

2) B is a domain.

3) B is normal if A is a noetherian normal domain.

Proof. 1) is easy to prove. Concerning 2) and 3), if $f(Y) = Yg(Y) + a$, $0 \neq a \in A$, and we put $W = Z - g(Y)$, then Y and W are indeterminates over A, and

$$A[Y,Z]/(YZ - f(Y)) = A[Y,Z]/(Y(Z - g(Y)) - a)$$
$$= A[Y,W]/(YW - a).$$

Therefore we can apply Lemma 1 and Lemma 2 directly. Q.E.D.

Consequently, as in the case of $\#\mathrm{Ind}(A) = 1$, an ASL domain with $\#\mathrm{Ind}(A) = 2$ is a normal (resp. Cohen-Macaulay, Gorenstein) ring if R is a noetherian normal (resp. Cohen-Macaulay, Gorenstein) ring. In particular, if A is an ASL domain which satisfies $\#\mathrm{Ind}(A) \leq 2$ over a field k, then A is a normal Gorenstein ring.

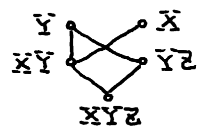
Supplement: Generalizing the method by which we can determine the posets on which there exist ASL domains which satisfy $\#\mathrm{Ind}(A) = 2$, it may be possible to determine the posets on which there exist ASL domains whose indiscrete parts Ind(A) are totally ordered sets.

In the case of #Ind(A) = 3, we have different posets from those in the case of #Ind(A) $\leqq$ 2.

For example,

$$A = R[X, Y, YZ] \subset R[X, Y, Z]$$

is an ASL domain on the poset




which satisfies #Ind(A) = 3.

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# On the Gorensteinness of Rees algebras over local rings

Shin Ikeda (Nagoya University)

Let $(A,m,k)$ be a Noetherian local ring and $I$ an ideal of $A$. We define

$$R_A(I) = \bigoplus_{n \geq 0} I^n$$

and call this graded $A$-algebra the Rees algebra of $I$. The purpose of this note is to give a summary of the theory of the canonical modules of graded rings and apply this theory to a characterization of the Gorensteinness of $R_A(I)$.

## 1. Canonical modules of graded rings.

Throughout this section $R = \bigoplus_{n \geq 0} R_n$ denotes a Noetherian graded ring.

Let $M = \bigoplus_{n \geq 0} M_n$ and $N = \bigoplus_{n \geq 0} N_n$ be graded $R$-modules. Let $H(R)$ be the category of graded $R$-modules whose morphisms are grade preserving $R$-homomorphisms. Let $n \in \mathbb{Z}$. We denote by $M(n)$ the graded $R$-module whose grading is given by $M(n)_m = M_{n+m}$ for all $m \in \mathbb{Z}$. For $n \in \mathbb{Z}$ we define

$$\underline{\mathrm{Hom}}_R(M, N)_n = \left\{ f \in \mathrm{Hom}_R(M, N) \; ; \; f(M_m) \subset N_{n+m} \text{ for all } m \in \mathbb{Z} \right\}$$

and

$$\underline{\mathrm{Hom}}_R(M, N) = \bigoplus_{n \in \mathbb{Z}} \underline{\mathrm{Hom}}_R(M, N)_n .$$

One can easily show that the category $H(R)$ is an abelian category with enough injectives and projectives. We define the functor $\underline{\mathrm{Ext}}^i_R(\ ,\ )$ to be the i-th derived functor of $\underline{\mathrm{Hom}}_R(\ ,\ )$ for $i \geq 0$. Let $I$ be a homogeneous ideal of $R$. We define

$$\underline{H}_I^i(\ ) = \varinjlim_n \underline{\mathrm{Ext}}_R(R/I^n\ ,\ )$$

and call this functor the i-th local cohomology functor with support in $I$.

In the rest of this section we assume that $R_0$ is a local ring with maximal ideal $m_0$ and we denote by $\hat{R}$ the ring $R \otimes_{R_0} \hat{R}_0$, where $\hat{R}_0$ is the completion of $R_0$. For an $R_0$-module $E$ we define $\underline{E}$ to be the graded $R_0$-module with grading given by $\underline{E}_0 = E$ and $\underline{E}_n = 0$ for $n \neq 0$. Let $E_{R_0}$ be the injective envelope of $R_0/m_0$ as an $R_0$-module. Regarding $R$ as a graded $R_0$-we define

$$\underline{E}_R = \underline{\mathrm{Hom}}_{R_0}(R\ ,\ \underline{E}_{R_0}).$$

Proposition (1.1) (1) $\underline{E}_R$ is an injective envelope of $R/M$ in the category $H(R)$, where $M$ is the unique maximal homogeneous ideal of $R$.

(2) $\underline{\mathrm{Hom}}_R(\underline{E}_R\ ,\ \underline{E}_R) = \hat{R}$.

Using this result we can prove a result corresponding to the Matlis duality by the same proof as in [M]. Let $d = \dim R$.

Definition (1) If $R_0$ is complete we define

$$\underline{K}_R = \underline{\mathrm{Hom}}_R(\underline{H}_M^d(R)\ ,\ \underline{E}_R)$$

and we call $\underline{K}_R$ the canonical module of $R$.

(2) If $R_0$ is not complete a graded $R$-module $\underline{K}_R$ is a canonical module of $R$ if

$$\underline{K}_R \otimes_R \hat{R} = \underline{K}_{\hat{R}}.$$

As in the local ring case, if a canonical module $\underline{K}_R$ exists then it is a finitely generated R-module and unique up to isomorphism.

Now we will give several properties of canonical modules.

Proposition (1.2) If $R_0$ is complete for any finitely generated graded R-module N we have

$$\underline{\mathrm{Hom}}_R(\underline{H}_M^d(N), \underline{E}_R) = \underline{\mathrm{Hom}}_R(N, \underline{K}_R).$$

Proposition (1.3) If $R_0$ is complete and R is Cohen-Macaulay then for all $0 \leq i \leq d$ and for any finitely generated graded R-module N we have

$$\underline{\mathrm{Ext}}_R^i(N, \underline{K}_R) = \underline{\mathrm{Hom}}_R(\underline{H}_M^{d-i}(N), \underline{E}_R).$$

Proposition (1.4) Let R be Cohen-Macaulay. Then, R is Gorenstein if and only if R has a canonical module $\underline{K}_R$ and $\underline{K}_R = R(n)$ for some $n \in \mathbb{Z}$.

Proposition (1.5) Let R be Cohen-Macaulay and let $R \to S$ be a finite homomorphism of graded rings. Suppose that R has a canonical module $\underline{K}_R$. Then

$$\underline{K}_S = \underline{\mathrm{Ext}}_R^r(S, \underline{K}_R),$$

where $r = \dim R - \dim S$.

Propositions (1.2) — (1.5) can be proved by the same arguments as in [GW].

2. The Gorensteinness of Rees algebras.

Throughout this section (A,m,k) denotes a Noetherian local ring of dim A = d. Let I be an ideal of A and we set

$$G_A(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}.$$

Let

$$\ell(I) = \dim G_A(I)/mG_A(I) .$$

$\ell(I)$ is called the analytic spread of $I$ and $\ell(I)$ is equal to the minimal number of generators of a minimal reduction of $I$ if the residue field $k$ is infinite.

The main result of this section is the following.

<u>Theorem</u> (2.1) Let $I$ be an ideal of $A$ such that $\ell(I) = ht(I)$ and grade $(I) \geq 2$. Suppose that $R_A(I)$ is Cohen-Macaulay. Then $R_A(I)$ is Gorenstein if and only if $G_A(I)$ has a canonical module $\underline{K}_{G_A(I)}$ and $\underline{K}_{G_A(I)} = G_A(I)(-2)$. In this case $A$ has a canonical module $K_A$ and $K_A = A$.

As an immediate consequece of Theorem (2.1) we have a result of S. Goto and Y. Shimoda [GS].

<u>Corollary</u> (2.2) ( Goto-Shimoda ) Let $A$ be Cohen-Macaulay with $\dim A \geq 2$. Then $R_A(m)$ is Gorenstein if and only if $G_A(m)$ is Gorenstein and $a(G_A(m)) = -2$, where $a(G_A(m))$ is the a-invariant of $G_A(m)$ (cf. [GW]). In this case $A$ is Gorenstein.

For the proof of Theorem (2.1) we need the following result in [HI].

<u>Proposition</u> (2.3) Let $I$ be an ideal of $A$ such that $\ell(I) = ht(I) > 0$. Then $R_A(I)$ is Cohen-Macaulay if and only if

(1) for $i < d$

$$\underline{H}_M^i(G_A(I))_n = \begin{cases} H_m^i(A) & \text{for } n = -1 \\ (0) & \text{for } n \neq -1 \end{cases}$$

(2) $\underline{H}_M^d(G_A(I))_n = (0)$ for $n \geq 0$,

where $M$ is the maximal homogeneous ideal of $R_A(I)$. In this case

$$\text{depth } A \geq \dim A/I + 1.$$

By Proposition (2.3) we can prove that if $R_A(I)$ is Cohen-Macaulay and grade $(I) \geq n$ then $A$ and $G_A(I)$ satisfy $(S_n)$. Note that $G_A(I)$ satisfies $(S_2)$ if and only if

$$G_A(I) = \underline{\text{Hom}}_{G_A(I)}(\underline{K}_{G_A(I)}, \underline{K}_{G_A(I)}).$$

<u>Sketch of the proof of Theorem (2.1)</u>

We may assume that $k$ is infinite and $A$ is complete. Let $(a_1,..,a_h)$ be a minimal reduction of $I$ ( $h = ht(I)$ ). We set $R = R_A(I)$, $G = G_A(I)$ and $\bar{R} = R/(a_1, a_1X)$, where $R_A(I)$ is identified with the subalgebra $A[IX]$ of the polynomial ring in one variable $A[X]$. Since $R$ is Cohen-Macaulay we have the exact sequences

$$0 \to \underline{H}_M^{d-1}(\bar{R}) \to H_m^d(A) \to \underline{H}_M^d(R/a_1XR) \to \underline{H}_M^d(\bar{R}) \to 0 \qquad (1)$$

$$0 \to \underline{H}_M^{d-1}(\bar{R}) \to \underline{H}_M^d(G)(-1) \to \underline{H}_M^d(R/a_1R) \to \underline{H}_M^d(\bar{R}) \to 0 \qquad (2).$$

Assume that $R$ is Gorenstein. By Prop.(1.3) and (1.4) we have

$$\underline{\text{Ext}}_R^2(\bar{R}, R) = \underline{\text{Hom}}_R(\underline{H}_M^{d-1}(\bar{R}), \underline{E}_R).$$

On the other hand one can prove that

$$\underline{\text{Ext}}_R^2(\bar{R}, R) = (a_1R :_R IR)/(a_1, a_1X) = (0)$$

by the assumption grade $(I) \geq 2$ . Hence $\underline{H}_M^{d-1}(\bar{R}) = (0)$ . Then, the exact sequence (2) gives $\underline{\mathrm{Hom}}_R(k, \underline{H}_M^d(G)) = \underline{k}(2)$. Since G satisfies $(S_2)$ we have $\underline{K}_G = G(-2)$.

Conversely assume that $\underline{K}_G = G(-2)$ . By a purely formal argument one can show that

$$\underline{\mathrm{Hom}}_R(k, \underline{H}_M^d(R/a_1XR))_n = (0) \quad \text{for} \quad n \neq 0 ,$$

$$\underline{H}_M^d(\bar{R})_n = (0) \quad \text{for} \quad n \geq 0$$

and

$$\underline{H}_M^{d-1}(\bar{R}) = (0) .$$

Then, Theorem (2.1) follows from the following lemma.

<u>Lemma</u> (2.4) If $\underline{K}_G = G(-2)$ then $K_A = A$ .

It is natural to ask whether the Gorensteinness of $R_A(I)$ implies that of A. In general this is not true . However we have the following results.

<u>Proposition</u> (2.5) Assume that $l_A(H_m^i(A)) < \infty$ for $i < d$ and $2\mathrm{ht}(I) \leq \dim R$. If $\ell(I) = \mathrm{ht}(I) \geq \mathrm{grade}(I) \geq 2$ and $R_A(I)$ is Gorenstein then A is Gorenstein.

<u>Proposition</u> (2.6) Let A be a Buchsbaum ring with $\dim A \geq 2$ and let q be a parameter ideal of A contained in $m^2$ . If $R_A(q^n)$ is Gorenstein for some $n > 0$ then A is Gorenstein.

Examples (1) Let $k$ be a field of characteristic 2 and $X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4$ indeterminates over $k$. We put

$$A = k[[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4]]/\underline{a} ,$$

where $\underline{a}$ is the ideal generated by $X_1Y_1 + X_2Y_2 + X_3Y_3$ , $Y_1^2$ , $Y_2^2$ , $Y_3^2$ , $Y_4^2$ , $Y_1Y_4$ $Y_2Y_4$ , $Y_3Y_4$ , $Y_1Y_2 - X_3Y_4$ , $Y_2Y_3 - X_1Y_4$ , $Y_1Y_3 - X_2Y_4$ .
Then $A$ is not Cohen-Macaulay ( $\dim A = 3$ , depth $A = 2$ ) but $R_A(m)$ is Gorenstein.

(2) Let $A$ be the same as in Example (1) and $T_1, \ldots, T_n$ indeterminates over $A$. We set $B = A[[T_1,\ldots,T_n]]$ and $I = mB$. Then $R_B(I) = R_A(m) \otimes_A B$ is Gorenstein since $R_B(I)$ is faithfully flat over $R_A(m)$. If $n \geq 3$ then $2ht(I) \leq \dim B$. Clearly $B$ is not Gorenstein. So, this example shows that without any restriction on the local cohomology of the local ring Proposition (2.5) does not hold.

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# Remarks on a Conjecture of Nakai

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Let k, R be commutative rings with 1 and let R be a k-algebra. A differential operator D of R/k of order $\leqq$ n is defined inductively as a k-linear map of R into itself such that for any a $\varepsilon$ R $[D, a] = Da - aD$ is a differential operator of order $\leqq n - 1$, where a differential operator of order 0 is a homothety by an element of R (cf. [2],[3]). An n-th order derivation of R/k in the sense of Nakai is the same notion as a differential operator of R/k of order $\leqq$ n which vanishes on 1 (cf. [7]). The set of differential operators of R/k of order $\leqq$ n is denoted by $\mathrm{Diff}^n(R/k)$ and the R-algebra of differential operators of R/k is denoted by Diff(R/k) and thus we have $\mathrm{Diff}(R/k) = \bigcup_{n=0}^{\infty} \mathrm{Diff}^n(R/k)$. Let Der(R/k) be the R-module of derivations of R/k and let diff(R/k) denote the subalgebra of Diff(R/k) which is generated by Der(R/k). We have $\mathrm{Diff}^1(R/k) = R \oplus \mathrm{Der}(R/k)$.

Let R be an affine domain over a field of characteristic 0 and let P be a prime ideal of R. It is known that if $R_P$ is regular then we have $\mathrm{Diff}(R_P/k) = \mathrm{diff}(R_P/k)$ ([2]). Y. Nakai asked the converse, that is, does the condition $\mathrm{Diff}(R_P/k) = \mathrm{diff}(R_P/k)$ imply the regularity of $R_P$ ? To our knowledge the only affirmative answer is given for the case of dim R = 1 ([6]) and we also have no counter examples. On the other hand the Nakai conjecture is closely related to the Zariski-Lipman conjecture which asserts that $R_P$-freeness of

$Der(R_P/k)$ implies the regularity of $R_P$. In fact Rego proved that the Nakai conjecture implies the Zariski-Lipman conjecture ([8]).

Our objective here is to investigate the Nakai conjecture in the case $R = \bigoplus_{i=0}^{\infty} R_i$ is graded by the non-negative integers N, $R_0 = k$, and $P = m$, where $m = \bigoplus_{i=1}^{\infty} R_i$ is the irrelevant maximal ideal. We represent R as T/A, where $T = k[x_1, \ldots ,x_s]$ is a polynomial ring in which each $x_i$ has weight $d_i$, and A is a graded prime ideal. We assume the following condition for a graded domain R.

(#) There exists an integer $d_0$ such that for every $d \geqq d_0$, $R^{(d)} = \bigoplus_{i=0}^{\infty} R_{di}$ is generated by $R_d = [R^{(d)}]_1$ over $R_0 = k$.

We call a differential operator D of R/k homogeneous of weight $\ell$ ($\ell \in Z$) if $D(R_i) \subset R_{i+\ell}$ for all i. In particular, the Euler derivation

$$I = \sum_{i=1}^{s} d_i x_i \frac{\partial}{\partial x_i}$$

(which is a derivation of R since A is graded) is homogeneous of weight 0. We denote by $Diff_\ell(R/k)$ the space of homogeneous differential operators of R/k of weight $\ell$ and set $Diff_\ell^n(R/k) = Diff^n(R/k) \cap Diff_\ell(R/k)$. It is easy to verify that $Diff_\ell(R/k)\, Diff_m(R/k) \subset Diff_{\ell+m}(R/k)$. Moreover we have the following:

(1) $Diff^n(R/k) = \bigoplus_{\ell=-\infty}^{\infty} Diff_\ell^n(R/k)$, (2) $Diff(R/k) = \bigoplus_{\ell=-\infty}^{\infty} Diff_\ell(R/k)$ ([1]).

The condition $Diff(R_m/k) = diff(R_m/k)$ is equivalent to $Diff(R/k) = diff(R/k)$ and hence the conjecture is as follows: if $Diff(R/k) = diff(R/k)$ then R is a polynomial ring over k. We may assume dim $R \geqq 2$, because the case of dim R = 1 is affirmative ([6]). Then we have

Theorem. Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a finitely generated graded domain over a field k of characteristic 0, where $R_0 = k$, $m = \bigoplus_{i=1}^{\infty} R_i$, and R satisfies the condition (#). Assume that R has an isolated singularity at m and depth $R_m \geqq 2$. If the only homogeneous derivation of R/k of weight $\leqq 0$ is the Euler derivation I, then we have $\mathrm{Diff}(R/k) \supsetneqq \mathrm{diff}(R/k)$.

The idea of proof is essentially due to that of [1]. The following lemma is a key to proof.

**Lemma.** Let R be a graded domain satisfying the conditions in Theorem. Then we have

$$\mathrm{Diff}^2_\ell(R/k) \supsetneqq I\,\mathrm{Diff}^1_\ell(R/k) + \mathrm{Diff}^1_\ell(R/k)$$

for all sufficiently large $\ell$.

**Remark.** Under the same notation as in Theorem, assume that R is generated by $R_1$ and R has an isolated singularity at m. By [9] any derivation of R/k maps m into itself. Hence R has no homogeneous derivations of negative weight.

As corollaries we have

**Corollary 1.** Let f be a homogeneous polynomial in $k[x_1, \ldots, x_s]$ and let $R = k[x_1, \ldots, x_s]/(f)$, where each $x_i$ has weight 1. Assume that $\mathrm{Spec}(R) - \{m\}$ is regular. If we have $\mathrm{Diff}(R/k) = \mathrm{diff}(R/k)$, then R is a polynomial ring, that is, f is linear.

**Corollary 2.** Let R be a two dimensional Cohen-Macaulay graded domain over the field $\mathbb{C}$ of complex numbers such that $R_0 = \mathbb{C}$ and R satisfies the condition (#). If $\mathrm{Diff}(R/k) = \mathrm{diff}(R/k)$, then R is a polynomial ring.

Corollary 2 is an immediate cosequence of the following

Theorem ([10]). Let R be a normal surface singularity with good $\mathbb{C}^*$-action which is not a cyclic quotient singularity. Then the only derivation of R of weight $\leqq 0$ is the Euler derivation.

For details we refer to [4].

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On the terminal toric singularities of dimension 3

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Let A be the local ring of a point of a normal algebraic variety of dimension n defined over $\mathbb{C}$ .

Definition (M. Reid) A is said to be canonical if there exist a resolution $f : Y \longrightarrow \operatorname{Spec} A$ of singularity and a positive integer r such that the direct image $f_* \omega_Y^{\otimes r}$ is invertible.

For a canonical local ring, the minimal positive integer r with this property is independent from the choice of the resolution, and it is called the index of the canonical singularity.

It was proved by Elkik that canonical singularities are rational and Cohen-Macaulay. In particular, a local ring A is rational and Gorenstein if and only if A is canonical of index one.

Let A be a canonical local ring of index r . Then for a resolusion $f : Y \longrightarrow \operatorname{Spec} A$ , we have $f^* f_* \omega_Y^{\otimes r} = \omega_Y^{\otimes r}(-Z)$ for an effective divisor $Z \subset Y$ .

Definition (M. Reid) In above notation, a canonical local ring A is said to be terminal if every prime divisor D of Y with dim f(D) $< n - 1$ is contained in the support of Z .

The case of 3-dimensional toric rings.

Here "toric ring" means the coordinate ring of an affine torus embedding. Let N be a free $\mathbb{Z}$-module of rank 3 , and let M be its dual.

Every toric ring is given as the semigroup ring $\mathbb{C}[M \cap \pi^\vee]$ for a rational polyhedral cone $\pi \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ of dimension 3 , where $\pi^\vee$ is the dual cone $\{x \in M_{\mathbb{R}} ; \langle x, a\rangle \geq 0$ , for every $a \in \pi\}$ . Let $\{\gamma_1, \ldots, \gamma_s\}$ be the set of one-dimensional faces of $\pi$ and let $a_i$ be the primitive element of $N$ with $\mathbb{R}_0 a_i = \gamma_i$ for $i = 1, \ldots, s$ where $\mathbb{R}_0 = \{c \in \mathbb{R} ; c \geq 0\}$ . The ring $A = \mathbb{C}[M \cap \pi^\vee]$ is known to be Cohen-Macaulay, and the dualizing module $K_A$ is identified with the ideal $\mathbb{C}[M \cap (\text{int } \pi^\vee)] = P_1 \cap \ldots \cap P_s$ where $P_i$ is the prime ideal $\mathbb{C}[M \cap (\pi^\vee \setminus \gamma_i^\perp)]$ for $i = 1, \ldots, s$ .

If $A$ is canonical of index $r$ , then the divisor $\operatorname{div}(K_A)$ is of order $r$ in the divisor class group $\mathrm{Cl}(A)$ . From the exact sequence

$$0 \longrightarrow M \longrightarrow \bigoplus_{i=1}^{s} \mathbb{Z}\operatorname{div}(P_i) \longrightarrow \mathrm{Cl}(A) \longrightarrow 0 \ ,$$
$$m \longmapsto \sum_{i=1}^{s} \langle m, a_i\rangle \operatorname{div}(P_i)$$

we know there exists $m \in M$ with $\langle m, a_i\rangle = r$ for every $i$ . It means there exists a coordinate of $N$ such that $a_i$ is equal to $a_i = (u_i, v_i, r)$ for $i = 1, \ldots, s$ .

<u>Theorem</u> (Reid, Danilov) The toric ring $A$ is canonical of index $r$ if and only if there exists a primitive element $m \in M$ such that $\langle m, a_i\rangle = r$ for $i = 1, \ldots, s$ and $\langle m, a\rangle \geq r$ for every non-zero $a \in N \cap \pi$ . Furthermore, it is terminal if and only if $\langle m, a\rangle > r$ for every $a \in N \cap \pi$ other than $a_1, \ldots, a_s$ and $0$ .

The terminal toric rings are determined as follows. (Frumkin, Morrison)

1). $s = 4$, $\{a_1, a_2, a_3, a_4\} = \{(0,0,1), (1,0,1), (0,1,1), (1,1,1)\}$,

2). $s = 3$, $\{a_1, a_2, a_3\} = \{(1,0,0), (0,0,1), (e,n,1)\}$ for non-negative integers $0 \leq e < n$ with $(e, n) = 1$,

for a coordinate of $N$.

This result is also stated in the following form. (see [R])

Let $\varepsilon = e^{2\pi i/n}$ and let $u : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ be the automorphism given by $u(x, y, z) = (\varepsilon^a x, \varepsilon^b y, \varepsilon^c z)$ for integers $a, b, c$. Assume $u$ is of order $n$. Then, the quotient $\mathbb{C}^3/(u)$ is terminal if and only if $(a, n) = 1$ and $b + c \equiv 0 \pmod{n}$ or its permutation in $a, b, c$.

I am going to give my own proof of this result.

It is easy to see that 1) is the unique three-dimensional non-simplicial cone which defines terminal toric ring. Hence the problem is to determine the tetrahedrons which are spanned by four integral points in $\mathbb{R}^3$ and contain no other integral points.

I gave in [I], a formula of the number of lattice points in the interior of a tetrahedron by counting the geometric genus of the algebraic surface defined by the sum of four monomials corresponding to the four vertices of the tetrahedron. I will use this formula for the proof.

From now on, we use $M_{\mathbb{R}}$ for the Euclidean space in which we consider the tetrahedron, since $M$ is the space of monomials.

For a finite set $F = \{v_0, \ldots, v_s\}$ of elements of $M$ with $v_1 - v_0$, $\ldots, v_s - v_0$ linearly independent, we denote by index(F) the index $[(\mathbb{R}(v_1-v_0) + \ldots + \mathbb{R}(v_s-v_0)) \cap M : \mathbb{Z}(v_s-v_0) + \ldots +(v_s-v_0)]$ .

Let $S = \{m_0, \ldots, m_3\}$ be a set of elements of $M \simeq \mathbb{Z}^3$ contained in no plane in $M_{\mathbb{R}} \simeq \mathbb{R}^3$ . We set

$\square$ : convex hull of $S$ , $n = \mathrm{index}(S)$ ,

$S_i$ : convex hull of $S \setminus \{m_i\}$, $b_i = \mathrm{index}(S \setminus \{m_i\})$ for $i = 0, 1, 2, 3$ ,

$E_{i,j}$ : convex hull of $S \setminus \{m_i, m_j\}$, $\ell_{i,j} = \mathrm{index}(S \setminus \{m_i, m_j\})$ , $0 \le i, j \le 3$.

For each $i = 0, \ldots, 3$ , there exists a unique primitive element $n_i$ of $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ such that $\nu_i$ is constant on the triangle $S_i$ and $\langle m_i, \nu_i \rangle > \langle S_i, \nu_i \rangle$ . We see easily that any three of $\{\nu_0, \ldots, \nu_3\}$ are linearly independent. For linearly independent primitive elements $u, v \in N$ , we denote by $n(u, v)$ the index $[(\mathbb{R}u + \mathbb{R}v) \cap N : \mathbb{Z}u + \mathbb{Z}v]$ and we denote by $q(u, v)$ the minimal non-negative integer $q$ with $(v + qu)/n \in N$ . Then we have $n(u, v) = n(v, u)$ and $q(u, v)q(v, u) \equiv 1 \pmod{n}$ . We set $n_{i,j} = n(v_i, v_j)$ and $q_{i,j} = q(v_i, v_j)$ for $0 \le i < j \le 3$ .

Theorem ([1]) The number of elements of $M$ contained in the interior of the tetrahedron $\square$ is given by the formula:

$$*) \quad \Big(2n - 3\sum_{i=0}^{3} b_i + \big(\sum_{i=0}^{3} b_i\big)^2/n - 12 - \sum_{0 \le i < j \le 3} \ell_{i,j}(\lambda(n_{i,j}, q_{i,j}) - 3 + 2/n_{i,j}\Big)/12 .$$

Where $\lambda(n, q) = a_1 + \ldots + a_s - 3s + (q + q^*)/n$ for the integers $a_1, \ldots,$ $a_s$ greater than one with $n/q = a_s - 1 \rfloor a_{s-1} - 1 \rfloor \ldots - 1 \rfloor a_1$ , and $q^*$ is the integer with $0 \le q^* < n$ and $qq^* \equiv 1 \pmod{n}$ . We understand $\lambda(1, 0) = 0$ .

We call $\lambda(n, q)$ the deviation of the pair $(n, q)$. It holds that $\lambda(n, q) = \lambda(n, q^*) = -\lambda(n, n-q)$ when $n > 1$.

Assume $\partial\square \cap M = \{m_0, m_1, m_2, m_3\}$ where $\partial\square$ is the boundary of the tetrahedron $\square$. Then since $b_i = 1$ for $i = 0, \ldots, 3$, $\ell_{i,j} = 1$, $n_{i,j} = n$ for $0 \le i < j \le 3$, we have

**) $$\#(\mathrm{int}\,\square \cap M) = \Big(2n - 6 + 4/n - \sum_{0\le i<j\le 3} \lambda(n, q_{i,j})\Big)/12$$

By the parallel translation, we assume $m_0 = 0$. Since the triangle spanned by $\{0, m_1, m_2\}$ contains no other points of $M$, we know $\{m_1, m_2\}$ is a part of a $\mathbb{Z}$-basis of $M$, Hence we can take a $\mathbb{Z}$-coordinate of $M$ such that $m_1 = (1, 0, 0)$, $m_2 = (0, 1, 0)$ and $m_3 = (p, q, n)$ with $0 \le p, q < n$. Since the triangle spanned by $\{0, m_2, m_3\}$ contains no other points of $M$, we know $p$ and $n$ are coprime. Similarly, $q$ and $n$ are also coprime. Since $\{m_3 - m_1, m_3 - m_2\}$ is a part of a $\mathbb{Z}$-basis of $M$, we know $p + q - 1$ and $n$ are coprime. We can calculate the $q_{i,j}$'s as follows:

$q_{0,1} \equiv (p + q - 1)^*p$, $q_{0,2} \equiv (p + q - 1)^*q$, $q_{0,3} \equiv -(p + q - 1)^*$, $q_{1,0} \equiv p^*(p + q - 1)$, $q_{1,2} \equiv -p^*q$, $q_{1,3} \equiv p^*$, $q_{2,0} \equiv q^*(p + q - 1)$, $q_{2,1} \equiv -q^*p$, $q_{2,3} \equiv q^*$, $q_{3,0} \equiv -(p + q - 1)$, $q_{3,1} \equiv p$, $q_{3,2} \equiv q$. (all in modulo $n$)

It is easy to check that they satisfy the following relations:

(1) $q_{i,j}q_{j,i} \equiv 1 \pmod n$ for all $0 \le i < j \le 3$.

(2) $\sum_{j\ne i} q_{i,j} \equiv 1 \pmod n$ for every $i = 0, \ldots, 3$.

(3) $q_{i,j}q_{j,k}q_{k,i} \equiv -1 \pmod n$ for every triple $(i, j, k)$ of distinct elements of $\{0, 1, 2, 3\}$.

Lemma 1. If one of $q_{i,j}$ is equal to one, then $\{m_1, m_2, m_3\}$ is equal to $\{(1, 0, 0), (0, 0, 1), (e, n, 1)\}$ in some order for an integer $0 \leq e < n$ with $(e, n) = 1$ and for a $\mathbb{Z}$-coordinate of $M$.

Proof. Assume $q_{0,1} \equiv (p + q - 1)*p \equiv 1 \pmod{n}$. Then we have $q = 1$ and hence $m_3 = (p, 1, n)$. Hence if we transpose the second and the third coordinates, we have $\{m_1, m_2, m_3\} = \{(1, 0, 0), (0, 1, 0), (p, n, 1)\}$. Other cases are similarly checked.

q. e. d.

Terminal Lemma (Frumkin, Morrison). Let $m_1, m_2, m_3$ be linearly independent elements of $M$. If the tetrahedron $\square$ spanned by $\{0, m_1, m_2, m_3\}$ contains no other points of $M$, then the set $\{m_1, m_2, m_3\}$ is equal to $\{(1, 0, 0), (0, 0, 1), (e, n, 1)\}$ for an integer $0 \leq e < n$ with $(e, n) = 1$ and for a $\mathbb{Z}$-coordinate of $M$.

We are going to give a proof of this lemma by using the formula **). By that formula, we know $\square \cap M = \{0, m_1, m_2, m_3\}$ if and only if

***) $$\sum_{0 \leq i < j \leq 3} \lambda(n, q_{i,j}) = (2n^2 - 6n + 4)/n .$$

By Lemma 1, it is enough to prove the following proposition.

Proposition 2. Let $n$ be a positive integer greater than one, and let $0 < q_{i,j} < n$, for $0 \leq i \neq j \leq 3$, be integers prime to $n$ satisfying the three conditions (1), (2), (3). Then the equality ***) holds if and only if $q_{i,j} = 1$ for some $0 \leq i < j \leq 3$.

Note that the conditions (1), (2), (3) are invariant by any permutation of the indices 0, 1, 2, 3 .

Although the "if" part of the proposition follows from Lemma 1, we can see it directly as follows. We may assume $q_{0,1} = 1$ . Let $q_{0,2}$ be $a$ . Then we have $q_{0,3} = n - a$ by (2). We can calculate easily by the conditions that $q_{1,2} = n - a$ , $q_{2,3} = 1$ and $q_{1,3} = a$ . We have $\lambda(n, q_{0,2}) + \lambda(n, q_{0,3}) = \lambda(n, q_{1,2}) + \lambda(n, q_{1,3}) = 0$ . Hence $\sum_{0\leq i<j\leq 3} \lambda(n, q_{i,j}) = 2\lambda(n, 1) = (2n^2 - 6n + 4)/n$ . (see the table of $\lambda(n, q)$ at the end of this paper)

For a convenience, we set $\phi(n) = (2n^2 - 6n + 4)/n$ .

<u>Lemma 3.</u> Proposition 2 holds for $n$ smaller than or equal to seven.

Proof. If $n = 2$ , then all $q_{i,j}$'s are 1 . If $n = 3$ , then $q_{i,j}$'s are 1 or 2 . Since $\lambda(3, 2) = -2/3$ and $\phi(3) = 4/3$ , some of $q_{i,j}$ must be 1 for the equality ***) holds. If $n = 4$ , then $q_{i,j}$'s are 1 or 3 . Since $\lambda(4, 3) = -3/2$ and $\phi(4) = 3$ , not all $q_{i,j}$'s are 3 . Assume $n = 5$ and non of $q_{i,j}$ is 1 . Since $\lambda(5, 2) = \lambda(5, 3) = 0$ and $\lambda(5, 4) = -12/5$ , we have $\sum_{0\leq i<j\leq 3} \lambda(n, q_{i,j}) \leq 0 < \phi(5) = 24/5$ . If $n = 6$ , then $q_{i,j}$'s are 1 or 5 . Since $\lambda(6, 5) = -10/3$ and $\phi(6) = 20/3$ , we know some of $q_{i,j}$'s are 1 . Assume $n = 7$ and non of $q_{i,j}$ is equal to 1 . We have $\lambda(7, 2) = \lambda(7,4) = 6/7$ , $\lambda(7,3) = \lambda(7, 5) = -6/7$ and $\lambda(7, 6) = -30/7$ . Hence we have $\sum_{0\leq i<j\leq 3} \lambda(n, q_{i,j}) \leq 36/7 < \phi(7) = 60/7$ .

q. e. d.

We set $\mathfrak{A} = \{(a_1, \ldots, a_r) ; r \geq 1 , a_1, \ldots, a_r \geq 2$ are integers$\}$ .

For an element $A = (a_1, \ldots, a_r) \in \mathfrak{A}$ with $A \neq (2)$ , we denote by $A^{\cdot}$ the element $(a_1, \ldots, a_r - 1) \in \mathfrak{A}$ if $a_r \geq 3$ and the element $(a_1, \ldots, a_{r-1}) \in \mathfrak{A}$ if $a_r = 2$ .

We introduce an order in $\mathfrak{A}$ as follows. For two elements $A = (a_1, \ldots, a_r)$ and $B = (b_1, \ldots, b_s)$ , we define $A \leq B$ if and only if $r \leq s$ , $a_1 = b_1 , \ldots, a_{r-1} = b_{r-1}$ and $a_r \leq b_r$ . The following facts are easily checked.

i) For an element $A \in \mathfrak{A}$ with $A \neq (2)$ , $A^{\cdot}$ is the maximum element of the set $\{B \in \mathfrak{A} ; B \leq A , B \neq A\}$ .

ii) The ordered set $\mathfrak{A}$ satisfies the descending chain condition.

For an element $A$ in $\mathfrak{A}$ , we denote

$$n(A) = \det \begin{pmatrix} a_1 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & a_r \end{pmatrix} , \qquad q(A) = \det \begin{pmatrix} a_1 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & a_{r-1} \end{pmatrix} .$$

It is well known and easily checked by induction on $r$ that $0 < q(A) < n(A)$ and $q(A)$ and $n(A)$ are relatively prime. Furthermore, from integers $q(A)$ and $n(A)$ , the element $A$ is recovered as the continued fraction $n(A)/q(A) = a_r - 1/a_{r-1} - 1/\cdots - 1/a_1$ . In this way, the set $\mathfrak{A}$ is naturally bijective to the set $\{(n, q) ; 0 < q < n , (q, n) = 1\}$ .

If $A \neq (2)$ , then we know from the continued fraction that $0 < q(A) < n(A)/2$ if $a_r = 2$ and $n(A)/2 < q(A) < n(A)$ if $a_r \geq 3$ .

We define an integral valued and a rational valued maps $\sigma$ and $\lambda$ from $\mathfrak{A}$ as follows: For $A = (a_1, \ldots, a_r) \in \mathfrak{A}$ ,

$\sigma(A) = a_1 + \dots + a_r - 3r + 1$ , and

$\lambda(A) = \sigma(A) + (q(A) + q^*(A))/n(A) - 1$ ,

where $q^*(A) = q(A^*)$ for $A^* = (a_r, \dots, a_1)$ . Since $q(A)q^*(A) \equiv 1 \pmod{n}$ [I], we know $\lambda(A)$ is equal to the deviation $\lambda(n(A), q(A))$ . We get the following easily from the definitions.

iii) $\sigma(A^{\cdot}) = \sigma(A) - 1$ if $0 < q(A) < n(A)/2$ ,

$\sigma(A^{\cdot}) = \sigma(A) + 1$ if $n(A)/2 < q(A) < n(A)$ ,

$\lambda(A) < \sigma(A) + 1$ , $\sigma(A) = \sigma(A^*)$ , $\lambda(A) = \lambda(A^*)$ .

For a positive integer $d$ , we denote by $\mathfrak{A}_d$ the subset $\{A \in \mathfrak{A}$ ; $q(A)$ or $q^*(A) = d\}$ .

Remark 4. For most of elements $A$ of $\mathfrak{A}$ , the absolute value of the deviation $\lambda(A)$ is small for $n(A)$ . At the end of this paper, we give a list of A's with relatively high $\lambda(A)$ .

Lemma 5. If $A \in \mathfrak{A}$ is not in $\mathfrak{A}_1$ , then we have $\lambda(A) \leq n(A)/2 - 1$ .

Proof. Since $\lambda(A) < \sigma(A) + 1$ , it is sufficient to show that $\sigma(A) \leq n(A)/2 - 2$ for every $A$ in $\mathfrak{A}\setminus\mathfrak{A}_1$ . Let $A$ be a minimal element of $\mathfrak{A}\setminus\mathfrak{A}_1$ with $\sigma(A) > n(A)/2 - 2$ . If $0 < q(A) < n(A)/2$ , then $\sigma(A^{\cdot}) = \sigma(A) - 1 > n(A^{\cdot})/2 - 2$ since $n(A^{\cdot}) = n(A) - q(A)$ and $q(A) \neq 1$ . This contradict the minimality of $A$ . If $n(A)/2 < q(A) < n(A)$ , then $\sigma(A^{\cdot}) = \sigma(A) + 1 > n(A^{\cdot})/2 -2$ . Hence $A^{\cdot} \in \mathfrak{A}_1$ and $q(A^{\cdot}) = 2q(A) - n(A) = 1$ . Then we know $A = (q, 2)$ for $q = q(A)$ , and this is impossible since $\sigma(A) = (n(A) - 5)/2 \leq n(A)/2 - 2$ by the table.

q. e. d.

Lemma 6. If $A \in \mathfrak{A}$ is not in $\mathfrak{A}_1$ nor $\mathfrak{A}_2$, then we have $\lambda(A) \leq n(A)/3 - 1$.

Proof. It is sufficient to show the lemma for $A \in \mathfrak{A} \setminus (\mathfrak{A}_1 \cup \mathfrak{A}_2)$ with $\lambda(A) > n(A)/3 - 2$. We take a minimal element $B$ in $\mathfrak{A} \setminus (\mathfrak{A}_1 \cup \mathfrak{A}_2)$ with $B \leq A$ and $\sigma(C) > n(C)/3 - 2$ for every $B \leq C \leq A$. If $0 < q(B) < n(B)/2$, then $\sigma(B^{\cdot}) = \sigma(B) - 1 > (n(B) - q(B))/3 - 2 = n(B^{\cdot})/3 - 2$ since $q(B) \neq 1, 2$. By the minimality of $B$, we know $B^{\cdot}$ is in $\mathfrak{A}_1 \cup \mathfrak{A}_2$. Since $q(B^{\cdot}) = q(B) \neq 1, 2$, we have $B^{\cdot} = (d, 2)$ for an integer $d \geq 3$ and $B = (d, 3)$. Since $\lambda(B) \leq n(B)/3 - 1$ by the table, we may assume $A \neq B$. Hence we have $(d, 3, 2) \leq A$ or $(d, 4) \leq A$. $C = (d, 3, 2) \leq A$ is impossible since $\sigma(C) \leq n(C)/3 - 2$ by the table. In case $C = (d, 4) \leq A$, we may assume $A \neq C$ since $\lambda(C) \leq n(C)/3 - 1$ by the table. Then we have $D = (d, 4, 2) \leq A$ or $D = (d, 5) \leq A$. But $D = (d, 4, 2)$ is impossible since then $\sigma(D) = (n(D) - 12)/7 \leq n(D)/3 - 2$. If $d \geq 4$, then we have $\sigma(D) = (n(D) + 1)/5 \leq n(D)/3 - 2$ for $D = (d, 5)$. Hence we have $D = (3, 5) \leq A$. Since $\lambda(D) \leq n(D)/3 - 1$ we may assume $A \neq D$. Furthermore, since $\lambda(E) \leq n(E)/3 - 1$ for every $E = (3, e)$ we may assume $A \neq (3, e)$ for every $e \geq 3$. This implies $F = (3, e, 2) \leq A$ for an $e$. This is impossible since $\sigma(F) = (n(F) - 13)/6 \leq n(F)/3 - 2$.

If $n(B)/2 < q(B) < n(B)$, then $\sigma(B^{\cdot}) = \sigma(B) + 1 > n(B^{\cdot})/3 - 2$. Hence by the minimality of $B$, we have $B^{\cdot} = (d), (d, 2)$ or $(2, d)$ for an integer $d \geq 2$. $B^{\cdot} = (d)$ is impossible since then $B = (d, 2) \in \mathfrak{A}_2$. If $B^{\cdot} = (2, d)$, then $B = (2, d, 2)$. This is not possible since then $\sigma(B) = (n(B) - 12)/4 \leq n(B)/3 - 2$ by the table. $B^{\cdot} = (d, 2)$ and $B =$

(d, 2, 2) is also impossible since then $\sigma(B) = (n(B) - 10)/3 \leq n(B)/3 - 2$ by the table.

q. e. d.

Lemma 7. If $A \in \mathfrak{A}$ is not in $\mathfrak{A}_1, \mathfrak{A}_2$ nor $\mathfrak{A}_3$, then we have $\lambda(A) \leq (3n(A) - 7)/10$ .

Proof. It is sufficient to show the lemma for $A \in \mathfrak{A} \setminus (\mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \mathfrak{A}_3)$ with $\sigma(A) > (3n(A) - 17)/10$ . Let $B$ be the minimal element of $\mathfrak{A} \setminus (\mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \mathfrak{A}_3)$ with $B \leq A$ and $\sigma(C) > (3n(C) - 17)/10$ for every $B \leq C \leq A$ . If $0 < q(B) < n(B)/2$ , then $\sigma(B^\cdot) = \sigma(B) - 1 > (3n(B^\cdot) - 17)/10 = (3n(B) - 17)/10 - 3q(B)/10$ since $q(B) \neq 1, 2, 3$ . By the minimality of $B$ and the fact $q(B^\cdot) = q(B) \neq 1, 2, 3$ , we know $B^\cdot = (d,2), (d,3)$ or $(d, 2, 2)$ for an integer $d \geq 3$ . $B^\cdot = (d, 2)$ is impossible since then $B = (d, 3) \in \mathfrak{A}_3$ . In case $B^\cdot = (d, 3)$ , then $B = (d, 4)$ . Then, since $\lambda(B) \leq (3n(B) - 7)/10$ by the table, we may assume $A \neq B$ . Hence we know $(d, 5) \leq A$ or $(d, 4, 2) \leq A$ . Neither is possible since $\sigma(D) = (n(D) - 12)/7 \leq (3n(D) - 17)/10$ for $D = (d, 4, 2)$ , and, for $d \geq 4$ , $\sigma(D) = (n(D) + 1)/5 \leq (3n(D) - 17)/10$ for $D = (d, 5)$ by the table. $B^\cdot = (d, 2, 2)$ and $B = (d, 2, 3)$ is also impossible since then $\sigma(B) = (n(B) - 12)/5 \leq (3n(B) - 17)/10$ .

If $n(B)/2 < q(B) < n(B)$ , then $\sigma(B^\cdot) = \sigma(B) + 1 > (3n(B^\cdot) - 17)/10$ . Hence we know $B^\cdot = (d), (d, 2), (2, d), (d, 3), (3, d), (d, 2, 2)$ or $(2, 2, d)$ for an integer $d \geq 2$ . $B^\cdot = (d)$ and $B^\cdot = (d, 2)$ are impossible since then $B = (d, 2) \in \mathfrak{A}_2$ or $B = (d, 2, 2) \in \mathfrak{A}_3$ , respectively. If $B^\cdot = (2, d)$ , then $B = (2, d, 2)$ and $\sigma(B) = (n(B) - 12)/4 \leq$

$(3n(B) - 17)/10$. If $B^{\cdot} = (d, 3)$, then $B = (d, 3, 2)$ and $\sigma(B) = (n(B) - 13)/5 \leq (3n(B) - 17)/10$. If $B^{\cdot} = (d, 2, 2)$, then $B = (d, 2, 2, 2)$ and $\sigma(B) = (n(B) - 17)/4$. If $B^{\cdot} = (2, 2, d)$, then $B = (2, 2, d, 2)$ and $\sigma(B) = (n(B) - 23)/6$.

q. e. d.

Lemma 8. Let $A_1, \ldots, A_6$ be elements of $\mathfrak{A}$ such that $n(A_1) = \ldots n(A_6) = n > 7$ and the equality $\lambda(A_1) + \ldots + \lambda(A_6) = \phi(n)$ holds. If $q(A_6) = n - 1$, then at least one of $A_1, \ldots, A_5$ is in $\mathfrak{A}_1$.

Proof. Since $\lambda(A_6) = -\lambda(n, 1) = (-n^2 + 3n - 2)/n$, we have $\lambda(A_1) + \ldots + \lambda(A_5) = (3n^2 - 9n + 6)/n$. If non of $A_1, \ldots, A_5$ is in $\mathfrak{A}_1$, then we have inequality $(3n^2 - 9n + 6)/n \leq 5n/2 - 5$ by Lemma 5. This inequality does not hold for $n > 7$.

q. e. d.

Lemma 9. Let $A_1, \ldots, A_4$ be elements of $\mathfrak{A} \setminus \mathfrak{A}_1$ such that $n(A_1) = \ldots = n(A_4) = n > 7$ and the equality $\lambda(A_1) + \ldots + \lambda(A_4) = \phi(n)$ holds. Then one of $A_1, \ldots, A_4$ is in $\mathfrak{A}_2$.

Proof. If non of $A_1, \ldots, A_4$ is in $\mathfrak{A}_2$, then we have $\phi(n) = (2n^2 - 6n + 4)/n \leq 4n/3 - 4$ by Lemma 6. This inequality does not hold for $n > 7$.

q. e. d.

The following lemma is crucial in the proof of Proposition 2.

<u>Lemma 10.</u> If elements $A_1, \ldots, A_6 \in \mathfrak{A} \setminus \mathfrak{A}_1$ with $n(A_1) = \ldots = n(A_6) = n > 7$ satisfy $\lambda(A_1) + \ldots + \lambda(A_6) = \phi(n)$, then either

i) at least two $A_i$ and $A_j$, $i \neq j$, are in $\mathfrak{A}_2$, or

ii) at least one $A_i$ is in $\mathfrak{A}_2$ and at least one $A_j$ is in $\mathfrak{A}_3$.

Proof. If non of $A_i$'s is in $\mathfrak{A}_2$, then, by Lemma 6, we have $\lambda(A_1) + \ldots + \lambda(A_6) \leq 6(n/3 - 1) = 2n - 6 < \phi(n)$ and which contradicts the assumption. Hence at least one of them is in $\mathfrak{A}_2$. Let it be $A_6$. If non of $A_1, \ldots, A_5$ is in $\mathfrak{A}_2$ nor $\mathfrak{A}_3$, then, by Lemma 7, we have $\lambda(A_1) + \ldots + \lambda(A_5) \leq 5(3n - 7)/10 = (3n - 7)/2$. Since $\lambda(A_6) < (n - 5)/2$ by the table, we have $\lambda(A_1) + \ldots + \lambda(A_6) \leq (3n - 7)/2 + (n - 5)/2 \leq 2n - 6$. This is a contradiction.

q. e. d.

<u>Lemma 11.</u> Let $A_1, \ldots, A_6$ be the elements of $\mathfrak{A} \setminus \mathfrak{A}_1$ such that $n(A_1) = \ldots = n(A_6) = n > 7$ and the equality $\lambda(A_1) + \ldots + \lambda(A_6) = \phi(n)$ holds. If $A_6 \in \mathfrak{A}_2$, $A_5 \in \mathfrak{A}_3$ and $q(A_4) = n - 6$, then one of $A_1$, $A_2$, $A_3$ is in $\mathfrak{A}_2$.

Proof. Since $(2, n) = (3, n) = 1$, we know $n \equiv 1, 5 \pmod 6$. We have $\lambda(A_6) = (n^2 - 6n + 5)/2n$, $\lambda(A_5) = (n^2 - 11n + 10)/3n$ and $\lambda(A_4) = -\lambda(n, 6) = (-n^2 + 38n - 37)/6n$ in case $m \equiv 1 \pmod 6$, and $\lambda(A_5) = (n^2 - 7n + 10)/3n$ and $\lambda(A_4) = (-n^2 - 2n - 37)/6n$ in case $m \equiv -1 \pmod 6$. Hence we know $\lambda(A_4) + \lambda(A_5) + \lambda(A_6) \leq (2n^2 - n - 1)/3n$ and $\lambda(A_1) + \lambda(A_2) + \lambda(A_3) \geq (4n^2 - 17n + 13)/3n$. If non of $A_1$, $A_2$, $A_3$ is in $\mathfrak{A}_2$, we have $(4n^2 - 17n + 13)/3n \leq n - 3$ by Lemma 6. This inequality does not hold for $n > 7$.

q. e. d.

Lemma 12. Let $A_1, \ldots, A_6$ be elements of $\mathfrak{A} \setminus \mathfrak{A}_1$ such that $n(A_1) = \ldots = n(A_6) = n > 7$ and the equality $\lambda(A_1) + \ldots + \lambda(A_6) = \phi(n)$ holds. If $A_5, A_6 \in \mathfrak{A}_2$ and $q(A_4) = n - 4$, then at least one of $A_1, A_2, A_3$ is in $\mathfrak{A}_2$.

Proof. Since $\lambda(A_6) = \lambda(A_5) = (n^2 - 6n + 5)/2n$ and $\lambda(A_4) = -\lambda(n, 4)$ is equal to $(-n^2 + 18n - 17)/4n$ if $n \equiv 1 \pmod 4$ and $(-n^2 + 6n - 17)/4n$ if $n \equiv 3 \pmod 4$ m we have $\lambda(A_4) + \lambda(A_5) + \lambda(A_6) \leq (3n^2 - 6n + 3)/4n$ and $\lambda(A_1) + \lambda(A_2) + \lambda(A_3) \geq (5n^2 - 18n + 13)/4n$. If non of $A_1, A_2, A_3$ is in $\mathfrak{A}_2$, then we have inequality $(5n^2 - 18n + 13)/4n \leq n - 3$ which does not hold for $n > 7$.

q. e. d.

Now we are going to prove Proposition 2.

By Lemma 3, we may assume $n > 7$. Assume non of $q_{i,j}$'s is equal to $1$. Then by Lemma 10, we know one of $q_{i,j}$'s is equal to $2$ and one other $q_{i',j'}$ with $\{i',j'\} \neq \{i, j\}$ is equal to $2$ or $3$. By renumbering the indices, we may assume $q_{0,1} = 2$ and one of the following holds:

1). $q_{0,2} = 2$, 2). $q_{2,0} = 2$, 3). $q_{1,2} = 2$, 4). $q_{2,1} = 2$, 5). $q_{2,3} = 2$,
6). $q_{0,2} = 3$, 7). $q_{2,0} = 3$, 8). $q_{1,2} = 3$, 9). $q_{2,1} = 3$, 10). $q_{2,3} = 0$.

We are going to show that each of these ten cases does not occur. Since $(2, n) = 1$, we know $n$ is odd. We set $n = 2d - 1$ for an integer $d \geq 5$.

1). By the condition (1) of the proposition, we have $q_{1,0} = d$, and by (3) we have $q_{1,2} \equiv - q_{0,2}q_{1,0} \equiv - 1 \pmod n$ . We know $q_{1,2} = n - 1$ , and this is impossible by Lemma 8.

2). We have $q_{2,1} \equiv - q_{0,1}q_{2,0} = - 4$ . Hence by Lemma 12, one more $q_{i,j}$ is equal to two. If one of $q_{0,3}$, $q_{2,1}$, $q_{2,3}$ is 2 , then we are reduced to the case 1) by a permutation of indices, and if one of $q_{3,0}$, $q_{3,1}$ is 2 , then we are reduced to the case 4). If $q_{1,2} = 2$ , then we know $n = 9$ by $q_{2,1}q_{1,2} = - 8 \equiv 1 \pmod n$ . This is impossible since we have $q_{2,3} = 3$ by (2) which is not coprime to n . Since $q_{1,0} = q_{0,2} = d \neq 2$ , we know $q_{1,3}$ or $q_{3,2}$ is equal to 2 . The later case is equivalent to the former case by the cyclic permutation (0, 2, 3, 1) of indices. If $q_{1,3} = 2$ , then we have $q_{0,3} \equiv - q_{1,3}q_{0,1} = - 4$ . Then we have $q_{0,1} + q_{0,2} + q_{0,3} = d - 2$ , and this is not equal to 1 modulo n .

3). This is reduced to the case 2) by the cyclic permutation (0, 2, 1) .

4). We have $q_{1,2} = d$ and $q_{0,2} \equiv - q_{0,1}q_{1,2} \equiv - 1$ . This is impossible by Lemma 8.

5). By (3), we have $q_{0,2}q_{2,3} = 2q_{0,2} \equiv - q_{0,3} \pmod n$ . On the other hand, we have $2 + q_{0,2} + q_{0,3} \equiv 1 \pmod n$ by (3). These two equations imply $q_{0,2} = 1$ .

6). $q_{0,1} = 2$ , $q_{0,2} = 3$ imply $q_{0,3} \equiv - 4 \pmod n$ by (2). Since $2q_{1,2} = q_{0,1}q_{1,3} \equiv - q_{0,3} \pmod n$ by (3) , we have $q_{1,3} = 2$ . Thus we are reduced to the case 1) by the cyclic permutation (1, 0, 2, 3) of indices.

7). We have $q_{2,1} \equiv - q_{0,1}q_{2,0} = - 6 \pmod n$ . Hence we are reduced to one of the cases 1), ..., 5) by Lemma 11.

8). Since $q_{0,2} \equiv - q_{0,1}q_{1,2} = - 6$ , this case is also reduced to one of the cases 1), ..., 5).

9). Since $(2, n) = (3, n) = 1$, we know $n \equiv \pm 1 \pmod 6$. Let $n = 6e - 1$ for a positive integer $e$. Then we get $q_{1,0} = 3e$, $q_{1,2} = 2e$, and they imply $q_{0,2} = 2e - 1$, $q_{2,0} = 3e - 2$ by (3). We know $q_{2,3} = 3e - 1$ by (2). Hence we have $q_{3,2} \equiv -2 \pmod n$ by (1). Next assume $n = 6e + 1$ for a positive integer $e$. Then we get $q_{1,0} = 3e + 1$, $q_{1,2} = 4e + 1$, $q_{0,2} = 4e$, $q_{2,0} = 3e - 1$ and $q_{2,3} = 3e$. Hence we get also $q_{3,2} \equiv -2 \pmod n$. In the both cases, we have $\lambda(n, q_{0,1}) + \lambda(n, q_{2,3}) = 0$. We are reduced to one of the cases 1), ..., 5) by Lemma 9.

10). Since $q_{2,1} \equiv -q_{0,1}q_{2,0} \equiv -2q_{2,0}$ by (3) and $q_{2,0} + q_{2,1} + 3 \equiv 1 \pmod n$ by (2), we have $q_{2,1} \equiv 2 \pmod n$. Hence this is the case 4).

Thus Proposition 2 and Terminal Lemma is proved.

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The list of $A = (a_1, \ldots, a_s)$ with relatively high deviation.

| A | d | q | σ | λ |
|---|---|---|---|---|
| (d) | n | 1 | n - 2 | $(n^2 - 3n + 2)/n$ |
| (2, d) | (n + 1)/2 | 2 | (n - 5)/2 | $(n^2 - 6n + 5)/2n$ |
| (3, d) | (n + 1)/3 | 3 | (n - 5)/3 | $(n^2 - 7n + 10)/3n$ |
| (2, 2, d) | (n + 2)/3 | 3 | (n - 10)/3 | $(n^2 - 11n + 10)/3n$ |
| (4, d) | (n + 1)/4 | 4 | (n - 3)/4 | $(n^2 - 6n + 17)/4n$ |
| (2, 2, 2, d) | (n + 3)/4 | 4 | (n - 17)/4 | $(n^2 - 18n + 17)/4n$ |
| (2, d, 2) | (n + 4)/4 | (n + 2)/2 | (n - 12)/4 | $(n^2 - 12n + 8)/4n$ |
| (2, 3, d) | (n + 2)/5 | 5 | (n - 13)/5 | $(n^2 - 15n + 26)/5n$ |
| (5, d) | (n + 1)/5 | 5 | (n + 1)/5 | $(n^2 - 3n + 26)/5n$ |
| (2, 4, d) | (n + 2)/7 | 7 | (n - 12)/7 | $(n^2 - 15n + 50)/7n$ |
| (3, 2, d) | (n + 3)/5 | 5 | (n - 12)/5 | $(n^2 - 15n + 26)/5n$ |
| (6, d) | (n + 1)/6 | 6 | (n + 7)/6 | $(n^2 + 2n + 37)/6n$ |
| (3, d, 2) | (n + 5)/6 | (n + 3)/2 | (n - 13)/6 | $(n^2 - 14n + 13)/6n$ |
| (2, 2, d, 2) | (n + 7)/6 | (n + 3)/2 | (n - 23)/6 | $(n^2 - 22n + 13)/6n$ |
| (4, d, 2) | (n + 6)/8 | (n + 4)/2 | (n - 10)/8 | $(n^2 - 12n + 20)/8n$ |

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# Remarks on rings of bounded module type

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Dedekind domainの一般化であるbounded module typeのringについて2，3の注意を与える。特に、そのhenselizationやクルル次元の低いものについて、考察する。環はすべて1をもつ可換環とする。また、$Z(R)$, $J(R)$, $\mathrm{max}(R)$, $\mathrm{minSpec}(R)$はそれぞれ、Rの零因子全体，Jacobson根基，極大スペクトル，極小スペクトルをあらわす。

## §1. 定義と例.

まず、bounded module typeの環の定義をする。

Def. 1. 環Rがbounded module typeのringとは、ある自然数$n$が存在して、任意の有限生成R-加群Mが$M \cong \bigoplus_{\lambda \in \Lambda} M_\lambda$（どの$M_\lambda$も生成元の個数が高々$n$）とかけるときをいう。

これは、Dedekind domainやspecial primary ringの一般化であるが、その構造は、今の所知られていないようで、非可換環にもつながっている。このような環はnormal ringだから、$k[X^2, X^3]$（$k$が体）は、bounded module typeのringではない（semi-normalでさえない）。

次に、bounded module typeのringと関連した環やR-加群の定義をする。

Def. 2. Mがarithmetical R-moduleとは、任意の有限生成部分加群$N$に対して、$S \subseteq N$なるすべての部分加群$S$に対して、Rのideal $\mathfrak{a}$が存在して$S = \mathfrak{a}N$とかけるものをいう。

このようなものの special case として、

Def. 3 M が *generalized multiplication R-module* とは、$K \subseteq N$ なる M の proper submodules に対して $K = \mathfrak{a}N$ ($\mathfrak{a}$ は R の ideal) となるときをいう。この Def. で "proper" をとり除いたとき、M は *multiplication R-module* という。

multiplication R-module でない generalized multiplication module の例は、quasi-cyclic group $\mathbb{Z}_{p^\infty}$ などがある。R が離散付値環で non-complete だがその henselization $^hR$ が complete の例が存在([6])するから、[5] より、このような R に対して {generalized multiplication $^hR$-module} = {indecomposable $^hR$-module} がいえて、$^hR$ が便利なことを示す一例である。

さて、bounded module type の ring より一般なものに arithmetical ring がある。これと同値なものは十幾つも知られている。

Def. 4. R が *arithmetical ring* とは R 自身 arithmetical R-module のときをいう。R が *multiplication ring* であることも同様な仕方で定義する。

最近 Anderson と Dobbs[1] の定義した condensed domain を定義しそれを使って Prüfer domain の同値な定義を与える。

Def. 5. R が *condensed domain* とは、R の任意の ideals $\mathfrak{a}$, $\mathfrak{b}$ に対して $\mathfrak{a}\mathfrak{b} = \{ab \mid a \in \mathfrak{a}, b \in \mathfrak{b}\}$ となる integral domain をいう。

Def. 6. R が *Prüfer domain* とは arithmetical domain R をいう。これは、R が locally condensed かつ locally GCD domain といっても同じである。

また、Prüfer domain より広い整域に *P-domain* ([3]) がある。これは、R の integral closure が Prüfer domain のときをいう。

P-domain や condensed domain の例について少し述べる (c.f. [1])。

例. $k$ を体とする。$R = k[[X^2, X^3]]$ は P-domain かつ condensed domain。しかし、seminormal ring ではない。

P-domain かつ condensed domain は Bezout domain を含む domain だがよく知らない。

[1] の結果より、ただちに次のことがいえる。

R を seminormal domain とする。このとき、R が condensed domain なら、

$\mathrm{Pic}(R[X_1, X_2, \cdots, X_n]) = \{0\}$. しかし逆については次の反例がある。

例. (c.f. [1]) Dを体でないGCD domainとすると $R = D[Y]$ は not condensed domain かつ normal domain. しかし $\mathrm{Pic}(R[X]) = \mathrm{Pic}(D[X, Y]) \cong \mathrm{Pic}(D) = \{0\}$.

## §2. (*)-ring

クルル次元が低いときの bounded module type の ring は以下で定義する (*)-ring になることが多い。

Def. 7 Rが(*)-ringとは、Rの各元 $x$ に対して巾等元 $e$ が存在して $x+e$ が非零因子のときをいう。これは、$\forall x \in Z(R) - J(R)$ に対して $x+e \notin Z(R)$ なる巾等元 $e$ が存在するといっても同じである。

例. ① $R[X]$ 又は $R\langle X\rangle = R[X]_S$ (Sは $R[X]$ の monic 多項式全体のつくる乗法域) が (*)-ring なら、Rもそうである。

②. Rが "$\forall x \in Z(R) - J(R)$ に対して $(x)$ が projective ideal" をみたす環なら、Rは(*)-ringである。特に、Rが p.p. ring i.e. すべての単項 ideal が projective ideal なら R は (*)-ring である。

③. $C(X)$ を $X = [0, 1]$ 上で定義された実数値連続関数環とする。この $C(X)$ は零因子が非常に多く (*)-ring ではない。

最も簡単な環であるブール環の一般化として、

Def. 8. Rが Boolean-like ring とは Rの標数が2でRの任意の2元 $x$, $y$ に対して $xy(1+x)(1+y) = 0$ となるものをいう。

Rが reduced Boolean-like ring は Boolean ring である。

命題 1. Rが Boolean-like ring かつ bounded module type の ring なら、$R = \mathbb{Z}/2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2\mathbb{Z} \oplus S_1 \oplus \cdots \oplus S_r$ となる。但し各 $S_i$ はクルル次元0の quasi-local ring で $S_i$ の元 $x$ は $x^2 = 0$ か又は $x^2 = 1$ である。従って R は (*)-ring である。

(証明は、Rのクルル次元が0で Spec R が noetherian space なることよりいえる。)

## §3. 主な結果

定理 2. R を bounded module type の ring とする。このとき、次のことが成立する。

(1). $R \cong \bigoplus_{i=1}^{n} R_i$ で Max($R_i$) は Noetherian space, Spec $R_i$ は irreducible で $R_i$ は Prüfer domain か又は w. gl. dim $R_i = \infty$ なる arithmetical ring である。

(2). (1)の $R_i$ に対して、(i). w. gl. dim $R_i = \infty$ (ii). $R_i$ は non-zero principal flat ideal を含む. (iii). $S_i$ は not reduced ring の 3つは同値である。

(3). (1)の分解において R の ideal $\mathfrak{a}$ で Spec($R_i/\mathfrak{a}R_i$) が domain $R_i$ に対して connected になり、non-domain である $R_i$ に対しては $\mathfrak{a}R_i \subseteq J(R_i)$ がみたされるならば、 $h(R, \mathfrak{a}) \cong \bigoplus_{i=1}^{n} h(R_i, \mathfrak{a}R_i)$ で $h(R_i, \mathfrak{a}R_i)$ は Prüfer domain か又は non-reduced arithmetical ring (従ってこのときは w. gl. dim $h(R_i, \mathfrak{a}R_i) = \infty$) のどちらかである。Max($h(R, \mathfrak{a})$) は Noetherian space である。

(4). (3)で更に domain である $R_i$ に対しても $\mathfrak{a}R_i \subseteq J(R_i)$ なら、Pic(R) $\cong$ Pic($h(R, \mathfrak{a})$)

(5). R のクルル次元が 0 なら、R は (*)-ring かつ すべての極大 ideal $\mathfrak{m}$ に対して、localization map $R \longrightarrow R_{\mathfrak{m}}$ が onto である。

(証明は省略するが、(1), (2) は [2], [9] の定理を使い、(3), (4) は Strano の定理 ([6] と [7]) を使っていえる。minSpec(R) が有限であることも使う。)

この定理よりクルル次元の低いものについて次のことがいえる。

系 3. R を von Neumann regular ring とする。このとき、次のことは同値である。

(1). R が bounded module type の ring である。

(2). R は有限個の体の直和である。

(3). 任意の有限生成 R-module は cyclic modules の直和である。

((2)と(3)の同値性は Pierce (1967年) による。)

系 4. R が Artinian ring とする。このとき、次の(1), (2) は同値である。

(1). R が bounded module type の ring である。

(2). $R = \bigoplus_{i=1}^{n} R_i$ ($R_i$ は体又は special primary ring である)

(これは、R は Z.P.I. ring だから浅野の結果より明らか)

次に、環 $R$ がネーター環で w. gl. dim $R \leq 1$ となる条件を述べる。

命題 5. 環 $R$ に対して次の (1) と (2) は同値である。

(1). $R$ のすべての ideal は高々 countably generated でかつ w. gl. dim $R \leq 1$ かつ locally noetherian ring である。また Spec $R$ は noetherian space.

(2). $R = \bigoplus_{i=1}^{n} R_i$ ($R_i$ は体 or Dedekind domain).

(証明には Vasconcellos [8] の定理を使う)。

最後に group ring と関連した次の結果を述べる。

命題 6. $R$ が ring で $G \neq \{0\}$ は torsion-free abelian group とする。group ring $R[G]$ が bounded module type の ring でクルル次元が 1 以下なら $R$ は有限個の体の直和になる。

($R \cong R[G]/\Delta$　$\Delta = (e-g \mid g \in G$　$e$: 単位元) を使えば仮定より Spec $R$ が noetherian space となる。$R$ は松田さんの結果より von Neumann regular ring だからいえる。)

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# $G/P$上のSchubert Calculusへの一つの試み

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§1. Notation　$G$を複素半単純リー群, $P$をparabolic部分群、$B$をBorel部分群とし、$G \supset P \supset B$を満たすとする。対応するリー環を$\mathfrak{g} \supset \mathfrak{p} \supset \mathfrak{b}$で表わす。$\mathfrak{g}$に含まれるcompact form $\mathfrak{k}$を固定し$*$-作用素を次式で定義する。

$$
\begin{array}{ccc}
* \; ; \; \mathfrak{g} = \mathfrak{k} + \sqrt{-1}\,\mathfrak{k} & \longrightarrow & \mathfrak{g} = \mathfrak{k} + \sqrt{-1}\,\mathfrak{k} \\
\Psi & & \Psi \\
X + \sqrt{-1}\,Y & \longrightarrow & X - \sqrt{-1}\,Y
\end{array}
$$

その時, $\mathfrak{h} = \mathfrak{b} \cap \mathfrak{b}^*$は$\mathfrak{g}$のカルタン部分環となり対応する極大トーラスを$T$で表わす。$\mathfrak{p}$に含まれる最大巾零イデアルを$\mathfrak{n}$とし$\mathfrak{n}, \mathfrak{n}^*$, $\mathfrak{g}_1 = \mathfrak{n} \cap \mathfrak{n}^*$に対応するリー群をそれぞれ$N$, $N^*$, $G_1$で表わす。$\mathfrak{g}$の次の直和分解はよく知

られている。

<u>Lemma 1</u> リー環 $\mathfrak{g}$ は次のように直和分解される。

1) $\mathfrak{g} = \mathfrak{n}^* + \mathfrak{g}_1 + \mathfrak{n}$,

$\mathfrak{p} = \mathfrak{g}_1 + \mathfrak{n}$.

2) $[\mathfrak{g}_1, \mathfrak{n}] \subset \mathfrak{n}$, $[\mathfrak{g}_1, \mathfrak{n}^*] \subset \mathfrak{n}^*$.

記号 $\Delta$ により root 系を表わし、$\Delta(\mathfrak{n}^*)$ により $\mathfrak{n}^*$ に対応したその部分系を表わす。各 $\alpha \in \Delta(\mathfrak{n}^*)$ に対しベクトル $(\mathfrak{g}_\alpha \ni) X_\alpha \neq 0$ を選び、$Z \in \mathfrak{n}^*$ を $Z = \sum_{\alpha \in \Delta(\mathfrak{n}^*)} z_\alpha X_\alpha$ と表わす。

$T$ の $G$ 内における normalizer を $N(T)$ で表わすと、$G$ の Weyl 群が $W = N(T)/T$ で、$P$ に対応した Weyl 部分群が $W_1 = N(T) \cap P/T$ で定義される。また $W' = W/W_1 = N(T)/N(T) \cap P$ とおく。群 $N(T)$ は $T$, $\mathfrak{h}$, $\Delta$ に次の規則で作用する。

1) $w \cdot \exp H \cdot w^{-1} = \exp(Ad(w)H)$

2) $(Ad(w)^* \alpha)(H) = \alpha(Ad(w)^{-1}H)$.

ここで $w \in N(T)$, $H \in \mathfrak{t}$, $\alpha \in \Delta$。しかし $T$ の元 $w$ はすべて自明に作用するので Weyl 群 $W$ が $T$, $\mathfrak{t}$, $\Delta$ に作用すると考えてよい。また簡単のため $Ad(w)$, $Ad(w)^*$ も同じ文字 $w$ で表わす。

§2. $G/P$ 上のベクトル場　最初に次の命題を示す。

命題2　極大トーラス $T$ を代数的均質空間 $X = G/P$ に作用させると、集合 $W'$ は $X$ 内の $T$ 固定点の集合として標準的に埋め込まれる。

証明　$\bar{g} \in X$ が $T$ の固定点

$$\Longleftrightarrow \quad g^{-1} T g \subset P$$

，ここで $g$ は $\bar{g}$ の $G$ における代表元。

一方 $g^{-1}Tg$ も $P$ に含まれる $G$ の極大トーラスより元 $p \in P$ があり $g^{-1}Tg = pTp^{-1}$ とできる。これは $gp \in N(T)$ を意味する。これより $\bar{g}$ は $W'$ の元 $\widehat{gP}$ を定義する。2つの固定

点 $\bar{g}, \bar{g}'$ が $W'$ 内で同じ元を定義したとすると、$gp = g'p'p''$, ${}^{\exists}p, {}^{\exists}p' \in P$, ${}^{\exists}p'' \in N(T) \cap P$ となり $\bar{g}$ と $\bar{g}'$ は $X$ 内で一致する。また $N(T)$ の元 $w$ をとると $\bar{w}$ に対応した元は $\hat{w}$ となり onto も証明できる。

ここで次の図式を考える。

$$
\begin{array}{ccccccc}
 & & G & & G/P & & G/P \\
 & & \cup & & \cup & & \cup \\
\mathfrak{n}^* & \xrightarrow{\ \phi\ } & N^* & \xrightarrow{\ \psi\ } & \bar{N}^* & \xrightarrow{\ w\ } & w\bar{N}^* \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
Z & \longrightarrow & \exp Z & \longrightarrow & \exp Z & \longrightarrow & w \exp Z
\end{array}
$$

リー環 $\mathfrak{n}^*$ は巾零のため写像 $\phi$ は 1:1 onto となる。また $N^* \cap P = \{I\}$ のため $\psi$ は 1:1 の写像となる。$w$ を左から掛けるという写像は明らかに 1:1 となる。従って $(w\bar{N}^*, \phi^{-1} \circ \psi^{-1} \circ w^{-1})$ は $T$ 固定点 $w \in W'$ の近傍の局所座標となる。

定理3 集合 $W' = W/W'$ は標準的な方法で $X = G/P$ に T 固定点の集合として埋め込まれ組 $(w\overline{N}^*, \phi^{-1}\circ\psi^{-1}\circ w^{-1})$ は点 $w \in W'$ の回りの座標近傍になる。そして $w\overline{N}^*, w \in W'$, は T 不変な Zariski 開集合となる。実際 $\exp H \in T$ を左から $w\overline{N}^*$ に掛けると局所座標 $\{z_\alpha(w\bar{n}^*)\}_{\alpha\in\Delta(n^*)}$ は $\{e^{(w\alpha)(H)}\cdot z_\alpha(w\bar{n}^*)\}_{\alpha\in\Delta(n^*)}$ に移る。また X は開集合の族 $\{w\overline{N}^*\}_{w\in W'}$ によりおおわれる。従って $X = G/P$ は標準的な局所座標系 $\{(w\overline{N}^*, \phi^{-1}\circ\psi^{-1}\circ w^{-1})\}_{w\in W'}$ を持つことになる。

注意4 $X = \mathbb{P}^n$ の場合上の局所座標系は inhomogeneous coordinate system と一致する。

証明 最初の部分は証明済み。$w_0$ を Weyl 群 W の中で最長の元とする。そのとき $w_0^{-1} N w_0 = N^*$ より $w_0\overline{N}^* = N w_0 P/P$ となり $w_0\overline{N}^*$ はブルアーセルの中で最大次元のものとなる。特に Zariski 開集合となる。そして $w\overline{N}^* = w w_0^{-1} w_0 \overline{N}^*$ より $w\overline{N}^*$ も Zariski 開集合となる。

$\exp Z \in N^*$ に対し

$$\exp H \cdot w \exp Z \cdot P = w w^{-1} \exp H w \cdot \exp Z w^{-1} \exp(-H) w \cdot P$$
$$= w \cdot \exp(w^{-1}(H)) \cdot \exp Z \cdot \exp(-w^{-1}(H)) \cdot P$$
$$= w \cdot \exp(Ad(\exp(w^{-1}(H))) Z) \cdot P$$
$$= w \cdot \exp(\mathrm{Exp}(ad(w^{-1}(H))) Z) \cdot P$$

であり、また

$$\mathrm{Exp}(ad(w^{-1}(H))) \cdot Z \in \mathfrak{n}^*$$

より

$$(\phi^{-1} \circ \psi^{-1} \circ w^{-1})(\exp H \cdot w \overline{\exp Z}) = \mathrm{Exp}(ad(w^{-1}(H))) \cdot Z$$

となる。そして

$$ad(w^{-1}(H)) \cdot Z = [w^{-1}(H), \sum_{\alpha \in \Delta(\mathfrak{n}^*)} z_\alpha X_\alpha]$$
$$= \sum_{\alpha \in \Delta(\mathfrak{n}^*)} \alpha(w^{-1}(H)) z_\alpha X_\alpha$$
$$= \sum_{\alpha \in \Delta(\mathfrak{n}^*)} (w\alpha)(H) z_\alpha X_\alpha$$

より

$$\mathrm{Exp}(ad(w^{-1}(H))) \cdot Z = \sum_{\alpha \in \Delta(\mathfrak{n}^*)} e^{(w\alpha)(H)} \cdot z_\alpha X_\alpha .$$

次に $X = \bigcup_{w \in W'} w \overline{N}^*$ を証明するために以下の

事実を[4]から引用する。

☆ $Y$をコンパクト・ケーラー多様体とし$H^1(Y,\mathbb{C})=0$を満たすとする。その時複素連結可解リー群$S$が$Y$に作用しているとすると、$S$が不変に保つ任意の部分解析集合には$S$の固定点が含まれる。

$X=G/P$の場合上の仮定をすべて満たし$S$として$T$をとることができる。$w\bar{N}^*$が$T$不変なzariski開集合より、補集合$X'=X-\bigcup_{w\in W'} w\bar{N}^*$は$T$不変な解析集合となる。仮に$X'$が空集合でないとすると上の事実より$X'$には$T$固定点が存在することになるがこれは矛盾.

$H\in \mathfrak{h}$に対し次の規則で$X$上の正則ベクトル場$V_H$を定義する。

$$(V_H f)(\bar{g}) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{f(\exp(\varepsilon H)\bar{g}) - f(\bar{g})\},$$

ここで$\bar{g}\in X$, $f$は$\bar{g}$の回りの局所関数。すると先の定理3よりベクトル場$V_H$は$w\bar{N}^*$上

で次のように書き下される

$$V_H = \sum_{\alpha \in \Delta(n^*)} (W\alpha)(H)\, z_\alpha \frac{\partial}{\partial z_\alpha}.$$

特にHがWeyl領域に属すれば $(W\alpha)(H) \neq 0$, $w \in W'$, $\alpha \in \Delta(n^*)$, より $V_H$ の零点は $W'$ と一致しそこで一次の消え方をする。

## §3. J.B. Carrell と D. Lieberman の結果

才2章で構成した $X = G/P$ 上のベクトル場 $V_H$ は J.B. Carrell と Dieberman の研究 [1],[2],[3] に対し有用な例を与えることを示す。そのために彼らの結果を簡単にふりかえることにする。

$X$ を $n$ 次元コンパクト・ケーラー多様体とする。$X$ には空でない単純に孤立した零集合 $Z$ を持つ正則ベクトル場 $V$ が存在すると仮定する。以後 $Z = \{z_1, z_2, \cdots, z_k\}$ と表わす。次の層の複体を $V$ により定義されたKoszul複体と呼ぶ。

$$\Omega^*: 0 \longrightarrow \Omega^n \xrightarrow{\partial} \Omega^{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^1 \xrightarrow{\partial} \Omega^0 = \mathcal{O}_X \to 0$$

ただし∂をVに関する縮約$i(V)$とする。$Z$の構造層は$\mathcal{O}_Z = \mathcal{O}_X / i(V)\Omega^1$となる。$\mathcal{H}^*$を複体$\Omega^*$のコホモロジー層とすると仮定より$\mathcal{H}^q=0$ $q \neq 0$, $\mathcal{H}^0 = \mathcal{O}_Z$となる。$\Omega^*$のハイパーコホモロジー$\mathbb{H}^*(X, \Omega^*)$を二重Čech複体$\check{C}^*(\mathcal{U}, \Omega^*)$を用いて計算すると、$\check{C}^*(\mathcal{U}, \Omega^*)$に自然に入る二種類のフィルトレーションに対応して$\mathbb{H}^{p+q}(X, \Omega^*)$に収束する次の二つのスペクトル列がある。

I) $'E_1^{p,q} = H^q(X, \Omega^{-q})$,

II) $''E_2^{p,q} = H^p(X, \mathcal{H}^q)$.

上に述べたことより従って$\mathbb{H}^r(X, \Omega^*) = 0$, $r \neq 0$, $\mathbb{H}^0(X, \Omega^*) = H^0(Z, \mathcal{O}_Z)$となる。そして$H^0(Z, \mathcal{O}_Z)$は$Z = \{s_1, \cdots, s_k\}$上の複素数値関数のつくる環とみなされる。そして次の事が知られている。[1], [2] 参照.

<u>Lemma 5</u> X, V を以上のとおりとするとき

1) スペクトル列 I) は $'E_1$ 項で退化、

2) 従ってスペクトル列 I), II) を比較すると

$$H^p(X, \Omega^q) = 0 \quad , \quad p \neq q。$$

3) スペクトル列 I) より環 $H^0(Z, \mathcal{O}_Z)$ に次を満たすフィルトレーションが入る。

i) $H^0(Z, \mathcal{O}_Z) = F_{-n} \supseteq F_{-n+1} \supseteq \cdots \supseteq F_0 \supseteq \{0\}$,

ii) $F_p \cdot F_q \subseteq F_{p+q}$,

iii) $F_{-p} / F_{-p+1} \cong H^p(X, \Omega^p)$,

iv) $H^*(X, \mathbb{C}) = gr\, H^0(Z, \mathcal{O}_Z) = \bigoplus_{p=0}^{n} F_{-p} / F_{-p+1}$.

この結果より次の二つのことが基本的な問題となる。

問題 I) E を X 上の正則ベクトル束とするときそれらのチャーン類は $H^0(Z, \mathcal{O}_Z)$ の中でどのように表わされるか。

II) Y を X 内の cycle とするときそれのポアンカレ dual は $H^0(Z, \mathcal{O}_Z)$ の中でどのように表わされるか。

問 I) に対しては以下のような解答があるが II) に関してはよくわからない。

<u>定義6</u> V 上のベクトル束 E に対し

$$\tilde{V}(f\cdot s) = V(f)\cdot s + f\cdot \tilde{V}(s)$$

を満たす $\mathbb{C}$-線型写像 $\tilde{V}: E \longrightarrow E$ が存在するとき、E を V-equivariant ベクトル束という。ここで f は $\mathcal{O}_X$ の local section, s は E の local section とする。

V-equivariant ベクトル束に対しては次のことが知られている。

<u>Lemma 7</u> E を V-equivariant ベクトル束とし $\tilde{V}$ の Z への制限を $\tilde{V}_Z$ と書くことにする。そのとき $\tilde{V}_Z$ は $H^0(Z, \mathcal{H}om(E,E)\otimes\mathcal{O}_Z)$ に属するが $(-1)^d\sigma_d(\tilde{V}_Z)$ は $F_{-d}\ (\subseteq H^0(Z,\mathcal{O}_Z))$ の元を定め E の d 次チャーン類を代表している。ここで $\sigma_d$ は次式で定義される。

$$\det|tI + A| = \sum_{d=0}^{r=\operatorname{rank}E} \sigma_d(A)\, t^{r-d}\ 。$$

## §4. 主要結果

定理8. $X = G/P$ としEを表現 $\phi: P \to GL(\mathbb{V})$ から誘導された均質ベクトル束とする。そのとき

1) ベクトル束Eは $V_H$-equivariant.

2) Eの $d$ 次チャーン類の $H^0(Z, \mathcal{O}_Z)$ の代表元として $w \in W'$ で値 $\sigma_d(d\phi(w^{-1}(H)))$ をとる関数を選べる。ここで $d\phi$ は $\phi$ の微分。

証明 ベクトル束Eは $G \times \mathbb{V}$ を同値関係 $(g, v) \sim (gp, \phi^{-1}(p)v)$, $g \in G$, $p \in P$, $v \in \mathbb{V}$, で割って得られる。従ってEの局所切断 $v$ は $G$ の或る開集合 $U$ 上の $\mathbb{V}$ に値をとる関数で $v(g) = \phi(p)v(gp)$, $g, gp \in U$, $p \in P$ を満たすものと思える。また $X$ 上の局所関数は $G$ 上の関数で $f(g) = f(gp)$ を満たすものと思える。そこで $v(g)$ に対し $(\tilde{V}_H v)(g)$ を次式で定義する。

$$(\tilde{V}_H v)(g) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\{v(\exp(\varepsilon H)g) - v(g)\}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\{\phi(p)v(\exp(\varepsilon H)gp) - \phi(p)v(gp)\}$$

$$= \phi(p) \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\{ v(\exp(\varepsilon H) gp) - v(gp) \}$$

$$= \phi(p) (\widetilde{V}_H v)(gp).$$

従って$(\widetilde{V}_H v)(g)$もEの局所切断となる。また

$$(\widetilde{V}_H (fv))(g) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\{ f(\exp(\varepsilon H) g) v(\exp(\varepsilon H) g) - f(g) v(g) \}$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\{ (f(\exp(\varepsilon H) g) - f(g)) v(\exp(\varepsilon H) g) \}$$

$$+ \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\{ f(g)(v(\exp(\varepsilon H) g) - v(g)) \}$$

$$= (V_H f)(g) v(g) + f(g) (\widetilde{V}_H v)(g)$$

となるのでEは確かに$V_H$-equivariantになる。

次にEの局所切断$v(g)$で集合$WN^*$に沿って定ベクトル$v$をとるものを選ぶ そのとき

$$(\widetilde{V}_H v)(w \exp Z) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\{ v(\exp(\varepsilon H) w \exp Z) - v(w \exp Z) \}$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\{ \phi(w^{-1} \exp(-\varepsilon H) w) v(w w^{-1} \exp(\varepsilon H) w \exp Z w^{-1} \exp(-\varepsilon H) w) - v(w \exp Z) \}$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\{ \phi(w^{-1} \exp(-\varepsilon H) w) v - v \}$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\{ \mathrm{Exp}(-\varepsilon\, d\phi(w^{-1}(H))) v - v \}$$

$$= - d\phi(w^{-1}(H)) v(w \exp Z).$$

Eの$W\bar{N}^*$の局所切断の基底としてこのようなものを選ぶと$w \in W'$において$\hat{V}_{H,z} = -d\phi(w^{-1}(H))$となる。従って

$$\begin{aligned}\det|tI - \hat{V}_{H,z}| &= \det(tI - (-d\phi)(w^{-1}(H))) \\ &= \sum_{d=0}^{r} (-1)^d \sigma_d(-d\phi(w^{-1}(H)))\, t^{r-d} \\ &= \sum_{d=0}^{r} \sigma_d(d\phi(w^{-1}(H)))\, t^{r-d}\end{aligned}$$

となり証明を終わる。

最後に次の問題を提出する。

<u>問題9</u> $X = G/P$には$W'$でパラメトライズされたgeneralized Schubert cycle $X_w$と呼ばれる基本的なcyclesが存在するが$X_w$のポアンカレ相双は$H^0(Z, \mathcal{O}_Z)$の中でどのように表わされるか？

仮に上の問題が解けると$X$のコホモロジー環$H^*(X, \mathbb{C})$の構造を調べるSchubert Calculusが完全にnumericalな問題に言い替えられることになる。$X$がグラスマンタタ様体の

場合の研究が[3],[5]にありWeyl群の誘導表現の問題と関係しているように思われるがなぜなのかわかっていない。

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# A Remark on Flatness over a Graded Ring

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The following results came out of a discussion of Manfred Herrmann and myself. Although they may be already known to some people, they do not seem to be well known.

Let $G$ be an abelian group, $R = \bigoplus_{g \in G} R_g$ be a graded ring of type $G$ and $M = \bigoplus_{g \in G} M_g$ be a graded R-module.

<u>Proposition</u> 1. The following are equivalent:

(1) M is R-flat;

(2) if $S: \quad \dots \to N \to N' \to N'' \to \dots$ is an exact sequence of graded R-modules and grade-preserving R-linear maps, then $S \otimes_R M$ is exact;

(3) $\mathrm{Tor}_1^R(M, N) = 0$ for all graded R-module N;

(4) $\mathrm{Tor}_1^R(M, R/\mathfrak{a}) = 0$ for all finitely generated homogeneous ideal $\mathfrak{a}$ of R.

<u>Proof</u>. (1) ⇒ (2) ⇒ (3) ⇒ (4) : trivial. (4) ⇒ (3) ⇒ (2) : same as in the non-graded case. To prove (2) ⇒ (1) we have only to show that if $\Sigma_1^n a_i x_i = 0$, $a_i \in R$, $x_i \in M$, then there are $b_{ij} \in R$ $(1 \leq i \leq n,\ 1 \leq j \leq r)$ and $y_j \in M$ $(1 \leq j \leq r)$ such that

$$(*) \qquad \sum_i a_i b_{ij} = 0 \quad (\text{all } j), \qquad x_i = \sum_j b_{ij} y_j \quad (\text{all } i).$$

We decompose $a_i$ and $x_i$ into homogeneous components: $a_i = \Sigma_{g \in G}\, a_{ig}$, $a_{ig} \in R_g$, $x_i = \Sigma_{h \in G}\, x_{ih}$, $x_{ih} \in M_h$. Then $\Sigma\, a_i x_i = 0$ is equivalent to

$$(\dagger) \qquad \sum_{h \in G} a_{i,g-h}\, x_{ih} = 0 \qquad (g \in G,\ 1 \leq i \leq n).$$

Consider free R-modules $F = \Sigma_{i=1}^n \Sigma_{g \in G} Re_{ig}$, $F' = \Sigma_{g \in G} Re'_g$ with bases $(e_{ig})$, $(e'_g)$ respectively, where $e_{ig}$ and $e'_g$ are homogeneous of degree $-g$. The linear map $\psi: F \to F'$ defined by $\psi(e_{ih}) = \Sigma_{g \in G}\, a_{i,g-h} e'_g$ is homogeneous of degree 0. Put $K = \mathrm{Ker}\,\psi$, $\psi_M = \psi \otimes 1_M$, and $\xi_i = \Sigma_h\, e_{ih} \otimes x_{ih}$. Then

$$\psi_M(\xi_i) = \Sigma_g\, e'_g \otimes (\Sigma_h\, a_{i,g-h}\, x_{ih}) = 0.$$

Since $0 \to K \otimes_R M \to F \otimes_R M \to F' \otimes_R M$ is exact, there exist finitely many elements $\beta_{ij} \in K$ and $y_j \in M$ $(1 \leq i \leq n,\ 1 \leq j \leq r)$ such that $\xi_i = \Sigma_j \beta_{ij} y_j$ $(1 \leq i \leq n)$. Then $\beta_{ij}$ must be a linear combination of $(e_{ih})_{h \in G}$, for each i. Let $\beta_{ij} = \Sigma_h b_{ijh} e_{ih}$, $b_{ijh} \in R$. Then $\Sigma_h a_{i,g-h} b_{ijh} = 0$ (all $g \in G$) and $x_{ih} = \Sigma_j b_{ijh} y_j$. Put $b_{ij} = \Sigma_h b_{ijh}$. Then $\Sigma_i a_i b_{ij} = 0$ (all j) and $x_i = \Sigma_j b_{ij} y_j$ (all i), as wanted.

<u>Proposition</u> 2. (Local criterion of flatness) Let R be a graded ring of type G, and I be an ideal of R (not necessarily homogeneous). Let M be a graded R-module (not necessarily finitely generated). Suppose that

(1) for every homogeneous ideal $\mathfrak{a}$ of R, the R-module $\mathfrak{a} \otimes_R M$ is I-adically separated;

(2) $M_0 = M/IM$ is flat over $R_0 = R/I$; and

(3) $\mathrm{Tor}_1^R(M, R_0) = 0$.

Then M is flat over R.

<u>Proof</u>. Derive $\mathrm{Tor}_1^R(M, R/\mathfrak{a}) = 0$ for homogeneous ideals $\mathfrak{a}$ by the usual proof (cf. Th.49 of my book Commutative Algebra), and apply Prop.1.

<u>Application</u> 1. (Tangential Flatness) Let A, B be rings and I (resp. J) be an ideal of A (resp. B). Let $\phi: A \to B$ be a ring homomorphism such that $\phi(I) \subseteq J$. Put $R := \mathcal{R}(I, A) = A[It, u]$, $S := \mathcal{R}(J, B) = B[Jt, u]$, where $u = t^{-1}$, and let R and S be graded as usual by $\deg(t) = 1$. Then $\phi$ induces degree-preserving ring-homomorphisms $\mathcal{R}(\phi): R \to S$ and $\mathrm{gr}(\phi): \mathrm{gr}_I(A) \to \mathrm{gr}_J(B)$. We have

$$A = R/(u-1)R, \quad B = S/(u-1)S, \quad \mathrm{gr}_I(A) = R/uR, \quad \mathrm{gr}_J(B) = S/uS,$$

and $\phi$ and $\mathrm{gr}(\phi)$ are obtained from $\mathcal{R}(\phi)$ by base-change.

<u>Proposition</u> 3. (B. Herzog) If $J \subseteq \mathrm{rad}(B)$ and if A and B are noetherian, then

$$\mathrm{gr}(\phi) \text{ is flat} \iff \mathcal{R}(\phi) \text{ is flat} \implies \phi \text{ is flat.}$$

<u>Proof</u>. Flatness of $\mathcal{R}(\phi)$ implies flatness of $\phi$ and $\mathrm{gr}(\phi)$ by base change. Suppose that $\mathrm{gr}(\phi)$ is flat, i.e. that S/uS is flat over R/uR. We have $\mathrm{Tor}_1^R(S, R/uR) = 0$ because u is a non-zero-divisor in S. If $\mathfrak{a}$ is a homogeneous ideal of R, then $\mathfrak{a} \otimes_R S$ is a finitely generated graded S-module, and as such it is u-adically separated. [For, if M is a finitely generated graded S-module and $N = \bigcap_{i=1}^{\infty} u^i M$, then N is a graded submodule of M and $(1-ux)N = 0$ for some $x \in S_1$. Then $ux \in J$ and so $1-ux$ is a unit in B (hence also in S). Therefore $N = 0$.] Thus S is R-flat by Proposition 2.

Remark. u-1 is also a non-zero-divisor in S, hence we have $Tor_1^R(S, R/(u-1)R) = 0$. But $\mathfrak{a}\otimes_R S$ is not necessarily separated in the (u-1)-adic topology. This is the reason why flatness of $\phi$ does not imply that of $\mathcal{R}(\phi)$. EXAMPLE: Let $A = k[[x^2]]$ and $B = k[[x]]$. Then B is free over A. Let $\mathfrak{a} = (x^2t, u)R$ and $w = x^2t\otimes 1 - u\otimes(xt)^2 \in \mathfrak{a}\otimes_R S$. Then $w \neq 0$ and $uw = 0$. Hence $w \in \bigcap_{i=1}^{\infty}(u-1)^i(\mathfrak{a}\otimes S)$.

Application 2. Let $A = \bigoplus_{n\geq 0} A_n$ and $B = \bigoplus_{n\geq 0} B_n$ be graded noetherian rings. Assume that $A_0$, $B_0$ are local rings with maximal ideals $\underline{m}$, $\underline{n}$, and put $\underline{M} = \underline{m} + A_+$, $\underline{N} = \underline{n} + B_+$. Let $f: A \to B$ be a ring homomorphism of degree 0 such that $f(\underline{m}) \subseteq \underline{n}$. Then the following are equivalent:

(1) B is A-flat;

(2) $B_{\underline{N}}$ is A-flat;

(3) $B_{\underline{N}}$ is $A_{\underline{M}}$-flat.

Proof. (1) $\Rightarrow$ (2): trivial. (2) $\Leftrightarrow$ (3): immediate from the definition of flatness. (2) $\Rightarrow$ (1): We apply Prop.2 to the case $(R,I,M) = (A,\underline{M},B)$. If $\mathfrak{a}$ is a homogeneous ideal of A then $\mathfrak{a}\otimes_A B$ is a finitely generated graded B-module, and every finitely generated graded B-module L is $\underline{N}$-adically separated. Since $A/\underline{M}$ is a field, it remains to check $Tor_1^A(B,k) = 0$, where $k = A/\underline{M}$. But $Tor_1^A(B, k)$ is a graded B-module and $0 = Tor_1^A(B_{\underline{N}}, k) = (Tor_1^A(B, k))_{\underline{N}}$. If L is a graded B-module, $L_{\underline{N}} = 0$ implies $L = 0$.

Remark. In "The category of Graded Modules", Math. Scand. 35(1974), Fossum and Foxby proved our Prop. 1 by Lazard's characterization of flat modules as direct limits of free modules.

# On the Algebraic Function Fields of Genus 0 which have no place of Degree 1.

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ここでは、体$k$上の代数函数体と言う時は、一変数を意味し、特に断らない場合、$k$をその定数体と仮定しないものといたします。

1. $K=k(x,y)$を有理函数体$k(x)$の2次拡大で、$k$の標数は2でないといたします。この時、$x$と$y$に適当な変換を施すことによって、$K=k(x,y)$、$y$の$k(x)$上の最小多項式を、$k[x]$の平方因子を持たないelement $F(x)$を用いて、$Y^2-F(x)$とすることが出来るので、$K$を始めからこの様な拡大と仮定してよい。この時、$k$が定数体であるためには$\deg F\geqq 1$が必要充分であることが分ります。以下$\deg F\geqq 1$とします。

2. 1の$K$について、そのPlaceの$k(x)$に関する分岐を調べることが出来て、結果は次で示す表の様に成ります。またこの結果によって$K$のGenus $g$は$g=[(\deg F-1)/2]$で与えられることが分ります、ここに[ ]はGauss記号。

$\mathfrak{p}_{P(x)}$ = the place of $k(x)$ defined by an irreducible element $P(x)\in k[x]$.

$\mathfrak{p}_\infty$ = the pole of $x$ of $k(x)$

$\mathfrak{P}$ = the place of $K$ which lies above $\mathfrak{p}_{P(x)}$ or $\mathfrak{p}_\infty$.

$e_\mathfrak{P}$ = the ramification index of $\mathfrak{P}$ with respect to $k(x)$.

$f_\mathfrak{P}$ = the relative degree of $\mathfrak{P}$ with respect to $k(x)$.

l.c. = the leading coefficient.

$V_{\mathfrak{p}}$ = the valuation defined by the place $\mathfrak{p}$.

| Places of $k(x)$ | | $\mathfrak{P}$ | $e_{\mathfrak{P}}$ | $f_{\mathfrak{P}}$ | $\deg \mathfrak{P}$ |
|---|---|---|---|---|---|
| $\mathfrak{p}_{p(x)}$ | $V_{\mathfrak{p}_{p(x)}}(F(x)) > 0$ | $\mathfrak{P}$ | 2 | 1 | $\deg P$ |
| | $V_{\mathfrak{p}_{p(x)}}(F(x)) = 0$ and | | | | |
| | $F(x) \equiv$ square (mod P) | $\mathfrak{P}_1, \mathfrak{P}_2$ | 1 | 1 | $\deg P$ |
| | $F(x) \not\equiv$ square (mod P) | $\mathfrak{P}$ | 1 | 2 | $2\deg P$ |
| $\mathfrak{p}_\infty$ | $\deg F$ is odd | $\mathfrak{P}$ | 2 | 1 | 1 |
| | $\deg F$ is even and | | | | |
| | l.c. of F is square | $\mathfrak{P}_1, \mathfrak{P}_2$ | 1 | 1 | 1 |
| | l.c. of F is not square | $\mathfrak{P}$ | 1 | 2 | 2 |

3. Genus 0 の代数函数体は、有理函数体またはその2次拡大であることが知られていまして、Degree 1 の Place を持つ場合が有理函数体で、そうでない場合は有理函数体として表わすことは出来ません。そこで、2. で述べた事によって次の様な定理が得られます。

定理. K を Degree 1 の Place を持たない Genus 0 の代数函数体で、$k$ をその定数体、標数は2でないとすると、K は次で与えられる。

㊍ $K = k(x, y)$, $\mathrm{Irr}(y, k(x), Y) = Y^2 - ax^2 - b$, $(a, b) = -1$.

ここに、

$$(a, b) = \begin{cases} 1, & \text{もし、2次方程式 } aX^2 + bY^2 - Z^2 = 0 \text{ が } k \text{ で非自明な解を持つ。} \\ -1, & \text{そうでない時。} \end{cases}$$

逆に、標数が2でない任意の体$k$に対して、⊛で与えられるKは$k$を定数体として持つGenus 0のDegree 1のPlaceを持たない代数函数体である。

また、Genus 0のDegree 1のPlaceを持たない2つの同じ定数体上の代数函数体がその定数体の上に同型であるための必要充分条件も、これらの結果と、少し手間は掛りますが、あまり難くない計算によって、求めることが出来ます。特にその過程で、⊛で与えられるKの$k$上の自己同型群$Aut_k(K)$は次の様に表わすことが出来る。

$$Aut_k(K) \cong G/\{\lambda E \mid \lambda \in k^*\}, \quad G = \left\{ J(V_0, V_1, V_2) \;\middle|\; \begin{array}{l} (V_0, V_1, V_2) \in GL(k, 3) \\ B(V_i, V_j) = 0 \ (i \neq j, \ i, j = 0, 1, 2) \\ \|V_0\| = -b\|V_2\|, \ \|V_1\| = -a\|V_2\| \end{array} \right\},$$

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix}, \quad Y = \begin{pmatrix} Y_0 \\ Y_1 \\ Y_2 \end{pmatrix}, \quad B(X, Y) = aX_0Y_0 + bX_2Y_2 - X_1Y_1 .$$

上でいう必要充分条件とか、$Aut_k(K)$については、あまりすっきりした結果ではありませんが、次で述べるExampleのために、一つの充分条件を上げますと、

$K = k(x, y)$, $y^2 = ax^2 + b$, $(a, b) = -1$、

$K_1 = k(x_1, y_1)$, $y_1^2 = a_1x_1^2 + b_1$, $(a_1, b_1) = -1$、の時、

i) $aa_1 \in k^2$, $(a_1, bb_1) = 1$ または ii) $a_1b$, $ab_1 \in k^2$ ならば、KとK$_1$は$k$上同型である。

Example 3−1、 $k = F_q$, $(2 \nmid q)$の場合、Kは有理函数体に限る。 証明は上の定理を用いる。

Example 3−2、 $k = \mathbb{R}$の場合、Kは有理函数体かまたは、$\mathbb{R}(x, y)$, $y^2 + x^2 + 1 = 0$で与えられる体に$\mathbb{R}$上同型である。 証明は上の定理とi)を用いる。

注、3−1, 3−2は$k$をKの定数体と仮定しなくともよい。

Example 3−3、 $k = Q_p$の場合、任意のPについて、Kは有理函数体かまたはそうでなくとも、すべて$Q_p$上同型である。証明は、( , )はHilbert's Symbolであるので、$k^*/k^{*2}$の代表元

のPairについて、そのHilbert's Symbolの計算及びi), ii)を用いる。

4. これ迄は、体の標数は2でないと仮定して来ましたが、そうでない場合に次の結果を得ることが出来ます。

Example または Theorem, 標数2の完全体の上のGenus 0の代数函数体は有理函数体に限る。

注、完全体という条件は少し弱められますが、あまり主張がすっきりしません。

# Rotthausの定理について

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§1. Introduction　EGA IV (7.4.8) に、次のような問がある。

問　ネター環 A とそのイデアル I について、
a) A/I が $\mathbb{P}$-ring、かつ
b) A が I-adic sep. complete
なら、A も $\mathbb{P}$-ring か？

ここでは、$\mathbb{P}$ = excellent に関する話題を中心に考える。以下、記号、定義、等、主に、Matsumura [M1] による。歴史をふりかえってみると、まず ① 1972年 Matsumura [M2], Nomura [No], Seydi らは、標数0の体(あるいは、ある条件をみたす標数0の excellent Dedekind ring) 上有限生成環のイデアルによる完備化が excellent になることを、正則点のヤコビアン判定法を用い、J-2 を導くことにより、示した。一方 ② 1973年 Marot [Ma1] は、$\mathbb{P}$ = universally japanese のとき、上述の問を肯定的に解いた。この Marot の結果と、①のヤコビアン判定法を(少し工夫して)適用し、③ 1974年、Valabrega [V] は、一般標数の体(および、標数0の excellent Dedekind ring) 上有限生成環のイデアルによる完備化が excellent になることを示した。(cf. [M3])。他方 ④ 1974年、André [A] は、quasi-excellent local ring A から、local ring B への formally smooth 準同型は、regular 準同型であることを示した。この André の結果を用い、⑤ 1978年、Rotthaus [R1] は、A が semi-local で、A/I が quasi-excellent なら、問が肯定的であることを示した。④の注）もっとも、A が標数0の体を含む場合や、A の剰余体 $k$ が、$k^p$ 上有限の場合には、すでに、Seydi、Brezuleanu-Radu らによって、ヤコビアン判定法の直接の応用として、示されていた。⑤の注）なお、この Rotthaus の結果は、ネター環 A が I-adic sep. complete で、A/I が G-ring かつ faithfully flat で quasi-excellent な A-algebra B が存在し、A/IA → B/IB が regular 準同型なら、A も quasi-excellent になることを示している。なお、

⑥ 1979年、Nishimura [N] は、A が semi-local で、$\mathbb{P}$ = normal（又は reduced）のときも、問は肯定的であるが、A が semi-local でなければ、A/I が $\mathbb{P}$-ring でも、A は必ずしも $\mathbb{P}$-ring にはならないことを示した。ところで、⑦ 1980年、Rotthaus [R2] は、Brodmann-Rotthaus [Bd-R] とあわせ、次の定理を証明した：

<u>Theorem</u>.（Rotthaus）ネター環 A と、そのイデアル I について、

a) $\dim A < \infty$（Krull 次元が有限），
b) A は universally catenary，
c) $A \supset \mathbb{Q}$（標数 0 の体を含む），
d) A は I-adic sep. complete，
e) A/I は excellent，

なら、A も excellent である。

<u>Theorem</u>.（Brodmann - Rotthaus）ネター環 A と、そのイデアル I について、

a) A は universally catenary，
b) $A \supset \mathbb{Q}$，
c) A は G-ring（i.e. $\mathbb{P}$ = regular），
d) A/I は J-2，かつ $I \subset \mathrm{rad}(A)$，

なら、A も J-2（すなわち excellent）である。

更に、⑦における key Prop. が $\mathbb{P}$ = normal に応用できることから、⑧ 1980年（又は 1981年）Brezuleanu - Rotthaus [B-R] では、次の結果を得ている。

<u>Theorem</u>（Brezuleanu - Rotthaus）ネター環 A と、そのイデアル I について、

a) $\dim A < \infty$，
b) A は universally catenary，
c) A は I-adic sep. complete，
d) A/I は universally japanese，
e) A/I は Z-ring（i.e. $\mathbb{P}$ = normal），

なら、A も Z-ring である。

なお、⑨ 1981年、Marot [Ma 2] は、標数 0 の体を含む local $\mathbb{P}$-ring $(A, \mathfrak{m})$ と、faithfully-flat A-algebra B について、A の closed pt. 上の fibre $B/\mathfrak{m}B$ が性質 $\mathbb{P}$ をみたせば、A から B への環準同型が $\mathbb{P}$-準同型であることを示し（cf. ④）、A が（標数 0 の体を含む）semi-local ring で、A/I が universally japanese $\mathbb{P}$-ring なら、問が肯定的であることを示した。（cf. ⑤）。
このように、上述の問に対するこの10年間の発展をかえりみると、近い将来、ほぼ満足のいく結果に到達しそうにおもわれる。

## §2. Key Proposition

以下, ネター環 $A$ と, そのイデアル $I$ を固定し, $A$ は, $I$-adic sep. complete とする。

**Notation** まず, $\mathrm{Max}(A) = \{ \mathfrak{m} \in \mathrm{Spec}\, A \mid \mathfrak{m} : \text{maximal} \}$ (i.e. $A$ の極大 ideal の集合), $\Gamma = \{ K \subseteq \mathrm{Max}(A) \mid |K| < \infty \}$ (i.e. $A$ の有限個の極大 ideals の集合の族) とする。このとき $\Gamma$ は, ふつうの包含関係で, filtered direct system になる。また, $\tilde{K} \in \Gamma$ に対し, $\Gamma(\tilde{K}) = \{ K \in \Gamma \mid K \supseteq \tilde{K} \}$ とかく。更に, $S_K = A - \bigcup_{\mathfrak{m} \in K} \mathfrak{m}$, $(K \in \Gamma)$, $A_K = S_K^{-1} A$ ($K$の元を極大イデアルにもつ) semi-local ring, $B_K = A_K^{*}$ ($=$ $A_K$ の $IA_K$-adic completion) とする。このとき, $K_1, K_2 \in \Gamma$, $K_1 \supseteq K_2$ なら, 次のような自然な準同型 $\psi_{K_1K_2}$, $\varphi_{K_1K_2}$ が存在する。

$$\begin{array}{ccc} A_{K_1} & \xrightarrow{\psi_{K_1K_2}} & A_{K_2} \\ \downarrow & \circlearrowleft & \downarrow \\ B_{K_1} & \xrightarrow{\varphi_{K_1K_2}} & B_{K_2} \end{array}$$

よって, $\{ B_K, \varphi_{K'K} \}_{K', K \in \Gamma}$ は, projective system になる。さて,

**Definition 2.1.** ($K_0 \in \Gamma$ を fix し) すべての $K \in \Gamma(K_0)$ について, $B_K$ の ideal $\mathfrak{b}_K$ が与えられているとする。いま, $\{\mathfrak{b}_K\}$ $(K \in \Gamma(K_0))$ が, 次の2条件をみたすとき, この集合を,(Aの) ideal-sequence という:

(2.1.1) 任意の $K', K \in \Gamma(K_0)$, $K' \supseteq K$ について, $\mathfrak{b}_{K'} = \mathfrak{b}_K \cap B_{K'}$,

(2.1.2) 任意の $K', K \in \Gamma(K_0)$, $K' \supseteq K$ について, $\mathfrak{b}_K = \mathfrak{b}_{K'} B_K$.

**Notation** いま, $\{\mathfrak{p}_K\}_{K \in \Gamma(K_0)}$ を, ideal-sequence で, 各 $\mathfrak{p}_K$ が prime なもの (このとき, prime-ideal-seq. とよぶ) を考える。さて, $A$ の非零元 $t(\neq 0)$ に対し

$$\Delta_K(t) = \{ Q_{Ki}(t) \in \mathrm{Spec}\, B_K \mid Q_{Ki}(t) : \mathfrak{p}_K + tB_K \text{ の極小 prime} \},$$

$$\Delta(t) = \{ \mathfrak{q}_{Ki}(t) \in \mathrm{Spec}\, A \mid \mathfrak{q}_{Ki}(t) = Q_{Ki}(t) \cap A \,;\, Q_{Ki}(t) \in \Delta_K(t) \}$$

とかく。このとき,

**Definition 2.2.** $A$ の prime-ideal-seq. $\{\mathfrak{p}_K\}_{K \in \Gamma(K_0)}$ が, 次の2条件をみたすとき, simple という:

(2.2.1) 任意の $A$ の非零元 $t(\neq 0)$ と, すべての $\mathfrak{q}_{Ki}(t) \in \Delta(t)$ について, $\mathrm{ht}\, \mathfrak{q}_{Ki}(t) = \mathrm{ht}(\mathfrak{p}_K + tB_K)$.

(2.2.2) 任意の $A$ の非零元 $t(\neq 0)$ について, $\Delta(t)$ は有限集合。

さて, Rotthaus の定理の Key Point となる prop. は:

Key Proposition: $A$をネター環, $I = xA$ $(x \neq 0)$ を principal ideal で, 次の条件をみたすとする:

a) $A$ は universally catenary domain,
b) $A$ は universally japanese,
c) $A$ は $xA$-adic sep. complete,
d) $A/xA$ は G-ring (又は Z-ring).
e) $\{\mathfrak{p}_K\}_{K \in \Gamma(K_0)}$ が simple prime-ideal-seq.

とする。このとき ht $\mathfrak{p}_K > 0$ なら $\mathfrak{p}_K \cap A \neq (0)$.

Outline of Proof. $\mathfrak{p}_K \cap A = (0)$ と仮定し矛盾をみちびく.

まず $C_K = \overline{(B_K/\mathfrak{p}_K)}$ (= $B_K/\mathfrak{p}_K$ の integral closure) とすると $K' \supseteq K$ について次の可換図を得る.

$$\begin{array}{ccc} B_{K'} & \xrightarrow{\varphi_{K'K}} & B_K \\ \downarrow & & \downarrow \\ B_{K'}/\mathfrak{p}_{K'} & \longrightarrow & B_K/\mathfrak{p}_K \\ \downarrow & & \downarrow \\ C_{K'} & \xrightarrow{\mu_{K'K}} & C_K \end{array}$$

よって $\{C_K, \mu_{K'K}\}_{K', K \in \Gamma(K_0)}$ も projective system になる.
そこで

$$D = \varprojlim_{K \in \Gamma(K_0)} C_K$$

と定義する。このとき $\{\mathfrak{p}_K\}$ が simple prime-ideal-seq. であること(および a), b), d)) より $D$ が Krull 環かつ $D/xD$ が $A$ 上 finite (module) であることが示される。(もっとも これらを示すのに 約10の それほど自明ではない sublemma を必要とする。詳細は Rotthaus [R2] 又は Marot [Ma 3] を参照のこと.) よって c) より $D$ 自身 $A$ 上 finite となり 単項 ideal $xD$, $xA$ の極小 prime の ht を比較すること および a) より 矛盾.

§3. Key Prop. の適用. Rotthaus の定理を示すにあたり まず noetherian induction および⑤(の注)より $A$ を domain, $I$ を prime (以下 $I$ を $P$ とも かく) とし $A$ の regular locus が non-empty open set を含むことを示せばよいことになる. まず $A_P$ が excellent をいう。このために $\widehat{A_P}$ の singular locus を定義する極小 prime の一つ $\hat{\mathfrak{p}}$ をとり (fix する), $B_K$ との intersection $\hat{\mathfrak{p}} \cap B_K$ を $\mathfrak{p}_K$ とおくと ある $K_0 \in \Gamma$ について この $\{\mathfrak{p}_K\}_{K \in \Gamma(K_0)}$ が simple-prime-ideal seq. であることを示せばよい。注) この際、上述の reduction に加え $\dim A < \infty$ の仮定より $A$ の Krull 次元に関する induction も いたるところで使う。
次に 各 $m \in \Gamma$ について $A_m$ が excellent (すなわち G-ring) をいう。

(上の $\hat{A}_{\mathfrak{p}}$ の場合と同じように)、$\hat{A}_{\mathfrak{m}}$ の singular locus を定義する極小 prime の一つ $\hat{\mathfrak{p}}_{\mathfrak{m}}$ をとり(fix する)、前と同様に、$\mathfrak{p}_k = \hat{\mathfrak{p}}_{\mathfrak{m}} \cap B_k$ ($k \in \Gamma(\mathfrak{m})$) とかくと、ある $k_0 \in \Gamma$ について、この $\{\mathfrak{p}_k\}_{k\in\Gamma(k_0)}$ が、また、simple prime-ideal-seq. であることを示せばよい。注) このこと(とくに、$\Delta(t)$ の有限性)を示すには、$A_{\mathfrak{p}}$ が excellent なので Hironaka resolution が適用できる。注) ここで、標数0が必要。しかし、この step では、$\dim A < \infty$ という仮定は使わない。以上より、A が G-ring であることが示される。
そこで、Brodmann-Rotthaus の定理を用い、Rotthaus の定理の証明が完成する。注) Brodmann-Rotthaus の定理においても、A の reg. locus の openness を示すのに、まず、noetherian induction により、$\mathrm{rad}(A)=Q$ を prime と考えてよいことを示し、$A_Q$ に Hironaka resolution を用いる(ので、やはり標数0が必要)。
一方、Z-ring に関しては、すでに② より、universally japanese なので、normalization は、いつもうまくいく。従って、標数一般でも、不等標数でも、定理は成立する。注) しかし、まず、$A_{\mathfrak{p}}$ が、Z-ring を示すのに、Key Prop. を用いるので、excellent の場合と同様に、$\hat{A}_{\mathfrak{p}}$ の non-normal locus を定義する極小 prime $\hat{\mathfrak{p}}$ から導かれる $\{\mathfrak{p}_k\}_{k\in\Gamma(k_0)}$ が、simple prime-ideal-seq. であることをいわねばならず、そこでの induction を用いる議論のために、$\dim A < \infty$ を必要とする。

§4. <u>その他</u>　1) Rotthaus の定理における証明法は、①,③ のように、直接 J-2 をいうのではなく、とりあえず G-ring になることを示し、その後、Hironaka resolution を利用し、reg. locus の openness を導く。はたして、J-2 を、もう少し直接に示すことはできないものだろうか? とくに、①,③(および④の注)より、ヤコビアン判定法のうまい利用が、案外有効であるようにおもわれる。たとえば、このような方法で、「A/I が、体(あるいは excellent Dedekind 環)上有限生成環で、A が I-adic sep. complete のとき、A は quasi-excellent になる」ことくらいなら、かんたんに示される(とおもう)。
2) さて、⑤～⑨ での考察を発展させ、一般の $\mathbb{P}$ について、Rotthaus の定理(や、Brezuleanu-Rotthaus の定理)と同様のことはいえないだろうか? つまり、再び最初の問にもどるわけだが、

**問**　ネター環 A と、そのイデアル I について、
a) A/I が、universally japanese, $\mathbb{P}$-ring, かつ、
b) A が、I-adic sep. complete
なら、A も $\mathbb{P}$-ring か?

上述の Key Prop. は、むしろこの問に対する重要な第一歩であるようにおもわれる。

注) もっとも $\mathbb{P}$ = regular の場合も、A/I における reg. locus の openness (i.e. J-2) の仮定がないと、標数0でさえ、まだわかっていない。

3) 最近、Brezuleanu から、3次元 local Z-ring で、J-2 をみたすが、G-ring ではない例が、作られた(1982年8月頃)と 伝えてきた。(10月25日付、手紙)。 以上。

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# Dualizing complex の存在について

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A を環(常に単位元を持つ可換な noether 環を考える)とする。finite injective dimension をもつ A-加群 $I^i$ の chain complex $I^{\cdot}$ で、$I^{\cdot}$ は下に有界かつ Homology module $H^j(I^{\cdot})$ が有限生成 A-加群となるもの $(i, j \in \mathbb{Z})$ は、次の条件を満たすとき dualizing complex と言われる;

どんな有限生成 A-加群 M についても、$\mathrm{Hom}_A(\mathrm{Hom}_A(M, I^{\cdot}), I^{\cdot})$ が derived category の中で M と quasi-isomorphic となる。

幾何学的環等、Gorenstein 環の準同型像となる環はすべて dualizing complex を持つのであるが、ここではどのくらい一般の環が dualizing complex を持つかを問題にする。

よく知られているように、Sharp [4] は、もし A が dualizing complex をもてば、A は acceptable ring である、すなわち

(1) 強素イデアル鎖条件を満たす。

(2) formal fiber は Gorenstein

(3) どんな有限生成 A-algebra B についても、Spec B の Gorenstein locus は Zarisky 位相で開集合となる。

の３つの条件を満たすことを示した。

一方、局所環（A, $\mathfrak{m}$）について、有限生成 A-加群 K が canonical module であるとは、$K\otimes_A\hat{A}$ が $\mathrm{Hom}_A(H^n_{\mathfrak{m}}(A), E)$ と同型となることである。但しここで $\hat{A}$ はAの完備化、$E=E(A/\mathfrak{m})$ は A-加群 $A/\mathfrak{m}$ の injective envelope、$n=\dim A$ で $H^n_{\mathfrak{m}}(A)$ は、Aの $n$ 次の local cohomology を表わす。なお、もしAが dualizing complex $I^{\cdot}$ を持つ場合には、K は $I^{\cdot}$ の homology module $H^{\cdot}(I^{\cdot})$ で 0でないもののうち、次数の最小のものとして得られる。

さて、これら知られた事実の逆について、得られた結果は次のようなものである。

定理 1　　環 A は次の条件を満たすとする。

(1) A は $(S_2)$ で $\dim A<\infty$。

(2) formal fiber は Gorenstein。

(3) A のどんな準同型像 B についても、Spec B の Cohen Macaulay locus は開集合。

(4) A は canonical module K をもつ。

この時、$H^0(I^{\cdot})=K$ となる A の dualizing complex $I^{\cdot}$ が存在する。

（K が A の canonical module であるとは、K は有限生成 A-加群であって、どの極大 ideal $\mathfrak{m}$ についても、$K_{\mathfrak{m}}$ が $A_{\mathfrak{m}}$

の canonical module であることである。)

定理 2. 局所環 A について. A が dualizing complex を持つ条件は.

(1) A の formal fiber は Gorenstein.

(2) A のどんな準同型像も Canonical module を持つ.

の 2 つを満たすことである。

(注意) 定理 1, 2 において acceptable 性と. canonical module の存在の両方の条件が重要である。実際 (a) canonical module を持たぬ acceptable ring の例 及び (b) acceptable でないが. canonical module をもつ環の例が存在する。だいたいどんなものかと言えば.

例 (a) 1980年12月に行なわれた第2回可換環論シンポジウムで話した方法を応用して(c.f. [2]). 可算体K上の algebra A で, A は一意分解局所整域, A の 0 でない素イデアル $\mathfrak{P}$ について. $A/\mathfrak{P}$ は有限生成 K-algebra の局所化かつ. A の完備化 $\hat{A}$ が

$$\hat{A} = K[[X_1, X_2, X_3, Y_1, Y_2, Y_3]] / (X_1Y_2 - X_2Y_1, X_1Y_3 - X_3Y_1, X_2Y_3 - X_3Y_2)$$

となるものを作る。正規環 A が canonical module をもてば. それは pure height 1 のイデアルと同型にならねばならぬから. 一意分解性より free となる。しかし, $\hat{A}$ の canonical module

$(x_1, y_1)\hat{A}$ は free でないから，A は canonical module を持たないことがわかる。A が acceptable ring であることは，$A/\mathfrak{q}$ の条件と，$\hat{A}$ の closed point 以外は regular であることからわかる。

(b) 体 K 上の次数付環 R で，$R_0 = K$ $R_+ = \mathfrak{m}$ により，完備化 $\widehat{R_{\mathfrak{m}}}$ が Cohen Macaulay でない一意分解環となるものが存在することを森氏［4］が示している。前述(a)の方法を使って，この R を定義する関係式を組み込むことにより，formal fiber が Cohen Macaulay とならない局所環 A で，完備化 $\hat{A}$ が一意分解環となるものを作る。A は，free な canonical module A をもつが，acceptable でない。

さて，もとにもどって，derived category の中で quasi-isomorphism を考えるというやっかいなことを避けるために，Sharp と Hall によって導入された fundamental dualizing complex $I^{\cdot}$ というものを考える［5］。すなわち，

(i) $I^i$ $(i \in \mathbb{Z})$ は，injective A-module

(ii) $I^{\cdot}$ は bounded complex

(iii) $H^i(I^{\cdot})$ $(i \in \mathbb{Z})$ は 有限生成 A-加群

(iv) $\bigoplus_{i \in \mathbb{Z}} I^i = \bigoplus_{\mathfrak{q} \in \operatorname{Spec} A} E(A/\mathfrak{q})$

ここで，$E(A/\mathfrak{q})$ は A-加群 $A/\mathfrak{q}$ の injective envelope。

もし、$\dim A < \infty$ であれば、Aの、dualizing complex が存在することと fundamental dualizing complex が存在することとは同値となる。

定理1の証明には、次の2つの命題が中心となる。

命題 3　Xは、そのどんな閉 scheme Y についても、Y の Cohen Macaulay locus が開集合となるような noetherian scheme とする。$\mathscr{I}^{\cdot}$ は、Xの各点 x で $\mathscr{I}^{\cdot}_x$ が $\mathcal{O}_{X,x}$ の fundamental dualizing complex となるような bounded complex とする。もし、Xの各既約成分Vについて、$\mathcal{H}om(\mathcal{O}_V, \mathscr{I}^{\cdot})$ の homology module のうちで、0とならない次数最低のものが、coherent な $\mathcal{O}_V$-加群 となるならば、$\mathscr{I}^{\cdot}$ は X の fundamental dualizing complex となる。
（$\mathscr{I}^{\cdot}$ が X の fundamental dualizing complex とは、X の affine covering $\{U_i\}$ が存在して、各 i について、$\Gamma(U_i, \mathscr{I}^{\cdot})$ が、$\Gamma(U_i, \mathcal{O}_X)$ の fundamental dualizing complex となることである）。

命題 4　局所環 $(A, \mathfrak{m})$ は、$(S_2)$ で $\dim A = n \geq 2$ かつ formal fiber は Gorenstein となるものとする。さて、

$$0 \to I^0 \to I^1 \to \cdots \to I^{n-1} \to 0$$

は、A-加群の injective complex で、$H^0(I^{\cdot}) = K$ は A の

canonical module かつ各 $\mathfrak{p} \in U = \operatorname{Spec} A - \{\mathfrak{m}\}$ について、$I^{\cdot} \otimes_A A_{\mathfrak{p}}$ は $A_{\mathfrak{p}}$ の fundamental dualizing complex になるものとする。このとき、A-準同型

$$d^{n-1} : I^{n-1} \longrightarrow E(A/\mathfrak{m}) = I^n$$

で、$(I^k, d^k \mid 0 \leq k \leq n)$ が A の dualizing complex となるものが存在する。

命題4は、主として命題3と次の事から導かれる。

補題 5（青山－　）　A は canonical module K を持つ局所環とする。この時、次の3つは同値。

(1)　$\operatorname{Hom}_A(K, K) \simeq A$

(2)　完備化 $\hat{A}$ は $(S_2)$

(3)　A は $(S_2)$

定理 1 の証明の概略

$E(A/\mathfrak{p})$ を略して $E(\mathfrak{p})$ と書くことにし、$I^i = \bigoplus_{ht\,\mathfrak{p}=i} E(\mathfrak{p})$ とおく。命題3を使えば、次の条件を満たす A-準同型の集合

$$\{f_{\mathfrak{p}\mathfrak{q}} : E(\mathfrak{p}) \longrightarrow E(\mathfrak{q}) \mid \mathfrak{p} \leq \mathfrak{q},\ ht\,\mathfrak{q}/\mathfrak{p} = 1\}$$

が存在することを示せばよいことがわかる。

(a)　$f^{\mathfrak{p}} = \prod_{\mathfrak{q}} f_{\mathfrak{p}\mathfrak{q}} : E(\mathfrak{p}) \longrightarrow \prod_{\mathfrak{q}} E(\mathfrak{q})$ について、

$f^{\mathfrak{p}}(E(\mathfrak{q})) \subseteq \bigoplus_{\mathfrak{q}} E(\mathfrak{q})$ となる。

(b)　$d^i = \bigoplus_{ht\,\mathfrak{p}=i} f^{\mathfrak{p}}$ について、$(I^{\cdot}, d^{\cdot})$ が chain complex

となる

(c) 各 $\mathfrak{z} \in \operatorname{Spec} A$ について $(I^{\cdot} \otimes A_{\mathfrak{z}}, d^{\cdot} \otimes A_{\mathfrak{z}})$ が $A_{\mathfrak{z}}$ の fundamental dualizing complex となる。

(d) $H^0(I^{\cdot}) = K$

そして、実際我々は、Kの injective envelope を取る事から始めて、命題4を使って帰納的にこのような A-準同型を定義できるのである。

定理1から定理2を得る方法は、A の 0 の primary decomposition を $0 = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ とした時、各 $A/\mathfrak{q}_i$ の $(S_2)$-化 $A_i$ は、定理1より dualizing complex を持つのであるが、それを $A/\mathfrak{q}_i$ そして A へ落としていくのである。落とす場合、それぞれ、環の fiber product となっていることを使うわけであるが、dualizing complex が落ちてゆくことを証明する中心となる命題は次のものである。

命題 6　$A_1 \longrightarrow A_0$, $A_2 \longrightarrow A_0$ を加群として有限生成になる環準同型とし、環の fiber 積 $A_1 \times_{A_0} A_2 = A$ の素イデアル $\mathfrak{z}$ について、A-加群 $A/\mathfrak{z}$ の injective envelope $E = E(A/\mathfrak{z})$ を考える。　$E_i = \operatorname{Hom}_A(A_i, E)$ $(i = 0, 1, 2)$ とおけば、E は A-加群の category で push out $E = E_1 \amalg_{E_0} E_2$ となる。

以上、証明の方向性ぐらいしか説明できなかったが、興味を持たれた方は［３］を読んで頂ければ幸いである。

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# Subrings of finitely generated rings

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Let $D \subseteq A$ be integral domains such that $D$ is noetherian and that $A$ is finitely generated over $D$. In this note we are mainly interested in a $D$-subalgebra $R$ of $A$ and we shall consider a problem asking when $R$ is again finitely generated over $D$. This problem is closely related to the fourteenth problem of Hilbert and many mathematicians gave various conditions for $R$ to be finitely generated over $D$. The purpose of this note is to give some new conditions when $D$ is a pseudo-geometric ring satisfying the following condition (C).

(C) Every normal locality over $D$ is analytically irreducible.

Throughout this note we fix the above notations and assumptions.

## 1. The ideal $A_D(R)$

We define a subset $A_D(R)$ of $R$ as follows :

$$A_D(R) = \{a \in R \mid R[1/a] \text{ is finitely generated over } D\} \cup \{0\}.$$

Then we have

<u>Lemma</u> 1.1. $A_D(R)$ is a non-zero radical ideal of $R$.

We omit the proof. The following lemma plays an important role.

<u>Lemma</u> 1.2. Let $P \in \mathrm{Spec}(R)$. Then we have $A_D(R) \not\subseteq P$ if and only if $R_P$ is a locality over $D$.

<u>Proof</u>. The "only if" part is obvious and we prove the "if" part. Assume that $R_P$ is a locality over $D$. Then there exist a subring $B$ of $R$ and a prime ideal $q$ of $B$ such that $B$ is finitely

generated over $D$ and $B_q = R_P$. Let $S = B \setminus q$. Then we have $S^{-1}B = S^{-1}R$. Let $0 \neq a \in A_D(R)$ and take an element $b$ of $B$ so that $R[1/a] \subseteq B[1/b]$. Let $F = \{Q \in \mathrm{Spec}(B) \mid \mathrm{depth}\, B_Q = 1$ and $B_Q \not\supseteq R\}$. Then, as is easily seen, we have $F \subseteq \mathrm{Ass}_B(B/bB)$ and hence $F$ is a finite set. Let $F = \{Q_1, \cdots, Q_m\}$. Obviously we have $Q_1 \cap \cdots \cap Q_m \not\subseteq P$. Let $s \in (Q_1 \cap \cdots \cap Q_m) \setminus P$ and let $\Delta = \{Q \in \mathrm{Spec}(B) \mid \mathrm{depth}\, B_Q = 1$ and $s \notin Q\}$. Then, by the choice of $s$, we have $B_Q \supseteq R$ for every $Q \in \Delta$. Hence we have $B[1/s] = \bigcap_{Q \in \Delta} B_Q \supseteq R$, which implies that $B[1/s] = R[1/s]$. Thus $s \in A_D(R) \setminus P$ and we proved $A_D(R) \not\subseteq P$ ∎

By virtue of this lemma, we have the following theorem.

Theorem 1.3. The following conditions are equivalent to each other.

(1) $R$ is finitely generated over $D$.

(2) $R_P$ is a locality over $D$ for every prime ideal $P$ of $R$.

(3) $R_p$ is finitely generated over $D_p$ for every prime ideal $p$ of $D$.

Proof. (1) ⇨ (3) ⇨ (2) is obvious. (2) ⇨ (1) is an immediate consequence of Lemma 1.2 ∎

## 2. A non-nullity criterion of $A_{D/p}(R/P)$

Let $P$ be a prime ideal of $R$ and let $p = P \cap D$. In general $A_{D/p}(R/P)$ may be a zero ideal even when $D$ is a field. For the later use, we give a condition for $A_{D/p}(R/P)$ to be non-zero in this section. For this purpose we recall the following definition.

Definition 2.1. Let $P \in \mathrm{Spec}(R)$ and let $p = P \cap D$. If

$$\mathrm{ht}(P) + \mathrm{tr.deg}_{D/p} R/P = \mathrm{ht}(p) + \mathrm{tr.deg}_D R,$$

then we say that $P$ satisfies the dimension formula relative to $D$.

Now we have the following

Lemma 2.2. Let $P$ be a prime ideal of $R$ with $\mathrm{ht}(P) = 1$. If $P$

satisfies the dimension formula relative to D, then $A_D(R) \not\subseteq P$.

Proof. Take a subring B of R such that (1) B is finitely generated over D (2) R is birational over B and (3) $ht(Q) = 1$, where $Q = P \cap B$. Let B' and R' be the derived normal rings of B and R, respectively. Then, by virtue of Krull-Akizuki's theorem, $B'_Q$ and $R'_Q$ are one-dimensional noetherian rings. Let $P'R'_Q$ be a maximal ideal of $R'_Q$ and let $Q' = P' \cap B'$. Since $B'_{Q'}$ is a discrete valuation ring we have $B'_{Q'} = R'_{P'}$. Therefore, by Theorem 1.3, we see that $R'_Q$ is finitely generated over $B'_Q$. Note that B' is a finite B-module. Thus we have $A_D(R') \cap S \neq \phi$, where $S = B \setminus Q$, by Lemma 1.2, which implies that $A_D(R) \cap S \neq \phi$. Hence we have $A_D(R) \not\subseteq P$ ∎

By making use of this lemma, we have the following theorem.

Theorem 2.3. If a prime ideal P of R satisfies the dimension formula relative to D, then we have $A_{D/p}(R/P) \neq (0)$, where $p = P \cap D$.

Proof. The assertion is easily verified by induction on $ht(P)$ ∎

3. Main theorems

First of all we prove the following

Lemma 3.1. Let D' and R' be the respective derived normal rings of D and R. If a prime ideal P' of R' satisfies the dimension formula relative to D' and if $R_{P' \cap R}$ is noetherian, then $R'_{P'}$ is a locality over D'.

Proof. Let $P = P' \cap R$. Take a subring B of R such that (1) B is finitely generated over D (2) R is birational over B (3) $tr.deg_{B/Q} R/P = 0$ and (4) $QR_P = PR_P$, where $Q = P \cap B$. Let B' be the derived normal ring of B and let $\bar{R} = R[B']$. Then $\bar{R}$ is a finite R-module. Let $\bar{P} = P' \cap \bar{R}$. Since $R_P$ is noetherian, we know that $\bar{R}_{\bar{P}}$ is also noetherian. Let $Q' = P' \cap B'$ and $p' = P' \cap D'$. Consider the ring extensions $D'_{p'} \subseteq B'_{Q'} \subseteq \bar{R}_{\bar{P}} \subseteq R'_{P'}$. Since $P'$ satisfies the dimension formula relative to $D'$ and since $B'_{Q'}$ and $\bar{R}_{\bar{P}}$ are noetherian, we can easily verify that both $\bar{P}$ and $Q'$ satisfy the dimension formula relative to $D'$. Thus we have

$$ht(\bar{P}) = ht(p') + tr.deg_{D'}\bar{R} - tr.deg_{D'/p'}\bar{R}/\bar{P},$$

$$ht(Q') = ht(p') + tr.deg_{D'}B' - tr.deg_{D'/p'}B'/Q'.$$

Therefore, by the choice of $B$, we see that $ht(\bar{P}) = ht(Q')$, i.e., $\dim \bar{R}_{\bar{P}} = \dim B'_{Q'}$. Let $K$ and $L$ be the quotient fields of $B'/Q'$ and $\bar{R}/\bar{P}$, respectively. We claim that $length_K \bar{R}_{\bar{P}}/Q'\bar{R}_{\bar{P}}$ is finite. In fact, since $Q'\bar{R}_{\bar{P}} \supseteq Q\bar{R}_{\bar{P}} = P\bar{R}_{\bar{P}}$ and $P\bar{R}_{\bar{P}}$ is a $\bar{P}\bar{R}_{\bar{P}}$-primary ideal, there exists a positive integer $n$ such that $\bar{P}^n\bar{R}_{\bar{P}} \subseteq Q'\bar{R}_{\bar{P}}$. Then we have

$$length_K \bar{R}_{\bar{P}}/Q'\bar{R}_{\bar{P}} \leqq length_{B'_{Q'}} \bar{R}_{\bar{P}}/\bar{P}^n\bar{R}_{\bar{P}}$$
$$= (length_K L)(length_{\bar{R}_{\bar{P}}} \bar{R}_{\bar{P}}/\bar{P}^n\bar{R}_{\bar{P}}).$$

Since $\bar{P}$ satisfies the dimension formula relative to $D'$, by Theorem 2.3, there exists a ring $E$ such that $E$ is finitely generated over $D'/p'$ and $\bar{R}/\bar{P} \subseteq E$. Let $S = (B'/p') \setminus \{0\}$. Then the ring extensions $B'/Q' \subseteq \bar{R}/\bar{P} \subseteq E$ implies $K = S^{-1}(B'/Q') \subseteq S^{-1}(\bar{R}/\bar{P}) \subseteq S^{-1}E$. Note that we have $S^{-1}(\bar{R}/\bar{P}) = \bar{R}_{Q'}/\bar{P}\bar{R}_{Q'}$ and $tr.deg_K \bar{R}_{Q'}/\bar{P}\bar{R}_{Q'} = 0$ by the choice of $B$. Since $K$ is a field, we know that $\bar{R}_{Q'}/\bar{P}\bar{R}_{Q'}$ is also a field, which implies that $\bar{R}_{Q'}/\bar{P}\bar{R}_{Q'} = L$. Thus we have $K \subseteq L \subseteq S^{-1}E$ and $tr.deg_K L = 0$. Note that $S^{-1}E$ is finitely generated over $K$. Hence $L$ is a finite algebraic extension field of $K$. Therefore $tr.deg_K L$ is finite. On the other hand, since $\bar{R}_{\bar{P}}$ is noetherian, it follows that $length_K \bar{R}_{\bar{P}}/\bar{P}^n\bar{R}_{\bar{P}}$ is finite. Moreover, since $B'_{Q'}$ is a normal locality over D, $B'_{Q'}$ is analytically irreducible by the assumption on D, and obviously, $B'_{Q'}$ and $\bar{R}_{\bar{P}}$ are birational to each other. Therefore we have $B'_{Q'} = \bar{R}_{\bar{P}}$ by Zariski's main theorem. This implies that $\bar{R}_{\bar{P}}$ is integrally closed, hence we have $\bar{R}_{\bar{P}} = R'_{\bar{P}}$. Thus $R'_{\bar{P}}$ is a local ring, and hence we have $R'_{\bar{P}} = R'_{P'}$. From these consideration, we have $R'_{P'} = B'_{Q'}$ and $R'_{P'}$ is a locality over D' ∎

Now we have the following theorem which is an immediate consequence of this lemma and Theorem 1.3.

<u>Theorem</u> 3.2. The following conditions are equivalent to each other.

(1) R is finitely generated over D.

(2) R is locally noetherian and the dimension formula holds between D' and R', where D' and R' are respective derived normal rings of D and R.

This theorem can be generalized as follows.

<u>Theorem</u> 3.3. If R is locally noetherian and if, for every maximal ideal M' of R', there exists a locality S over R' such that S dominates $R'_{M'}$ and that the maximal ideal $\mathfrak{m}$ of S satisfies the dimension formula relative to D', then R is finitely generated over D.

This theoerm follows from Theorem 3.2 and the following two lemmas.

<u>Lemma</u> 3.4. $A_D(R[X_1,\cdots,X_n]) = A_D(R)[X_1,\cdots,X_n]$ for every positive integer n, where $X_1,\cdots,X_n$ are indeterminates.

<u>Lemma</u> 3.5. Let $\mathfrak{p}$ be a prime ideal of A and let $P = \mathfrak{p} \cap R$. If $\mathfrak{p}$ satisfies the dimension formula relative to D, then there exists a positive integer n such that the prime ideal $P[X_1,\cdots,X_n]$ of $R[X_1,\cdots,X_n]$ satisfies the dimension formula relative to D.

The proofs of these lemmas are omitted.

Corollary 3.6. If $R$ is locally noetherian and if the natural map $\mathrm{Spec}(A') \longrightarrow \mathrm{Spec}(R')$ is surjective, then $R$ is finitely generated over $D$.

Furthermore we can prove the following theorem.

Theorem 3.7. Assume that we have $\dim D[1/a] = \dim D$ for every non-zero element $a$ of $D$. If $R$ is locally noetherian and if its integral closure $R'$ in the quotient field is equi-dimensional, then $R$ is finitely generated over $D$.

This is one of the natural generalizations of that given in [1]. We omit the proof.

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Castelnuovo's regularity of graded rings and generic Cohen-Macaulay algebras

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We denote by $A = \bigoplus_{n \geqslant 0} A_n$ a noetherian graded algebra over a field $k = A_0$. For a graded A-module $M = \bigoplus_{n \in \mathbb{Z}} M_n$ (notation: $M_n = [M]_n$), the i-th local cohomology module $H^i_P(M)$ of M with support in $P = A_+$ is also a graded A-module. Fix an integer m. We say M is m-regular if $[H^i_P(M)]_j = 0$ whenever $i + j > m$, and we define $\mathrm{reg}(M) = \inf\{ m \in \mathbb{Z};\ M$ is m-regular $\}$ (the regularity of M) and $a(M) = \mathrm{reg}(M) - \dim(M)$. Our aim is to study the relationship between this invariant reg(A) and the structure of a graded ring A. For this purpose, the following theorem is fundamental:

Castelnuovo's lemma. Assume that A is homogeneous (i.e., $A = k[A_1]$) and $M = \bigoplus_{n \geqslant 0} M_n$ is finitely generated. (In this case, $\mathrm{reg}(M) \geqslant 0$.) If $[H^i_P(M)]_{-i} = [H^0_P(M)]_i = 0$ for all $i > 0$, then M is 0-regular and M is generated by $M_0$.

(A variant of this theorem: Generalized Castelnuovo's lemma (Mumford). Let X be a projective variety, D an ample Cartier divisor on X such that $\mathrm{Bs}|D| = \emptyset$ and F a coherent $\mathcal{O}_X$-module. If $H^i(X, F(-iD)) = 0$ for all $i > 0$, then $H^i(X, F(jD)) = 0$ whenever $i + j \geqslant 0$, $i > 0$ and $A_1 M_j = M_{j+1}$ for all $j \geqslant 0$, where $A_i = H^0(X, \mathcal{O}(iD))$ and $M_j = H^0(X, F(jD))$.)

We give some examples.

(1) If $A$ is homogeneous, then $\mathrm{reg}(A) = 0$ if and only if $A$ is a polynomial ring.

(2) If $a \in P$ is a homogeneous $M$-regular element, then $\mathrm{reg}(M/aM) = \mathrm{reg}(M) + \deg(a) - 1$. Hence, if $A$ is a complete intersection of type $(e_1,\ldots,e_r)$, then $\mathrm{reg}(A) = \sum_{i=1}^{r} e_i - r$.

(3) If $A$ is Gorenstein, then $K_A = A(a(A))$, where $K_A$ is the canonical module of $A$.

(4) If $A$ is Cohen-Macaulay, then $K_A$ is also Cohen-Macaulay and $\mathrm{reg}(K_A) = \dim(A)$.

(5) If $X$ is a smooth non-hyperelliptic projective curve, then its canonical ring $A = A(X, K) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}(nK))$ is a normal Gorenstein homogeneous algebra with $\mathrm{reg}(A) = 3$.

(6) If $X$ is a Fano variety (char $k = 0$), then its anti-canonical ring $A = A(X, -K)$ is a Gorenstein algebra with $\mathrm{reg}(A) = \dim(A) - 1$.

(7) If $X$ is an algebraic surface (char $k = 0$) whose canonical divisor $K$ is ample, then we have $\mathrm{reg}\, A(X, K) = 4$ and $A$ is Cohen-Macaulay if and only if the irregularity $q(X)$ of $X$ is zero (and in this case $A(X, K)$ is Gorenstein).

(8) Let $X$ be an abelian variety and $D$ a very ample divisor on $X$. Then $A = A(X, D) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}(nD))$ is a Buchsbaum algebra with $\mathrm{reg}(A) = \dim(A)$, and $A$ is Cohen-Macaulay if and only if $X$ is an elliptic curve (and in this case $A$ is Gorenstein).

For Cohen-Macaulay homogeneous algebras, there is an important relation between regularity and the "postulation formula" for their Hilbert functions, and using this relation we can often calculate the regularity: Let $M$ be a Cohen-Macaulay graded module over a homogeneous $k$-algebra $A$. Then, for an integer $m$, we have $a(M) < m$ if and only if $H(M, n) = h(M, n)$ for all $n \geqslant m$ (resp. $H(M, m) = h(M, m)$) if and only if $\deg F(M, T) < m$, where $H(M, n) = \dim_k M_n$, $h(M, n)$ and $F(M, T) = \sum_{n \in \mathbb{Z}} H(M, n)T^n$ are the Hilbert function, the Hilbert polynomial and the Hilbert series of $M$ respectively. Therefore, if we write $F(M, T) = f_M(T)/(1 - T)^d$, $d = \dim(M)$, $f_M(T) \in \mathbb{Z}[T, T^{-1}]$, then we have $\operatorname{reg}(M) = \deg f_M(T)$.

We can generalize this theorem in various ways:

(1) Let $H$ be a function from $\mathbb{Z}$ to $\mathbb{Z}$. Then $H$ is a polynomial function (i.e., there exists a polynomial $h \in \mathbb{Q}[T]$ such that $H(n) = h(n)$ for all sufficiently large $n$) and $H(n) = 0$ for all sufficiently small $n$ if and only if $F(T) \overset{=}{\text{def}} \sum_{n \in \mathbb{Z}} H(n)T^n = f(T)/(1 - T)^d$ for some $d \in \mathbb{Z}$, $d \geqslant 0$ and $f(T) \in \mathbb{Z}[T, T^{-1}]$. Moreover, in this case, we have $d = \deg h + 1$ (we put $\deg 0 = -1$), and for an integer $m$, we have $H(n) = h(n)$ for all $n \geqslant m$ if and only if $\deg F(T)$ $(= \deg f(T) - d)$ $< m$. Therefore, if $M$ is a finitely generated graded module over a homogeneous $k$-algebra $A$, then for an integer $m$, we have $H(M, n) = h(M, n)$ for all $n \geqslant m$ if and only if $\deg F(M, T) < m$.

(2) Suppose that $M$ is Cohen-Macaulay ($A$ is not necessarily homogeneous). Then we have $a(M) = \deg F(M, T)$.

(3) For a graded ring $A$, an integer $s > 0$ is called a <u>period</u> for $A$ if the s-ple Veronesean subring $A^{(s)} = \bigoplus_{n \geq 0} A_{ns}$ is homogeneous. For a polarized variety $(X, D)$, an integer $s$ is called a period for $D$ if $s$ is a period for $A(X, D)$, i.e., $sD$ is normally generated in the terminology of Mumford.

<u>Problem</u>. Find (good) periods for a given graded ring $A$ or a given polarized variety $(X, D)$.

<u>Examples</u>. (1) Every graded ring has a period. For example, if $A = k[x_1,\dots,x_v]$, where $x_i$ are homogeneous, then $s = (v - 1)\mathrm{l.c.m.}\{ \deg(x_i); 1 \leq i \leq v \}$ is a period for $A$.

(2) (Mumford) If $(X, D)$ is a polarized variety with $\mathrm{Bs}|D| = \emptyset$, then every $n \geq \mathrm{reg}\, A(X, D)$ is a period for $D$.

(3) (Mumford-Fujita) If $(X, D)$ is a polarized curve, then every $n \geq (2p_a(X) + 1)/\deg D$ is a period for $D$.

(4) (Bombieri) Let $X$ be an algebraic surface over $\mathbb{C}$ such that $K$ is ample. If $(K^2) \geq 5$ and $p_g \geq 3$, then every $n \geq 6$ is a period for $A(X, K)$.

Let $M$ be a finitely generated graded $A$-module and let $s$ be a period for $A$. Then there exist $h_i \in \mathbb{Q}[T]$, $0 \leq i < s$ such that $H(M, n) = h_i(n)$ for any sufficiently large $n$ such that $n \equiv i \bmod s$. Moreover, we have $\dim(M) = \max\{ \deg h_i ; 0 \leq i < s \} + 1$ (Shah, Shukla). If $M$ is Cohen-Macaulay, then for an integer $m$, we have $a(M) < ms$ if and only if $H(M, ns + i) = h_i(n)$ for every $n \geq m$ and $0 \leq i < s$. When $A$ is <u>almost homogeneous</u>, i.e., $A^{(s)}$ is homogeneous for any sufficiently large $s$, then all polynomials $h_i$ $(0 \leq i < s)$ are equal to some polynomial $h \in \mathbb{Q}[T]$ (we call this polynomial the Hilbert polynomial of $M$), and we have $a(M) < ms$ if and only if $H(M, n) = h(n)$ for every $n \geqslant ms$.

Now we evaluate the regularity of homogeneous algebras. We give upper bounds and lower bounds for regularity, and in the cases for which the given bounds are attained, we get the notions of stretched Cohen-Macaulay algebras and extremal Cohen-Macaulay algebras etc. First we consider upper bounds. Let $A$ be a homogeneous $k$-algebra.

(1) If $A$ is Buchsbaum (resp. Cohen-Macaulay), then

$$\mathrm{reg}(A) \leqslant e(A) + \dim(A) - \mathrm{emb}(A) + I(A)$$

(resp. $\mathrm{reg}(A) \leqslant e(A) + \dim(A) - \mathrm{emb}(A)$).

If the equality holds, then we say $A$ is a <u>stretched Buchsbaum algebra</u> (resp. a <u>stretched Cohen-Macaulay algebra</u>). For example, if $A$ is a Buchsbaum algebra with $\mathrm{emb}(A) = e(A) + \dim(A) + I(A) - 1$ and is not regular or with $\mathrm{emb}(A) = e(A) + \dim(A) + I(A) - 2$, then $A$ is a stretched Buchsbaum algebra. We can determine the structure of the artinian stretched Cohen-Macaulay algebras, but we don't write down the equations defining them here.

(2) Suppose that $A$ is Gorenstein. Then

$\mathrm{reg}(A) = 1$ if and only if $A$ is a quadric hypersurface,

$\mathrm{reg}(A) = 2$ if and only if $\mathrm{emb}(A) = e(A) + \dim(A) - 2$,

$\mathrm{reg}(A) = 3$ if and only if $\mathrm{emb}(A) = e(A)/2 + \dim(A) - 1$,

and if $\mathrm{reg}(A) \geqslant 3$ and $A$ is not a hypersurface, then we have

$$\mathrm{reg}(A) \leqslant e(A)/2 + \dim(A) - \mathrm{emb}(A) + 2.$$

(This bound is not the best possible one. It seems difficult to give the precise upper bound for Gorenstein algebras.)

(3) (Castelnuovo's bound) If $A$ is a normal Cohen-Macaulay homogeneous algebra over an algebraically closed field, then

$$\mathrm{reg}(A) \leqslant \min\{ k;\ k \geqslant (e(A) - 1)/(\mathrm{emb}(A) - \dim(A)) \}.$$

Next we consider lower bounds for regularity in terms of the degree of defining equations of homogeneous algebras. For a homogeneous algebra $A = S/I$, where $S = k[X_1,\dots,X_v]$, $v = \mathrm{emb}(A)$, we put $i(A) = \min\{ t;\ I_t \neq 0 \}$ (the <u>initial degree</u> of $A$), i.e., $i(A)$ is the minimal degree of defining equations of $A$. Then we have $\mathrm{reg}(A) \geqslant i(A) - 1$ and the case for which the equality holds is characterized by the structure of minimal free resolution of $A$, i.e., $A$ has a linear resolution in the sense of Goto. If $A$ is Gorenstein and is not a hypersurface, then we have $\mathrm{reg}(A) \geqslant 2(i(A) - 1)$, and a similar structure theorem for minimal free resolutions is known when the equality $\mathrm{reg}(A) = 2(i(A) - 1)$ holds. According to Schenzel, we call a Cohen-Macaulay algebra with $\mathrm{reg}(A) = i(A) - 1$ (resp. a Gorenstein algebra with $\mathrm{reg}(A) = 2(i(A) - 1)$) an <u>extremal Cohen-Macaulay algebra</u> (resp. an <u>extremal Gorenstein algebra</u>). The Betti numbers in minimal free resolutions of extremal algebras can be completely determined.

Now we introduce the notion of generic Cohen-Macaulay algebras. Let $k$ be an algebraically closed field and let $P_1,\dots,P_s$ be a finite set of points in $\mathbb{P}^r(k)$. For an integer $n > 0$, consider the n-ple Veronesean embedding $v_n\colon \mathbb{P}^r \longrightarrow \mathbb{P}^{N-1}$, $N =$

$\binom{n+r}{r}$, $v((x_0: \dots :x_r)) = (x_0^{i_0} \cdots x_r^{i_r})$, $i_0 + \dots + i_r = n$, and denote by $A_{n,s}$ the N×s-matrix obtained by arranging the coordinate vectors of $v_n(P_1),\dots,v_n(P_s)$. Then we say $P_1,\dots,P_s$ are <u>in generic position</u> in $\mathbb{P}^r$ if rank $A_{n,s} = \min\{ N, s \}$ for all $n > 0$ (Orecchia, 1981).

<u>Examples</u>. (1) The following sets of points are in generic position: Any two points in $\mathbb{P}^r$. Any finite set of points in $\mathbb{P}^1$. Any four points in $\mathbb{P}^2$ which are not collinear.

(2) $r + 1$ points in $\mathbb{P}^r$ ($r \geqslant 2$) are in generic position if and only if they are not on a hyperplane.

(3) 6 points in $\mathbb{P}^2$ are in generic position if and only if they are not on a conic. 10 points in $\mathbb{P}^3$ are in generic position if and only if they are not on a quadric.

(Caution: There is another important notion of a finite set of points which are <u>in general position</u> in $\mathbb{P}^r$. We don't discuss this notion here.)

Let $P_1,\dots,P_s$ be a finite set of points in $\mathbb{P}^r$ and let A be the homogeneous coordinate ring of the set $\{P_1,\dots,P_s\}$. (A is a one-dimensional reduced homogeneous k-algebra.) Then $P_1,\dots,P_s$ are in generic position in $\mathbb{P}^r$ if and only if $H(A, n) = \min\{ s, \binom{n+r}{n} \}$ for all $n \geqslant 0$. Generalizing this condition, we get the notion of generic Cohen-Macaulay algebras:

<u>Theorem</u>. For a Cohen-Macaulay homogeneous algebra A, the following conditions are equivalent:

(1) $\mathrm{reg}(A) = i(A) - 1$ or $i(A)$.

(2) $\mathrm{reg}(A) = \min\{\, n \in \mathbf{Z};\ e \leqslant \binom{v - d + n}{n} \,\}$ and

$i(A) = \min\{\, n \in \mathbf{Z};\ e < \binom{v - d + n}{n} \,\}$.

(3) $e = \binom{v - d + m}{m}$ or $\binom{v - d + m - 1}{m - 1} < e < \binom{v - d + m}{m}$.

(Here $e = e(A)$, $v = \mathrm{emb}(A)$, $d = \dim(A)$ and $m = \mathrm{reg}(A)$.)

(4) If $A$ is artinian,

$$A = \frac{k[X_1,\ldots,X_v]}{(X_1,\ldots,X_v)^{m+1}} \quad \text{or} \quad \frac{k[X_1,\ldots,X_v]}{((X_1,\ldots,X_v)^{m+1},\ V)},$$

where $V$ is a subspace such that $0 \subsetneq V \subsetneq k[X_1,\ldots,X_v]_m$ .

(5) If $\dim(A) = 1$,

$H(A, n) = \min\{\, e,\ \binom{v + n - 1}{n} \,\}$ for all $n \in \mathbf{Z}$.

If these conditions are satisfied, we say $A$ is a <u>generic Cohen-Macaulay algebra</u>. Moreover, if $\mathrm{reg}(A) = i(A) - 1$ (resp. $i(A)$), then we say $A$ is a generic Cohen-Macaulay algebra of type I (resp. type II). As the condition (5) shows, this notion generalizes the notion of a finite set of points which are in generic position.

If $A$ is a generic Cohen-Macaulay algebra of type I, i.e., an extremal Cohen-Macaulay algebra, then we have

$$F(A, T) = \frac{\sum_{n=0}^{t-1} \binom{r-1+n}{n} T^n}{(1 - T)^d},$$

$b_i(A) = \binom{t-1+r}{t-1+i}\binom{t+i-2}{i-1}$, $1 \leqslant i \leqslant r$ (the Betti numbers in the minimal free resolution of $A$),

$e(A) = \binom{r+t-1}{r}$, and

$r(A) = \binom{r+t-2}{r-1}$ (the Cohen-Macaulay type of $A$),

where $t = i(A)$, $r = v - d$.

If $A$ is a generic Cohen-Macaulay algebra of type II, then

$$F(A, T) = \frac{\sum_{n=0}^{t-1} \binom{r+n-1}{n} T^n + \left(e - \binom{r+t-1}{r}\right) T^t}{(1 - T)^d}, \text{ and}$$

$r(A) = e - \binom{r+t-1}{r}$, where $e = e(A)$.

Note that if $A$ is artinian, we have $r(A) = H(A, m)$ in both cases, and this fact is useful to construct Gorenstein homogeneous algebras from these algebras. Namely, let $A$ be an artinian homogeneous $k$-algebra with $reg(A) = m$, and put $E = \underline{E}_A(k)(m+1)$. Then $B = A \times E$ is an artinian Gorenstein graded $k$-algebra and $B$ is homogeneous if and only if $r(A) = H(A, m)$ (Stanley).

Examples. Every hypersurface is a generic Cohen-Macaulay algebra of type I. If $A$ is a Cohen-Macaulay algebra with $emb(A) = e(A) + dim(A) - 2$, then $A$ is a generic Cohen-Macaulay algebra of type II. Conversely, if $A$ is Gorenstein, $A$ is a generic Cohen-Macaulay algebra if and only if $A$ is a hypersurface or $emb(A) = e(A) + dim(A) - 2$.

For more details and references, see Ooishi [3].

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(December, 1982)

# "Generalized analytic independence について。

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§0 G. Valla と W. Bruns の論文を中心に，generalized analytic independence について述べる。
以下，環は可換なネーター環で，単位元を持つとする。
またイデアルは，(1) と異なるとする。
まず，generalized analytic independence の定義は，

Definition 0,1 (G. Valla (4)).

$a_1, \cdots, a_n$ は，環 $A$ の元，$\mathfrak{a}$ は，$A$ のイデアルとする。
$a_1, \cdots, a_n$ が $\mathfrak{a}$-independent とは，$A[X_1, \cdots, X_n]$ の斉次式 $F(X_1, \dots, X_n)$ が，$F(a_1, \cdots, a_n) = 0$ のとき，$F$ の係数は，$\mathfrak{a}$ の元であるときをいう。

イデアル $\mathfrak{a}$ についての Rees algebra を

$$R_A(\mathfrak{a}) = \bigoplus_{i=0}^{\infty} \mathfrak{a}^i \qquad (\mathfrak{a}^0 = A) \text{ とする。}$$

Proposition 0, 2.

$\mathfrak{a} = (a_1, \dots, a_n) \subseteq \mathfrak{b}$ は，それぞれ環 $A$ のイデアルとする。このとき，次は同値となる。

(1) $a_1, \cdots, a_n$ は，$\mathfrak{b}$-independent.

(2) 自然な写像で $(A/\mathfrak{b})[X_1, \dots, X_n] \simeq R_A(\mathfrak{a}) \otimes_A A/\mathfrak{b}$.

次に，ここでの主題となる，$\sup \mathfrak{a}$ を導入する。

Definition.

$\mathfrak{a}$ は，環 $A$ のイデアルとする。

$\sup \mathfrak{a} = \sup\{n \mid a_1, \cdots, a_n \in \mathfrak{a} : \mathfrak{a}\text{-independent}\}$
とおく。

Remark 0.3. (4) より任意のイデアル $\mathfrak{a}$ について、
$\operatorname{grade} \mathfrak{a} \leq \sup \mathfrak{a} \leq \operatorname{ht} \mathfrak{a}$ が成り立つ。
また、特に、$\mathfrak{a}$ がラディカルイデアルのときは、
$\sup \mathfrak{a} = \operatorname{ht} \mathfrak{a}$ となる。

§1. ここでは、G, Valla (5) の結果を中心に述べる。

Theorem 1.1.
$\mathfrak{a} = (a_1, \ldots, a_n)$ は、環 A のイデアル、
$\mathfrak{p}$ は、$\mathfrak{a}$ 上の minimal prime ideal とする。
このとき、$\operatorname{ht} \mathfrak{p} = n$ と $a_1, \cdots, a_n$ は、$\mathfrak{p}$-independent
とは、同値である。

Remark 1.2.
Th. 1.1 と Prop. 0.2 より、$\mathfrak{a} = (a_1, \cdots, a_n)$, $\mathfrak{p}$ は、$\mathfrak{a}$ 上
minimal prime ideal とすると、$\operatorname{ht} \mathfrak{p} = n$ と、
$(A/\mathfrak{p})[X_1, \cdots, X_n] \simeq \bigoplus_{\nu=0} \mathfrak{a}^\nu/\mathfrak{p}\mathfrak{a}^\nu$ (自然な写像で)
は、同値となる。

Remark 1.3.
イデアル $\mathfrak{a} = (a_1, \cdots, a_n) \subseteq$ 環 A, $n \geq 2$ とする。
吉田憲一さんの示唆より、次の short exact sequence
を考える、

$$0 \longrightarrow I \longrightarrow A[X_1, \cdots, X_n] \longrightarrow \bigoplus_{\nu=0}^{\infty} \mathfrak{a}^\nu/\mathfrak{a}^{\nu+1} \longrightarrow 0$$

(写像は、自然なものとする。)

C(I) は、I の元の係数で生成される A のイデアルとする。
このとき、$a_1, \cdots, a_n$ は、C(I)-independent となっている。

Remark 0,3 より $\mathrm{ht}\, C(I) \geq n$ (ただし $C(I)=(1)$ のとき $\mathrm{ht}\, C(I) = \infty$.). もし $\mathrm{ht}\, C(I) > n$ なら, Th. 1.1 より $\mathfrak{a}$ の minimal prime $\mathfrak{p}$ は, $\mathrm{ht}\,\mathfrak{p} < n$ となるのがわかる。

Corollary 1.4.

$\mathfrak{a} = (a_1, \cdots, a_n)$ は, 環 A のイデアルとする。
このとき, $\mathrm{ht}\,\mathfrak{a} = n$ と $a_1, \cdots, a_n$ は, $\sqrt{\mathfrak{a}}$ - independent とは, 同値である。

Proposition 1.5.

環 A について, 次は, 同値である。

(1) A は $S_1$ (i.e, (0) は, unmixed である。)。

(2) 任意の単項イデアル $(a)$ について,
$\sup(a) = \mathrm{ht}(a)$ 。

Theorem 1.6.

$(A, \mathfrak{m})$ は, 局所環 とする。
もし, $\mathfrak{p} \in \mathrm{Ass}(A)$ で, $\dim A/\mathfrak{p} \leq 1$ となるものが, 存在すれば, 任意のイデアル $\mathfrak{a}$ について,
$\sup \mathfrak{a}^n \leq 1$ for some $n \in \mathbb{N}$ 。

N. V. Trung (3) において, 次の結果が得られている。
$(A, \mathfrak{m})$ は, 局所環, 任意の $\mathfrak{p} \in \mathrm{Ass}(A)$ について,
$\dim A/\mathfrak{p} \geq \sup \mathfrak{m}^n$ for $n >> 0$ 。

§2, このセクションは, W. Bruns (2) を中心に述べる。
Remark 0,3 より 環 A は, Cohen - Macaulay であることと, 任意のイデアル $\mathfrak{a}$ について $\sup \mathfrak{a} = \mathrm{grade}\,\mathfrak{a}$ とは, 同値である。

W. Bruns は, 任意のイデアル $\mathfrak{a}$ について $\mathrm{ht}\,\mathfrak{a} = \sup \mathfrak{a}$ となる環を特徴づけた。

また, N, V, Trung (11) も別な方法によって同様な結果を得ている。

Definition 2.1.

$\mathfrak{a}$ は, 環 A のイデアルとする。

$\sup^\infty \mathfrak{a} = \inf\{\sup \mathfrak{a}^n : n \geq 1\}$ とおく。

(6) において $\mathrm{Ass}_A(A/\mathfrak{a}^n A)$ は, 十分大きい $n$ について一定な集合となることが得られている。

それを $\mathrm{Ass}^\infty \mathfrak{a}$ とおく。

Proposition 2.2.

A は, 環, $\mathfrak{a}$ は, A のイデアル, $S$ は, flat A-algebra とする。このとき,

$\sup^\infty \mathfrak{a} \leq ht(\mathfrak{a}S + \mathfrak{q})/\mathfrak{q}$ for $^\forall \mathfrak{q} \in \mathrm{Ass}(S)$

が成り立つ。

$\mathfrak{p}$ は, 環 A の素イデアルとする。$A_\mathfrak{p}$ の $\mathfrak{p}A_\mathfrak{p}$-adic completion を $\widehat{A_\mathfrak{p}}$ とおく。

Theorem 2.3.

$\mathfrak{a}$ は, 環 A のイデアルとする。このとき, 次が成り立つ,

$\sup \mathfrak{a} \geq \min\{ht(\mathfrak{a}\widehat{A_\mathfrak{p}} + \mathfrak{q})/\mathfrak{q} : \mathfrak{p} \in \mathrm{Ass}\, A/\mathfrak{a}, \mathfrak{q} \in \mathrm{Ass}\widehat{A_\mathfrak{p}}\}$

$\sup^\infty \mathfrak{a} = \min\{ht(\mathfrak{a}\widehat{A_\mathfrak{p}} + \mathfrak{q})/\mathfrak{q} ; \mathfrak{p} \in \mathrm{Ass}^\infty \mathfrak{a}, \mathfrak{q} \in \mathrm{Ass}\widehat{A_\mathfrak{p}}\}$。

この Th, 2, 3 と Prop. 2, 2 より, 次の結果が得られる。

Corollary 2.4

A は, 環とする。次は, 同値

(1) 任意のイデアル $\mathfrak{a}$ について,

$\sup \mathfrak{a} = ht\, \mathfrak{a}$ ,

(2) A は，locally unmixed (i.e $\dim \widehat{A_{\mathfrak{p}}}/\mathfrak{q} = ht\, \mathfrak{p}$ for ${}^{\forall}\mathfrak{p} \in Spec(A)$, $\mathfrak{q} \in Ass\, \widehat{A_{\mathfrak{p}}}$).

また，$\mathfrak{a}$ を環 A のイデアルとし，$\mathfrak{a}_a$ を A の中での $\mathfrak{a}$ の整閉包とおく。

L, J, Ratliff (10) は，次の結果を得ている。

Theorem. A は，環とする。次は，同値

(1) 任意のイデアル $\mathfrak{a}$ について，

$$\sup \mathfrak{a}_a = ht\, \mathfrak{a}.$$

(2) A は，locally quasi-unmixed.

環 A のイデアル $\mathfrak{a}$ について，

$$V(\mathfrak{a}) = \{ \mathfrak{p} \in Spec(A) \ ; \ \mathfrak{p} \supseteq \mathfrak{a} \}$$

$$P_i(\mathfrak{a}) = \bigcap_{\mathfrak{p} \in Ass(A)\ s.t.\ \dim A/\mathfrak{p} \leqq i} \mathfrak{p}$$

$$U_i(\mathfrak{a}) = \bigcup_{n=1}^{\infty} (0 \ ; \ P_i(\mathfrak{a})^n \cap \mathfrak{a}^n)$$

$$\left( = \bigcap \{ \mathfrak{q} \mid \mathfrak{q} \text{ は primary components of } (0) \text{ で } \mathfrak{p} = \sqrt{\mathfrak{q}} \text{ は，} \mathfrak{p} \notin V(\mathfrak{a}) \text{ かつ } \dim A/\mathfrak{p} > i \} \right)$$

とおく。ただし，$\phi = \{ \mathfrak{p} \in Ass\, A \mid \mathfrak{p} \notin V(\mathfrak{a}),\ \dim A/\mathfrak{p} > i \}$ のとき，$U_i(\mathfrak{a}) = A$ とする。

N, V, Tsung (11) は，次の結果を得ている。

Theorem. A は，環とし，$\mathfrak{a}$ は，イデアルとする。

$$\sup \mathfrak{a} = \inf \{ i \in \mathbb{N} \ ; \ U_i(\mathfrak{a}\widehat{A_{\mathfrak{p}}}) \not\subseteq \mathfrak{a}\widehat{A_{\mathfrak{p}}} \text{ for } \mathfrak{p} \in Ass\, A/\mathfrak{a} \}.$$

そして，この結果より，N, V, Tsung (11) は，Cor. 2.4 とその他を得ている。

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*CANONICAL DUALITY FOR BUCHSBAUM MODULES*
*-An application of Goto's lemma on Buchsbaum modules*

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Let $(A,\mathfrak{m},k)$ be a local ring and M a finitely generated A-module of dimension d.

DEFINITION. A finitely generated A-module K is called the canonical module of M (denoted by $K_M$) if the completion $\hat{K}$ is isomorphic to $\mathrm{Hom}_A(H^d_{\mathfrak{m}}(M), E_A(k))$.

In his talk at the first of this series of symposium on Commutative Algebra in 1978, the author mentioned the following problem:If M is a Buchsbaum module, then is $K_M$ also a Buchsbaum module?

For lower dimensional cases, it is easy to see the validity: indeed, if $d \leq 2$ $K_M$ is always a Cohen-Macaulay;if d=3, since $\mathrm{depth}_A K_M \geq 2$, it suffices to show that $\mathfrak{m}H^2_{\mathfrak{m}}(K_M) = 0$ and this follows from the following general lemma.

LEMMA. [Proposition (5.1),2] Let M be a finitely generated A-module with finite local cohomology, i.e., $H^i_{\mathfrak{m}}(M)$ is of finite length for all $i \neq d = \dim M$. Then we have:

(i) there exists an exact sequence

$$0 \longrightarrow D^0D^0(M) \rightarrow \hat{M} \longrightarrow D^dD^d(M) \rightarrow D^0D^1(M) \longrightarrow 0;$$

(ii) $D^iD^d(M) \cong D^0D^{d-i+1}(M)$, for i=2,...,d;

(iii) $D^1D^d(M) = D^0D^d(M) = 0$,

where $D^i(*) := \mathrm{Hom}_A(H^i(*), E_A(k))$.

As to the general cases, P. Schenzel gave the affirmative answer using the characterization of Buchsbaum modules in terms of dualizing complex.

In the talk, the author gave quite an elementary proof, making best use of a lemma on Buchsbaum rings given by S. Goto.

LEMMA. ([Goto]). Let $M$ be a Buchsbaum module of dimension $d$ and $a_1, \ldots, a_r$ be a subsystem of parameters for $M$ and $n$ be an integer $\geq 2$. Then we have

$$U_M(a_1^n, \ldots, a_r^n) = \sum_{I \subseteq \{1,\ldots,r\}} a_I^{n-1} U_M(a_i;\ i \in I),$$

where $U_M(\mathfrak{a})$ denotes the unmixed component of a primary decomposition of $\mathfrak{a}M$ in $M$, $a_I = \prod_{i \in I} a_i$ and $a_\emptyset = 1$.

We now give a brief sketch of the proof of the following:

THEOREM. *The canonical module* $K_M$ *of a Buchsbaum module* $M$ *is also a Buchsbaum module.*

Proof. We may assume that $A = \hat{A}$. Induction on $d = \dim M$. Let $d \geq 3$ and we may still more assume that depth $M > 0$ since $H^d_{\mathfrak{m}}(M) \cong H^d_{\mathfrak{m}}(M/H^0_{\mathfrak{m}}(M))$. Let $a_1, \ldots, a_d$ be any s.o.p. for $K_M$. We have an exact sequence

$$0 \rightarrow K_M/aK_M \longrightarrow K_{(M/aM)} \xrightarrow{\pi} D^{d-1}(M) \longrightarrow 0.$$

Consider the long exact sequence of Koszul homology modules with respect to $\underline{a'} = \{a_2, \ldots, a_d\}$,

$$H_1(\underline{a'}; K_{M'}) \longrightarrow H_1(\underline{a'}; V) \longrightarrow K_M/(\underline{a})K_M \longrightarrow K_{M'}/(\underline{a'})K_{M'} \rightarrow V \longrightarrow 0$$

with $M' = M/aM$ and $V = D^{d-1}(M)$. If we have that the mapping

$$H_1(\underline{a'}; \pi) : H_1(\underline{a'}; K_{M'}) \longrightarrow H_1(\underline{a'}; V)$$

is a zero map, then the equality

$$l_A(K_M/(\underline{a})K_M) = l(K_{M'}/(\underline{a'})K_{M'}) + (d-1)(\dim_k V)$$

holds. On the other hand, we have

$$e_0(\underline{a}; K_M) = e_0(\underline{a'}; K_M/aK_M) = e_0(\underline{a'}; K_{M'})$$

hence, by the induction assumption, we can conclude that the difference

$$l_A(K_M/(\underline{a})K_M) - e_0(\underline{a}; K_M)$$

does not depend on the choice of the s.o.p. $\underline{a} = \{a_1, \ldots, a_d\}$ for $M$; which is the definition of the Buchsbaumness of $K_M$.

Consider the direct system $\left\{H^i(a_1^n,\dots,a_d^n;M),\ \phi^i_{n,n+1}\right\}$ with the limit $H^i_{\mathfrak{m}}(M)$, where

$$H^i(\underline{a};M) := H^i(\mathrm{Hom}_A(K.(\underline{a};A),M)) \cong H_{d-i}(\underline{a};M).$$

Let $\underline{a}^n := \{a_1^n,\dots a_d^n\}$ and $\underline{a'}^n := \{a_2^n,\dots,a_d^n\}$. There induced a commutative diagram from the exact sequence

$$0 \longrightarrow M \longrightarrow M \xrightarrow{\pi} M/aM \longrightarrow 0.$$

$$\begin{array}{ccccc}
\longrightarrow & H^{d-1}(\underline{a}^n;M) & \longrightarrow & H^{d-1}(\underline{a}^n;M') & \longrightarrow \\
 & \| & & \| & \\
 & H_1(\underline{a}^n;M) & \xrightarrow{H_1(\underline{a}^n;\pi)} & H_0(\underline{a'}^n;M')\oplus H_1(\underline{a'}^n;M') & \\
 & \downarrow \phi^n_M & & \downarrow \phi^n_{M'} & \\
\longrightarrow & H^{d-1}_{\mathfrak{m}}(M) & \xrightarrow{\lambda} & H^{d-1}_{\mathfrak{m}}(M') & \text{—}
\end{array}$$

Let $(f_2,\dots,f_d)\in Z_1(a_2,\dots,a_d;K_{M'})$, namely, for $j=2,\dots,d$, $f_j\in K_{M'} = \mathrm{Hom}_A(H^{d-1}_{\mathfrak{m}}(M/aM),E_A(k))$ and

$$\textstyle\sum_{j=2}^{d} a_j f_j = 0 \qquad (\#).$$

It suffices to show that for any $j=2,\dots,d$, $f_j\circ\lambda = 0$. Let $z\in H^{d-1}_{\mathfrak{m}}(M)$. Then there exists $(u,v)\in Z_1(\underline{a}^n;M)\subset K_0(\underline{a'}^n;M)\oplus K_1(\underline{a'}^n;M)$ such that $\phi^n_M(|u,v|) = z$. Note at first that the cycle condition leads that $a_1^n\, u \in (a_2^n,\dots,a_d^n)M$; hence

$$u \in U_M(a_2^n,\dots,a_d^n) \qquad (\#\#).$$

We claim that $f_j\circ\phi^n_{M'}\circ H_1(\underline{a}^n;\pi)(|u,v|) = 0$. Here we use the notation $|c|$ for the homology class of a cycle c. Let $|\bar{u},\bar{v}| = (|\bar{u}|,|\bar{v}|) := H_1(\underline{a}^n;\pi)(|u,v|)\in H_0(\underline{a'}^n;M')\oplus H_1(\underline{a'}^n;M')$. It is not so hard to see that

$$\phi^{n,n+1}_{M'}(|\bar{u},\bar{v}|) = (|(a_2\dots a_d)\tilde{u}|,|\tilde{0}|)\in H_0(\underline{a'}^{n+1};M')\oplus H_1(\underline{a'}^{n+1};M').$$

So it suffices to show that $f_j\circ\phi^n_{M'}(|\bar{u}|,|\bar{v}|) = f_j\circ\phi^{n+1}_{M'}(|a_2\dots a_d\tilde{u}|,|\tilde{0}|) = 0$, with $|\tilde{u}|\in H_0(a_2^{n+1},\dots a_d^{n+1};M/aM) = M/(a,a_2^{n+1},\dots,a_d^{n+1})M$.

By Goto's lemma, from (##) we have the following expression:

$$u = \textstyle\sum_{I\subsetneq\{2,\dots,d\}} a_I^{n-1} u_I$$

with $u_I\in U_M(a_i;i\in I)$. We must show that for any subset I of $\{a_2,\dots,a_d\}$ $f_j\circ\phi^{n+1}_{M'}(|(a_2\dots a_d)a_I^{n-1}\tilde{u}_I|) = 0$.

If $I \neq \{2,\ldots,d\}$, there exists $j \notin I$, hence

$$(a_2\cdots a_d)a_I^{n-1}u_I = (\prod_{\substack{1\neq j \\ 1\notin I}} a_1) a_I^n a_j u_I \in (a_i^{n+1}; i\in I)M \subset (a_2^{n+1},\ldots,a_d^{n+1})M.$$

This means that

$$|(a_2\cdots a_d)a_I^{n-1}\tilde{u}_I| = 0 \quad \text{in} \quad M/(a,a_2^{n+1},\ldots,a_d^{n+1})M.$$

For $I=\{2,\ldots,d\}$,

$$f_j \circ \phi_{M'}^{n+1}(|(a_2\cdots a_d)^{n-1}a_I\tilde{u}_I|) = f_j\circ\phi_{M'}^{n+1}(|(a_2\cdots a_d)^n\tilde{u}_I|)$$

$$= a_j f_j \circ \phi_{M'}^{n+1}(|(a_2\cdots \hat{a}_j \cdots a_d)^n a_j^{n-1}\tilde{u}_I|),$$

by the cycle condition (#),

$$= -\sum_{i\neq j} a_i f_i \circ \phi_{M'}^{n+1}(|(a_2\cdots \hat{a}_j \cdots a_d)^n a_j^{n-1}\tilde{u}_I|)$$

$$= -\sum_{i\neq j} f_i \circ \phi_{M'}^{n+1}(|(a_2\cdots \hat{a}_i \hat{a}_j \cdots a_d)^n a_j^{n-1} a_i^{n+1}\tilde{u}_I|).$$

Since

$$|a_i^{n+1}\tilde{u}_I| = 0 \quad \text{in } M/(a_2^{n+1},\ldots,a_d^{n+1})M',$$

we conclude in this case also that

$$f_j\circ\phi_{M'}^{n+1}(|(a_2\cdots a_d)^{n-1}a_I\tilde{u}_I|) = 0$$

as required.

Q.E.D.

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(Dec. 1982)

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# When Is The Product Of Modules Flat Over The Product Ring?

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This is an exposition of some of the ideas in my paper: Preservation of Flatness for the Product of Modules over the Product of Rings, Journal of Algebra, 74 (1982) 159-205; which I will refer to as [PF].

Suppose $\{R^\lambda\}_{\mathcal{L}}$ is a collection of rings and for each $\lambda \in \mathcal{L}$ $M^\lambda$ is a right $R^\lambda$-module. Then $\underline{M} = \Pi M^\lambda$ is a right $\underline{R} = \Pi R^\lambda$-module. In general if each $M^\lambda$ is a flat $R^\lambda$-module then $\underline{M}$ is NOT a flat $\underline{R}$-module. We shall present necessary and sufficient conditions on $\{R^\lambda\}_{\mathcal{L}}$ such that $\underline{M}$ is a flat $\underline{R}$-module when each $M^\lambda$ is a flat $R^\lambda$-module. Here is a sample result:

THEOREM: $\underline{a}$. Suppose there is a cofinite subset $\mathcal{L}' \subset \mathcal{L}$ (i.e. $\#(\mathcal{L} \smallsetminus \mathcal{L}') < \infty$) and for each $\lambda \in \mathcal{L}'$ $R^\lambda$ is commutative and satisfies one of the next three conditions (which condition may vary with $\lambda$):

i. $R^\lambda$ is a principal ideal domain (PID),

ii. $R^\lambda$ is a polynomial ring in one variable over a PID,

iii. $R^\lambda$ is a local ring of global dimension two or less,

then $\underline{M}$ is a flat $\underline{R}$-module when each $M^\lambda$ is a flat $R^\lambda$-module.

$\underline{b}$. If for an infinite number of $\lambda$ $R^\lambda$ is a polynomial ring in 3 or more variables over an algebraically closed field then there exists $\{M^\lambda\}_{\mathcal{L}}$ where each $M^\lambda$ is a finite rank free $R^\lambda$-module and $\underline{M}$ is NOT a flat $\underline{R}$-module.

Suppose $\sum_1^t n_j r_j = 0$ is a length $t$ relation in a right

R-module N. An exposé of the relation is a pair of subsets $\{b_i\}_1^T \subset N$ and $\{\mu_{ij}\}_{i=1,\dots T}^{j=1,\dots t} \subset R$ where

$$n_j = \sum_{i=1}^{T} b_i \mu_{ij} \qquad j = 1,\dots t$$

$$0 = \sum_{j=1}^{t} \mu_{ij} r_j \qquad i = 1,\dots T$$

$T$ is the size of the exposé.

It is well known that a module is flat if and only if all relations in the module have exposés. The possible lengthening from $t$ to $T$ is the obstruction to flatness of $\underline{M}$. This is true because a length $t$ relation in $\underline{M}$ is equivalent to having a length $t$ relation in $M^\lambda$ for each $\lambda$. If each of these had an exposé with a common bound $N$ on the size, the exposés could be put together to be the components of a size $N$ exposé of the original relation in $\underline{M}$.

By mapping free modules onto flat modules it can be seen that the lengthening from $t$ to $T$ is as bad as possible in free modules. Support controls lengthening as shown by the next theorem. But first:

DEFINITION: A left R-module $L$ is $T$ supported if for each finite subset $\mathcal{S} \subset L$ there is a submodule $S_{\mathcal{S}}$ generated by $T$ or fewer elements with $\mathcal{S} \subset S_{\mathcal{S}}$.

THEOREM: Suppose $\{r_j\}_1^t \subset R$, $F$ is a free left R-module with basis $\{x_j\}_1^t$ and $\phi: F \to R$ is a module map with $\phi(x_j) = r_j$ for $j = 1,\dots t$.

a. If $\operatorname{Ker} \phi$ is $T$ supported and $\Sigma_1^t n_j r_j = 0$ is a relation in a right R-module $N$ which has an exposé then the relation has a size $T$ exposé.

b. The following are equivalent:

i. $\operatorname{Ker} \phi$ is $T$ supported.

ii. All relations with coefficients $\{r_j\}_1^t$ in a free rank $T+1$ right R-module have size $T$ exposés.

DEFINITION: If F is a free rank $t$ left R-module and Ker $\phi$ is T supported for all R-module maps $\phi: F \to R$ then the left support presentation of $t$ elements of R is less than or equal to T. This will be abbreviated to: Sup $\text{Pres}_t\ R \leq T$.

The previous theorem gives:

THEOREM: Sup $\text{Pres}_t\ R \leq T$ if and only if all length $t$ relations in flat right R-modules have size T exposés.

Example: (Thanks W.V. Vascancelos.)

a. If R is a PID or polynomial ring in one variable over a PID then Sup $\text{Pres}_t\ R = t$ .

b. If R is a commutative local ring of global dimension two or less then Sup $\text{Pres}_t\ R = t$.

c. If R is a polynomial ring in three or more variables over an algebraically closed field then Sup $\text{Pres}_t\ R = \infty$ for $t \geq 3$ .

SUPPORT

Support behaves like "number of generators" with respect to exact sequences. (The notion of Measure Function in [PF, §5, p. 188] is the axiomatization of the similarity.) The idea of finite presentation may be defomed in terms of finite support in the same way as is usually done with finite number of generators. T generated modules are T supported. Every T supported module is the direct limit of its T generated submodules. The direct limit of T supported modules is again T supported. If $\gamma: R \to S$ is a map of commutative rings then S is a 1 supported R-module if and only if $\gamma$ induces a surjective map from a localization of R onto S. These and other elementary results about support are in [PF, §2, p175]. One other elementary result is that support controls the length of the summation needed to express elements in the tensor product.

Using support we can give necessary and sufficient conditions for the product of flat modules to remain flat over the product ring.

THEOREM: Suppose $\{R^{\lambda}\}_{\mathcal{L}}$ is a collection of rings. The following conditions are equivalent:

a. If $M^{\lambda}$ is a flat right $R^{\lambda}$-module for each $\lambda$ then $\underline{M}$ is a flat $\underline{R}$-module.

b. All products of finite rank free right $R^{\lambda}$-modules are flat $\underline{R}$-modules.

c. There is a product of finite rank free right $R^{\lambda}$-modules which is a flat $\underline{R}$-module.

d. For each $t \in \mathbb{N}$ there is a bound $T_t \in \mathbb{N}$ where $\text{Sup Pres}_t\ R^{\lambda} \leq T_t$ for almost all $\lambda$.

e. For each $t \in \mathbb{N}$ there is $T'_t \in \mathbb{N}$ and cofinite $\mathcal{L}_t \subset \mathcal{L}$ where $\text{Sup Pres}_t\ (\prod_{\mathcal{L}_t} R^{\lambda}) \leq T'_t$ .

From parts (d) and (e) of the above theorem you might guess that:

$$\text{Sup Pres}_t\ (\text{Product}) \leq T$$

$$\Longleftrightarrow$$

$$\text{Sup Pres}_t\ (\text{Each Factor}) \leq T$$

I do not know if this is true but it is true that $\text{Sup Pres}_t$ (Product) is finite if and only if the set $\{\text{Sup Pres}_t\ (\text{Each Factor})\}$ is bounded. I found it difficult to work with $\text{Sup Pres}_t$ of a product. To help study $\text{Sup Pres}_t$ of a product two related notions are useful.

DEFINITION: $\text{Sup Div}_t\ R \leq T$ if for all $t-1$ generated left ideals $I \subset R$ and all $r \in R$, $(I:r)$ is $T$ supported.

$((I:r) = \{s \in R \mid sr \in I\}.)$

DEFINITION: For $t,v \in \mathbb{N}$, $\text{Sup}\ {}_t\text{Int}_v\ R \leq T$ if for all $t$ generated left ideals $I \subset R$ and all $v$ generated left ideals $J \subset R$ the ideal $I \cap J$ is $T$ supported.

Sup Pres, Sup Div and Sup Int are monotonically increasing and are related by the:

COMPARISON THEOREM: a. The following conditions are equivalent:

i. $\text{Sup Pres}_t\ R < \infty$

ii. $\text{Sup Div}_t\ R < \infty$

iii. $\text{Sup Div}_1\ R < \infty$ and $\text{Sup}\ {}_{t-1}\text{Int}_1\ R < \infty$.

b. If the conditions in (a) hold then

i. $\text{Sup Div}_t\ R \leq \text{Sup Pres}_t\ R$
$\leq \sum_{i=1}^{t} \text{Sup Div}_i\ R \leq t(\text{Sup Div}_t\ R)$

ii. $\text{Sup Pres}_t\ R \leq t(\text{Sup Div}_1\ R) + \sum_{i=2}^{t} \text{Sup}\ {}_{i-1}\text{Int}_1\ R$
$\leq t(\text{Sup Div}_1\ R) + (t-1)(\text{Sup}\ {}_{t-1}\text{Int}_1\ R)$

iii. For $1 \leq s < t$ : $\text{Sup}\ {}_s\text{Int}_{t-s}\ R \leq \text{Sup Pres}_t\ R$

Sup Div behaves well with respect to product.

THEOREM: $\text{Sup Div}_t\ (\Pi\ R^\lambda) \leq T$ if and only if $\text{Sup Div}_t\ R^\lambda \leq T$ for each $\lambda$.

The ideas for Sup Pres, Sup Div and Sup Int are inspired by S.U. Chase's paper: Direct Products of Modules, Transactions American Mathematics Society, 97 (1960) 457-473; which I will refer to as [DP]. In [DP] Chase considers the number of generators rather than the support of the modules which occur in Sup Pres, Sup Div and Sup Int. Thus although Chase never formally defined them he was working with Gen Pres, Gen Div, and Gen Int. His techniques for working with and comparing Gen Pres, Gen Div and Gen Int are valid when "Gen" is replaced by "Sup" or any other measure function.

## MEASURE FUNCTIONS

DEFINITION: A measure function is a function $\mathfrak{m}$ from left R-modules to the set $\{0,1,2,\ldots\} \cup \{\infty\}$ where if $\{0\} \to L \to M \to N \to \{0\}$ is an exact sequence of left R-modules then

1. $\mathfrak{m}\, N \leq \mathfrak{m}\, M$
2. $\mathfrak{m}\, M \leq \mathfrak{m}\, L + \mathfrak{m}\, N$

It follows from the definition that $\mathfrak{m}\, M = \mathfrak{m}\, M'$ if $M \cong M'$. Minimal number of generators gives the measure function $\mathfrak{m}_1$ where

$$\mathfrak{m}_1\, M = \begin{cases} \infty & \text{if } M \text{ is not finitely generated, otherwise} \\ 0 & \text{if } M = \{0\}\text{ , otherwise} \\ \text{minimal number of generators of } M & \end{cases}$$

Minimal support gives the measure function $\mathfrak{m}_2$ which is defined similarly to $\mathfrak{m}_1$ but with generation replaced by support. For any measure function $\mathfrak{m}$ one can define $\mathfrak{m}$ Pres, $\mathfrak{m}$ Div and $\mathfrak{m}$ Int similarly to how we defined Sup Pres, Sup Div and Sup Int. For example $\mathfrak{m}\ {}_t\mathrm{Int}_v\ R \leq T$ if for all $t$ generated left ideals $I \subset R$ and all $v$ generated left ideals $J \subset R$ the ideal $I \cap J$ satisfies $\mathfrak{m}(I \cap J) \leq T$ .

The Comparison Theorem holds for general measure functions. Hence for each measure function $\mathfrak{m}$ we get three closely related types of $\mathfrak{m}$-dimension

$\mathfrak{m}$ Pres, $\mathfrak{m}$ Div, $\mathfrak{m}$ Int

One aspect of [DP] is the study and application of these dimensions for $\mathfrak{m}$ = Gen , and one aspect of [PF] is the study and application of these dimensions for $\mathfrak{m}$ = Sup . Chase's paper is a gold mine of ideas for further development as well as interesting and important in itself.

One last comment about measure functions. The usual definition of finite presentation is in terms of finite generation of certain modules. One can define $\mathcal{M}$ finite presentation in terms of $\mathcal{M}$ finiteness of the same modules. $\mathcal{M}$ finite presentation has the same properties as ordinary finite presentation and the usual proof with Schanuel's Lemma still applies.

## IDEAS FOR FURTHER DEVELOPMENT

How does flatness behave with respect to the inverse limit of modules over the inverse limit of rings? Perhaps the theory for inverse limits will be similar to the theory for products.

Another measure function besides Gen and Sup is given in [PF, p 190, (5.4)]. What are some other measure functions and their applications? For example for each $n \in \mathbb{N}$ is there a measure function $\mathcal{M}_n$ where $\mathcal{M}_n \operatorname{Pres}_t R$ is finite if $R$ is a polynomial ring over a field in $n$ or fewer variables but $\mathcal{M}_n \operatorname{Pres}_t R$ is infinite for large $t$ if $R$ is a polynomial ring in more than $n$ variables over an algebraically closed field?

On a conjecture of Davis and Geramita

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Abstract

I would like to talk about the following

Conjecture (Davis-Geramita [2, 3]). Let M be a maximal ideal in a polynomial ring $A = R[T_1, T_2, \ldots, T_n]$ $(n > 0)$ over a regular ring R. Then M can be generated by a regular sequence.

At the present time this conjecture remains open, though it is solved affirmatively in several cases. An expansion of Hilbert's Nullstellensatz is achieved if this is true. In my lecture I will give a historical note on the above conjecture together with remarks about some common reduction techniques among affirmative cases.

## §0 序

Rを可換なネーター環，IをRのidealとするとき $\mathrm{ht}(I)$ と $\nu(I)$ (= the minimal number of generators for I) の間には，$\nu(I) \geqq \mathrm{ht}(I)$ なる不等式がある (Krull's principal ideal theorem)。Rが体k上の多項式環 $k[T_1, \ldots, T_n]$ で Mをその極大idealとすると，$\nu(M) = \mathrm{ht}(M) = n$ であることが古典的に知られている (Hilbertの零点定理)。DavisとGeramitaは[2], [3]の中で，Hilbertの零点定理の拡張ともいえる次の問題を提起した。

Conjecture　R: regular ring. $A = R[T_1, \ldots, T_n]$ $(n > 0)$ : R上のn変数多項式環. $\Longrightarrow$ $\nu(M) = \mathrm{ht}(M)$ for every maximal ideal M of A

i.e M is generated by a regular sequence

この予想は未だ完全には解決されていないが，多くの場合は正しいことがわかっている。このノートでは，今までに得られてきた結果とそれに用いられる共通の手法について簡単な紹介を行う。

## §1 Davis and Geramita's theorems

ネーター環 $R$ に於いて、任意の極大 ideal $M$ について $\nu(M)=ht(M)$ が成り立つとき $R$ を strongly regular と呼ぼう。strongly regular ring は regular ring であることは容易にわかる。逆は正しくない。何故なら non factorial Dedekind domain は strongly regular でないからである。換約して言うと予想は「regular ring 上の多項式環は strongly regular であるか？」といい直すことができる

最初に次の定理に注目しよう。

定理 (S. Endô [1])　$R$: 1次元 semi local domain. $R$ の the normalization $\bar{R}$ は $R$ 上 finite とする。$A=R[T]$ を $R$ 上の一変数多項式環、このとき次は同値である。

(1)　$\nu(M)=\nu(MA_M)$ for all maximal ideals $M$

(2)　$R$ は seminormal

(3)　任意の有限生成 projective $A$-module は free である。

この定理により $R$ が1次元 semi local Dedekind domain のときは $A=R[T]$ は strongly regular であることがわかる。また $R$ の正則性を除くと予想は正しくないことがわかる。例えば $k[[t^2,t^3]]$ ($k$ は体) は seminormal でないから $k[[t^2,t^3]][T]$ の極大 ideal $M$ で $\nu(M)>\nu(MA_M)=ht(M)$ なるものが存在する。実際、簡単な計算により $M=(t^2T^4-1,\ t^3T^2-t^2)$ は極大 ideal で $\nu(M)=2$, $ht\,M=1$ である。$R$ が正則のとき、従って $A$ が正則のときは $\nu(MA_M)=ht(M)$ であるから、一般のネーター環 $A$ の極大 ideal $M$ について $\nu(M)$ と

$\nu(MA_M)$の関係を調べておくことが必要になる。全く一般の場合は $\nu(MA_M) \leqq \nu(M) \leqq \nu(MA_M)+1$ が成り立つ（[3]参照）

以下に述べる定理1，定理2は Davis と Geramita によって与えられた基本的かつ重要なものである。先ず，それらの定理に共通な記号法を定めておこう。

Notation; $R$ はネーター環．$A=R[T_1,\dots,T_n]$ $(n>0)$ は $R$ 上の $n$ 変数多項式環．$M$ を $A$ の極大 ideal．$P=M\cap R$ とおく。

このとき，$ht(M)=ht(P)+n$．$R/P$ は $\dim R/P \leqq 1$ の semi local ring であることに注意しておく。

定理1　$P$: maximal ideal of $R \Longrightarrow \nu(M)=\nu(MA_M)$.

定理2　$n \geqq 2 \Longrightarrow \nu(M)=\nu(MA_M)$.

証明は省くが，定理2は $n=2$ の場合に示せば十分であり，次の lemma を使って定理1に帰着させる．

Lemma. $M$ を $R[X,Y]$ の極大 ideal とするとき，自然数 $s$ を十分大きくとれば $M\cap R[X-Y^s]$ は $R[X-Y^s]$ の極大 ideal となる。

さて $D$ を non semi local な Dedekind domain とすれば，$D$ は Hilbert ring であるから定理1の仮定をみたす。それ故 $D[T_1,\dots,T_n]$ は strongly regular である．前ページで述べたことと合わせて $\dim R=1$ のときは予想は正しい。また定理2により $n\geqq 2$ ならば予想は正しい。

§2　The case of $n=1$ and $\dim R \geqq 2$

定理3（[5]）　$R$: 2次元 regular $\Longrightarrow R[T]$ は strongly regular.

証明は P. Murthy の idea に依る．大雑把にいうと，ht 2 の極大 ideal M についてのみ議論すればよく，この場合，Serre の補題から $\exists L \longrightarrow M \longrightarrow 0$ : exact with L a projective A-module of rank 2 それ故 L は free である。故に $\nu(M) = 2 = ht(M)$ である。

さて $\dim R = d \geq 3$ の場合，この予想が肯定的に解決される時は次の steps でなされるのが望ましいと思われるがどうだろうか？

(1) 極大 ideal M を homomorphic image にもつ A-projective module L of rank d を見出すこと（$ht(M) = d$ なる M についてのみ考えれば良いことに注意）。

(2) この L は free である。(つまり Bass-Quillen 予想は真)

最近 S. M. Bhatwadekar は次の 2 種類の regular local ring R に対して，予想は正しいことを示した（[6] 参照）

(a) a local ring of an affine algebra over an infinite perfect field.

(b) a power series ring over a field.

(a) について言えば，local affine algebra の議論を駆使し複雑な reduction の後 定理 2 に帰着させている。(b) は割合単純な証明で，やはり定理 2 に帰着させている。

この予想の完全な解決にはまだ道のりがあり，筆者には反例を見つけるにしても容易でないような気がします．上に述べた (1), (2) は K-理論との関連にまたねばなりません．以下に references をあげておく．

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On $\mathfrak{F}$-modules and balanced big Cohen-Macaulay modules

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We shall study certain modules (over a local ring) whose localizations by some pre-assigned prime ideals are Cohen-Macaulay. Our results will be applied to balanced big Cohen-Macaulay modules studied by R. Y. Sharp [8].

Preliminaries. $(A, \underline{m})$ will always be a (Noetherian) local ring. For elements $a_1, a_2, \ldots, a_k$ of $A$, we write $\underline{a}_k = (a_1, a_2, \ldots, a_k)$ which means the sequence of these elements or also the ideal generated by them. If $k = 0$, we put $\underline{a}_k = (0)$, the zero ideal. Let $M$ be an $A$-module. A sequence $\underline{a}_k = (a_1, \ldots, a_k)$ is a poor $M$-sequence if $a_i$ is regular on $M/\underline{a}_{i-1}M$ for $i = 1, \ldots, k$, and is an $M$-sequence if, in addition, $\underline{a}_k M \neq M$. $H_i(\underline{a}_k, M)$ denotes the i-th homology module of the Koszul complex generated by $\underline{a}_k = (a_1, \ldots, a_k)$ over $M$. If $M$ is finitely generated and $P \in \mathrm{Spec}(A)$,

$$\mathrm{ht}_M P = \text{the } M\text{-height of } P = \dim_{A_P} M_P .$$

We use the following abbreviations :

f.g. = finitely generated,

C.M. = Cohen-Macaulay,

s.o.p. = system of paramaters,

and s.s.o.p. = subsystem of parameters.

Lemma (0.1) Let $M$ be a f.g. $A$-module, $\underline{a}_k = (a_1, \ldots, a_k)$ an s.s.o.p. for $M$ $(0 \leqq k \leqq \dim M)$ and $P \in \mathrm{Supp}(M)$ such that $\underline{a}_k \subseteq P$ and $\dim M = \mathrm{ht}_M P + \dim A/P$.

Then $k \leqq ht_M P$ and the canonical images $a_1/1, \ldots, a_k/1$ of $a_1, \ldots, a_k$ in $A_P$ form an s.s.o.p. for $M_P$ .

## 1. $\mathfrak{F}$-rings and $\mathfrak{F}$-modules.

Definition (1.1) Let $M$ be a f.g. A-module and $\mathfrak{F}$ any subset of Spec(A) . We call $M$ an $\mathfrak{F}$-module if

$$\mathfrak{F} \cap \mathrm{Supp}(H_1(\underline{a}_k, M)) = \emptyset$$

for each s.s.o.p. $(a_1, \ldots, a_k)$ for $M$ . The local ring $A$ is called an $\mathfrak{F}$-ring if $A$ itself is an $\mathfrak{F}$-module.

Remarks (1.2) (1) $M$ is an $\mathfrak{F}$-module if and only if $M$ is an $\mathfrak{F}^*$-module, where $\mathfrak{F}^* = \{ P \in \mathrm{Spec}(A) \ ; \ P \subseteq {}^{\exists} Q \in \mathfrak{F} \}$ .

(2) If $\underline{m} \in \mathfrak{F}$ , $M$ is an $\mathfrak{F}$-module if and only if $M$ is C.M..

(3) If $\mathfrak{F} = \mathrm{Supp}(M) - \{ \underline{m} \}$ , an $\mathfrak{F}$-module is an f-module in the sense of [6] and conversely.

Definition (1.3) Let $M$ be an A-module and $\mathfrak{F}$ a subset of Spec(A) . A sequence $\underline{a}_k = (a_1, \ldots, a_k)$ of elements of $\underline{m}$ is an M-sequence with respect to $\mathfrak{F}$ if each ideal $(\underline{a}_{i-1}M \underset{A}{:} (\underline{a}_{i-1}M \underset{M}{:} a_i))$ is contained in no member of $\mathfrak{F}$ for $i = 1, \ldots, k$. This means that $(a_1/1, \ldots, a_k/1)$ is a poor M-sequence for all $P$ in $\mathfrak{F}$.

Remarks (1.4) (1) In the case $\underline{m} \in \mathfrak{F}$ , an M-sequence with respect to $\mathfrak{F}$ is nothing but a poor M-sequence in $\underline{m}$ .

(2) If $\mathfrak{F} = \mathrm{Supp}(M) - \{ \underline{m} \}$ , an M-sequence with respect to $\mathfrak{F}$ is just a filter-M-regular sequence in the sense of [6] or [9].

Theorem (1.5) Let $M$ be a f.g. A-module and $\mathfrak{F}$ a subset of Supp(M) . Then the following conditions are equivalent :

(i) $M$ is an $\mathfrak{F}$-module.

(ii) Each s.s.o.p. for $M$ is an M-sequence with respect to $\mathfrak{F}$ .

(iii) For each $P$ in $\mathfrak{F}$ , $M_P$ is a C.M. $A_P$-module and

$$\dim M = ht_M P + \dim A/P \ .$$

(iv) For any s.s.o.p. $\underline{a}_k = (a_1, \dots, a_k)$ for $M$ $(0 \leqq k \leqq \dim M)$ and any $P$ in $\mathfrak{F}$ with $\underline{a}_k \subseteq P$, it holds

$$\dim M = \mathrm{ht}_M P + \dim A/P$$

and $\dim A_P/QA_P = \mathrm{ht}_M P - k$

where $Q$ is any element of $\mathrm{Ass}_A(M/\underline{a}_k M)$ with $Q \subseteq P$.

Proof. (i) $\Leftrightarrow$ (ii). Use the following exact sequence for $k \geqq 2$ :

$$0 \to H_1(\underline{a}_{k-1}, M)/a_k H_1(\underline{a}_{k-1}, M) \to H_1(\underline{a}_k, M) \to (\underline{a}_{k-1}M : a_k)/\underline{a}_{k-1}M \to 0$$

(e.g. [7], Chap. 1, Prop. 1).

(ii) $\Rightarrow$ (iii). The assumption implies

$$\mathrm{depth}_{A_P} M_P \geqq k \geqq \dim_{A_P} M_P \quad \text{where} \quad k = \dim M - \dim A/P .$$

(iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii). Use (0.1).

Corollary (1.6) Assume that $A$ is a homomorphic image of a Gorenstein ring and let $M$ be a f.g. A-module and $\mathfrak{F}$ a subset of $\mathrm{Supp}(M)$. Equivalent conditions :

(i) $M$ is an $\mathfrak{F}$-module.

(ii) $\mathfrak{F} \subseteq \mathrm{Supp}(T^d(M)) - \bigcup_{i<d} \mathrm{Supp}(T^i(M))$

where $d = \dim M$ and $T^i(M) = \mathrm{Hom}(H^i_{\underline{m}}(M), E(A/\underline{m}))$.

Proof. Use the so-called local duality ([5], Satz 1).

Corollary (1.7) Assume that a f.g. A-module $M$ is an $\mathfrak{F}$-module for a subset $\mathfrak{F}$ of $\mathrm{Supp}(M)$. If an element $a$ of $\underline{m}$ forms an s.s.o.p. for $M$, then the $A/aA$-module $M/aM$ is an $\mathfrak{F}_a$-module, where $\mathfrak{F}_a = \{ P/aA \ ; P \in \mathfrak{F} \text{ and } a \in P \}$.

Corollary (1.8) Assume the local ring $A$ is an $\mathfrak{F}$-ring for a subset $\mathfrak{F}$ of $\mathrm{Spec}(A)$. Let $P$ be any element of $\mathfrak{F}$ and $Q$ a prime ideal of $A$ such that $Q \subseteq P$. Then it holds

$$\mathrm{ht}(P/Q) = \mathrm{ht}\, P - \mathrm{ht}\, Q = \dim A/Q - \dim A/P .$$

## 2. Balanced big Cohen-Macaulay modules.

We first recall some definitions in Sharp [8].

A (not necessarily f.g.) A-module $M$ is a big C.M. module if some s.o.p. for $A$ is an $M$-sequence and is a balanced big C.M. module if each s.o.p. for $A$ is an $M$-sequence. The supersupport of a balanced big C.M. A-module $M$ is defined to be the set $\mathrm{Supersupp}(M) = \{ P \in \mathrm{Spec}(A) \ ; \ P \in \mathrm{Ass}(M/\underline{a}_k M)$ for some $M$-sequence $\underline{a}_k \}$. Sharp's results in [8] together with the well-known theory of injective modules show that, if $M$ is a balanced big C.M. A-module and if

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \dots$$

is a minimal injective resolution of $M$, then each associated prime ideal $P$ with $E^i$ (is a member of $\mathrm{Supersupp}(M)$ and) satisfies

$$\mathrm{ht}\, P = \dim A - \dim A/P \leqq i \quad (i = 0, 1, \dots, \dim A) .$$

Here we shall consider the converse.

Theorem (2.1) Let $\mathfrak{F}^0$ and $\mathfrak{F}$ be subsets of $\mathrm{Spec}(A)$ such that $\mathfrak{F}^0 \subseteq \mathfrak{F}$ and $\mathfrak{F}^0$ consists of only prime ideals $P$ with $\dim A = \dim A/P$. Assume that $A$ is an $\mathfrak{F}$-ring and let $M$ be an A-module and $n$ an integer $\geqq 0$. If a minimal injective resolution

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \dots$$

of $M$ satisfies $\mathrm{Ass}_A(E^0) \subseteq \mathfrak{F}^0$ and $\mathrm{Ass}_A(E^i) \subseteq \mathfrak{F}$ $(i = 1, \dots, n)$ and an element $a$ of $A$ forms an s.s.o.p. for $A$, then $a$ is $M$-regular and the finite sequence

$$0 \longrightarrow \mathrm{Hom}_A(A/aA, D) \longrightarrow \mathrm{Hom}_A(A/aA, E^1) \longrightarrow \dots \longrightarrow \mathrm{Hom}_A(A/aA, E^{n+2})$$

is a part of a minimal injective resolution of the $A/aA$-module $\mathrm{Hom}_A(A/aA, D) \cong M/aM$, where $D = \mathrm{Im}(E^0 \longrightarrow E^1)$.

Proof. For each $i = 0, 1, \dots, n$, our assumption and (1.5) imply that the multiplication $a^{\cdot}$ by $a$ induces a surjective endomorphism of $E^i$, that is, $aE^i = E^i$. For $i = 0$, in particular, since $\mathrm{Ass}(E^0) \subseteq \mathfrak{F}^0$, $a$ is regular on $E^0$ as well as on $M$ and hence $a^{\cdot} : E^0 \cong E^0$. Putting $D = \mathrm{Coker}(M \longrightarrow E^0) = \mathrm{Im}(E^0 \longrightarrow E^1)$, we have $\mathrm{Hom}_A(A/aA, D) \cong (M :_{E^0} a)/M \cong M/aM$.

The exact sequence

$$0 \to D \to E^1 \to E^2 \to \dots$$

induces

$$0 \to \mathrm{Hom}_A(A/aA,\ D) \to \mathrm{Hom}_A(A/aA,\ E^1) \to \dots \to \mathrm{Hom}_A(A/aA,\ E^{n+2})$$

which is clearly exact at $\mathrm{Hom}(A/aA, D)$ and $\mathrm{Hom}(A/aA, E^1)$. As to the exactness at the other vertices it suffices to see the exactness of the sequence

$$K^1 \to K^2 \to \dots \to K^{n+2}$$

where $K^i = \mathrm{Ker}(a^\cdot : E^i \to E^i)$ and the arrows are induced by $E^i \to E^{i+1}$. Then the exactness at $K^i$ $(2 \leqq i \leqq n+2)$ follows from chasing a diagram, using the surjectivity of $a^\cdot$ on $E^{i-2}$.

In the following we put $d = \dim A$ and, for each integer $k \geqq 0$,

$$\mathcal{U}^k(A) = \{ P \in \mathrm{Spec}(A)\ ;\ d - \dim A/P \leqq k \}$$

and

$$\mathcal{V}^k(A) = \{ P \in \mathrm{Spec}(A)\ ;\ \mathrm{ht}\, P = d - \dim A/P \leqq k \}.$$

Proposition (2.2) Let $M$ be an $A$-module. If, for each s.s.o.p. $(a_1, \dots, a_k)$ for $A$ $(0 \leqq k \leqq d)$, $\underline{a}_k M \neq M$ and $\mathrm{Ass}_A(M/\underline{a}_k M) \subseteq \mathcal{U}^k(A)$, then $M$ is a balanced big C.M. module.

Theorem (2.3) Assume that $A$ is a $\mathcal{V}^{d-2}(A)$-ring and let $M$ be a big C.M. $A$-module. Then the following conditions are equivalent :

(i) $M$ is a balanced big C.M. module.

(ii) If

$$0 \to M \to E^0 \to E^1 \to \dots$$

is a minimal injective resolution of $M$, it holds $\mathrm{Ass}_A(E^i) \subseteq \mathcal{V}^i(A)$ for $i = 0, 1, \dots, d$.

Proof. We show (ii) $\Rightarrow$ (i). Let $(a_1, \dots, a_d)$ be any s.o.p. for $A$. Clearly $\underline{a}_d M \neq M$. It suffices to see $a_i$ is regular on $M/\underline{a}_{i-1}M$ or $\mathrm{Ass}_A(M/\underline{a}_i M) \subseteq \mathrm{Ass}(E^i)$ for $i = 1, \dots, d$. These assertions are verified by applying (1.7) and (2.1) inductively to a minimal injective resolution of $M$.

In [8] Sharp conjectured that, if $M$ is a balanced big C.M. A-module, the localization $M_P$ by any prime ideal $P$ in Supersupp(M) is again a balanced big C.M. $A_P$-module. We give a weak answer to this. (Note that the conjecture is clearly true in the case $A$ is C.M..)

Proposition (2.4) Assume that $A$ is a $\mathcal{V}^{d-2}(A)$-ring and that, for any $P$ in $\mathcal{V}^{d-1}(A)$ and any $Q$ in $\mathcal{V}^{d-2}(A)$ with $Q \subseteq P$, $ht(P/Q) = ht\ P - ht\ Q$. Then, if $M$ is a balanced big C.M. A-module, $M_P$ is a balanced big C.M. $A_P$-module for every $P$ in Supersupp(M) .

Proof. We see $M_P$ is a big C.M. $A_P$-module by Sharp's results (2.2), (3.2) and (3.3) in [8]. Then the assertion follows from (2.3) and (1.8).

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# Some Characterizations of Smoothness

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Let $A$ be a (not necessarily noetherian) commutative ring, $B$ an $A$-algebra, and $I$ an ideal of $B$. We say that $B$ is $I$-smooth (resp. $I$-unramified) over $A$ if for any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ f\downarrow & & \downarrow v \\ C & \xrightarrow{g} & C/N \end{array}$$

where $C$ is an $A$-algebra, $N$ is an ideal of $C$ such that $N^2=0$, and $v$ is a ring homomorphism such that $v(I^n)=0$ for some $n$, there exists at least one (resp. at most one) homomorphism $\varphi: B \longrightarrow C$ such that $f=\varphi\circ u$ and $v=g\circ\varphi$. If $B$ is $I$-smooth and $I$-unramified over $A$, we say that $B$ is $I$-etale over $A$. In particular, if $I=0$, we say shortly that $B$ is smooth (resp. unramified, resp. etale) over $A$.

Now in terms of differential modules, we can restate smoothness and unramifiedness as follows:

<u>Proposition 1.</u> (i)(cf.[3, §25]) $B$ is unramified over $A$ iff $\Omega_{B/A}=0$.

(ii) When $A$ and $B$ are noetherian, $B$ is smooth over $A$ iff $\Omega_{B/A}$ is a projective $B$-module and the ring homomorphism $A \longrightarrow B$ is regular.

Now let $k$ be a ring and $A$ be a quotient ring of a noetherian smooth k-algebra (e.g. $k$ is noetherian and $A$ is of essentially finite type over $k$ ). Then for $P \in \mathrm{Spec}(A)$ , $A_P$ is smooth over $k$ iff $A_P$ is $PA_P$-smooth over $k$ . More generally, we have the following proposition under the above situation:

Proposition 2. Let $I$ be an ideal of $A$ . Then $A$ is $I$-smooth over $k$ iff $V(I) \subseteq \{\mathfrak{m} \in \mathrm{Max}(A) \mid A_{\mathfrak{m}} \text{ is } \mathfrak{m}A_{\mathfrak{m}}\text{-smooth over } k\}$ .

But in general, since k-algebra is not a quotient ring of a noetherian smooth k-algebra, it is difficult to show the difference between I-smoothness and smoothness. For example, for a ring $A$ and an ideal $I$ of $A$ , $A[[X_1,\ldots,X_n]]$ is $\sum_{i=1}^{n} X_i A[[X_1,\ldots,X_n]]$ -smooth over $A$ and I-adic completion $(A,I)^{\wedge}$ is I-smooth over $A$ . But since these rings are not quotient rings of smooth A-algebras in general, it is hard to show whether these are smooth over $A$ or not.

Now in [3] , H. Matsumura asks

(I) what is the difference between smoothness and I-smoothness ?

(II) when is a ring $A[[X_1,\ldots,X_n]]/\mathfrak{a}$ smooth over $A$ ?

We will study his problems when $A$ is a noetherian ring. For Problem (I) , Proposition 1 is an answer. Concerning with Problem (II) , we list up three problems :

(A) When is $A[[X_1,\ldots,X_n]]$ smooth over $A$ ?

(B) When is $A[[X_1,\ldots,X_n]]/\mathfrak{a}$ smooth over $A$ in the case $\mathfrak{a} \neq 0$ ?

In particular

(C) when is $(A,I)^{\wedge}$ smooth over $A$ ?

We will give answers to these problems under some assumptions.

## §1. Problem (A).

**Theorem 3.** Let $A$ be a noetherian ring containing a field $k$. Then the following are equivalent :

(i) $A[[X_1,\dots,X_n]]$ is smooth over $A$ for every $n \geq 1$ ;

(ii) $A[[X_1,\dots,X_n]]$ is smooth over $A$ for some $n \geq 1$ ;

(iii) $ch(k) = p > 0$ and $A$ is a finite $A^p$-algebra.

Sketch of the proof. We have only to prove (ii) $\Rightarrow$ (iii). First, we show it when $A$ is a field. We consider the following exact sequence :

$$\Omega_{A[X]/A} \otimes_{A[X]} A[[X]] \xrightarrow{\varphi} \Omega_{A[[X]]/A} \longrightarrow \Omega_{A[[X]]/A[X]} \longrightarrow 0 \ .$$

where $X = \{X_1,\dots,X_n\}$. Now since $\Omega_{A[[X]]/A} \otimes_{A[[X]]} (A[[X]]/(X))$ $\cong \bigoplus_{i=1}^{n} (A[[X]]/(X))dX_i$ and since $\Omega_{A[[X]]/A}$ is a free $A$-module by our assumption, $\Omega_{A[[X]]/A} \cong \bigoplus_{i=1}^{n} A[[X]]dX_i$. So $\varphi$ is an isomorphism and $\Omega_{A[[X]]/A[X]} = 0$. Therefore $\Omega_{A((X))/A(X)} = 0$. Since $\text{tr.deg}_{A(X)} A((X))$ $= \infty$, we have $ch(A) = p > 0$ and $A^p((X^p))[A(X)] = A((X))$. Then it is easy to see that $[A : A^p] < \infty$.

In general case, using noetherian induction and Marot's theorem on Nagata rings, we can prove the theorem.

## §2. Problem (B).

Let $A$ be a noetherian ring and $P \in \text{Spec}(A)$. We say that $P$ satisfies SC if $k(P)$ satisfies one of the following two conditions :

(i) $ch(k(P)) = 0$ ;

(ii) $ch(k(P)) = p > 0$ and $[k(P) : k(P)^p] = \infty$.

Moreover we define the natural map $\pi : A[[X_1,\dots,X_n]] \longrightarrow A$ such that, for $f \in A[[X_1,\dots,X_n]]$, $\pi(f)$ is the constant term of $f$. Then we have :

Proposition 4. Let $A$ be a noetherian ring and $X=\{X_1,\dots,X_n\}$ be variables over $A$. Let $\mathfrak{a}$ be an ideal of $A[[X]]$, and assume that every $\mathfrak{m} \in \mathrm{Max}(A)$ containing $\pi(\mathfrak{a})$ satisfies SC. (It is possible that $\mathrm{ch}(k(P)) \neq \mathrm{ch}(k(Q))$ for $P,Q \in \mathrm{Max}(A)$.) Then if $R=A[[X]]/\mathfrak{a}$ is smooth over $A$, $R \cong (A, \pi(\mathfrak{a}))^\wedge$.

So we can reduce Problem (B) to Problem (C) under SC condition.

§3. Problem (C).

Let $A$ be a noetherian ring and $I$ be an ideal of $A$. Put $\hat{A}=(A,I)^\wedge$ and $A^h=(A,I)^h$. When $A$ contains a field $k$, we can find criteria for $\hat{A}$ to be smooth over $A$.

Theorem 5. Assume that $\mathrm{ch}(k)=0$. Then the following conditions are equivalent :

(i) $\hat{A}$ is smooth over $A$ ;

(ii) $\hat{A}$ is etale over $A$ ;

(iii) $\hat{A}$ is unramified over $A$, and $A \longrightarrow \hat{A}$ is normal ;

(iv) $A^h \cong \hat{A}$.

Theorem 6. Assume that $\mathrm{ch}(k)=p>0$ and that $A/P$ is N-1 (i.e. the derived normal ring of $A/P$ is a finite $A/P$-module) for all $P \in \mathrm{Min}(A)$. Then the following conditions are equivalent :

(i) $\hat{A}$ is smooth over $A$ ;

(ii) $\hat{A}$ is etale over $A$ ;

(iii) $\hat{A}$ is unramified over $A$, and $A \longrightarrow \hat{A}$ is normal ;

(iv) $\hat{A}^p[\bar{A}] = \hat{A}$ where $\bar{A}$ is the homomorphic image of $A$ in $\hat{A}$, and $A \longrightarrow \hat{A}$ is reduced ;

(v) $\hat{A}^p[A^h] = \hat{A}$, and $A \longrightarrow \hat{A}$ is reduced.

<u>Corollary 7.</u> Assume that $A$ is a noetherian normal Z-ring (i.e. all formal fibres are geometrically normal) and $ch(k) = p > 0$. Then $\hat{A}$ is smooth over $A$ iff $Q(\hat{A}^p[A]) = Q(\hat{A})$.

Moreover, from Th.6, we can give another proof of Kunz's theorem :

<u>Corollary 8.</u> (cf.[1] or [2, (42.B) Th.108]) Let $A$ be a noetherian ring containing a field of characteristic $p > 0$. Then if $A$ is a finite $A^p$-module, $A$ is a G-ring.

Finally we will construct non-trivial examples such that $\hat{A}$ is smooth over $A$. Let $k$ be a field. We distinguish three cases.

Case (I): $ch(k) = 0$. In [4, (11,3) Ex.3], it is shown that there exists a DVR $A$ which we want. We will sketch the construction.

Let $X$ be a variable over $k$ and $B$ a transcendence base of $k((X))$ over $k(X)$. Put $A = k[[X]] \cap k(X)(B)$. Then $(A,(X))$ is a DVR and $\hat{A} \cong k[[X]] \cong A^h$. In particular, $\hat{A}$ is smooth over $A$.

Case (II): $ch(k) = p > 0$ and $[k : k^p] < \infty$. Let $X$ be a variable over $k$, and put $A = k[X]_{(X)}$. Then $\hat{A} = k[[X]]$ and $A$ is a finite $A^p$-algebra. It is easy to show that $\hat{A}$ is smooth over $A$.

Case (III): $ch(k) = p > 0$ and $[k : k^p] = \infty$. Imitating the construction in Case (I), we will construct a desirable example such that $A$ is not finite over $A^p$.

Let $X$ be a variable over $k$. Then $k((X))$ is separable over $k(X)$. Let $B$ be a p-basis of $k((X))$ over $k(X)$. Then $k((X))$ is separable over $k(X)(B)$ by [2,(38.E)]. Put $A = k[[X]] \cap k(X)(B)$. Then it follows easily that $(A,(X),k)$ is an excellent DVR such that $\hat{A} = (A,(X))^{\wedge} = k[[X]]$ and $A \neq \hat{A}$. Moreover, since $[k : k^p] = \infty$, $A$ is not a finite $A^p$-module. Since $Q(\hat{A}^p[A]) = k^p((X^p))[k(X)(B)] = k((X)) = Q(\hat{A})$, $\hat{A}$ is smooth over $A$ by Cor. 7. Therefore $A$ is the example which we want.

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Standard systems of parameters

of generalized Cohen-Macaulay modules

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INTRODUCTION

In 1965, Buchsbaum posed a conjecture which, roughly speaking, states that given a finitely generated module M over a local ring A, then the difference

$$I(\underline{q};M) := l(M/\underline{q}M) - e(\underline{q};M)$$

between length and multiplicity takes a constant value for all parameter ideals $\underline{q}$ of M. This is not true [V]. However, in [SV1] and [SV2], Stückrad and Vogel found out that modules satisfying this conjecture enjoy many interesting properties which are similar to the ones of Cohen-Macaulay (abbr. C-M) modules, and gave them the name Buchsbaum modules. That led in [CST] to the study of modules M for which the difference $I(\underline{q};M)$ is bounded above by an invariant of M. It turned out that M satisfies this condition iff

$$l(H^i_{\underline{m}}(M)) < \infty$$

for i = 0,...,dim M - 1, where $H^i_{\underline{m}}(M)$ denotes the ith local cohomology module of M relative to the maximal ideal $\underline{m}$ of A. Since M is a C-M module iff $H^i_{\underline{m}}(M) = 0$ for i = 0,...,dim M - 1, one call such modules generalized C-M modules.

The class of generalized C-M modules is rather large. It is known that if A is a factor ring of a C-M ring, then M is a generalized C-M module iff $M_{\underline{p}}$ is a C-M module with

$$\dim M_{\underline{p}} = \dim M - \dim A/\underline{p}$$

for all $p \in \mathrm{Supp}(M) \setminus \{m\}$ [CST]. Hence it is easy to verify that most geometric local rings, e.g. of isolated singularities or of the vertex of the affine cone over projective curves, are generalized C-M rings.

Although the theory of Buchsbaum modules has been rapidly developed by works of Goto, Schenzel, Stuckrad, Vogel, little has been done in the theory of generalized C-M modules.

If one is acquainted enough with the few references on generalized C-M modules [CST], [S1], [G3], one would notice that almost all properties of systems of parameters (abbr. s.o.p) of Buchsbaum modules also hold for s.o.p of generalized C-M modules which are contained in a sufficiently large power of $\underline{m}$. For instance, if M is a generalized C-M module and if $\underline{q} \subseteq \underline{m}^n$ for n sufficiently large, then $I(\underline{q};M)$ attains a maximal constant value $I(M)$. So, with regard to the origin of generalized C-M modules and the above notice, it is of interest to study s.o.p of M with $I(\underline{q};M) = I(M)$, where q is the corresponding parameter ideal. Such s.o.p will be called standard s.o.p.

The aim of this report is to show that standard s.o.p play an important role in the theory of generalized C-M modules and that by their help, one can derive the theory of Buchsbaum modules as part of the theory of generalized C-M modules.

Now we will describe the main results in accordance with the organization of this report.

In Section 1 we shall see that standard s.o.p may be characterized by different ways. First, using the notation of filter-regular sequences of [CST] we can define standard s.o.p of M without the explicit assumption that M is a generalized C-M module. As a consequence, we get an interesting criterion saying that M is a generalized C-M module iff M has a standard s.o.p. Further, we can also characterize standard s.o.p by means of local cohomology. It follows that standard s.o.p are just s.o.p which are standard sequences in the sense of [B], hence the name. In particular, we can show that standard s.o.p are absolutely superficial, i.e. they are d-sequences which have been proved lately as very useful for different topics of the theory of modules [H1], [H2], [T3], [HSV].

In Section 2 we shall see that standard s.o.p may be used well to study Hilbert-Samuel (abbr. H-S) functions. First, inspired of the characterization of absolutely superficial s.o.p by means of H-S functions in [T3], we give a polynomial bounding above the H-S function of an arbitrary s.o.p of a generalized C-M module and show that they coincide iff this s.o.p is standard. Similarly, we can also estimate the H-S functions of a generalized C-M module M with respect to an arbitrary ideal $\underline{a}$ of A with $l(M/\underline{a}M) < \infty$ . In particular, M will behave very well if $l(M/\underline{a}^nM)$ attains some extreme value for some n. As a consequence, we can extend results of Sally [S] and [G1] on C-M or Buchsbaum rings with maximal embedding dimension for the case of modules.

In Section 4 and Section 5 we shall show that the associated graded module and the Rees module (arithmetical blowing-up) of a generalized C-M module relative to a standard s.o.p or to an ideal whose H-S function behave extremely are generalized C-M modules and that their local cohomology modules may be computed explicitly. As a consequence, we can give conditions for these graded modules to be C-M modules. For Buchsbaum rings, most results of these Sections have been already known by Goto and Shimoda [G1], [G2], [GS].

The author would like to thank Brodmann, Goto, and Schenzel for making their works available. He is also indebted to Goto and Suzuki for inviting him to this symposium.

## 1. CHARACTERIZATIONS

From now on, $a_1,\dots,a_d$ will be a s.o.p of M and $\underline{q}$ the ideal $(a_1,\dots,a_d)$. In order to simplify the notations, we further put $\underline{q}_i = (a_1,\dots,a_i)$, $i = 1,\dots,d-1$, and $\underline{q}_0 = 0$ (the zeroideal).

<u>DEFINITION 1.1</u>. $a_1,\dots,a_d$ is called a <u>standard system of parameters</u> of M if the following conditions are satisfied:

(i) By every permutation, $a_1,\dots,a_d$ is a filter-regular sequence, i.e. $a_i \notin \underline{p}$ for all $\underline{p} \in \mathrm{Ass}(M/\underline{q}_{i-1}M) \setminus \{\underline{m}\}$ for all $i = 1,\dots,d$.

(ii) $I(a_1^2,\dots,a_d^2;M) = I(\underline{q};M)$.

This definition of standard s.o.p is different from the one given in the introduction of this report. It does not explicitly contain the assumption that M is a generalized C-M module but leads to the same notation by the following result.

<u>THEOREM 1.2</u>. $a_1,\dots,a_d$ is a standard s.o.p of M iff one of the following equivalent conditions is satisfied:

(i) $I(a_1^{n_1},\dots,a_d^{n_d};M) = I(\underline{q};M)$ for all positive integers $n_1,\dots,n_d$.

(ii) M is a generalized C-M module and $I(\underline{q};M) = I(M)$.

(iii) $\underline{q}H^i_{\underline{m}}(M/\underline{q}_jM) = 0$ for all non-negative integers i, j with $i+j < d$.

To check whether a given s.o.p of M is standard one can use Definition 1.1, Theorem 1.2 (i) or (ii). It should be mentioned that if $a_1,\dots,a_d$ is a filter-regular M-sequence, which is always satisfied if M is a generalized C-M module [CST], then

$$I(\underline{q};M) = l(\underline{q}_{d-1}M : a_d/\underline{q}_{d-1}M).$$

Hence to check Definition 1.1 and Theorem 1.2 (i) is rather easy. If one knows the local cohomology modules of M, Theorem 1.2 (ii) is more convenient because

$$I(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} l(H^i_{\underline{m}}(M)).$$

Theorem 1.2 (ii) further yields the following simple characterization of generalized C-M modules by means of only one s.o.p:

COROLLARY 1.3. M is a generalized C-M module iff M has a standard s.o.p.

Theorem 1.2 (iii) justify the name standard s.o.p. Namely, in [B] Brodmann call a sequence $b_1,\dots,b_r$ of elements of $\underline{m}$ a $\underline{m}$-standard M-sequence if

$$(b_1,\dots,b_r)H^i_{\underline{m}}(M/(b_1,\dots,b_i)M) = 0$$

for all non-negative integers i, j with

$$i + j < \max\{n;\ l(H^t_{\underline{m}}(M)) < \infty \text{ for all } t < n\}.$$

So standard s.o.p are just $\underline{m}$-standard s.o.p in this sense.

From Theorem 1.2 one can easily deduce the following consequences:

COROLLARY 1.4. Suppose that $a_1,\dots,a_d$ is a standard s.o.p. Then

(i) $a_1,\dots,a_d$ is a standard s.o.p of $M/(\bigcup_{i=1}^{\infty} 0_M : m^i)$.

(ii) $a_2,\dots,a_d$ is a standard s.o.p of $M/a_1M$ if $d > 1$.

COROLLARY 1.5. Suppose that $a_1,\dots,a_d$ is a standard s.o.p of M. Then

(i) $\underline{q}_{i-1}M : a_i = \bigcup_{n=0}^{\infty} \underline{q}_{i-1}M : \underline{m}^n$ for all $i = 1,\dots,d$.

(ii) $a_1,\dots,a_d$ is a d-sequence of M, i.e.

$$\underline{q}_{i-1}M : a_i a_j = \underline{q}_{i-1}M : a_i$$

for all $j \geqq i$, $i = 1,\dots,d$.

(iii) $a_1,\dots,a_d$ is absolutely superficial, i.e.

$$[(\underline{q}^{n+1},\underline{q}_{i-1})M : a_i] \cap \underline{q}M = (\underline{q}^n,\underline{q}_{i-1})M$$

for all $n \geqq 0$, $i = 1,\dots,d$.

(vi) $(\underline{q}_{i-1}M : a_i) \cap \underline{q}(a_i,\dots,a_d)^n M = \underline{q}_{i-1}(a_i,\dots,a_d)^n M$ for all $n \geqq 0$, $i = 1,\dots,d$.

It should be mentioned that all the above statements of Corollary 1.5 are equivalent to each other. See [T3] for more informations.

In the course of this report we shall see that many numerical invariants of a s.o.p attain their maximal values if it is a standard s.o.p. Here is only an example:

PROPOSITION 1.6. Suppose that M is a generalized C-M module. Then

$$l(H^i_{\underline{m}}(M/\underline{q}_j M)) \leqq \sum_{n=i}^{i+j} \binom{i+j}{n} l(H^n_{\underline{m}}(M))$$

for all non-negative integers i, j with $i+j < d$. Equalities hold above for all such i, j iff $a_1,\dots,a_d$ is a standard s.o.p of M.

Now we will establish the ubiquity of standard s.o.p for generalized C-M modules. First, inspired of Schenzel's results on cohomological annihilators [S2], [S3], we get the following result:

PROPOSITION 1.7. Suppose that M is a generalized C-M module. Let $\underline{a}_i$ denote the annihilator of $H^i_{\underline{m}}(M)$, $i = 0,\dots,d-1$. Then every s.o.p of M contained in the product ideal

$$\underline{a}_M := \prod_{i=0}^{d-1} \underline{a}_i^{\binom{d-1}{n}}$$

is a standard s.o.p of M.

Of course, $\underline{a}_M$ is a $\underline{m}$-primary ideal. So we can say that every standard s.o.p of a generalized C-M module contained in a sufficiently large power of $\underline{m}$ is standard.

In particular, we can characterize $\underline{m}$-primary ideals which contain only standard s.o.p.

THEOREM 1.8. Let $\underline{a}$ be a $\underline{m}$-primary ideal. Then the following conditions are equivalent:

(i) Every s.o.p of M contained in $\underline{a}$ is standard.

(ii) There exists a generating set S for $\underline{a}$ such that every d element subset of S forms a standard s.o.p of M.

(iii) The natural homomorphism $H^i(\underline{a};M) \longrightarrow H^i_{\underline{m}}(M)$ is surjective for $i = 0,\dots,d-1$, where $H^i(\underline{a};M)$ denotes the ith Koszul cohomology module of M with respect to $\underline{a}$.

See [T1] for the existence of such a generating set S for $\underline{a}$ as in Theorem 1.8 (ii). Theorem 1.8 (iii) is due to an idea of Goto.

COROLLARY 1.9. Suppose that $a_1,\dots,a_d$ is a standard s.o.p of M. Then every s.o.p of M contained in $\underline{q}$ is standard too.

COROLLARY 1.10. M is a Buchsbaum module iff one of the following equivalent conditions is satisfied:

(i) There exists a generating set S for $\underline{m}$ such that every d element subset of S forms a standard s.o.p of M.

(ii) The natural homomorphism $H^i(\underline{m};M) \longrightarrow H^i_{\underline{m}}(M)$ is surjective for $i = 0,\dots,d-1$.

It is interesting to notice the difference between Corollary 1.3 and Corollary 1.10 (i). Corollary 1.10 (ii) is known under the name "Surjectivity criterion" and plays an important role in the theory of Buchsbaum modules.

## 2. HILBERT-SAMUEL FUNCTIONS

In [T3] we have shown that the H-S function $l(M/\underline{q}^{n+1}M)$ of an arbitrary s.o.p $a_1,\ldots,a_d$ of M is bounded above by a polynomial of the form

$$\sum_{i=0}^{d} \binom{n+d-i}{d-i} e_i(\underline{q};M),$$

where $e_i(\underline{q};M)$ are well-determined invariants of $a_1,\ldots,a_d$, and that they coincide iff $a_1,\ldots,a_d$ is absolutely superficial. In that case, if M is a generalized C-M module, one can express $e_i(\underline{q};M)$ explicitely in terms of local cohomology. Inspired of this fact, we get a similar result on H-S functions of s.o.p of generalized C-M modules as follows.

From now on, M will always be a generalized C-M module.

<u>THEOREM 2.1</u>. For all $n \geqq 0$,

$$l(M/\underline{q}^{n+1}M) \leqq \binom{n+d}{d} e(\underline{q};M) + \sum_{i=1}^{d} \sum_{j=0}^{d-i} \binom{n+d-i}{d-i} \binom{d-i-1}{j-1} l(H^j_{\underline{m}}(M)).$$

Equality holds for some fixed n iff the following conditions are satisfied:

(i) $\underline{q}^{n+1}M \cap (\bigcup_{i=1}^{\infty} 0_M:\underline{m}^i) = 0$.

(ii) $a_1,\ldots,a_d$ is a standard s.o.p for $M/(\bigcup_{i=1}^{\infty} 0_M:\underline{m}^i)$.

Equality holds for all $n \geqq 0$ iff $a_1,\ldots,a_d$ is a standard s.o.p of M.

Theorem 2.1 may be used to estimate other H-S functions of M, following some ideas of [T2] and [T3].

PROPOSITION 2.2. Let $\underline{a}$ be an ideal of A with $l(M/\underline{a}M) < \infty$. Let $r \geqq 0$, $s \geqq 1$ be arbitrary integers. Then, for all $n = 1$,

$$(*) \quad l(M/\underline{a}^{ns+r}M) \leqq \binom{n+d-1}{d} s^d e(\underline{a};M) + \sum_{i=1}^{d} \sum_{j=0}^{d-i} \binom{n+d-i-1}{d-i} \binom{d-i-1}{j-1} l(H^i_{\underline{m}}(M)) + \binom{n+d-1}{d-1} l(M/\underline{a}^r M + \bigcup_{i=1}^{\infty} 0_M : \underline{m}^i).$$

If $A/\underline{m}$ is an infinite field, (*) is an equality for some fixed n iff for all elements $a_1, \ldots, a_d \in \underline{a}^s \setminus \underline{a}^{s+1}$ whose initial forms in $G_{\underline{a}}(A)$ form a homogeneous s.o.p of the associated graded module $G_{\underline{a}}(M) := \bigoplus_{i=0} \underline{a}^i M/\underline{a}^{i+1}M$, the following conditions are satisfied:

(i) $\underline{q}^n \underline{a}^r M = \underline{a}^{ns+r}M$.

(ii) $\underline{q}^n \underline{a}^r M \cap (\bigcup_{i=1}^{\infty} 0_M : \underline{m}^i) = 0$.

(iii) $a_1, \ldots, a_d$ is a standard s.o.p for $\bar{M} := M/(\bigcup_{i=1}^{\infty} 0_M : \underline{m}^i)$.

(iv) $\underline{q}_{d-1}\bar{M} : a_d \subseteq \underline{a}^r \bar{M}$ by every permutation of $a_1, \ldots, a_d$.

It should be mentioned that the hypothesis $A/\underline{m}$ being an infinite field does not cause us any problem.

The condition that (*) is an equality for some n is very strong. In that case, one gets a lot of informations about the structure of M relative to $\underline{a}$. For example:

COROLLARY 2.3. Suppose that (*) is an equality for some n. Then

(i) $\underline{a}^r H^i_{\underline{m}}(M) = 0$ for $i = 1, \ldots, d-1$.

(ii) Every s.o.p of $\bar{M}$ or of $\underline{a}^r \bar{M}$ contained in $\underline{a}^s$ is standard.

(iii) Every form of $\bar{M}[X_1, \ldots, X_d]$ vanishing at $a_1, \ldots, a_d$ has all its coefficients in $\underline{a}^r \bar{M}$.

From Corollary 2.3 one can deduce that $r \leqq s$ if (*) is an equality. Hence (*) is not the best possible upper bound for

$l(M/\underline{a}^{ns+r}M)$ if $r > s$.

Note that Corollary 2.3 (iii) just means that $a_1,\ldots,a_d$ are $\underline{a}^r\bar{M}$-independent. This notation is originally due to Valla , cf. [T3] and [T4].

In particular, from Proposition 2.2 we get the following interesting application in Algebraic Geometry:

COROLLARY 2.4. Let $C \subset \mathbb{P}^3$ be a projective curve. Let A denote the local ring of the vertex of the affine cone over C. Let H(n) denote the H-S polynomial of A with respect to its maximal ideal $\underline{m}$. Then

$$H(n) + l(H^1_{\underline{m}}(M)) \geqq 0$$

for all integers n.

It should be mentioned that if A is a Buchsbaum ring, i.e. C is arithmetically Buchsbaum, one can replace $l(H^1_{\underline{m}}(M))$ of the above inequality by $e_A + 1$, where $e_A$ denotes the multiplicity of A (the degree of C), cf. [G2].

If $r = s = n = 1$, Proposition 2.2 becomes more interesting. In this case, we have

$$l(M/\underline{a}^2M) \leqq e(\underline{a};M) + I(M) + d\,l(M/\underline{a}M + \bigcup_{i=1}^{\infty} 0_M:\underline{m}^i).$$

From this it follows, e.g. for $M = A$ and $\underline{a} = \underline{m}$, that

$$l(\underline{m}/\underline{m}^2) \leqq e_A + I(A) + d - 1$$

which gives an upper bound for the embedding dimension of a generalized C-M ring, cf. [A], [G1], [S], [T2]. If equality happens in the last inequality, we say that A is of maximal embedding dimension. Sally [S] and Goto [G1] have found that C-M and Buchsbaum rings with maximal embedding dimension behave very well. Now we will consider the analog case in the theory of generalized C-M modules.

PROPOSITION 2.5. Let $A/\underline{m}$ be an infinite field. Let $\underline{a}$ be an ideal of A with $l(M/\underline{a}M) < \infty$. Then

$$l(M/\underline{a}^2M) = e(\underline{a};M) + I(M) + dl(M/\underline{a}M)$$

iff $\bigcup_{i=1}^{\infty} 0_M:\underline{m}^i \subset \underline{a}M$ and for all elements $a_1,\dots,a_d \in \underline{a} \setminus \underline{a}^2$ whose initial forms in $G_{\underline{a}}(A)$ form a homogeneous s.o.p of $G_{\underline{a}}(M)$, the following conditions are satisfied:

(i) $\underline{q}\,\underline{a}M = \underline{a}^2M$.

(ii) $a_1,\dots,a_d$ is a standard s.o.p of M.

(iii) $\underline{q}_{d-1}M : a_d \subseteq \underline{a}M$.

COROLLARY 2.6. Suppose that

$$l(M/\underline{a}^2M) = e(\underline{a};M) + I(M) + dl(M/\underline{a}M).$$

Then

(i) $$l(M/\underline{a}^{n+1}M) = \binom{n+d-1}{d} e(\underline{a};M) + \sum_{i=1}^{d}\sum_{j=0}^{d-i} \binom{n+d-i-1}{d-i}\binom{d-i-1}{j-1} l(H^i_{\underline{m}}(M)) + \binom{n+d-1}{d-1} l(M/\underline{a}M)$$

for all n = 0.

(ii) $\underline{a}H^i_{\underline{m}}(M) = 0$ for $i = 0,\dots,d-1$.

(iii) Every s.o.p of M or of $\underline{a}M$ contained in $\underline{a}$ is standard.

In particular, from Corollary 2.6 (iii) one can deduce that there do not exist generalized C-M non-Buchsbaum rings with maximal embedding dimension. This follows from the following

COROLLARY 2.7. Suppose that

$$l(M/\underline{m}^2M) = e(\underline{m};M) + I(M) + dl(M/\underline{m}M).$$

Then M is a Buchsbaum module.

Note that $e(\underline{a};M) + I(M) + dl(M/\underline{a}M + \bigcup_{i=1}^{\infty} 0_M:\underline{m}^i)$ is the best possible upper bound for $l(M/\underline{a}^2M)$ but $l(M/\underline{a}^2M)$ = this bound would only imply that the factor module $M/(\bigcup_{i=1}^{\infty} 0_M:\underline{m}^i)$ behaves well.

## 3. ASSOCIATED GRADED MODULES

Let P denote the maximal graded ideal of $G_{\underline{q}}(A)$. Then our main result in this Section may be formulated as follows.

<u>THEOREM 3.1</u>. $a_1,\ldots,a_d$ is a standard s.o.p of M iff the initial forms of $a_1,\ldots,a_d$ in $G_{\underline{q}}(A)$ form a standard s.o.p of $[G_{\underline{q}}(M)]_P$. In this case, $[G_{\underline{q}}(M)]_P$ is a generalized C-M module with

$$[H_P^i(G_{\underline{q}}(M))]_n = \begin{cases} 0 & \text{if } n \neq -i, \\ H_{\underline{m}}^i(M) & \text{if } n = -i, \end{cases}$$

$i = 0,\ldots,d-1$, and $[H_P^d(G_{\underline{q}}(M))]_n = 0$ if $n > -d$.

The case M = A being a Buchsbaum ring was already known in [G2], where one also showed that $[G_{\underline{q}}(A)]_P$ is again a Buchsbaum ring. Using the same method of [G2], we can generalize this result as follows (due to a suggestion of Goto).

<u>COROLLARY 3.2</u>. $[G_{\underline{q}}(M)]_P$ is a Buchsbaum module iff $\underline{m}H_{\underline{m}}^i(M/\underline{q}_jM) = 0$ for all non-negative integers i, j with $i+j < d$.

Note that M is called a quasi-Buchsbaum module if $mH_{\underline{m}}^i(M) = 0$ for $i = 0,\ldots,d-1$ or if every s.o.p $a_1,\ldots,a_d$ of M contained in $\underline{m}^2$ is a weak M-sequence, i.e. $\underline{q}_{i-1}M : a_i = \underline{q}_{i-1}M : \underline{m}$ for $i = 1,\ldots,d$. Then from Corollary 3.2 we immediately get the following interesting characterization of quasi-Buchsbaum modules:

<u>COROLLARY 3.3</u>. M is a quasi-Buchsbaum module iff there exists some parameter ideal $\underline{q}$ such that $[G_{\underline{q}}(M)]_P$ is a Buchsbaum module.

Of course, M is a C-M or Buchsbaum module iff $[G_{\underline{q}}(M)]_P$ is a C-M or Buchsbaum module for some or every parameter ideal $\underline{q}$ of M, respectively.

From Theorem 3.1 we also get the following generalization of results of Sally [S] and [Gl] on C-M and Buchsbaum rings with maximal embedding dimension:

PROPOSITION 3.4. Let $\underline{a}$ be an ideal of A with $l(M/\underline{a}M) < \infty$ and

$$l(M/\underline{a}^2M) = e(\underline{a};M) + I(M) + dl(M/\underline{a}M).$$

Then $[G_{\underline{a}}(M)]_P$ is a generalized C-M module with

$$[H_P^i(G_{\underline{a}}(M))]_n = \begin{cases} 0 & \text{if } n \neq 1-i, \\ H_{\underline{m}}^i(M) & \text{if } n = 1-i, \end{cases}$$

$i = 0,\dots,d-1$, and $[H_P^d(G_{\underline{a}}(M))]_n = 0$ if $n > 1-d$, where P denotes the maximal graded ideal of $G_{\underline{a}}(A)$.

COROLLARY 3.5. Suppose that M is a C-M module. Let $\underline{a}$ be an ideal of A with $l(M/\underline{a}M) < \infty$ and

$$l(M/\underline{a}^2M) = e(\underline{a};M) + dl(M/\underline{a}M).$$

Then $G_{\underline{a}}(M)$ is a C-M module.

COROLLARY 3.6. Suppose that

$$l(M/\underline{m}^2M) = e(\underline{m};M) + I(M) + dl(M/\underline{m}M).$$

Then $G_{\underline{m}}(M)$ is a graded Buchsbaum module.

## 4. REES MODULES

In the following we denote by $R_{\underline{a}}(M)$ the Rees module $\bigoplus_{i=0}^{\infty} \underline{a}^iM$ of M relative to an ideal $\underline{a}$ of A, which is also known under the name "arithmetical blowing-up". In [B], standard sequences were just introduced in order to study Rees modules.

Rees modules are closely connected with symmetric modules. Let

$S_{\underline{a}}(M)$ denote the symmetric module of M relative to $\underline{a}$. Then it is known that there is a natural homomorphism from $S_{\underline{a}}(M)$ onto $R_{\underline{a}}(M)$ which turns to be an isomorphism if $\underline{a}$ is generated by a d-sequence of M [H1], [T3]. Hence from Corollar 1.5 we get the following

PROPOSITION 4.1. Suppose that $a_1,\dots,a_d$ is a standard s.o.p of M. Then $R_{\underline{q}}(M) \cong S_{\underline{q}}(M)$.

Let Q denote the maximal graded ideal of $R_{\underline{q}}(A)$. Then our main result in this Section may be formulated as follows.

THEOREM 4.2. Suppose that $a_1,\dots,a_d$ is a standard s.o.p of M. Then $[R_{\underline{q}}(M)]_Q$ is a generalized C-M module with

$$[H_Q^0(R_{\underline{q}}(M))]_n = \begin{cases} H_{\underline{m}}^0(M) & \text{if } n = 0, \\ 0^0 & \text{if } n \neq 0, \end{cases}$$

$$[H_Q^i(R_{\underline{q}}(M))]_n = \begin{cases} H_{\underline{m}}^i(M) & \text{if } -1 \geqq n \geqq 2-i, \\ 0 & \text{else}, \end{cases}$$

$i = 1,\dots,d$, and $[H_Q^{d+1}(R_{\underline{q}}(M))] = 0$ if $n \geqq 0$.

By the statement of Theorem 4.2 we always have $H_Q^1(R_{\underline{q}}(M)) = 0$ and if $d \geqq 2$, $H_Q^2(R_{\underline{q}}(M)) = 0$. Thus, from Theorem 4.2 we can easily derive the following generalization of the main result of [GS] which dealt only with the case that M = A is a Buchsbaum ring:

COROLLARY 4.3. $R_{\underline{q}}(M)$ is a C-M module iff the following conditions are satisfied:

(i) $H_{\underline{m}}^i(M) = 0$ for $i \neq 1,d$.

(ii) $a_1,\dots,a_d$ is a standard s.o.p of M.

For the module-version of generalized C-M rings with maximal embedding dimension we have the following results:

PROPOSITION 4.4. Suppose that $d > 1$. Let $\underline{a}$ be an ideal of $A$ with $l(M/\underline{a}M) < \infty$ and

$$l(M/\underline{a}^2M) = e(\underline{a};M) + I(M) + d\,l(M/\underline{a}M).$$

Then $[R_{\underline{a}}(M)]_Q$ is a generalized C-M module with

$$[H_Q^0(R_{\underline{a}}(M))]_n = \begin{cases} H_{\underline{m}}^0(M) & \text{if } n = 0,1, \\ 0 & \text{if } n \neq 0,1, \end{cases}$$

$$[H_Q^1(R_{\underline{a}}(M))]_n = \begin{cases} H_{\underline{m}}^1(M) & \text{if } n = 0, \\ 0 & \text{if } n \neq 0, \end{cases}$$

$$[H_Q^i(R_{\underline{a}}(M))]_n = \begin{cases} H_{\underline{m}}^{i-1}(M) & \text{if } -1 \geqq n \geqq 3-i, \\ 0 & \text{else}, \end{cases}$$

for $2 = i = d$, and $[H_Q^{d+1}(R_{\underline{a}}(M))]_n = 0$ if $n \geqq 0$, where $Q$ denotes the maximal graded ideal of $R_{\underline{a}}(A)$.

By the statement of Proposition 4.4 we always have $H_Q^2(R_{\underline{a}}(M)) = 0$ and, if $d > 2$, $H_Q^3(R_{\underline{a}}(M)) = 0$. Note that for the case $d = 1$ we have the same formula for $H_Q^0(R_{\underline{a}}(M))$ and $H_Q^1(R_{\underline{a}}(M)) = 0$. Then from Proposition 4.4 we immediately get the following

COROLLARY 4.5. Let $\underline{a}$ be an ideal as in Proposition 4.4. Then $R_{\underline{a}}(M)$ is a C-M module iff $H_{\underline{m}}^i(M) = 0$ for $i \neq 2,d$.

In particular, using some recent result of Goto in [G3] we can show that the Rees ring of a Buchsbaum ring with maximal embedding dimension is again a graded Buchsbaum ring. That may be formulated in a more general statement as follows.

COROLLARY 4.6. Suppose that

$$l(M/\underline{m}^2M) = e(\underline{m};M) + I(M) + d\,l(M/\underline{m}M).$$

Then $R_{\underline{m}}(M)$ is a graded Buchsbaum module.

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## A note on standard systems of parameters for generalized C-M modules

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The purpose of this short note is just to mention that in equi-characteristic case the theory of Buchsbaum rings possibly includes the one of so-called standard systems of parameters for generalized Cohen-Macaulay modules.

More explicitly, let $A$ be a Noetherian local ring with maximal ideal $m$ and assume that $A$ contains a field as a subring. Let $M$ be a finitely generated A-module of dimension $d$. We denote by $\hat{A}$ (resp. $\hat{M}$) the m-adic completion of $A$ (resp. $M$). Let $a_1, a_2, \dots, a_d$ be a system of parameters for $M$ and put

$$R = k[[a_1, a_2, \dots, a_d]]$$

in $\hat{A}$, where $k$ denotes a coefficient field of $\hat{A}$. Then as is well-known, the ring $R$ is isomorphic to a formal power series ring with $d$ variables over $k$ and the R-module $\hat{M}$ is finitely generated. Moreover we have

<u>Theorem</u>. <u>The following conditions are equivalent to each other.</u>

(1) <u>The local cohomology modules</u> $H_m^i(M)$ <u>are finitely generated for all</u> $i \neq d$ <u>and the equality</u>

$$l_A(M/qM) - e_M(q) = \sum_{i=0}^{d-1} \binom{d-1}{i} \cdot l_A(H_m^i(M))$$

<u>holds, where</u> $q = (a_1, a_2, \dots, a_d)$.

(2) $\hat{M}$ <u>is a Buchsbaum R-module.</u>

(3) <u>The idealization</u> $R \ltimes \hat{M}$ <u>is a Buchsbaum ring.</u>

<u>Proof</u>. (1) $\Longrightarrow$ (2) First of all we like to show that

$$q.H_m^i(M/(a_1, a_2, \dots, a_j)M) = (0)$$

if $i + j < d$. If $d = 1$, we get from the exact sequence

$$0 \longrightarrow H_m^0(M) \longrightarrow M \longrightarrow M/H_m^0(M) \longrightarrow 0$$

an exact sequence

$$0 \longrightarrow H_m^0(M)/a_1H_m^0(M) \longrightarrow M/a_1M \longrightarrow M/(a_1M + H_m^0(M)) \longrightarrow 0$$

of A-modules. Consequently $l_A(M/a_1M) = e_M(a_1A) + l_A(H_m^0(M)/a_1H_m^0(M))$, and hence $a_1H_m^0(M) = (0)$ as $l_A(H_m^0(M)/a_1H_m^0(M)) = l_A(H_m^0(M))$ by assumption (1). Now assume that $d \geq 2$ and let $1 \leq k \leq d$ be a fixed integer. We put $M' = M/a_kM$ and $q' = (a_1, \dots, \hat{a}_k, \dots, a_d)A$. Then

$$\sum_{i=0}^{d-1}\binom{d-1}{i} l_A(H_m^i(M)) = l_A(M/qM) - e_M(q)$$
$$= l_A(M'/q'M') - e_{M'}(q')$$
$$\leq \sum_{i=0}^{d-2}\binom{d-2}{i} l_A(H_m^i(M'))$$
$$\leq \sum_{i=0}^{d-1}\binom{d-1}{i} l_A(H_m^i(M))$$

and therefore we get

$$a_k . H_m^i(M) = (0)$$

for all $i \neq d$ *) (hence $q.H_m^i(M) = (0)$ for all $i \neq d$ ) and that $l_A(M'/q'M') - e_{M'}(q') = \sum_{i=0}^{d-2}\binom{d-2}{i} l_A(H_m^i(M'))$ . Thus the induction on $d$ tells us that $q.H_m^i(M/(a_1, a_2, \dots, a_j)M) = (0)$ if $i + j < d$ .

Now let $\underline{n} = (a_1, a_2, \dots, a_d)R$ . Then if $i + j < d$, $n.H_n^i(\hat{M}/(b_1, b_2, \dots, b_j)\hat{M}) = (0)$ for any permutation $(b_1, b_2, \dots, b_d)$ of $a_1, a_2, \dots, a_d$ and therefore the induction on $d$ yields that $\hat{M}$ is a Buchsbaum R-module**).

(2)$\Longleftrightarrow$(3) This is well-known.

(2)$\Longrightarrow$(1) As $H_m^i(M) = H_n^i(\hat{M})$ , the A-module $H_m^i(M)$ is finitely generated for any $i \neq d$ . On the other hand as $k = R/n = \hat{A}/\hat{m}$ , we see that

$$l_A(M/qM) - e_M(q) = l_{\hat{A}}(\hat{M}/q\hat{M}) - e_{\hat{M}}(q\hat{A})$$
$$= l_R(\hat{M}/q\hat{M}) - e_{\hat{M}}(n)$$
$$= \sum_{i=0}^{d-1}\binom{d-1}{i} l_R(H_n^i(\hat{M}))$$
$$= \sum_{i=0}^{d-1}\binom{d-1}{i} l_A(H_m^i(M)) .$$

This completes the proof of Theorem.

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*) c.f. (2.6) (2) in : Blowing-up of Buchsbaum rings, Commutative Algebra: Durham 1981, London Mathematical Society Lecture Note Series 72, 140-162.

**) c.f. (2.12) in : Noetherian local rings with Buchsbaum associated graded rings, to appear in J. Alg.

# Quasi-Buchsbaum rings obtained by idealizations

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Abstract and introduction.

This note is devoted to introducing the results given in paper [4] which is preparated now together with S. Goto. The aim of our research is to determine the criterions for local rings obtained by idealizations to be quasi-Buchsbaum and Buchsbaum and evaluate the distance between quasi-Buchsbaum rings and Buchsbaum rings using such rings.

Let $A$ be a Noetherian local ring with maximal ideal $m$ and $M$ a finitely generated A-module. We denote by $A \ltimes M$ the idealization of $M$ (over $A$) which is the direct sum $A \oplus M$ as an A-module and endowed the multiplication defined by $(a, x).(b, y) = (ab, ay + bx)$, where $a, b \in A$ and $x, y \in M$ ([7]). As is well-known, $A \ltimes M$ is a Noetherian local ring with maximal ideal $m \times M$ and $\dim A \ltimes M = \dim A$. If $A \ltimes M$ is a Buchsbaum ring, then so are $A$ and $M$ and $\dim M = 0$ or $\dim A$. In case $\dim M = 0$, it is already given by [2] that $A \ltimes M$ is Buchsbaum if and only if so is $A$ and $M$ is a vector space over $A/m$, i.e., $m.M = (0)$. So we will discuss in this note the remaining case $\dim M = \dim A$.

In Section 1, we shall study the criterions for local rings obtained by idealizations to be quasi-Buchsbaum (resp. Buchsbaum). We get the following

Theorem (1.2). Let $A$ be a Buchsbaum ring of $\dim A = d > 0$ and $M$ a Buchsbaum A-module of $\dim M = \dim A$. Then the following two conditions are equivalent.

(1) $A \ltimes M$ is a quasi-Buchsbaum (resp. Buchsbaum) ring.

(2) For some (resp. every) system $\{a_1, a_2, \ldots, a_d\}$ of parameters for $A$, the equality

$$[(a_1, \ldots, a_{d-1}) : a_d].M = (a_1, \ldots, a_{d-1}).M$$

holds.

As is given in [6],[8] and [9], the canonical module $K_A$ of a Buchsbaum ring A, if there exists, is also a Buchsbaum A-module and $\dim K_A = \dim A$. We shall discuss the idealizations of the canonical modules in Section 2 and also we shall construct Buchsbaum rings which have the canonical modules such that its idealizations are also Buchsbaum. The typical examples of local rings obtained by idealizations are given in Section 3.

## §1. Criterions for idealizations to be quasi-Buchsbaum and Buchsbaum.

以下，この§では $A$ は Buchsbaum 局所環で $\dim A = d > 0$，$\mathfrak{m}$ はその極大イデアルとする。さらに，$M$ は Buchsbaum $A$-加群で，自然な単射 $a \mapsto (a, 0)$ によって $A$ は $A \ltimes M$ の部分環とみなす。

<u>補題</u> (1.1). $\{a_1, \cdots, a_d\}$ は $A$ の s.o.p. とする。すると次の条件は同値である。

(1) $a_1, \cdots, a_d$ は $A \ltimes M$ の中で weak-sequence をなす。
i.e., 各 $1 \leq i \leq d$ に対して，

$$(\mathfrak{m} \times M) \cdot [(a_1, \cdots, a_{i-1}) \underset{A \ltimes M}{:} a_i] = (a_1, \cdots, a_{i-1}) \cdot (A \ltimes M).$$

(2) $$[(a_1, \cdots, a_{d-1}) \underset{A}{:} a_d] \cdot M = (a_1, \cdots, a_{d-1}) M.$$

<u>定理</u> (1.2). 次は同値。

(1) $A \ltimes M$ は quasi-Buchsbaum 環（あるいは Buchsbaum 環）である。

(2) Aの適当な（あるいは，すべての）s.o.p. $\{a_1, \cdots, a_d\}$ に対して，

$$[(a_1, \cdots, a_{d-1}) \underset{A}{:} a_d] \cdot M = (a_1, \cdots, a_{d-1}) M .$$

系 (1.3). 次は同値。

(1) A は Cohen-Macaulay 環.

(2) $\dim M = \dim A$ であるすべての Buchsbaum A-加群 M に対して，$A \ltimes M$ は quasi-Buchsbaum.

(3) $\dim M = \dim A$ であるすべての Buchsbaum A-加群 M に対して，$A \ltimes M$ は Buchsbaum.

注意 (1.4). $A = \hat{A}$ ならば，M は適当な Buchsbaum ring のイデアルと同一視できる。実際，CM 環 R で $\dim R = \dim A$，$A = R/\sim$ となるものをとり，イデアル化 $R \ltimes M$ を考えればよい。

命題 (1.5).

(1) 各 $0 \leq i \leq d-1$ について，$H^i_m(A) = (0)$ かまたは $H^i_m(M) = (0)$ ならば，$A \ltimes M$ は quasi-Buchsbaum 環である.

(2) 整数 $0 \leq r \leq d$ について，$H^i_m(A) = (0)$ $(i \geq r, i \neq d)$ でかつ $H^i_m(M) = (0)$ $(i \leq r-2)$ ならば，$A \ltimes M$ : Buchsbaum 環 $\Longleftrightarrow$ $A \ltimes M$ : quasi-Buchsbaum 環である.

系 (1.6). A は canonical module $K_A$ をもつものとする。もし depth $K_A > \frac{d+1}{2}$ ならば, $A \ltimes K_A$ は Buchsbaum 環である。(cf. §2).

## §2. Idealization of canonical modules.

この§では, A は Buchsbaum 局所環で $\dim A = d > 0$, $\mathfrak{m}$ はその極大イデアルとし, さらに A は canonical module $K_A$ を持つものとする。[6, 8] や [9] が示すように, $K_A$ は Buchsbaum A-加群で $\dim K_A = \dim A$ である。そこで, $K_A$ のイデアル化を考える。

補題 (2.1). $\{a_1, \cdots, a_d\}$ は A の s.o.p. とする。すると, 各 $1 \leqq t \leqq d$, $\ell \geqq 2$ に対して,

$$[(a_1^\ell, \cdots, a_d^\ell) :_A (a_1, \cdots, a_t)] = \sum_{i=1}^{t} [(a_1^\ell, \cdots, \widehat{a_i^\ell}, \cdots, a_d^\ell) : a_i] + \left\{\bigcap_{i=1}^{t} (a_1^\ell, \cdots, a_i^{\ell-1}, \cdots, a_d^\ell)\right\}.$$

定理 (2.2). 次は同値。

(1) $A \ltimes K_A$ は quasi-Buchsbaum 環 (あるいは, Buchsbaum 環)。

(2) 適当な (あるいは, すべての) A の s.o.p. $\{a_1, \cdots, a_d\}$ に対して,

$$[(0) : [(a_1, \cdots, a_{d-1}) : a_d]]_{H_\mathfrak{m}^d(A)} = [(0) : (a_1, \cdots, a_{d-1})]_{H_\mathfrak{m}^d(A)}.$$

(3) 適当な（あるいは，すべての）A の s.o.p $\{a_1, \cdots, a_d\}$ に対して，

$$\left[(a_1, \cdots, a_{d-1}) \underset{A}{:} a_d\right]\left\{\bigcap_{i=1}^{d-1}(a_1^{\ell}, \cdots, a_i^{\ell+1}, \cdots, a_d^{\ell})\right\} \subseteq \sum_{i=1}^{d}\left[(a_1^{\ell}, \cdots, \hat{a_i^{\ell}}, \cdots, a_d^{\ell}) : a_i\right]$$

がほとんどすべての $\ell \gg 0$ について成立する。

$\dim A = 3$ のときは特別な様相を呈する。

<u>定理</u> (2.3) $\dim A = 3$ とする。次は同値。

(1) $A \ltimes K_A$ は quasi-Buchsbaum 環である。

(2) $A \ltimes K_A$ は Buchsbaum 環である。

(3) A の s.o.p $\{a, b, c\}$ で，

$$U_A(a,b)\,U_A(b,c) = (a,b)\,U_A(b,c) + (b,c)\,U_A(a,b)$$

をみたすものが存在する。ただし $U_A(\cdot)$ は準素分解において unmixed 成分を表わす。

<u>命題</u> (2.4). $p > 0$ は素数，$ch(A) = p$ とする。A は F-pure, i.e., $F : A \to A$ ($F(a) = a^p$) が pure，とする。すると，

$A \ltimes K_A$ : Buchsbaum $\Longleftrightarrow$ $A \ltimes K_A$ : quasi-Buchsbaum.

さて，canonical module のイデアル化がまた Buchsbaum 環となるような Buchsbaum 環の例は (3.2) でも与えるが，実際に非常に

たくさん存在する。

定理 (2.5). $d \geqq 2$ と $h_1, \cdots, h_{d-1} \geqq 0$ は整数とする。このとき、次の3つの条件をみたす Buchsbaum 局所整域 A が存在する。

(i) $\dim A = d$ でかつ $\dim_{A/\mathfrak{m}} H^i_{\mathfrak{m}}(A) = h_i \quad (1 \leqq i \leqq d-1)$, ただし $\mathfrak{m}$ は A の極大イデアルとする。

(ii) A は canonical module $K_A$ をもつ。

(iii) A と $K_A$ は Buchsbaum 環である。

さらに、(iv) もし $h_1 = 0$ なら、A は normal にとれる。

この定理を示すには、次数付き環の議論を必要とする。以下、$R = \bigoplus_{n \geq 0} R_n$ は次数付き Noether 環で $\dim R = d$, $R_0 = k$ (体) とし、$M = R_+$ とおく。よく知られているように、R は canonical module $K_R = (H^d_M(R))^*$ をもつ [5], ただし $(\ )^*$ は graded $k$-dual を表わす。

補題 (2.6). すべての $i \neq d$ について、$H^i_M(R)$ は有限生成であるならば、 $H^i_M(K_R) \cong (H^{d-i+1}_M(R))^* \quad (2 \leqq i \leqq d-1)$.

さらに，$A = R_M$ とおけば，A は canonical module $K_A = (K_R)_M$ をもつ Krull 次元 d の Noether 局所環である。
$a(R) = \max\{n \in \mathbb{Z} \mid [H_M^d(R)]_n \neq (0)\}$ とおくと，

<u>補題</u> (2.7). $t_0, \ldots, t_{d-1}$ は次の条件をみたす整数とする；
各 $0 \leqq i \leqq d-1$ について，

(i) $t_i \leq t_{i+1} + 1$ ,

(ii) $[H_M^i(R)]_n = (0) \quad (n \neq t_i)$ .

もし，$a(R) < \min\{t_i + t_j \mid i+j = d+1\}$ ならば，$A \ltimes K_A$ は Buchsbaum 環である。

定理 (2.5) は (2.6) と (2.7) と [3 の §5] から得る。

§3. Examples.

(3.1). $A \ltimes \mathfrak{m}$ は Buchsbaum 環である。

(3.2). R は Cohen-Macaulay 環で $\dim R > 0$ ，R は canonical module をもつものとする。さらに，M は Buchsbaum R-加群で $\dim M = \dim R$ とする。今，

$$A = R \ltimes M$$

とおけば，A は Buchsbaum 局所環で canonical module $K_A$ を

持ち，ともに $A \ltimes K_A$ は Buchsbaum 環である。

(3.3). $R = k[[X_1, \cdots, X_6]]$ ($k$ は体), $\mathfrak{a} = (X_1, X_2, X_3) \cap (X_2, X_3, X_4) \cap (X_3, X_4, X_5) \cap (X_4, X_5, X_6) \cap (X_5, X_6, X_1) \cap (X_6, X_1, X_2)$ とおく。今 $A = R/\mathfrak{a}$ とおけば，$A$ は 3 次元の Buchsbaum 局所環で canonical module $K_A$ をもち，ともに $A \ltimes K_A$ は Buchsbaum 環である。

(3.4). $k$ は体，$k[X, Y, Z]$，$k[V, W]$ は多項式環とする。今

$$R = \left( k[X, Y, Z] / (X^3 + Y^3 + Z^3) \right) \# \, k[V, W]$$

，ここで $\#$ は Segre 積を表わす，

$$M = R_+$$

とおけば，$A = R_M$ は 3 次元 Buchsbaum 局所環で canonical module $K_A$ を持ち，ともに $A \ltimes K_A$ は Buchsbaum 環である。$\hat{A}$ は $(S_2)$ をみたすから，$B = A \ltimes K_A$ は quasi-Gorenstein [1], i.e., $K_B \cong B$，よって，$B \ltimes K_B$ はもはや Buchsbaum 環ではない。

(3.5). $R = k[[X_1, \cdots, X_d, Y_1, \cdots, Y_d]]$ ($k$ は体) とする。今，

$$A = R / (X_1, \cdots, X_d) \cap (Y_1, \cdots, Y_d)$$

とおき，ともに

$$M = A / \mathfrak{m}(X_1, \cdots, X_d)$$

とおけば，$A$, $M$ は共に Buchsbaum で $\dim M = \dim A$ である。

ところが，このとき $A \ltimes M$ は quasi-Buchsbaum 環ではあるが Buchsbaum 環ではない。

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# Locally Simple Extensions of Rings

Osaka Univ. Ken-ichi Yoshida

In this talk, we study the following obstraction ideal of flatness with respect to ring extension $A/R$.

Definition. $F_R(A) := \{a \in R \mid a \neq 0, A[1/a]/R[1/a]$ is flat$\} \cup \{0\}$.

Let $R$ be a noetherian domain and let $A$ be finitely generated over $R$. Then we have

(i) $F_R(A)$ is a non-zero radical ideal, and

(ii) for $p \in \operatorname{Spec} R$, $p \not\supset F_R(A)$ if and only if $A_p/R_p$ is flat.

If $A/R$ is a birational extension, then the prime divisor of $F_R(A)$ is a prime ideal of depth one.

Even if $A$ is integral over $R$, the same result does not hold.

Example. There exists a non-Cohen Macaulay complete local domain $A$ of dimension 3. Hence there exists a subring $R$ of $A$ such that $R$ is regular and $A$ is a finite $R$-module. If we can show that the prime divisor of $F_R(A)$ is a prime ideal of depth one, then $A/R$ is flat since $R$ is normal, so $A$ is Cohen Macaulay. This is a contradiction.

Therefore we look for a condition that our assertion holds.

Lemma. Let R be a noetherian domain and let A be a finite extension of R. If A is locally simple extension over R, then the prime divisor of $F_R(A)$ is a prime ideal of depth one.

W.Vasconcelos had the following result in the paper [ Simple flat extensions, J. Algebra, 16, 105-107];

Let S be a simple extension of R and let

$$0 \longrightarrow I \longrightarrow R[X] \longrightarrow S \longrightarrow 0 \quad \text{(exact)}.$$

Then, S is a flat extension of R if and only if I is a projective ideal of R[X], and the ideal of R generated by the coefficients of the polynomials in I [ the so-called content of I, notation: c(I)] is generated by an idempotent element of R.

Hence if R is an integral domain and S is a flat extension of R, then c(I) = (0) or R. And the radical ideal $\sqrt{c(I)}$ does not depend on the choice of the generator in S/R. Indeed, it is easily seen that, for $p \in \operatorname{Spec} R$, $p \not\supset c(I)$ if and only if Spec $A_p/R_p$ is a finite set (may be empty). Hence we have that a simple extension A/R is quasi-finite if and only if c(I) = R.

Lemma. Assume that R is a local domain and the residue field is an infinite field. Let A be a simple extension of R.

If the extension A/R is quasi-finite, then there exists an element $\alpha$ of A such that $\alpha$ is integral over R and $A = R[\alpha][1/\alpha]$.

Applying this lemma, we have the following;

Proposition. Let $R$ be a noetherian domain and let $A$ be finitely generated over $R$. Assume that the extension $A/R$ is locally simple extension and quasi-finite. Define

$\Delta := \{P \in \operatorname{Spec} A \mid A_P/R_{P\cap R}$ is not flat$\}$. Then $\Delta$ is a closed set of Spec $A$. If $P$ is the generic point of an irreducible component in $\Delta$, then we have depth $R_{P\cap R} = 1$.

Theorem. Under the above conditions, the prime divisor of $F_R(R)$ is a prime ideal of depth one.

On the Gorensteinness of the variety of complexes

Yuji Yoshino ( Nagoya University )

Let $R$ be a Noetherian ring and let $\{n_0,n_1,..,n_m\}$ $\{k_1,k_2,..,k_m\}$ be two sequences of integers satisfying $m > 0$ , $k_i \geqq 0$ and $n_i \geqq k_i + k_{i+1}$ with $k_0 = k_{m+1} = 0$ . We consider the m-ple of matrices $( X^{(1)}, X^{(2)},.., X^{(m)} )$, where $X^{(s)} = ( x_{ij}^{(s)} )$ is an $n_{s-1} \times n_s$ matrix of indeterminates over $R$ $(s=1,2,..,m)$. We now define an R-algebra $B_R\binom{n_0,n_1,\dots,n_m}{k_1,..,k_m}$ as a factor ring of $R[\, x_{ij}^{(s)} \mid$ all s, i and j $]$ by an ideal generated by all the elements of matrices $X^{(s-1)}X^{(s)}$ and the determinants of the minors of $X^{(s)}$ of size $(k_s+1)$ $(s=1,2,..,m)$.

$B_R\binom{n_0,n_1,\dots,n_m}{k_1,..,k_m}$ is the homogeneous coordinate ring of the variety parameterizing all the complexes of the form;

$$0 \to R^{n_m} \xrightarrow{f_m} R^{n_{m-1}} \xrightarrow{f_{m-1}} \cdots\cdots \to R^{n_1} \xrightarrow{f_1} R^{n_0} \to 0$$

where $\mathrm{rank}(f_i) = k_i$ . It is hence called the ( Buchsbaum-Eisenbud ) algebra of variety of complexes. We should notice here that if $m = 1$ , then the variety of complexes are nothing but determinantal varieties.

The purpose of the present note is to provide a necessary and sufficient condition for $B_R\binom{n_0,n_1,\dots,n_m}{k_1,..,k_m}$ to be Gorenstein. To this end we need some elementary properties of the varieties of complexes. Particularly its Hodge algebra structure is one of the most important among them.

Definition of Hodge algebra. Let $H$ be a finite poset (= partially ordered set). A monomial on $H$ is an element of $\mathbb{N}^H$. A subset $\Sigma \subset \mathbb{N}^H$ is called an ideal of monomials if $M \in \Sigma$ and $N \in \mathbb{N}^H$ implies $MN \in \Sigma$ . A monomial $M$ is called standard with respect to $\Sigma$ if $M \notin \Sigma$ . A generator of an ideal $\Sigma$ is an element of $\Sigma$ which is not divisible by any other element of $\Sigma$ .

Now let A be an R-algebra and suppose that there is an injection $\phi : H \to A$. We call A a Hodge algebra generated by $\phi(H)$ and governed by $\Sigma$ if the following axioms are satisfied;

(H-1) A is a free R-module admitting the set of standard monomials ( with respect to $\Sigma$ ) as basis.

(H-2) If $N \in \Sigma$ is a generator and

(*) $N = \sum_i r_{N,i} M_{N,i}$ $\quad ( 0 \neq r_{N,i} \in R )$

is the unique expression for $N \in A$ as a linear combination of standard monomials, then for each $x \in H$ which divides N and for each $M_{N,i}$ , there is $y_{N,i} \in H$ which divides $M_{N,i}$ and satisfies $y_{N,i} < x$ in H .

The relations (*) are called the straightening relations for A .

Now let us return to the question of the variety of complexes. The symbol $[i_1,i_2,..,i_t|j_1,j_2,..,j_t]_s$ will denote the element of $B_R(^{n_0,n_1,..,n_m}_{k_1,..,k_m})$ which is given by the determinant of the minor of the matrix $X^{(s)}$ whose rows are those of indices $i_1,i_2,..,i_t$ and whose columns are those of indices $j_1,j_2,..,j_t$ .

Let H be a set consisting of all the determinants $\{ [i_1,i_2,..,i_t|j_1,j_2,..,j_t]_s \mid 1 \leqq s \leqq m , 1 \leqq t \leqq k_s , 1 \leqq i_1 < i_2 <.. < i_t \leqq n_{s-1} , 1 \leqq j_1 < ... < j_t \leqq n_s \}$ .

We partially order H by the following: When $x = [i_1,i_2,..,i_t|j_1,j_2,..,j_t]_s$ and $x' = [i'_1,i'_2,..,i'_{t'}|j'_1,j'_2,..,j'_{t'}]_{s'}$ , x and x' are incomparable if $s \neq s'$ , while if $s = s'$ , then $x \leqq x'$ if $t \geqq t'$ and $i_u \leqq i'_u$ , $j_u \leqq j'_u$ for $u=1,2,..,t'$.

The product $xx'$ ( $s \leqq s'$ ) is said to be standard if one of the following holds;

(1) $s' > s + 1$ ,

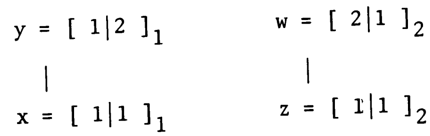
(2) $s' = s$ , and x and x' are comparable in the partial order on H ,

(3) $s' = s + 1$ , $n_s - t \geqq t'$ and writing $u_1 < u_2 < ... < u_{n_s-t}$ for the complement of $\{ j_1,j_2,..,j_t \}$ in $\{ 1,2,..,n_s\}$ , we have $u_{n_s-t-v+1} \geqq i'_{t'-v+1}$ for $v=1,2,..,t'$.

We define an arbitrary product $x_1x_2...x_n$ of minors to be standard if each $x_ix_j$ is standard in suitable order. Finally we set $\Sigma$ as a set of all non-standard monomials.

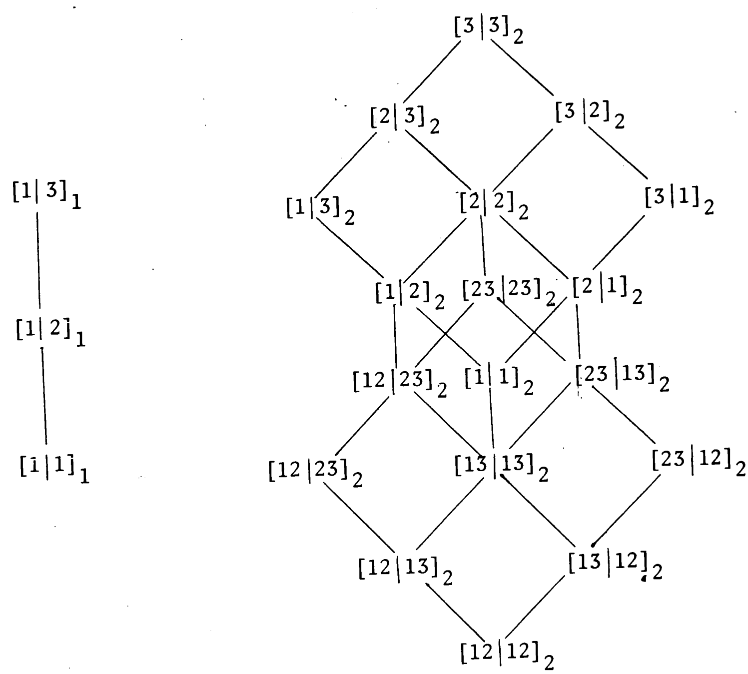
Then the theorem of DeConcini and Strickland says that $B_R\binom{n_0, n_1, \ldots, n_m}{k_1, \ldots, k_m}$ is a Hodge algebra over R generated by H and governed by $\Sigma$ .

<u>Examples.</u> (1) If we consider $B_R\binom{1,2,1}{1,1} = R[\,(x, y)\,,\,\binom{z}{w}\,]\,/\,(xz+yw)$, the poset H is described in the following:




Although yw is a non-standard monomial, we have the equality yw = - xz where xz is standard and $x < y$ , $z < w$ . In this manner one can easily see that $B_R\binom{1,2,1}{1,1}$ is in fact a Hodge algebra over R .

(2) The poset of $B_R\binom{1,3,3}{1,2}$ is ;




A non-standard monomial $[1|3]_1[3|3]_2$ can be written as a linear combination of standard monomials in the following way ; $[1|3]_1[3|3]_2 = -\,[1|1]_1[1|3]_2 - [1|2]_1[2|3]_2$ and we have also the equality $[1|2]_1[23|23]_2 = -\,[1|1]_1[13|23]_2$ where the left hand side is non-standard but the right is standard, and so on.

Using the Hodge algebra structure one can prove that $B_R\binom{n_0, n_1, \ldots, n_m}{k_1, \ldots, k_m}$ is Cohen-Macaulay ( resp. normal ) if and only if R is Cohen-Macaulay ( resp. normal). This result was proved by DeConcini — Strickland, and independently by Huneke. So it seems to be natural to ask when $B_R\binom{n_0, n_1, \ldots, n_m}{k_1, \ldots, k_m}$ is Gorenstein. Our main result about this question is the following

<u>Theorem 1</u>. Assume that $k_i > 0$ ( i=1,2,..,m ). If we denote $t_i = n_i - k_i - k_{i+1}$ ( i=0,1,..,m ), then $B_R\binom{n_0, n_1, \ldots, n_m}{k_1, \ldots, k_m}$ is Gorenstein if and only if R is Gorenstein and one of the following conditions holds;

(1) $t_0 = t_1 = \ldots = t_m$

(2) $t_0 = 0\ ,\ t_1 = t_2 = \ldots = t_m$

(3) $t_m = 0\ ,\ t_0 = t_1 = \ldots = t_{m-1}$

(4) $t_0 = t_m = 0\ ,\ t_1 = t_2 = \ldots = t_{m-1}$ .

<u>Examples</u>. (1) $B_R\binom{1,2,1}{1,1}$ is Gorenstein whenever R is Gorenstein, for $t_0 = t_1 = t_2 = 0$ .

(2) $B_R\binom{1,3,3}{1,2}$ is not Gorenstein, since $t_0 = t_1 = 0$ and $t_2 = 1$ .

(3) Applying our theorem to the determinantal case, we will see that $B_R\binom{n_0, n_1}{k_1}$ is Gorenstein if and only if R is Gorenstein and $(n_0 - n_1)(n_0 - k_1)(n_1 - k_1) = 0$.

To prove Theorem 1 I needed to compute the divisor class group of $B_R\binom{n_0, n_1, \ldots, n_m}{k_1, \ldots, k_m}$ explicitly in normal case. The consequence I got is ;

<u>Theorem 2</u>. Let R be a normal domain. Then there is a group isomorphism ;

$\mathrm{Cl}(\ B_R\binom{n_0, n_1, \ldots, n_m}{k_1, \ldots, k_m}) \simeq \mathrm{Cl}(R) \oplus \mathbf{Z}^h$ , where $h = \#\{i \mid 0 < k_i < n_i\ ,\ t_{i-1} > 0\} + \#\{i \mid 0 < k_i < n_i\ ,\ t_i = t_{i-1} = 0\}$ .

<u>Examples</u>. (1) $\mathrm{Cl}(\ B_R\binom{1,2,1}{1,1}\ ) \simeq \mathrm{Cl}(R) \oplus \mathbf{Z}$ .

(2) $\mathrm{Cl}(\ B_R\binom{1,3,3}{1,2}\ ) \simeq \mathrm{Cl}(R) \oplus \mathbf{Z}$ .

(3) If $m = 1$ , then $\mathrm{Cl}(\ B_R\binom{n_0, n_1}{k_1}\ ) \simeq \mathrm{Cl}(R) \oplus \mathbf{Z}^h$ , where $h = 1$ ( if $0 < k_1 < \min(\ n_0, n_1\ )$ ), 0 ( otherwise ).

We shall close this note with showing the outline of the proof of Theorem 1. For the detail we refer the reader to our paper [3].

<u>Outline of the proof of Theorem 1.</u>

<u>$1^{st}$ step.</u> We may assume that $R$ is a field. ( This reduction is quite easy and elementary.)

<u>$2^{nd}$ step.</u> Proving the following

<u>Proposition.</u> Assume that $k_i \geqq 2$ for some $i$ . Then $B_R\left(\begin{smallmatrix} n_0, & n_1, \ldots, & n_{i-1}, & n_i, \ldots, & n_m \\ & k_1, \ldots\ldots, & & k_i, \ldots, & k_m \end{smallmatrix}\right)$ is Gorenstein if and only if $B_R\left(\begin{smallmatrix} n_0, & n_1, \ldots, & n_{i-1}-1, & n_i-1, \ldots, & n_m \\ & k_1, \ldots\ldots, & & k_i-1, \ldots, & k_m \end{smallmatrix}\right)$ is.

( For the proof of this proposition we use ( the proof of ) Theorem 2.) Continueing this process, we may assume that $k_1 = k_2 = \ldots\ldots = k_m = 1$.

<u>$3^{rd}$ step.</u> Computing the Poincaré series of $B_R\left(\begin{smallmatrix} n_0, & n_1, \ldots, & n_m \\ & 1, \ldots, & 1 \end{smallmatrix}\right)$ explicitly and apply the following theorem of Stanley :

<u>Theorem.</u> Let $B$ be an $\mathbb{N}$-graded algebra over a field $k = B_0$ and suppose that $B$ is a Cohen-Macaulay integral domain of dimension d. If $P(\lambda)$ is the Poicaré series defined by $P(\lambda) = \sum_{n=0}^{\infty} \dim_k(B_n).\lambda^n \in \mathbf{Z}[[\lambda]]$ , then $B$ is Gorenstein if and only if for some $a \in \mathbf{Z}$ , $P(1/\lambda) = (-1)^d \lambda^a P(\lambda)$ .

We should notice that the Poicaré series will be obtained by counting the number of standard monomials. For instance if we consider $B_R\left(\begin{smallmatrix} n_0, & n_1, & n_2 \\ & 1, & 1 \end{smallmatrix}\right)$ in case $m = 2$ , then it has standard monomials as an R-base, and all the elements of the R-base of $B_R\left(\begin{smallmatrix} n_0, & n_1, & n_2 \\ & 1, & 1 \end{smallmatrix}\right)$ of degree $n$ are described in the following diagram ;

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_u & b_u \end{bmatrix}_1 \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \\ \vdots & \vdots \\ c_v & d_v \end{bmatrix}_2$$

where $1 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_u \leqq n_0$ , $1 \leqq b_1 \leqq b_2 \leqq \ldots \leqq b_u \leqq n_1$ , $1 \leqq c_1 \leqq c_2 \leqq \cdots \leqq c_v \leqq n_1$ , $1 \leqq d_1 \leqq d_2 \leqq \cdots \leqq d_v \leqq n_2$ , and $c_v < n_1$ whenever $b_u = n_1$ , and $n = u + v$ . Thus the number of standard monomials of degree $n$ is given by

$$c^{(n)} = \sum_{u+v=n} \binom{u+n_0-1}{n_0-1} \left\{ \binom{u+n_1-2}{n_1-2} \binom{v+n_1-1}{n_1-1} + \binom{u+n_1-2}{n_1-1} \binom{v+n_1-2}{n_1-2} \right\} \binom{v+n_2-1}{n_2-1}$$

After some computations, we shall have the following equality as a result.

$$(1-\lambda)^{n_0+2n_1+n_2-3} P(\lambda) = \left\{ \sum_{j\geqq 0} \binom{n_0-1}{j} \binom{n_1-2}{j} \lambda^j \right\} . \left\{ \sum_{j\geqq 0} \binom{n_1-1}{j} \binom{n_2-1}{j} \lambda^j \right\}$$
$$+ \left\{ \sum_{j\geqq 0} \binom{n_0}{j} \binom{n_1-2}{j-1} \lambda^j \right\} . \left\{ \sum_{j\geqq 0} \binom{n_1-2}{j} \binom{n_2-1}{j} \lambda^j \right\} ,$$

where both sides of the equality are polynomials with the coefficients in $\mathbb{Z}$ . Then one can easily verify that $P(\lambda)$ satisfies the equality in the theorem of Staley if and only if one of the following holds ; (1) $n_0 + 1 = n_1 = n_2 + 1$ , (2) $n_0 = 1$ and $n_1 = n_2 + 1$ , (3) $n_2 = 1$ and $n_0 + 1 = n_1$ , (4) $n_0 = n_2 = 1$ .
This implies Theorem 1 in case $m = 2$ .

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## Problem Session

Problem Sessionは11月5日の夜開かれました。8時頃から2時間半あまり熱心な討論が交され，6人の方からあわせて13の問題が提起されました。

1. 浅沼 照雄（富山大） Problem 1, 2.
2. 石川 武志（都立大） Problem 3, 4.
3. 松村 英之（名大） Problem 5, 6.
4. N.V. Trung（Hanoi大／名大） Problem 7, 8, 9, 10.
5. 吉田 憲一（阪大） Problem 11, 12.
6. M.E. Sweedler（Cornell大／筑波大） Problem 13.

Sweedler氏から「全員が順々に前に出て話をしていき，気おくれがして話すことのできない人は，十分勇気がでるまで目の前のビールを飲むことにしよう」というユーモアたっぷりな提案が飛び出すなど，Sessionはなごやかな雰囲気の中で進められました。残念ながら時間が足りず全員が発言するということは実現できませんでしたが，大変有意義なSessionとなりました。以下に提起された問題の詳細を報告致します。

## Problems posed in the Problem Session

### Teruo Asanuma

Let $R[X,Y]$ be a polynomial ring over a commutative ring $R$. Let $I$ be an ideal of $R$ and put $\bar{R} = R/I$.

Problem 1. Let $f, g$ be elements of $\bar{R}[X,Y]$ such that $\bar{R}[X,Y] = \bar{R}[f,g]$. Then can we find elements $F, G$ in $R[X,Y]$ so that $f = F \bmod IR[X,Y]$, $g = G \bmod IR[X,Y]$ and $R[X,Y] = R[F,G]$ ?

Problem 2. When is the canonical homomorphism $Sl_2(R) \longrightarrow Sl_2(\bar{R})$ onto?

In general this map is not surjective. For example, let $R = \mathbb{R}[X,Y]$ and $I = (X^2 + Y^2 - 1)R$. Then $Sl_2(R)$ contains no element corresponding to $\begin{pmatrix} \bar{X} & \bar{Y} \\ -\bar{Y} & \bar{X} \end{pmatrix}$ in $Sl_2(\bar{R})$.

### Takeshi Ishikawa

For an ideal $I$ of an Artinian local ring $(R,m)$, let $T(I) = [l_R(R) - l_R(0 : I)]/l_R(I)$ (here $l_R(.)$ stands for length). We put

$$T(R) = \sup_I T(I),$$

where $I$ runs over ideals of $R$. Then we can show the following

Propostion. (1) Let $r = l_R(0 : m)$. Then $1/r \leqq T(I) \leqq r$ for any ideal $I$ of $R$ and hence $1 \leqq T(R) \leqq r$.

(2) The following conditions are equivalent to each other: (a) $T(R) = r$. (b) $T(I) = r$ for some ideal $I$ of $R$. (c) $R$ is a Gorenstein ring.

Notice that $R$ is not a Gorenstein ring even if $T(R) = 1$. For example, let $R = k[X_1,X_2, \ldots , X_n]/(X_1,X_2, \ldots , X_n)^2$ ( $k$ a field). Then $T(R) = 1$, since $T(I) = 1/l_R(I)$ for any ideal $I$ of $R$. On the other hand $r = n$, as is well-known.

Problem 3. Let $S = k[X_1,X_2, \ldots , X_n]$ be a polynomial ring over a field $k$ and $M = (X_1,X_2, \ldots , X_n)S$. Then is it true that $T(S/I) = 1$ for any $M$-primary ideal $I$ of $S$ ?

I like to mention that for any integer $r \geqq 2$ and for any real number $t > 0$, there exists an Artinian local ring $R$ such that

$$r - t < T(R) < r.$$

Examples are easily constructed.

Problem 4. Let $R$ be a Noetherian local ring which is not necessarily Artinian. Let $T(R) = \sup_q T(R/q)$, where $q$ runs over parameter ideals in R . Explore this invariant $T(R)$ of R . For example, is $T(R)$ finite? Compare the numbers $T(R)$ and $T(R[X]/(X^2))$ .

Hideyuki Matsumura

Let $(A,m)$ be a Noetherian local ring.

Let D be a derivation on A . Then we say that D is integrable if there exists a homomorphism $E : A \longrightarrow A[\![t]\!]$ such that

$$E(a) = a + tD(a) \mod t^2 .$$

Assume that A contains no field and put $p = \mathrm{ch}(A/m)$ . Then any derivation D on A induces a derivation $\bar{D}$ on $\bar{A} = A/pA$ if $D(p) = 0$ and it is easy to show that $\bar{D}$ is non-integrable if so is D .

Problem 5. Find any example of derivation D on A such that D is not integrable but $\bar{D}$ is integrable.

Let $A = k[\![X_1, X_2, \ldots, X_n]\!]$ be a formal power series ring over a field k . Let P be a prime ideal of A and assume that P is a complete intersection, say $P = (X_1, X_2, \ldots, X_r)$ ( $r = \dim A_P$ ).

Problem 6. Does there exist an integer $N = N(P) > 0$ with the following property?

If $g_1, g_2, \ldots, g_r$ are elements in A such that $g_i = f_i$ mod $(X_1, X_2, \ldots, X_r)^N$ for any $1 \leqq i \leqq r$ and if $\mathrm{ht}_A(g_1, g_2, \ldots, g_r) = r$ , then $(g_1, g_2, \ldots, g_r)$ is a prime ideal in A .

Ngo V. Trung

A system $a_1, a_2, \ldots, a_s$ of elements in a Noetherian local ring $(A,m)$ is called a weak sequence if

$$(a_1, \ldots, a_{i-1}) : a_i = (a_1, \ldots, a_{i-1}) : m$$

for any $1 \leqq i \leqq s$ . A weak sequence $a_1, a_2, \ldots, a_s$ is said to be maximal if all the $a_i$"s are in m and $a_1, a_2, \ldots, a_s, a$ cannot be a weak sequence for any element a in m .

Problem 7 (W. Vogel, 1973). Do the length of two maximal weak sequences coincide with each other?

If $l_A(0 : m^2) > 2.l_A(0 : m)$ , the maximal ideal m contains no weakly regular element.

Problem 8. Is the converse also true?

Problem 9. Let A be a generalized Cohen-Macaulay local ring. Then does there always exist a generalized Cohen-Macaulay local ring B such that $A = B/bB$ for some element b of B with $\dim B/bB = \dim B - 1$ ? (c.f. Note by S. Goto below.)

Problem 10. Let I be an ideal in a Cohen-Macaulay UFD. Then is the ring $G_I^{\bullet}(R)$ Cohen-Macaulay, if it is an integral domain?

Ken-ichi Yoshida

Let A denote an affine ring over a field k of characteristic 0 and $\Omega_k(A)$ the module of k-differentials of A .

Problem 11 (S. Suzuki). Characterize the associated prime ideals P of $\Omega_k(A)$ in terms of $A_P$ .

Problem 12. Is $\Omega_k(A)$ torsionfree if A is normal?

In his letter of November 24, 1982, Yoshida informed to the editor that the deformation theory might be applicable to Problem 11 in case A is local.

Problem 13. What Makes Symmetric Powers Vanish?

Moss E. Sweedler Tsukuba/Cornell

We all know that having n-1 generators causes the $n^{th}$ exterior power of a module to be zero. What causes the $n^{th}$ symmetric power to be zero? I know no examples of a module M where the $n^{th}$ symmetric power of M is zero but the $n^{th}$ tensor power and n-1 symmetric power of M are not zero. I would be interested to see such examples or better yet a characterization of modules with this property. It may be true that $\otimes^n M \neq \{0\} \neq S^{n-1} M$ implies $S^n M \neq \{0\}$. Proving this would also be a satisfactory answer.

For tensor powers the examples I know where $\otimes^n M = \{0\}$ but $\otimes^{n-1} M \neq \{0\}$ are for $n = 2$ . What happens for higher n?

A note on Trung's problem

Shiro Goto (Nihon University)

The purpose of this note is to give a negative answer to the following question posed by Trung at the Problem Session:

*Let* $A$ *be a generalized Cohen-Macaulay local ring. Then does there always exist a generalized Cohen-Macaulay local ring* $B$ *such that* $A \cong B/bB$ *for some element* $b$ *of* $B$ *with* $\dim B/bB = \dim B - 1$ ?

My answer is

*Proposition. Let* $A$ *be a Buchsbaum local ring of* $e(A) = 2$ *and assume that* $A$ *is not a Cohen-Macaulay ring. Then if* $\operatorname{depth} A > 0$,

$$A \ncong B/bB$$

*for any generalized Cohen-Macaulay local ring* $B$ *and for any element* $b$ *of* $B$ *such that* $\dim B/bB = \dim B - 1$.

*Proof*. Assume that $A \cong B/bB$ for some generalized Cohen-Macaulay local ring $B$ and for some parameter $b$ of $B$. We put $d = \dim A$. Then $d \geqq 2$, because $\operatorname{depth} A > 0$ and $A$ is not a Cohen-Macaulay ring by our standard assumption. If $d > 2$, then we get an exact sequence

$$\begin{aligned} 0 \longrightarrow [(0) : b]_B \longrightarrow H_n^0(B) \xrightarrow{\ b\ } H_n^0(B) &\longrightarrow H_n^0(B/bB) \\ \longrightarrow H_n^1(B) \xrightarrow{\ b\ } H_n^1(B) &\longrightarrow H_n^1(B/bB) \\ &\ \vdots \\ \longrightarrow H_n^i(B) \xrightarrow{\ b\ } H_n^i(B) &\longrightarrow H_n^i(B/bB) \longrightarrow \cdots \end{aligned}$$

of local cohomology modules relative to the maximal ideal $n$ of $B$ (c.f. |2, (2.6)|). Therefore $H_n^i(B) = (0)$ for all $i \neq d + 1$, as the $B$-modules $H_n^i(B)$ ( $i \neq d + 1$ ) are of finite length and as $H_n^i(B/bB) = (0)$ for any $i \neq 1, d$ (c.f. |1, (1.1)|). Thus $d = 2$, i.e., $\dim B = 3$. Notice that by the above argument, the element $b$ is a non-zero-divisor of $B$ also in this case.

After enlarging, we may assume that the field $B/n$ is infinite.

Similarly we may assume that $B$ is complete. Now choose elements $b_2$, $b_3$ of $n$ so that $\bar{n}^{r+1} = (b_2, b_3)\bar{n}^r$ for some $r \geqq 0$, where $\bar{n} = n/bB$. Then as $A = B/bB$ is a Buchsbaum ring of $e(A) = 2$ and depth $A > 0$, we get by |1, (1.2)| that $\bar{n}^2 = (b_2, b_3)\bar{n}$ and that $v(B/bB) = 4$. Therefore $v(B) = 5$, and $b_1 = b$, $b_2$, $b_3$ is a part of a minimal system of generators for $n$. Hence $n^2 = (b_1, b_2, b_3)n$ as $n^2 \subset (b_1, b_2, b_3)$, and consequently $e(B) = 2$.

Claim 1. $(b_1, b_2) \cap n^i = (b_1, b_2)n^{i-1}$ and $b_1B \cap n^i = b_1 n^{i-1}$ for any integer $i$.

Proof. Let $x \in (b_1, b_2) \cap n^i$. We like to show that $x \in (b_1, b_2)n^{i-1}$. We may assume that $i \geqq 3$ and that our assertion holds for $i - 1$. Since $(b_1, b_2, b_3) \cap n^i = (b_1, b_2, b_3)n^{i-1}$, we write $x = b_1x_1 + b_2x_2 + b_3x_3$ with $x_i \in n^{i-1}$. Then $x_3 \in [(b_1,b_2) : b_3] \cap (b_1, b_2, b_3)$ as $n^{i-1}$ is contained in $(b_1, b_2, b_3)$. Moreover since $[(b_1,b_2) : b_3] \cap (b_1,b_2,b_3) = (b_1,b_2)$ (c.f. |2, (2.4)|, recall that $B/bB$ is Buchsbaum), we get that $x_3 \in (b_1,b_2) \cap n^{i-1}$. Thus the induction works. Similarly we can prove the second assertion by the first one.

Let $G = G_n^{\cdot}(B)$ and $M = G_+$, the irrelevant maximal ideal of $G$. We put $f_i = b_i \mod n^2$ ( $i = 1, 2, 3$ ). Then as $b_1$, $b_2$ is a $B$-regular sequence, by Claim 1 we see that $f_1$, $f_2$ is a $G$-regular sequence and that $G/f_1G \cong G_{n/bB}^{\cdot}(B/bB)$. Therefore $[H_M^i(G)]_p = (0)$ if $p > 1 - i$ since $[H_M^i(G/f_1G)]_p = (0)$ for $p > 1 - i$ (c.f. |1, (2.9)|).

Claim 2. $\dim_k G_p = p^2 + 3p + 1$ ( $p \geqq 0$ ).

Proof. We put $R = k[f_1, f_2, f_3]$ in $G$. Then we may express $G \cong R \oplus E$ with some graded $R$-submodule $E$ of $G$. Notice that $\mathrm{rank}_R E = 1$ as $\mathrm{rank}_R G = 2$ and that $E$ is generated by two linear forms since $\dim_k [G/(f_1, f_2, f_3)G]_1 = 2$ and $[G/(f_1, f_2, f_3)G]_i = (0)$ for all $i > 1$. Moreover we see that $[H_P^i(E)]_p = (0)$ for $p > 1 - i$, because $[H_M^i(G)]_p = (0)$ for $p > 1 - i$ as is remarked above (here $P = R_+$ ). Therefore the graded $R$-module $E$ has a resolution of the following form

$$0 \longrightarrow R(-2) \longrightarrow R(-1) \oplus R(-1) \longrightarrow E \longrightarrow 0$$

(c.f. |3, (5.6)|). Hence $\dim_k E_p = (p^2 + 3p)/2$ for $p \geqq 0$ and our claim is now obvious.

Let $S = k[X_1, \ldots, X_5]$ be a polynomial ring with 5 variables over the field $k = B/n$ and let us express $G = S/J$ for some graded ideal $J$ of $S$. Then we have the following

Claim 3. The graded $S$-module $G$ has a resolution of the form

$$0 \longrightarrow S(-4) \longrightarrow S^4(-3) \longrightarrow S^4(-2) \longrightarrow S \longrightarrow G \longrightarrow 0.$$

Proof. As depth $G = 2$ and $[H_M^i(G)]_p = (0)$ for $p > 1 - i$, we know by |3, (5.7)| that the graded minimal free resolution of $G$ over $S$ has the following form:

$$0 \longrightarrow S(-4)^{r_3} \longrightarrow S(-3)^{r_2} \longrightarrow S(-2)^{r_1} \longrightarrow S \longrightarrow G \longrightarrow 0 .$$

The fact that $r_1 = r_2 = 4$ and $r_3 = 1$ follows from Claim 2.

We are now ready to finish the proof. Choose a complete regular local ring $R$ of dimension 5 so that $B \cong R/I$ for some ideal $I$ of $R$. Then by |4, (2.6)| and Claim 3 we see that $B$ has a minimal free resolution of the following form:

$$0 \longrightarrow R \longrightarrow R^4 \longrightarrow R^4 \longrightarrow R \longrightarrow B \longrightarrow 0 .$$

Take the R-dual $\mathrm{Hom}_R(*,R)$ of this resolution and we get an exact sequence

$$R^4 \xrightarrow{f} R \longrightarrow \mathrm{Ext}_R^3(B,R) \longrightarrow 0.$$

As $\mathrm{Ext}_R^3(B,R) \cong \mathrm{Hom}_B(H_n^2(B), E_B(B/n))$, the length $l_R(R/\mathrm{Im}\ f)$ must be finite which claims that $\dim R \leqq 4$ —— this is a contradiction since $\dim R = 5$ by our choice.

In |1| we can find several examples of Buchsbaum rings which satisfy the requirements in the above proposition. Therefore Trung's guess is not true in general and it is hoped to characterize the class of generalized Cohen-Macaulay local rings that satisfy his requirement.

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