G-prime and G-primary G-ideals on G-schemes

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1 Introduction

This report is a preliminary version, and a more detailed final version will be published elsewhere.

Let A be a $\mathbb{Z}^n$-graded ring, I a prime ideal (resp. radical ideal, primary ideal) of A, and I^* the homogeneous ideal generated by the all homogeneous elements of I. Then it is well-known that I^* is again a prime ideal (resp. radical ideal, primary ideal). In particular, if P is a prime ideal of A, then the local ring A_{P^*} makes sense. In particular, the following theorem makes sense.

Theorem 1.1. *Let $M = \mathbb{Z}^n$, A be an M-graded noetherian ring, and P a prime ideal of A. If A_{P^*} is Cohen–Macaulay (resp. Gorenstein, complete intersection, regular), then so is A_P.*

This theorem was conjectured by Nagata [8] for the case that $n = 1$ for the Cohen–Macaulay property, and solved by Hochster–Ratliff [5], Matijevic–Roberts [7], Matijevic [6], Aoyama–Goto [1], and Avramov–Achiles [2], affirmatively.

If M is a finitely generated abelian group with torsion elements and A is M-graded, then even if P is a prime ideal, P^* may not be a prime. However, a homogeneous ideal of the form P^* has some special interest. For homogeneous ideals I and J, if $IJ \subset P^*$, then either $I \subset P^*$ or $J \subset P^*$. Our start of this research is to consider a substitute of A_{P^*} in this context.

More generally, let S be a scheme, G an S-group scheme, and X a noetherian G-scheme, where a G-scheme means an S scheme on which G acts. We assume that the second projection $p_2 : G \times X \to X$ is flat of finite type. Under these settings, we define a G-prime, G-primary, and G-radical G-ideals. As we will see, these are natural generalization of prime, primary, and radical ideals, respectively. We study some important properties of G-stable closed subschemes defined by G-primary ideals. Moreover, we generalize Theorem 1.1.

Utilizing this research, we can remove the assumption that G is smooth with connected fibers from the talk of Ohtani [9] given at the 29th Symposium on Commutative Algebra in Japan. This will be discussed elsewhere.

2 G-prime ideals

Let S, G, and X be as in the introduction.

Definition 2.1 (Mumford). A G-linearized $\mathcal{O}_X$-module (an equivariant $(G, \mathcal{O}_X)$-module) is a pair $(\mathcal{M}, \Phi)$ such that $\mathcal{M}$ is an $\mathcal{O}_X$-module, and $\Phi : a^*\mathcal{M} \to p_2^*\mathcal{M}$ is an isomorphism of $\mathcal{O}_{G\times X}$-modules such that

$$(\mu \times 1_X)^*\Phi : (\mu \times 1_X)^* a^*\mathcal{M} \to (\mu \times 1_X)^* p_2^*\mathcal{M}$$

agrees with

$$\begin{aligned}(\mu \times 1_X)^* a^*\mathcal{M} \xrightarrow{\cong} (1_G \times a)^* a^*\mathcal{M} \xrightarrow{\Phi} (1_G \times a)^* p_2^*\mathcal{M} \\ \xrightarrow{\cong} p_{23}^* a^*\mathcal{M} \xrightarrow{\Phi} p_{23}^* p_2^*\mathcal{M} \xrightarrow{\cong} (\mu \times 1_X)^* p_2^*\mathcal{M},\end{aligned}$$

where $p_{23} : G \times G \times X \to G \times X$ is the projection. In this case, we sometimes say that $\mathcal{M}$ is a G-linearized $\mathcal{O}_X$-module with Φ its structure map.

Definition 2.2. A morphism $\varphi : (\mathcal{M}, \Phi) \to (\mathcal{N}, \Psi)$ of G-linearized $\mathcal{O}_X$-modules is a morphism $\varphi : \mathcal{M} \to \mathcal{N}$ such that $\Psi \circ (a^*\varphi) = (p_2^*\varphi) \circ \Phi$.

Thus we have a category of G-linearized $\mathcal{O}_X$-modules in a natural way.

Definition 2.3. Let $(\mathcal{M}, \Phi)$ be a G-linearized $\mathcal{O}_X$-module. We say that $\mathcal{N}$ is an equivariant $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$ if $\mathcal{N}$ is an $\mathcal{O}_X$-submodule of $\mathcal{M}$, and $\Phi(a^*\mathcal{N}) = p_2^*\mathcal{N}$ (note that a and p_2 are flat). If, moreover, $\mathcal{M} = \mathcal{O}_X$, then we say that $\mathcal{N}$ is a G-*ideal* of $\mathcal{O}_X$.

If $\mathcal{N}$ is an equivariant $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$, then $(\mathcal{N}, \Phi|_{\mathcal{N}})$ is a G-linearized $\mathcal{O}_X$-module, and the inclusion $\mathcal{N} \hookrightarrow \mathcal{M}$ is a morphism of G-linearized $\mathcal{O}_X$-modules. Conversely, if $\varphi : \mathcal{N} \to \mathcal{M}$ is a morphism of G-linearized $\mathcal{O}_X$-modules, then the image of φ is an equivariant $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$.

The following is [4, Corollary 12.8, Lemma 12.12].

Theorem 2.4. The category $\mathrm{Qch}(G, X)$ of quasi-coherent G-linearized $\mathcal{O}_X$-modules is a locally noetherian abelian category, and $(\mathcal{M}, \Phi)$ is a noetherian object of $\mathrm{Qch}(G, X)$ if and only if $\mathcal{M}$ is coherent. The forgetful functor $F_X : \mathrm{Qch}(G, X) \to \mathrm{Qch}(X)$ given by $(\mathcal{M}, \Phi) \mapsto \mathcal{M}$ is faithful exact, and admits a right adjoint.

(Quasi-) coherent G-linearized $\mathcal{O}_X$-modules are closed under various ring-theoretic operations.

Lemma 2.5. Let $\mathcal{M}$, $\mathcal{N}$, $\mathcal{L}$ be in $\mathrm{Qch}(G, X)$, $\mathcal{I}$ be a G-ideal, and $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, and $\mathcal{M}_\lambda$ be quasi-coherent equivariant $(G, \mathcal{O}_X)$-submodules of $\mathcal{M}$. Let $\mathcal{L}$ and $\mathcal{M}_3$ be coherent. Then the following modules have structures of quasi-coherent G-linearized $\mathcal{O}_X$-modules: $\underline{\mathrm{Tor}}_i^{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$, $\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{L}, \mathcal{M})$, $\underline{H}^i_{\mathcal{I}}(\mathcal{M}) \cong \varinjlim \underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M})$, the Fitting ideal $\underline{\mathrm{Fitt}}_j(\mathcal{L})$, $\mathcal{M}_1 \cap \mathcal{M}_2$, $\sum_\lambda \mathcal{M}_\lambda$, $\mathcal{I}\mathcal{M}_1$, $\mathcal{M}_1 : \mathcal{M}_3$, and $\mathcal{M}_1 : \mathcal{I}$.

Let $\mathcal{M}$ be in $\mathrm{Qch}(G, X)$, and $\mathfrak{m}$ be an $\mathcal{O}_X$-submodule of $\mathcal{M}$. The sum of all quasi-coherent equivariant $(G, \mathcal{O}_X)$-submodules of $\mathcal{M}$ contained in $\mathfrak{m}$ is denoted by $\mathfrak{m}^*$. $\mathfrak{m}^*$ is the largest quasi-coherent equivariant $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$ contained in $\mathfrak{m}$.

Let $Y = V(\mathfrak{a})$ be a closed subscheme of X. Then $Y^* := V(\mathfrak{a}^*)$ is the smallest G-stable closed subscheme of X containing Y.

From now on, all ideals and G-ideals are required to be coherent. All modules and G-linearized modules are required to be quasi-coherent.

Lemma 2.6. Let $\mathcal{M}$ be in $\mathrm{Qch}(G, X)$, $\mathfrak{m}$, $\mathfrak{n}$, and $\mathfrak{m}_\lambda$ be $\mathcal{O}_X$-submodules of $\mathcal{M}$, and $\mathcal{N}$ be a coherent equivariant $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$. Let $\mathcal{I}$ be a G-ideal of $\mathcal{O}_X$. Then we have: **1)** $(\bigcap_\lambda \mathfrak{m}_\lambda^*)^* = (\bigcap_\lambda \mathfrak{m}_\lambda)^*$; **2)** $\mathfrak{m}^* \cap \mathfrak{n}^* = (\mathfrak{m} \cap \mathfrak{n})^*$; **3)** $(\mathfrak{m} : \mathcal{N})^* = \mathfrak{m}^* : \mathcal{N}$; **4)** $(\mathfrak{m} : \mathcal{I})^* = \mathfrak{m}^* : \mathcal{I}$.

3 G-prime and G-radical G-ideals

Lemma 3.1. Let $\mathcal{P}$ be a G-ideal of $\mathcal{O}_X$. Then the following are equivalent.

- There exists some ideal $\mathfrak{p}$ of $\mathcal{O}_X$ such that $\mathfrak{p}$ is prime (i.e., $V(\mathfrak{p})$ is integral) and $\mathfrak{p}^* = \mathcal{P}$.
- $\mathcal{P} \neq \mathcal{O}_X$, and if $\mathcal{I}$ and $\mathcal{J}$ are G-ideals of $\mathcal{O}_X$ and $\mathcal{I}\mathcal{J} \subset \mathcal{P}$, then $\mathcal{I} \subset \mathcal{P}$ or $\mathcal{J} \subset \mathcal{P}$.

Definition 3.2. If the equivalent conditions in the lemma are satisfied, we say that $\mathcal{P}$ is a *G-prime* G-ideal.

Definition 3.3. Let $\mathcal{I}$ be a G-ideal of $\mathcal{O}_X$. Then $V_G(\mathcal{I})$ denotes the set of G-prime ideals containing $\mathcal{I}$. We set $\sqrt[G]{\mathcal{I}} := (\bigcap_{\mathcal{P} \in V_G(\mathcal{I})} \mathcal{P})^*$, and call $\sqrt[G]{\mathcal{I}}$ the *G-radical* of $\mathcal{I}$.

Lemma 3.4. Let $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{P}$ be G-ideals of $\mathcal{O}_X$. Then we have: **1)** $\mathcal{I} \subset \sqrt[G]{\mathcal{I}} \subset \sqrt{\mathcal{I}}$, $\sqrt[G]{\mathcal{I}} = \sqrt{\mathcal{I}}^*$. **2)** If $\mathcal{I} \supset \mathcal{J}$, then $\sqrt[G]{\mathcal{I}} \supset \sqrt[G]{\mathcal{J}}$. **3)** $\sqrt[G]{\mathcal{I}\mathcal{J}} = \sqrt[G]{\mathcal{I} \cap \mathcal{J}} = \sqrt[G]{\mathcal{I}} \cap \sqrt[G]{\mathcal{J}}$. **4)** $\sqrt[G]{\sqrt[G]{\mathcal{I}}} = \sqrt[G]{\mathcal{I}}$. **5)** If $\mathcal{P}$ is a G-prime, then $\sqrt[G]{\mathcal{P}} = \mathcal{P}$.

Lemma 3.5. Let $\mathcal{I}$ be a G-ideal of $\mathcal{O}_X$. Then the following are equivalent. **1)** $\mathcal{I} = \sqrt[G]{\mathcal{I}}$; **2)** $\mathcal{I}$ is the intersection of finitely many G-prime G-ideals; **3)** There exists some ideal $\mathfrak{a}$ of $\mathcal{O}_X$ such that $\mathfrak{a}$ is radical (i.e., $V(\mathfrak{a})$ is reduced), and $\mathfrak{a}^* = \mathcal{I}$.

If the equivalent conditions in the lemma are satisfied, then we say that $\mathcal{I}$ is *G-radical*. A G-prime G-ideal is G-radical.

4 G-primary submodules

From now on, until the end of this report, let $\mathcal{M}$ be a coherent G-linearized $\mathcal{O}_X$-module, and $\mathcal{N}$ its coherent equivariant $(G, \mathcal{O}_X)$-submodule.

Definition 4.1. We say that $\mathcal{N}$ is *G-primary* if $\mathcal{N} \neq \mathcal{M}$, and for any coherent equivariant $(G, \mathcal{O}_X)$-submodule $\mathcal{L}$ of $\mathcal{M}$, either $\mathcal{N} : \mathcal{L} = \mathcal{O}_X$ or $\mathcal{N} : \mathcal{L} \subset \sqrt[G]{\mathcal{N} : \mathcal{M}}$ holds.

If $\mathcal{N}$ is G-primary, then $\mathcal{P} = \sqrt[G]{\mathcal{N} : \mathcal{M}}$ is G-prime. In this case, we say that $\mathcal{N}$ is $\mathcal{P}$-G-primary.

Lemma 4.2. For a prime ideal $\mathfrak{p}$ of $\mathcal{O}_X$, $\mathfrak{p}^*$ is G-prime. For a radical ideal $\mathfrak{a}$ of $\mathcal{O}_X$, $\mathfrak{a}^*$ is G-radical. If $\mathfrak{n}$ is a $\mathfrak{p}$-primary $\mathcal{O}_X$-submodule of $\mathcal{M}$, then $\mathfrak{n}^*$ is a $\mathfrak{p}^*$-G-primary submodule of $\mathcal{M}$. For a G-primary submodule $\mathcal{N}$ of $\mathcal{M}$, there exists some primary $\mathcal{O}_X$-submodule $\mathfrak{n}$ of $\mathcal{M}$ such that $\mathfrak{n}^* = \mathcal{N}$.

An expression

$$\mathcal{N} = \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_r$$

is called a *G-primary decomposition* if this equation holds, and each $\mathcal{M}_i$ is a G-primary submodule of $\mathcal{M}$. We say that the decomposition is *minimal* if $\mathcal{N} \neq \bigcap_{j \neq i} \mathcal{M}_j$ for any i, and $\sqrt[G]{\mathcal{M}_i : \mathcal{M}}$ is distinct.

Proposition 4.3. $\mathcal{N}$ has a minimal G-primary decomposition.

Proof (sketch). Let

$$\mathcal{N} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r$$

be a usual primary decomposition. Then

$$\mathcal{N} = \mathcal{N}^* = (\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r)^* = \mathfrak{m}_1^* \cap \cdots \cap \mathfrak{m}_r^*$$

is a G-primary decomposition. We can make it minimal, as usual. □

Theorem 4.4. The set

$$\mathrm{Ass}_G(\mathcal{M}/\mathcal{N}) = \{ \sqrt[G]{\mathcal{M}_i : \mathcal{M}} \mid i = 1, \ldots, r \}$$

is independent of the choice of minimal G-primary decomposition

$$\mathcal{N} = \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_r,$$

and depends only on $\mathcal{M}/\mathcal{N}$.

We call an element of $\mathrm{Ass}_G(\mathcal{M}/\mathcal{N})$ a *G-associated G-prime*. The set of minimal elements of $\mathrm{Ass}_G(\mathcal{M}/\mathcal{N})$ is denoted by $\mathrm{Min}_G(\mathcal{M}/\mathcal{N})$, and its element is called a *G-minimal G-prime*. An element of the set $\mathrm{Ass}_G(\mathcal{M}/\mathcal{N}) \setminus \mathrm{Min}_G(\mathcal{M}/\mathcal{N})$ is called a *G-embedded G-prime*.

Theorem 4.5. Let

$$\mathcal{N} = \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_r$$

be a minimal G-primary decomposition and

$$\mathcal{M}_i = \mathfrak{m}_{i,1} \cap \cdots \cap \mathfrak{m}_{i,s_i}$$

a minimal primary decomposition. Then

$$\mathcal{N} = \bigcap_{i=1}^{r} (\mathfrak{m}_{i,1} \cap \cdots \cap \mathfrak{m}_{i,s_i})$$

is a minimal primary decomposition.

Proposition 4.6. A G-primary submodule $\mathcal{N}$ of $\mathcal{M}$ does not have an embedded prime. For each minimal prime $\mathfrak{p}$ of $\mathcal{M}/\mathcal{N}$, we have $\mathfrak{p}^* = \sqrt[G]{\mathcal{N} : \mathcal{M}}$.

Corollary 4.7. We have

$$\mathrm{Ass}(\mathcal{M}/\mathcal{N}) = \coprod_{i=1}^{s} \mathrm{Ass}(\mathcal{M}/\mathcal{M}_i) = \coprod_{\mathcal{P} \in \mathrm{Ass}_G(\mathcal{M}/\mathcal{N})} \mathrm{Ass}(\mathcal{O}_X/\mathcal{P})$$

and

$$\mathrm{Ass}_G(\mathcal{M}/\mathcal{N}) = \{\mathfrak{p}^* \mid \mathfrak{p} \in \mathrm{Ass}(\mathcal{M}/\mathcal{N})\}$$

Corollary 4.8. $\mathrm{Ass}(\mathcal{M}/\mathcal{N}) = \mathrm{Min}(\mathcal{M}/\mathcal{N})$ if and only if $\mathrm{Ass}_G(\mathcal{M}/\mathcal{N}) = \mathrm{Min}_G(\mathcal{M}/\mathcal{N})$.

5 Smooth group schemes and Group schemes with connected fibers

For some groups, the notion of G-prime G-ideal agrees with that of G-ideal which is a prime ideal.

Lemma 5.1. Assume that G is S-smooth. If $\mathfrak{a}$ is a radical ideal of $\mathcal{O}_X$, then $\mathfrak{a}^*$ is also radical. In particular, any G-radical G-ideal is radical.

Corollary 5.2. Assume that G is S-smooth. If $\mathcal{I}$ is a G-ideal of $\mathcal{O}_X$, then $\sqrt{\mathcal{I}} = \sqrt[G]{\mathcal{I}}$. In particular, $\sqrt{\mathcal{I}}$ is a G-radical G-ideal.

Lemma 5.3. Assume that $G \to S$ has connected fibers. If $\mathfrak{q}$ is a primary ideal of $\mathcal{O}_X$, then $\mathfrak{q}^*$ is also primary. In particular, a G-primary G-ideal is primary.

Corollary 5.4. Assume that $G \to S$ has connected fibers. If $\mathcal{I}$ is a G-ideal, then a minimal G-primary decomposition of $\mathcal{I}$ is also a minimal primary decomposition.

Corollary 5.5. Assume that $G \to S$ is smooth with connected fibers. If $\mathfrak{p}$ is a prime, then $\mathfrak{p}^*$ is also a prime. Any G-prime G-ideal is a prime. For a G-ideal $\mathcal{I}$ of $\mathcal{O}_X$, any associated prime of $\mathcal{I}$ is a G-prime G-ideal.

6 G-stable closed subschemes defined by G-primary G-ideals

Theorem 6.1. Let 0 be G-primary in $\mathcal{O}_X$. Then the dimension of the fiber of $p_2 : G \times X \to X$ is constant.

Theorem 6.2. Let 0 be G-primary in $\mathcal{O}_X$. If X has an affine open covering $(\operatorname{Spec} A_i)$ such that each A_i is **Hilbert, universally catenary, and for any minimal prime P of A_i, the heights of maximal ideals of A_i/P are the same** (for example, X is of finite type over a field or $\mathbb{Z}$). Then the dimensions of the irreducible components of X are the same.

Remark 6.3. There is an example of $G = X$ such that the dimensions of the irreducible components are different. The **bold face** assumptions are necessary. The **bold face** property is preserved by of-finite-type extensions.

The following is a generalization of Theorem 1.1.

Theorem 6.4. Let $y \in X$ and $Y = \bar{y}$. Let η be the generic point of an irreducible component of Y^*. Then: **1)** $\dim \mathcal{O}_{X,y} \geq \dim \mathcal{O}_{X,\eta}$. **2)** If $\mathcal{M}_\eta$ is maximal Cohen–Macaulay (resp. of finite injective dimension, projective dimension m, $\dim - \operatorname{depth} = n$, torsionless, reflexive, G-dimension g), then so is $\mathcal{M}_y$. **3)** If $\mathcal{O}_{X,\eta}$ is a complete intersection, then so is $\mathcal{O}_{X,y}$. **4)** If G is smooth and $\mathcal{O}_{X,\eta}$ is regular, then $\mathcal{O}_{X,y}$ is regular. **5)** Assume that G is smooth and X is a locally excellent $\mathbb{F}_p$-scheme. If $\mathcal{O}_{X,\eta}$ is weakly F-regular (resp. F-regular, F-rational), then so is $\mathcal{O}_{X,y}$.

Some special cases of Theorem 6.4 was proved by the author [3], and the author and M. Ohtani (unpublished).

Consider the case $S = \operatorname{Spec} \mathbb{Z}$, $G = \mathbb{G}_m^n$, and $X = \operatorname{Spec} A$ is affine. Then A is a $\mathbb{Z}^n$-graded ring.

Corollary 6.5. Let A be a locally excellent $\mathbb{Z}^n$-graded $\mathbb{F}_p$-algebra. Let P be a prime ideal of A, and let P^* be the prime ideal generated by homogeneous elements of P. If A_{P^*} is weakly F-regular (resp. F-regular, F-rational), then so is A_P.

Corollary 6.6. Let Y be a G-stable closed subscheme of X defined by a G-primary G-ideal. If η and ζ are generic points of irreducible components of Y, then $\dim \mathcal{O}_{X,\eta} = \dim \mathcal{O}_{X,\zeta}$.

X is said to be G-*artinian* if every G-prime of $\mathcal{O}_X$ is a G-minimal G-prime of 0.

Corollary 6.7. A G-artinian G-scheme is Cohen–Macaulay.

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