GENERALIZED F-SIGNATURE OF INVARIANT SUBRINGS

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ABSTRACT. It is known that a certain invariant subring R has finite F-representation type. Thus, we can write the R-module ${}^eR$ as a finite direct sum of finitely many R-modules. In such a decomposition of ${}^eR$, we pay attention to the multiplicity of each direct summand. For the multiplicity of free direct summand, there is the notion of F-signature defined by C. Huneke and G. Leuschke and it characterizes some singularities. In this paper, we extend this notion to non free direct summands and determine the explicit values of them.

1. INTRODUCTION

Throughout this paper, we suppose that k is an algebraically closed field of prime characteristic $p > 0$, and V is a d-dimensional k-vector space. Let $G \subset \mathrm{GL}(V)$ be a finite subgroup such that the order of G is not divisible by p, and G contains no pseudo-reflections. Let S be a symmetric algebra of V. We denote the invariant subring of S under the action of G by $R := S^G$. Sometimes we denote p^e by q. Since $\operatorname{char} R = p > 0$, we can define the Frobenius map $F : R \to R$ $(r \mapsto r^p)$ and also define the e-times iterated Frobenius map $F^e : R \to R$ $(r \mapsto r^{p^e})$ for $e \in \mathbb{N}$. For any R-module M, we denote the module M with its R-module structure pulled back via F^e by ${}^eM$. That is, ${}^eM$ is just M as an abelian group, and its R-module structure is given by $r \cdot m := F^e(r)m = r^{p^e}m$ for all $r \in R$, $m \in M$.

In our assumption, it is known that the invariant subring R has finite F-representation type (or FFRT for short). The notion of FFRT is defined by K. Smith and M. Van den Bergh [SVdB].

Definition 1.1. *We say that R has finite F-representation type by $\mathscr{N}$ if there is a finite set $\mathscr{N}$ of isomorphism classes of finitely generated R-modules, such that for any $e \in \mathbb{N}$, the R-module ${}^eR$ is isomorphic to a finite direct sum of elements of $\mathscr{N}$.*

More explicitly, finitely many finitely generated R-modules which form such a finite set $\mathscr{N}$ are described as follows.

Proposition 1.2. ([SVdB, Proposition 3.2.1]) *Let $V_0 = k, V_1, \cdots, V_n$ be a complete set of non-isomorphic irreducible representations of G and we set $M_i := (S \otimes_k V_i)^G$ $(i = 0, 1, \cdots, n)$. Then R has FFRT by $\{M_0 \cong R, M_1, \cdots, M_n\}$.*

From this proposition, we can decompose ${}^eR$ as follows

$$ {}^eR \cong R^{\oplus c_{0,e}} \oplus M_1^{\oplus c_{1,e}} \oplus \cdots \oplus M_n^{\oplus c_{n,e}}. $$

Now, we want to investigate the multiplicities $c_{i,e}$. For the multiplicity of the free direct summand, that is, for the multiplicity $c_{0,e}$, there is the notion of F-signature defined by C. Huneke and G. Leuschke.

Definition 1.3. ([HL, Definition 9]) *The F-signature of R is $s(R) := \lim_{e \to \infty} \frac{c_{0,e}}{p^{de}}$, if it exists.*

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Note that K. Tucker showed its existence under more general settings [Tuc, Theorem 4.9] (see also Proposition 3.1). And it is known that this numerical invariant characterizes some singularities. For example, $s(R)=1$ if and only if R is regular [HL, Corollary 16] (see also [Yao2, Theorem 3.1]), and $s(R)>0$ if and only if R is strongly F-regular [AL, Theorem 0.2]. In our situation, the F-signature of the invariant subring R is determined as follows and it implies that R is strongly F-regular.

Theorem 1.4. ([WY, Theorem 4.2]) *The F-signature of the invariant subring R is*

$$s(R)=\frac{1}{|G|}.$$

Remark 1.5. In [WY], this theorem is proved in terms of minimal relative Hilbert-Kunz multiplicity. And Y. Yao showed that it coincides with the F-signature [Yao2, Remark 2.3 (4)].

Now, we extend this notion to other direct summands. Namely, we investigate the multiplicities $c_{i,e}$ $(i=1,\cdots,n)$ and determine the limit $\lim\limits_{e\to\infty}\dfrac{c_{i,e}}{p^{de}}$. In order to determine this limit, we have to care about the next two problems first.

- For each $e\in\mathbb{N}$, are the multiplicities $c_{i,e}$ determined uniquely?
- Does the limit $\lim\limits_{e\to\infty}\dfrac{c_{i,e}}{p^{de}}$ exist?

In Section 2, we will show the uniqueness of the multiplicities. In Section 3, we will show the existence of the limit and determine the limit (see Theorem 3.4).

2. Uniqueness of decomposition

In this section, we show the uniqueness of the multiplicities. Firstly, we introduce the notion of Frobenius twist (e.g. [Jan]).

Definition 2.1. *For k-vector space V and $e\in\mathbb{Z}$, we define k-vector space ${}^eV$ as follows*

- *${}^eV$ is the same as V as an additive group;*
- *the action of $\alpha\in k$ on ${}^eV$ is $\alpha\cdot v=\alpha^{p^e}v$.*

An element $v\in V$, viewed as an element of ${}^eV$, is sometimes denoted by ${}^ev$. Thus $\alpha\cdot{}^ev={}^e(\alpha^{p^e}v)$. By the composition $G\hookrightarrow \mathrm{GL}(V)\xrightarrow{\phi}\mathrm{GL}({}^eV)$, ${}^eV$ is also a representation of G, where ϕ is given by $\phi(g)({}^ev)={}^e(gv)$ for $g\in G$ and $v\in V$. We call this representation the Frobenius twist of V. Sometimes we denote this representation by $V^{(-e)}$.

Let $v_1,\cdots,v_d$ be a basis of V. For this basis, we suppose that a representation of G is defined by

$$g\cdot v_j=\sum_{i=1}^{d}f_{ij}(g)v_i\quad(g\in G,\ f_{ij}:G\to k).$$

Namely, a matrix representation of V is described by $\big(f_{ij}(g)\big)$. Since k is an algebraically closed field, the basis $v_1,\cdots,v_d$ also form a basis of ${}^eV$, and the action of G on ${}^eV$ is described as follows

$$g\cdot{}^ev_j={}^e(g\cdot v_j)={}^e(\sum_{i=1}^{d}f_{ij}(g)v_i)=\sum_{i=1}^{d}f_{ij}(g)^{p^{-e}}({}^ev_i).$$

From this observation, a matrix representation of the Frobenius twist ${}^eV$ is described by $\left((f_{ij}(g))^{p^{-e}}\right)$, that is, each component of the matrix representation of ${}^eV$ is the p^{-e}-th power of the original one.

In order to show the uniqueness of the multiplicities, we prove the following.

Proposition 2.2. *For $e \geq 1$, $c_{0,e}, \cdots, c_{n,e} \geq 0$, the following decompositions are equivalent*

(1) ${}^eR \cong M_0^{\oplus c_{0,e}} \oplus M_1^{\oplus c_{1,e}} \oplus \cdots \oplus M_n^{\oplus c_{n,e}}$ *as R-modules;*
(2) ${}^eS \cong (S \otimes_k V_0)^{\oplus c_{0,e}} \oplus (S \otimes_k V_1)^{\oplus c_{1,e}} \oplus \cdots \oplus (S \otimes_k V_n)^{\oplus c_{n,e}}$ *as (G,S)-modules;*
(3) ${}^eS/\mathfrak{m}^eS \cong V_0^{\oplus c_{0,e}} \oplus V_1^{\oplus c_{1,e}} \oplus \cdots \oplus V_n^{\oplus c_{n,e}}$ *as G-modules;*
(4) *there exist $\alpha_{ij} \in \frac{1}{q}\mathbb{Z}_{\geq 0}$ such that* ${}^eS \cong \bigoplus_{i=0}^{n} \bigoplus_{j=1}^{c_{i,e}} (S \otimes_k V_i)(-\alpha_{ij})$
as $\frac{1}{q}\mathbb{Z}$-graded (G,S)-modules;
(5) *there exist $\alpha_{ij} \in \frac{1}{q}\mathbb{Z}_{\geq 0}$ such that* ${}^eR \cong \bigoplus_{i=0}^{n} \bigoplus_{j=1}^{c_{i,e}} M_i(-\alpha_{ij})$
as $\frac{1}{q}\mathbb{Z}$-graded R-modules.

Remark 2.3. A similar correspondence holds for more general situation up to the action of the e-th Frobenius kernel of a group scheme [Has]. For the case of a finite group G, the e-th Frobenius kernel of G is trivial. Thus, we may ignore it in our context.

To prove this proposition, the next theorem plays the central role. This is proved in [Has] for more general settings using some geometric settings. For convenience of the reader, we give a short and simple proof using a result of Iyama and Takahashi [IT] or Leuschke and Wiegand [LW]. The two-dimensional case is very well-known as a theorem of Auslander [Aus]. See also [Yos, Chapter 10].

Theorem 2.4. *If G contains no pseudo-reflections, then the functor* $\mathrm{Ref}(G,S) \to \mathrm{Ref}(R)$ $(M \mapsto M^G)$ *is an equivalence, where* $\mathrm{Ref}(G,S)$ *is the category of reflexive (G,S)-modules and* $\mathrm{Ref}(R)$ *is the category of reflexive R-modules. The quasi-inverse is $N \mapsto (S \otimes_R N)^{**}$.*

The same functors give an equivalence ${}^*\mathrm{Ref}(G,S) \to {}^*\mathrm{Ref}(R)$, *where* ${}^*\mathrm{Ref}(G,S)$ *is the category of $\mathbb{Z}[1/p]$-graded reflexive (G,S)-modules and* ${}^*\mathrm{Ref}(R)$ *is the category of $\mathbb{Z}[1/p]$-graded reflexive R-modules.*

Proof. Let $S*G$ denote the twisted group algebra. It is $\bigoplus_{g \in G} S \cdot g$ as an S-module (with the free basis G), and the multiplication is given by $(sg)(s'g') = (s(gs'))(gg')$. A (G,S)-module and an $S*G$-module are one and the same thing. As a (G,S)-module, $S*G$ and $S \otimes_k kG$ are the same thing, where kG is the group algebra (the left regular representation) of G over k. So $\mathrm{Hom}_S(S*G,S) \cong S \otimes_k k[G] \cong S \otimes_k kG \cong S*G$, where $k[G] = (kG)^*$ is the k-dual of kG (the left regular representation).

Let us denote by S' the R-module S with the trivial G-module structure. Note that $S' \to (S \otimes_k k[G])^G$ given by $s \mapsto \sum_{g \in G} gs \otimes e_g$ is an isomorphism, where $\{e_g \mid g \in G\}$ is the dual basis of $k[G]$, dual to G, which is a basis of kG. Note that $g'e_g = e_{g'g}$.

For $M \in \mathrm{Ref}(G,S)$, M^G is certainly reflexive. Indeed, there is a presentation

$$(S*G)^u \to (S*G)^v \to M^* \to 0. \qquad (2.1)$$

Applying $(?)^G \circ \mathrm{Hom}_S(?,S)$,

$$0 \to M^G \to (S')^v \to (S')^u$$

is exact. As it is easy to see that S' satisfies the (S_2)-condition as an R-module (that is, for $P \in \operatorname{Spec} R$, if $\operatorname{depth}_{R_P}(S'_P) < 2$, then S'_P is a maximal Cohen–Macaulay R_P-module), so is M^G, and it is reflexive.

On the other hand, it is obvious that $(S \otimes_R N)^{**}$ is a reflexive (G,S)-module, since it is a dual of some S-finite (G,S)-module.

Let $u : N \to ((S \otimes_R N)^{**})^G$ be the map given by $u(n) = \lambda(1 \otimes n)$, where $\lambda : S \otimes_R N \to (S \otimes_R N)^{**}$ is the canonical map. We show that u is an isomorphism. To verify this, since both N and $((S \otimes_R N)^{**})^G$ are reflexive, it suffices to show that

$$u_P : N_P \to (((S \otimes_R N)^{**})^G)_P \cong ((S_P \otimes_{R_P} N_P)^{**})^G$$

is an isomorphism for $P \in \operatorname{Spec} R$ with $\dim R_P \leq 1$ (cf. [LW, Lemma 5.11]). Then N_P is a free module, and we may assume that $N_P = R_P$ by additivity. This case is trivial.

Let $\varepsilon : (S \otimes_R M^G)^{**} \to M$ be the composite

$$(S \otimes_R M^G)^{**} \xrightarrow{a^{**}} M^{**} \xrightarrow{\lambda^{-1}} M,$$

where $a : S \otimes_R M^G \to M$ is given by $a(s \otimes m) = sm$. We show that ε is an isomorphism. Since $(S \otimes_R M^G)^*$ and M are reflexive, it suffices to show that $a^* : M^* \to (S \otimes_R M^G)^*$ is an isomorphism. By the five lemma and the existence of the presentation of the form (2.1), we may assume that $M = S \otimes_k k[G]$. Then a^* is identified with the map

$$S * G \cong (S \otimes_k k[G])^* \xrightarrow{a^*} (S \otimes_R (S \otimes_k k[G])^G)^* \cong (S \otimes_R S')^* \cong \operatorname{Hom}_R(S', S).$$

It is easy to see that this map is given by $sg \mapsto (s' \mapsto s(gs'))$. This is an isomorphism by [IT, Theorem 4.2] or [LW, Theorem 5.12].

As u and ε are isomorphisms, $M \mapsto M^G$ and $N \mapsto (S \otimes_R N)^{**}$ are quasi-inverse each other, and hence they are category equivalences.

The graded version is proved similarly. □

By using this theorem, we give the proof of Proposition 2.2.

Proof of Proposition 2.2. The equivalence of (1) and (2), (4) and (5) follow from Theorem 2.4, and (3) is obtained by applying $(- \otimes_S k)$ to (2). If we forget the grading from (4), then we obtain (2).

$$\begin{array}{ccccc} (1) & \overset{\text{Thm.2.4}}{\Longleftrightarrow} & (2) & \overset{\otimes_S k}{\Longrightarrow} & (3) \\ & & \Big\Uparrow \ \text{forget grading} & & \\ & & (4) & \underset{\text{Thm.2.4}}{\Longleftrightarrow} & (5) \end{array}$$

So we will show (3) $\Rightarrow$ (4). If we consider ${}^eS/\mathfrak{m}^eS$ as a $\frac{1}{q}\mathbb{Z}$-graded G-module, then we can write

$${}^eS/\mathfrak{m}^eS \cong \bigoplus_{i=0}^{n} \bigoplus_{j=1}^{c_{i,e}} V_i(-\alpha_{ij})$$

for some $\alpha_{ij} \in \frac{1}{q}\mathbb{Z}_{\geq 0}$. Then as in the proof of [SVdB, Proposition 3.2.1], we have ${}^eS \cong S \otimes_k ({}^eS/\mathfrak{m}^eS)$, and (4) follows. □

Especially, the decomposition (3) appears in Proposition 2.2 is unique. Thus, we obtain the next statement as a corollary.

Corollary 2.5. *Each M_i is indecomposable and the multiplicities $c_{i,e}$ are determined uniquely.*

In Proposition 2.2 and Corollary 2.5, the condition "G contains no pseudo-reflections" is essential. If G contains a pseudo-reflection, then there is a counter-example as follows.

Example 2.6. *Let $S = k[x,y]$ be a polynomial ring, where $(\operatorname{char} k, |G|) = 1$. Set $G = \left\langle \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$, that is G is a symmetric group $\mathfrak{S}_2$, and, $V_0 = k, V_1 = \operatorname{sgn}$ are irreducible representations of G. (Note that σ is a pseudo-reflection.) Then, $R := S^G \cong k[x+y, xy]$. Since R is a polynomial ring, ${}^eR \cong R^{\oplus p^{2e}}$. On the other hand,*

$$M_1 := (S \otimes_k V_1)^G = \{ f \in S \mid \sigma \cdot f = (\operatorname{sgn}\ \sigma) f \} = (x-y)R \cong R.$$

So ${}^eR$ also decompose as ${}^eR \cong M_1^{\oplus p^{2e}}$. Therefore, the uniqueness doesn't hold in this case.

3. Generalized F-signature of invariant subrings

In this section, we show the existence of the limit and determine it.

In our case, the invariant subring R has FFRT. Thus, the existence of the limit $\lim_{e\to\infty} \frac{c_{i,e}}{p^{de}}$ is guaranteed by the next proposition. So we can define this limit.

Proposition 3.1. ([SVdB, Proposition 3.3.1], [Yao1, Theorem 3.11]) *If R has FFRT, then for $i = 0, 1, \cdots, n$, the limit $\lim_{e\to\infty} \frac{c_{i,e}}{p^{de}}$ exists.*

Remark 3.2. In [SVdB], this proposition is proved under the assumption "R is strongly F-regular and has FFRT". After that, Y. Yao showed the condition of strongly F-regular is unnecessary [Yao1]. Note that the existence of the limit for free direct summands (i.e. F-signature) is proved under more general settings as we showed before.

Definition 3.3. *We call this limit the generalized F-signature of M_i with respect to R and denote it by*

$$s(R, M_i) := \lim_{e\to\infty} \frac{c_{i,e}}{p^{de}}.$$

The main theorem in this paper is the following.

Theorem 3.4 (Main theorem)**.** *Let the notation be as above. Then for all $i = 0, \cdots, n$ one has*

$$s(R, M_i) = \frac{\dim_k V_i}{|G|} = \frac{\operatorname{rank}_R M_i}{|G|}.$$

Remark 3.5. The second equation follows from $\dim_k V_i = \operatorname{rank}_R M_i$ clearly.

The case that $i = 0$ is due to [WY, Theorem 4.2], as we have seen before (Theorem 1.4). And a similar result holds for finite subgroup scheme of SL_2 [HS, Lemma 4.10].

Remark 3.6. From this theorem, we can see that each indecomposable MCM R-modules in the finite set $\{R, M_1, \cdots, M_n\}$ actually appear in ${}^eR$ as a direct summand for sufficiently large e (see also [TY, Proposition 2.5]).

In order to prove this theorem, we introduce the notion of the Brauer character. In the representation theory of finite groups over $\mathbb{C}$, the character gives us very effective method to distinguish each representation. But now, we are in a positive characteristic field k, not in $\mathbb{C}$. So the character in the original sense doesn't work well. Therefore we have to modify it for applying to our context. For this purpose, we introduce the Brauer character (for more details, refer to some textbooks e.g. [CR], [Wei]).

As we assume that $m := |G|$ is not divisible by p, there is a primitive mth root of unity in k, and thus buth $\mu_m(k) = \{\omega \in k^\times \mid \omega^m = 1\}$ and $\mu_m(\mathbb{C}) = \{\omega \in \mathbb{C}^\times \mid \omega^m = 1\}$ are the cyclic groups of order m. Fix a group isomorphism $\Phi : \mu_m(k) \to \mu_m(\mathbb{C})$.

Definition 3.7. *For a kG-module V, the Brauer character χ_V of V is the function $\chi_V : G \to \mathbb{C}$ given by*

$$\chi_V(g) := \sum_{i=1}^{d} \Phi(\omega_i) \in \mathbb{C} \quad (g \in G),$$

where $\omega_1, \cdots, \omega_d$ are the eigenvalues of g.

The following proposition is well-known for the original character over $\mathbb{C}$. And this kind of formula also holds for the Brauer character.

Proposition 3.8. *Let V, W be kG-modules and $g \in G$, then*

(1) $\chi_{V\otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$.
(2) $\chi_{V\oplus W}(g) = \chi_V(g) + \chi_W(g)$.
(3) $\chi_{V^*}(g) = \overline{\chi_V(g)}$, *where the bar denotes the conjugate of a complex number.*
(4) $\chi_V(1_G) = \dim_k V$.
(5) $\dim_k V^G = \dfrac{1}{|G|} \sum_{g\in G} \chi_V(g)$.
(6) $\dim_k \mathrm{Hom}_G(V, W) = \dfrac{1}{|G|} \sum_{g\in G} \overline{\chi_V(g)} \cdot \chi_W(g)$.

Proof. The statements (1)–(4) follow easily from the definition. (6) follows from (1), (3), and (5). So we only prove (5). If we show (5) for a particular choice of Φ, then (5) is true for arbitrary choice, say Φ', because we can write $\Phi' = \alpha \circ \Phi$, where α is some automorphism of $\mathbb{Q}(\omega)$ over $\mathbb{Q}$, where ω is a primitive mth root of unity in $\mathbb{C}$. Let R be the ring of Witt vectors over k. Note that R is a complete DVR (discrete valuation ring). Let t be its uniformizing parameter. We identify R/tR with k. Let $\bar{\omega}$ be a fixed primitive mth root of unity in k. By Hensel's lemma, it is easy to see that $\bar{\omega}$ lifts to a primitive mth root of unity in R uniquely, say to ω. Note that V is a kG-module, and hence is an RG-module. Let $V_R \to V$ be the projective cover as an RG-module, which exists (note that RG is semiperfect). Note that $V_R/tV_R = V$, and V_R is an R-free module of rank $\dim_k V$.

Let $R_0 = \mathbb{Z}[\omega]$ be the subring of R generated by ω. Then regarding R_0 as a subring of $\mathbb{C}$, we have that $\tilde{\chi}_V$ is a Brauer character of V, where $\tilde{\chi}_V(g) = \mathrm{trace}_{V_R}(g)$ (the trace makes sense, since V_R is a finite free R-module). Let $\gamma = \dfrac{1}{|G|} \sum_{g\in G} g \in RG$. Then it is easy to see that γ is a projector from any RG-module M to M^G. In particular, the G-invariance $(?)^G$ is an exact functor on the category of RG-modules. It follows that $V^G = (V_R/tV_R)^G \cong V_R^G/tV_R^G = k \otimes_R V_R^G$. Let $U := (1-\gamma)V_R$. Then $V_R = V_R^G \oplus U$, and γ is the identity map on V_R^G and zero on U. So $\dfrac{1}{|G|} \sum_{g\in G} \tilde{\chi}_V(g) = \mathrm{trace}_{V_R}(\gamma) = \mathrm{rank}_R V_R^G = \dim_k V^G$. This is what we wanted to prove. □

So we are now ready to prove the main theorem.

Proof of Theorem 3.4. Firstly, there is $e_0 \geq 1$ such that the group ring $\mathbb{F}_{q_0}G$ is isomorphic to the direct product of total matrix rings over $\mathbb{F}_{q_0}$, where $q_0 = p^{e_0}$. Namely,

$$\mathbb{F}_{q_0}G \cong \mathrm{Mat}_{r_1}(\mathbb{F}_{q_0}) \times \cdots \times \mathrm{Mat}_{r_m}(\mathbb{F}_{q_0}), \quad (r_1, \cdots, r_m \in \mathbb{N}).$$

Since the component of matrix representation of Frobenius twist is p^{-e}-th power of the original one, so if we take an appropriate basis, then any component of matrix representation is in the finite field $\mathbb{F}_{q_0}$. Thus, if $e = e_0 t$, then we can consider ${}^eM \cong M$ for any G-module M.

Since we know the existence of the limit, it suffices to show the subsequence $\{\frac{c_{i,e_0t}}{p^{de_0t}}\}_{t\in\mathbb{N}}$ converge on $(\dim_k V_i)/|G|$. So we prove

$$\lim_{t\to\infty} \frac{c_{i,e_0t}}{p^{de_0t}} = \frac{\dim_k V_i}{|G|}.$$

For $e = e_0t$, we obtain ${}^eS/\mathfrak{m}^eS \cong {}^e(S/\mathfrak{m}^{[q]}) \cong S/\mathfrak{m}^{[q]}$. And ${}^eS/\mathfrak{m}^eS$ is also isomorphic to the finite direct sum of irreducible representations (cf. Proposition 2.2). By Proposition 3.8 (6), the multiplicity $c_{i,e}$ is described as follows.

$$c_{i,e} = \dim_k \mathrm{Hom}_G(V_i, S/\mathfrak{m}^{[q]}) = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_{V_i}(g)} \cdot \chi_{S/\mathfrak{m}^{[q]}}(g).$$

Set $g \in G$ and suppose that the order of g is m. Then there is a basis $\{x_1, \cdots, x_d\}$ of V such that each x_i is an eigenvector of g and we can write $g \cdot x_i = \omega_i x_i$ with $\omega_i = \omega^{\delta_i}$ for some $0 \leq \delta_i < m$, where ω is a primitive m-th root of unity. In this situation

$$\{x_1^{\lambda_1} \cdots x_d^{\lambda_d} \mid 0 \leq \lambda_1, \ldots, \lambda_d < q\} \subset \bigoplus_{l=0}^{(q-1)d} \mathrm{Sym}_l V$$

is a basis of $S/\mathfrak{m}^{[q]}$. As each $x_1^{\lambda_1} \cdots x_d^{\lambda_d}$ is an eigenvector of g with the eigenvalue $\omega_1^{\lambda_1} \cdots \omega_d^{\lambda_d}$, we have

$$\chi_{S/\mathfrak{m}^{[q]}}(g) = \sum_{0\leq\lambda_1,\ldots,\lambda_d<q} \Phi(\omega_1^{\lambda_1} \cdots \omega_d^{\lambda_d}) = \prod_{i=1}^{d}(1 + \theta_i + \cdots + \theta_i^{q-1}),$$

where $\theta_i := \Phi(\omega_i)$.

(i) In case $g = 1$, by Proposition 3.8 (5),

$$\frac{\overline{\chi_{V_i}(g)} \cdot \chi_{S/\mathfrak{m}^{[q]}}(g)}{q^d} = \frac{\dim_k V_i \cdot q^d}{q^d} = \dim_k V_i.$$

(ii) In case $g \neq 1$, we may assume $\theta_d \neq 1$. Then

$$\left| \frac{\overline{\chi_{V_i}(g)} \cdot \chi_{S/\mathfrak{m}^{[q]}}(g)}{q^d} \right| \leq \frac{|\overline{\chi_{V_i}(g)}|}{q^d} \prod_{i=1}^{d-1}(|1| + |\theta_i| + \cdots + |\theta_i|^{q-1}) \cdot \left| \frac{1-\theta_d^q}{1-\theta_d} \right|$$

$$\leq \frac{\dim_k V_i}{q} \cdot \frac{2}{|1-\theta_d|} \xrightarrow{t\to\infty} 0.$$

The first inequation is obtained by applying the triangle inequality. Since $|\theta_i| \leq 1$, we can obtain the second inequation.

From previous arguments, we may only discuss in case $g = 1$. Thus, we conclude

$$\lim_{e\to\infty} \frac{c_{i,e}}{q^d} = \lim_{e\to\infty} \frac{1}{q^d} \cdot \frac{1}{|G|} \sum_{g\in G} \overline{\chi_{V_i}(g)} \cdot \chi_{S/\mathfrak{m}^{[q]}}(g) = \frac{\dim_k V_i}{|G|}.$$

□

Next, we consider the decomposition of ${}^eM_i$. Since each MCM R-module M_i appears in ${}^{e'}R$ for sufficiently large $e' \gg 0$ as a direct summand, it also decompose as

$${}^eM_i \cong M_0^{\oplus d^i_{0,e}} \oplus M_1^{\oplus d^i_{1,e}} \oplus \cdots \oplus M_n^{\oplus d^i_{n,e}}.$$

In this situation, we define the limit

$$s(M_i, M_j) := \lim_{e\to\infty} \frac{d^i_{j,e}}{p^{de}}$$

and call it the generalized F-signature of M_j with respect to M_i. The next corollary immediately follows from Theorem 3.4 and [SVdB, Proposition 3.3.1, Lemma 3.3.2].

Corollary 3.9. *Let the notation be as above. Then for all $i, j = 0, \cdots, n$ one has*

$$s(M_i, M_j) = (\dim_k V_i) \cdot s(R, M_j) = \frac{(\dim_k V_i) \cdot (\dim_k V_j)}{|G|} = \frac{(\operatorname{rank}_R M_i) \cdot (\operatorname{rank}_R M_j)}{|G|}.$$

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