

Acyclicity of flat complexes over a noetherian ring

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Dedicated to Professor Masayoshi Nagata on his eightieth birthday

Abstract

Let R be a noetherian commutative ring, and

$$\mathbb{F} : \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow 0$$

a flat R -complex. We prove that if $\kappa(\mathfrak{p}) \otimes_R \mathbb{F}$ is acyclic for any $\mathfrak{p} \in \operatorname{Spec} R$, then \mathbb{F} is acyclic, and $H_0(\mathbb{F})$ is R -flat. It follows that if \mathbb{F} is a (possibly unbounded) complex of flat R -modules and $\kappa(\mathfrak{p}) \otimes_R \mathbb{F}$ is exact for any $\mathfrak{p} \in \operatorname{Spec} R$, then $\mathbb{G} \otimes_R \mathbb{F}$ is exact for any R -complex \mathbb{G} . If, moreover, \mathbb{F} is a complex of projective R -modules, then it is null-homotopic (follows from Neeman's theorem).

1. Introduction

Throughout this paper, R denotes a noetherian commutative ring. The symbol \otimes without any subscript means \otimes_R . For $\mathfrak{p} \in \operatorname{Spec} R$, let $?(\mathfrak{p})$ denote the functor $\kappa(\mathfrak{p}) \otimes ?$, where $\kappa(\mathfrak{p})$ is the field $R_{\mathfrak{p}} / \mathfrak{p}R_{\mathfrak{p}}$. An R -complex of the form

$$\mathbb{F} : \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \rightarrow 0$$

is said to be acyclic if $H_i(\mathbb{F}) = 0$ for any $i > 0$.

It has been known that, for an R -linear map of R -flat modules $\varphi : F_1 \rightarrow F_0$, if $\varphi(\mathfrak{p})$ is injective for any $\mathfrak{p} \in \text{Spec } R$, then φ is injective and $\text{Coker } \varphi$ is R -flat (see [1, Lemma I.2.1.4] and Corollary 6).

In this paper, we prove:

Theorem 1. *Let*

$$\mathbb{F} : \dots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \rightarrow 0$$

be an R -flat complex. If $\mathbb{F}(\mathfrak{p})$ is acyclic for any $\mathfrak{p} \in \text{Spec } R$, then \mathbb{F} is acyclic, and $H_0(\mathbb{F})$ is R -flat. In particular, $M \otimes \mathbb{F}$ is acyclic for any R -module M .

The case of a map $\varphi : F_1 \rightarrow F_0$ is a special case of the theorem such that $F_i = 0$ for any $i > 1$.

By the theorem, it follows immediately that if \mathbb{F} is an (unbounded) R -flat complex and $\mathbb{F}(\mathfrak{p})$ is exact for any \mathfrak{p} , then \mathbb{F} is K -flat and exact. Combining this and Neeman's result, we can also prove that an (unbounded) projective R -complex \mathbb{P} is null-homotopic if $\mathbb{P}(\mathfrak{p})$ is exact for any $\mathfrak{p} \in \text{Spec } R$.

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2. Acyclicity of flat complexes

We give a proof of Theorem 1.

Proof of Theorem 1. It suffices to prove that $R/I \otimes \mathbb{F}$ is acyclic for any ideal I of R . Indeed, if so, then considering the case that $I = 0$, we have that \mathbb{F} is acyclic so that it is a flat resolution of $H_0(\mathbb{F})$. Since $R/I \otimes \mathbb{F}$ is acyclic for any ideal I , we have that $\text{Tor}_i^R(R/I, H_0(\mathbb{F})) = 0$ for any $i > 0$. Thus $H_0(\mathbb{F})$ is R -flat. So $\text{Tor}_i^R(M, H_0(\mathbb{F})) = 0$ for any $i > 0$, and the last assertion follows.

Assume the contrary, and let I be maximal among the ideals J such that $R/J \otimes \mathbb{F}$ is not acyclic. Then replacing R by R/I and \mathbb{F} by $R/I \otimes \mathbb{F}$, we may assume that $R/I \otimes \mathbb{F}$ is acyclic for any nonzero ideal I of R , but \mathbb{F} itself is not acyclic.

Assume that R is not a domain. There exists some filtration

$$0 = M_0 \subset M_1 \subset \dots \subset M_r = R$$

such that for each i , $M_i/M_{i-1} \cong R/P_i$ for some $P_i \in \text{Spec } R$. Since each P_i is a nonzero ideal, $R/P_i \otimes \mathbb{F}$ is acyclic. So $M_i \otimes \mathbb{F}$ is acyclic for any i . In

particular, $\mathbb{F} \cong M_r \otimes \mathbb{F}$ is acyclic, and this is a contradiction. So R must be a domain.

For each $x \in R \setminus 0$, there is an exact sequence

$$0 \rightarrow \mathbb{F} \xrightarrow{x} \mathbb{F} \rightarrow R/Rx \otimes \mathbb{F} \rightarrow 0.$$

Since $R/Rx \otimes \mathbb{F}$ is acyclic, we have that $x : H_i(\mathbb{F}) \rightarrow H_i(\mathbb{F})$ is an isomorphism for any $i > 0$. In particular, $H_i(\mathbb{F})$ is a K -vector space, where $K = \kappa(0)$ is the field of fractions of R . So

$$H_i(\mathbb{F}) \cong K \otimes H_i(\mathbb{F}) \cong H_i(K \otimes \mathbb{F}) = H_i(\mathbb{F}(0)) = 0 \quad (i > 0),$$

and this is a contradiction. \square

Let A be a ring. A complex \mathbb{F} of left A -modules is said to be K -flat if the tensor product $\mathbb{G} \otimes_A^\bullet \mathbb{F}$ is exact for any exact sequence \mathbb{G} of right A -modules, see [3].

For a chain complex

$$\mathbb{H} : \cdots \rightarrow H_{i+1} \xrightarrow{d_{i+1}} H_i \xrightarrow{d_i} H_{i-1} \rightarrow \cdots$$

of left or right A -modules, we denote the complex

$$\cdots \rightarrow H_{i+1} \rightarrow \text{Ker } d_i \rightarrow 0$$

by $\tau_{\geq i} \mathbb{H}$ or $\tau^{\leq -i} \mathbb{H}$. Since $\mathbb{G} \cong \varinjlim \tau^{\leq n} \mathbb{G}$, \mathbb{F} is K -flat if and only if $\mathbb{G} \otimes_A^\bullet \mathbb{F}$ is exact for any exact sequence \mathbb{G} of right A -modules bounded above (i.e., $\mathbb{G}_{-i} = \mathbb{G}^i = 0$ for $i \gg 0$). A flat complex \mathbb{F} of left A -modules is K -flat if it is bounded above, as can be seen easily from the spectral sequence argument. A null-homotopic complex \mathbb{F} is K -flat, since $\mathbb{G} \otimes_A \mathbb{F}$ is null-homotopic for any complex \mathbb{G} .

Lemma 2. *Let A be a ring, and*

$$\mathbb{F} : \cdots \rightarrow F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \rightarrow \cdots$$

a complex of flat left A -modules. Then the following are equivalent.

- 1** $M \otimes_A \mathbb{F}$ is exact for any right A -module M .
- 2** \mathbb{F} is exact, and $\text{Im } d_i$ is flat for any i .

3 \mathbb{F} is K -flat and exact.

Proof. **1** \Rightarrow **2** Obviously, $\mathbb{F} \cong A \otimes_A \mathbb{F}$ is exact. Thus

$$\mathbb{F}' : \cdots \rightarrow F_{i+1} \rightarrow F_i \rightarrow 0$$

is a flat resolution of $\text{Im } d_i$, where F_{n+i} has the homological degree n in \mathbb{F}' . For any $i \in \mathbb{Z}$,

$$\text{Tor}_1^A(M, \text{Im } d_i) \cong H_1(M \otimes_A \mathbb{F}') \cong H_{i+1}(M \otimes_A \mathbb{F}) = 0$$

for any right A -module M . Thus $\text{Im } d_i$ is A -flat.

2 \Rightarrow **1** For any $i \in \mathbb{Z}$,

$$H_{i+1}(M \otimes_A \mathbb{F}) \cong H_1(M \otimes_A \mathbb{F}') \cong \text{Tor}_1^A(M, \text{Im } d_i) = 0,$$

where \mathbb{F}' is as above.

1,2 \Rightarrow **3** We prove that $\mathbb{G} \otimes_A \mathbb{F}$ is exact for any bounded above complex \mathbb{G} of right A -modules. Since $\mathbb{F} \cong \varinjlim \tau^{\leq n} \mathbb{F}$ and $\tau^{\leq n} \mathbb{F}$ satisfies **2** (and hence **1**), we may assume that \mathbb{F} is also bounded above. Then by an easy spectral sequence argument, $\mathbb{G} \otimes_A \mathbb{F}$ is exact.

3 \Rightarrow **1** Let \mathbb{P} be a projective resolution of M . Since \mathbb{P} is a bounded above flat complex of left A^{op} -modules and \mathbb{F} is an exact complex of right A^{op} -modules, $\mathbb{P} \otimes_A \mathbb{F}$ is exact. Let \mathbb{Q} be the mapping cone of $\mathbb{P} \rightarrow M$. Then $\mathbb{Q} \otimes_A \mathbb{F}$ is also exact, since \mathbb{Q} is exact and \mathbb{F} is K -flat. By the exact sequence of homology groups

$$H_i(\mathbb{P} \otimes_A \mathbb{F}) \rightarrow H_i(M \otimes_A \mathbb{F}) \rightarrow H_i(\mathbb{Q} \otimes_A \mathbb{F}),$$

we have that $M \otimes_A \mathbb{F}$ is also exact. □

Corollary 3. *Let*

$$\mathbb{F} : \cdots \rightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \rightarrow \cdots$$

be a (possibly unbounded) flat complex of R -modules. If $\mathbb{F}(\mathfrak{p})$ is exact for any $\mathfrak{p} \in \text{Spec } R$, then \mathbb{F} is K -flat and exact.

Proof. By Lemma 2, it suffices to show that for any $n \in \mathbb{Z}$ and any R -module M , $H_n(M \otimes \mathbb{F}) = 0$. But this is trivial by Theorem 1 applied to the complex

$$\cdots \rightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \rightarrow 0.$$

□

The following was proved by A. Neeman [2, Corollary 6.10].

Theorem 4. *Let A be a ring, and \mathbb{P} a complex of projective left A -modules. If \mathbb{P} is K -flat and exact, then \mathbb{P} is null-homotopic.*

By Corollary 3 and Theorem 4, we have

Corollary 5. *Let \mathbb{P} be a complex of R -projective modules. If $\mathbb{P}(\mathfrak{p})$ is exact for any $\mathfrak{p} \in \text{Spec } R$, then \mathbb{P} is null-homotopic.*

The following also follows.

Corollary 6 (cf. [1, Lemma I.2.1.4]). *Let $\varphi : F_1 \rightarrow F_0$ be an R -linear map between R -flat modules. Then the following are equivalent.*

1 φ is injective and $\text{Coker } \varphi$ is R -flat.

2 φ is pure.

3 $\varphi(\mathfrak{p})$ is injective for any $\mathfrak{p} \in \text{Spec } R$.

Proof. **1** \Rightarrow **2** \Rightarrow **3** is obvious. **3** \Rightarrow **1** is a special case of Theorem 1. \square

Corollary 7 ([1, Corollary I.2.1.6]). *Let F be a flat R -module. If $F(\mathfrak{p}) = 0$ for any $\mathfrak{p} \in \text{Spec } R$, then $F = 0$.*

Proof. Consider the zero map $F \rightarrow 0$, and apply Corollary 6. We have that this map is injective, and hence $F = 0$. \square

Corollary 8. *Let $\varphi : F_1 \rightarrow F_0$ be an R -linear map between R -flat modules. If $\varphi(\mathfrak{p})$ is an isomorphism for any $\mathfrak{p} \in \text{Spec } R$, then φ is an isomorphism.*

Proof. By Corollary 6, φ is injective and $C := \text{Coker } \varphi$ is R -flat. Since $C(\mathfrak{p}) \cong \text{Coker}(\varphi(\mathfrak{p})) = 0$ for any $\mathfrak{p} \in \text{Spec } R$, we have that $C = 0$ by Corollary 7. \square

Corollary 9. *Let M be an R -module. If $\text{Tor}_i^R(\kappa(\mathfrak{p}), M) = 0$ for any $i > 0$, then M is R -flat. If $\text{Tor}_i^R(\kappa(\mathfrak{p}), M) = 0$ for any $i \geq 0$, then $M = 0$.*

Proof. For the first assertion, Let \mathbb{F} be a projective resolution of M , and apply Theorem 1. The second assertion follows from the first assertion and Corollary 7. \square

Corollary 10. *Let M be an R -module. If $\text{Ext}_R^i(M, \kappa(\mathfrak{p})) = 0$ for any $i > 0$ (resp. $i \geq 0$), then M is R -flat (resp. $M = 0$).*

Proof. This is trivial by Corollary 9 and the fact

$$\text{Ext}_R^i(M, \kappa(\mathfrak{p})) \cong \text{Hom}_{\kappa(\mathfrak{p})}(\text{Tor}_i^R(\kappa(\mathfrak{p}), M), \kappa(\mathfrak{p})). \quad \square$$

3. Some examples

Example 11. There is an acyclic projective complex

$$\mathbb{P} : \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow 0$$

over a noetherian commutative ring R such that $H_0(\mathbb{P})$ is R -flat and $h_0(\mathfrak{p}) := \dim_{\kappa(\mathfrak{p})} H_0(\mathbb{P}(\mathfrak{p}))$ is finite and constant, but $H_0(\mathbb{P})$ is neither R -finite nor R -projective.

Proof. Set $R = \mathbb{Z}$, $M = \sum_p (1/p)\mathbb{Z}$, and \mathbb{P} to be a projective resolution of M . Then M is R -torsion free, and is R -flat. Since $M_{(p)} = (1/p)\mathbb{Z}_{(p)}$, $h_0(\mathfrak{p}) = 1$ for any $\mathfrak{p} \in \text{Spec } \mathbb{Z}$. A finitely generated nonzero torsion-free \mathbb{Z} -submodule of \mathbb{Q} must be rank-one free, but M is not a cyclic module, and is not rank-one free. This shows that M is not R -finite. As R is a principal ideal domain, any R -projective module is free. If M is projective, then it is free of rank $h_0((0)) = 1$. But M is not finitely generated, so M is not projective. \square

Remark 12. Let (R, \mathfrak{m}) be a noetherian *local* ring, F a flat R -module, and c a non-negative integer. If $\dim_{\kappa(\mathfrak{p})} F(\mathfrak{p}) = c$ for any $\mathfrak{p} \in \text{Spec } R$, then $F \cong R^c$, see [1, Corollary III.2.1.10].

Remark 13. Let

$$\mathbb{P} : 0 \rightarrow P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \xrightarrow{d^2} \cdots$$

be an R -flat complex such that P^0 is R -projective. Assume that $\mathbb{P}(\mathfrak{p})$ is acyclic (i.e., $H^i(\mathbb{P}(\mathfrak{p})) = 0$ for any $i > 0$) and $h_{\mathbb{P}}^0(\mathfrak{p}) := \dim_{\kappa(\mathfrak{p})} H^0(\mathbb{P}(\mathfrak{p}))$ is finite for any $\mathfrak{p} \in \text{Spec } R$. If $h_{\mathbb{P}}^0$ is a locally constant function on $\text{Spec } R$, then $H^0(\mathbb{P})$ is R -finite R -projective, and $M \otimes \mathbb{P}$ is acyclic and the canonical map $M \otimes H^0(\mathbb{P}) \rightarrow H^0(M \otimes \mathbb{P})$ is an isomorphism for any R -module M , see [1, Proposition III.2.1.14]. If, moreover, \mathbb{P} is an R -projective complex, then $\text{Im } d^i$ is R -projective for any $i \geq 0$, as can be seen easily from Theorem 4.

Example 14. Let M be an R -module. Even if $M(\mathfrak{p}) = 0$ for any $\mathfrak{p} \in \text{Spec } R$, M may not be zero. Even if $\text{Tor}_1^R(\kappa(\mathfrak{p}), M) = 0$ for any $\mathfrak{p} \in \text{Spec } R$, M may not be R -flat.

Indeed, let (R, \mathfrak{m}, k) be a d -dimensional regular local ring, and E the injective hull of k . Then

$$\text{Tor}_i^R(\kappa(\mathfrak{p}), E) \cong \begin{cases} k & (i = d \text{ and } \mathfrak{p} = \mathfrak{m}) \\ 0 & (\text{otherwise}) \end{cases}$$

E is not R -flat unless $d = 0$.

Proof. Since $\text{supp } E = \{\mathfrak{m}\}$, $\text{Tor}_i^R(\kappa(\mathfrak{p}), E) = 0$ unless $\mathfrak{p} = \mathfrak{m}$.

Let $\mathbf{x} = (x_1, \dots, x_d)$ be a regular system of parameters of R , and \mathbb{K} the Koszul complex $K(\mathbf{x}; R)$, which is a minimal free resolution of k . Note that \mathbb{K} is self-dual. That is, $\mathbb{K}^* \cong \mathbb{K}[-d]$, where $\mathbb{K}^* = \text{Hom}_R^\bullet(\mathbb{K}, R)$, and $\mathbb{K}[-d]^n = \mathbb{K}^{n-d}$. So

$$\begin{aligned} \text{Tor}_i^R(k, E) &\cong H^{-i}(\mathbb{K} \otimes E) \cong H^{-i}(\mathbb{K}^{**} \otimes E) \cong H^{-i}(\text{Hom}_R^\bullet(\mathbb{K}[-d], E)) \\ &\cong H^{-i}(\text{Hom}_R^\bullet(k[-d], E)) \cong \begin{cases} k & (i = d) \\ 0 & (i \neq d) \end{cases} . \end{aligned}$$

□

Example 15. There is a projective complex \mathbb{P} over a noetherian commutative ring R such that for each $\mathfrak{m} \in \text{Max}(R)$, $\mathbb{P}(\mathfrak{m})$ is exact, but \mathbb{P} is not exact, where $\text{Max}(R)$ denotes the set of maximal ideals of R .

Proof. Let R be a DVR with its field of fractions K , and \mathbb{P} a projective resolution of K . □

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