F-purity of homomorphisms, strong F-regularity, and F-injectivity

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1. Introduction

These notes are a summary of the results in [5].

In [6], a generalization of Matijevic–Roberts theorem was proved, and as a corollary, we have the following [6, Corollary 7.10].

1.1 Theorem. *Let p be a prime number, and A a $\mathbb{Z}^n$-graded noetherian ring. Let P be a prime ideal of A, and P^* be the prime ideal of A generated by the homogeneous elements of P. If A_{P^*} is excellent of characteristic p and is weakly F-regular* (*resp. F-regular, F-rational*), *then A_P is weakly F-regular* (*resp. F-regular, F-rational*). *If A_P is excellent of characteristic p and is weakly F-regular, then A_{P^*} is weakly F-regular.*

The proof relies on the smooth base change and the flat descent of (weak) F-regularity and F-rationality.

It is natural to ask the same problem for F-purity, strong F-regularity, and Cohen–Macaulay F-injectivity. This question was posed by Ken-ichi Yoshida. The purpose of these short notes is to give an answer to this question. On the way, we also consider the problem of the openness of the loci of strong F-regularity and CMFI (Cohen–Macaulay F-injective) property. The openness of the F-rational locus is discussed in [12]. Related to the smooth base change of F-purity, we define F-purity of a homomorphism between noetherian rings of characteristic p. It is not characterized by the F-purity of (geometric) fibers. We discuss when an F-pure homomorphism is flat.

Strong F-regularity was defined for an F-finite noetherian ring of characteristic p by Hochster and Huneke [8]. This notion was generalized to those for a general noetherian ring of characteristic p in different ways by Hochster [7] and Hochster–Huneke [10]. The author does not know if the two definitions agree. So we name Hochster's definition "strongly F-regular" and Hochster–Huneke's "very strongly F-regular." As the name shows, very strongly F-regular implies strongly F-regular in general. For local rings, F-finite rings, and algebras essentially of finite type over excellent local rings, these two notions coincide. We prove "F-pure base change" of strong F-regularity, generalizing the smooth base change.

2. F-purity of homomorphisms

It is sometimes useful to promote a property of a ring to that of a homomorphism of rings. This idea is due to Grothendieck.

2.1 Definition (Grothendieck). Let $\mathbb{P}$ be one of the properties: Cohen–Macaulay, Gorenstein, l.c.i., reduced, normal, and regular. We say that a homomorphism $f : A \to B$ of noetherian rings is $\mathbb{P}$, if f is flat, and the fiber ring $B \otimes_A \kappa(P)$ is geometrically $\mathbb{P}$ for any $P \in \operatorname{Spec} A$, that is, $B \otimes_A K$ satisfies $\mathbb{P}$ for any finite extension field K of $\kappa(P)$.

Weakening the flatness condition, Avramov and Foxby generalized this definition, see [2], [3], [4].

2.2 Remark. Let $\mathbb{P}$ be as in Definition 2.1. A composite of $\mathbb{P}$ morphisms is again $\mathbb{P}$. If $f : A \to B$ is $\mathbb{P}$ and A is $\mathbb{P}$, then B is $\mathbb{P}$. If $f : A \to B$ is $\mathbb{P}$ and A' is an A-algebra, then the base change $f' : A' \to B'$ is $\mathbb{P}$, provided both A' and B' are noetherian. If $f : A \to B$ is a homomorphism, A' is a faithfully flat A-algebra, and if the base change $f' : A' \to B'$ is $\mathbb{P}$, then f is $\mathbb{P}$.

It is natural to ask if Grothendieck's idea applies to F-singularities.

2.1 Theorem (Aberbach–Enescu [1]). *Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a flat local homomorphism of noetherian local rings of characteristic p. If A is Cohen–Macaulay F-injective and $B/\mathfrak{m}B$ is geometrically Cohen–Macaulay F-injective, then B is Cohen–Macaulay F-injective.*

Thus we may define a Cohen–Macaulay F-injective homomorphism to be a flat homomorphism with CMFI geometric fibers.

2.3 Example (Singh [11]). There is a flat local homomorphism $f : A \to B$ with A a DVR, f has geometrically F-regular fibers, but B is not F-pure.

Because of the example, probably it is not appropriate to define an F-pure homomorphism to be a flat homomorphism with geometrically F-pure fibers.

(2.4) To define an F-pure homomorphism, we uitilize Radu–André homomorphisms.

Let R be a ring of characteristic p. Let $F_R^e : R \to {}^eR$ be the eth Frobenius map given by $F_R^e(x) = x^{p^e}$, where the ring R, considered as an R-algebra via the structure map F_R^e, is denoted by ${}^eR$. An element $a \in R$, viewed as an element of ${}^eR = R$, is denoted by ${}^ea$.

2.5 Definition. For a homomorphism $f : A \to B$ of commutative rings of characteristic p, we define

$$\Psi_e(f) = \Psi_e(A, B) : B \otimes_A {}^eA \to {}^eB$$

by $\Psi_e(f)(b \otimes {}^ea) = {}^e(b^{p^e}a)$, and call it the eth *Radu–André homomorphism* (or the eth relative Frobenius map).

2.2 Theorem (Radu–André). *Let $f : A \to B$ be a homomorphism of noetherian rings of characteristic p. Then the following are equivalent: 1) f is regular; 2) $\Psi_e(f)$ is flat for some $e \geq 1$; 3) $\Psi_e(f)$ is flat for every $e \geq 1$.*

The absolute case (i.e., the case that $A = \mathbb{F}_p$) is due to Kunz.

Using Radu–André homomorphism, we define the F-purity of homomorphisms.

2.6 Definition. A homomorphism $f : A \to B$ of rings of characteristic p is said to be *F-pure* if the Radu–André homomorphism $\Psi_1(f) : B \otimes_A {}^1A \to {}^1B$ is pure.

Thus a homomorphism $f : A \to B$ of rings of characteristic p is F-pure if it is regular. We list some consequences of the definition.

2.7 Lemma. *Let $f : A \to B$ and $g : B \to C$ be homomorphisms between $\mathbb{F}_p$-algebras.*

1) If f and g are F-pure, then so is gf. 2) A is F-pure if and only if the unique map $\mathbb{F}_p \to A$ is F-pure. 3) If gf is F-pure and g is pure, then f is F-pure. 4) If A is F-pure and f is F-pure, then B is F-pure. 5) A pure subring of an F-pure ring is F-pure. 6) Let A' be an A-algebra, and $B' = B \otimes_A A'$. If f is F-pure, then the base change $A' \to B'$ is also F-pure. 7) If $A \to A'$ is a pure homomorphism and $A' \to B' = B \otimes_A A'$ is F-pure, then f is F-pure.

F-purity over a field can be described as follows.

2.3 Theorem. *Let K be a field of characteristic p, and A a K-algebra. Then the following are equivalent.*

1. *B is noetherian, and $K \to B$ is F-pure.*
2. *For any $e \geq 1$, $B \otimes_K {}^eK$ is noetherian and F-pure.*
3. *For some $e \geq 1$, $B \otimes_K {}^eK$ is noetherian and F-pure.*
4. *B is noetherian, and B is geometrically F-pure over K.*

2.8 Corollary. *If $f : A \to B$ is an F-pure homomorphism between noetherian rings of characteristic p, then the fiber $B \otimes_A \kappa(P)$ is geometrically F-pure for each $P \in \operatorname{Spec} A$.*

The converse of the corollary is false, see (2.3).

2.9 Definition. A homomorphism $f : A \to B$ of rings of characteristic p is said to be *Dumitrescu* if $\Psi_1(f) : B \otimes_A {}^1A \to {}^1B$ is ${}^1A$-pure.

2.4 Theorem (Dumitrescu). *For a flat homomorphism $f : A \to B$ of noetherian rings of characteristic p, the following are equivalent.*

1. *f is Dumitrescu.*
2. *f is reduced.*

It is natural to ask, is a Dumitrescu homomorphism flat? The author does not know the answer. Some partial results follows.

2.5 Theorem. *Let $f : A \to B$ be a homomorphism of noetherian rings of characteristic p. If f is Dumitrescu and the image of $\operatorname{Spec} B \to \operatorname{Spec} A$ contains all maximal ideals of A, then f is pure.*

2.10 Corollary. *A Dumitrescu local homomorphism between noetherian local rings of characteristic p is pure.*

2.11 Definition. Let $f : A \to B$ be a homomorphism of noetherian rings. We say that f is almost quasi-finite if each fiber $B \otimes_A \kappa(P)$ is finite over $\kappa(P)$. This is equivalent to say that for each $Q \in \operatorname{Spec} B$, $\kappa(Q)$ is a finite extension of $\kappa(P)$, where $P = Q \cap A$.

A quasi-finite homomorphism (finite-type homomorphism with zero dimensional fibers) is almost quasi-finite. A localization is almost quasi-finite. A composite of almost quasi-finite homomorphisms is almost quasi-finite. A base change of an almost quasi-finite homomorphism is again almost quasi-finite, provided it is a homomorphism between noetherian rings.

2.6 Theorem (Watanabe, H–). *Let $f : A \to B$ be an almost quasi-finite homomorphism between noetherian rings of characteristic p. Then the following are equivalent: 1) f is F-pure; 2) f is Dumitrescu; 3) f is regular.*

The case that both A and B are domains and f is finite is due to K.-i. Watanabe.

2.7 Theorem. *Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a Dumitrescu local homomorphism between noetherian local rings of characteristic p. If $t \in \mathfrak{m}$, A is normally flat along tA, and $A/tA \to B/tB$ is flat, then f is flat.*

2.12 Corollary. *Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a Dumitrescu local homomorphism between noetherian local rings of characteristic p. If $t \in \mathfrak{m}$ is a nonzerodivisor of A and $A/tA \to B/tB$ is flat, then f is flat.*

2.13 Corollary. *Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a Dumitrescu local homomorphism between noetherian local rings of characteristic p. If A is regular, then f is flat.*

3. Strong F-regularity and Cohen–Macaulay F-injectivity

3.1 Definition. Let R be a noetherian ring of characteristic p. We say that R is

1. (Hochster, [7]) *strongly F-regular* if any R-submodule of any R-module is tightly closed.
2. (Hochster–Huneke, [10]) *very strongly F-regular* if for any $c \in R^\circ$, there exists some $e \geq 1$ such that ${}^e cF^e : R \to {}^e R \quad (x \mapsto {}^e(cx^{p^e}))$ is R-pure.

The name "very strongly F-regular" was new in [5]. This was introduced to distinguish the notion from the strong F-regularity by Hochster. The author does not know if these two definitions agree. They agree with the original definition of Hochster–Huneke [8], if the ring is F-finite.

3.2 Lemma. *Let R be a noetherian ring of characteristic p. Then the following are equivalent.*

1. *R is strongly F-regular.*
2. *For any multiplicatively closed subset S of R, R_S is strongly F-regular.*
3. *For any maximal ideal $\mathfrak{m}$ of R, $R_\mathfrak{m}$ is strongly F-regular.*

4. *For any maximal ideal $\mathfrak{m}$ of R, the R-submodule 0 of $E_R(R/\mathfrak{m})$ is tightly closed.*

5. *For any maximal ideal $\mathfrak{m}$ of R, the $R_{\mathfrak{m}}$-submodule 0 of $E_R(R/\mathfrak{m})$ is tightly closed.*

6. *For any maximal ideal $\mathfrak{m}$ of R, $R_{\mathfrak{m}}$ is very strongly F-regular.*

3.3 Remark. Let R be a noetherian ring of characteristic p.

1. If R is very strongly F-regular and S is a multiplicatively closed subset of R, then R_S is very strongly F-regular.

2. If R is very strongly F-regular, then it is strongly F-regular.

3. If R is strongly F-regular, then it is F-regular.

4. If R is F-rational Gorenstein, then it is strongly F-regular.

3.4 Lemma. *Let S be a noetherian normal ring, and R its cyclically pure subring. Then R is noetherian normal, and R is a pure subring of S.*

The following is a generalization of [9, (4.12)] (the assumption $R^\circ \subset S^\circ$ is dropped). For the proof, we use the lemma above.

3.5 Proposition. *Let S be a noetherian ring of characteristic p, and R its cyclically pure subring. If S is very strongly F-regular (resp. strongly F-regular, F-regular, weakly F-regular), then so is R.*

The following is proved using the Γ-construction developed in [10]. A similar result for F-rationality is proved by Vélez [12].

3.1 Theorem. *Let R be an excellent local ring of characteristic p, and A an R-algebra essentially of finite type. Then the strongly F-regular locus and the Cohen–Macaulay F-injective locus of A are Zariski open in* $\operatorname{Spec} A$.

Using a similar technique, Hoshi proved the following.

3.2 Theorem (Hoshi). *Let R be an excellent local ring of characteristic p, and A an R-algebra essentially of finite type. Then the F-pure locus of A is Zariski open in* $\operatorname{Spec} A$.

The following is an "F-pure base change" of strong F-regularity, which is stronger than the smooth base change.

3.3 Theorem. *Let $\varphi : A \to B$ be a homomorphism of noetherian rings of characteristic p. Assume that A is a strongly F-regular domain. Assume that the generic fiber $Q(A) \otimes_A B$ is strongly F-regular, where $Q(A)$ is the field of fractions of A. If φ is F-pure and B is locally excellent, then B is strongly F-regular.*

4. Matijevic–Roberts type theorem

M. Miyazaki and the author proved the following form of Matijevic–Roberts type theorem.

4.1 Theorem. *Let S be a scheme, G a smooth S-group scheme of finite type, X a noetherian G-scheme, $y \in X$, $Y := \overline{\{y\}}$, Y^* the smallest G-stable closed subscheme of X containing Y. Let η be the generic point of an irreducible component of Y^*. Let $\mathcal{C}$ and $\mathcal{D}$ be classes of noetherian local rings. Assume:*

1. (Smooth base change) *If $A \to B$ is a regular (i.e., flat with geometrically regular fibers) local homomorphism essentially of finite type and $A \in \mathcal{C}$, then $B \in \mathcal{D}$.*
2. (Flat descent) *If $A \to B$ is a regular local homomorphism essentially of finite type and $B \in \mathcal{D}$, then $A \in \mathcal{D}$.*

If $\mathcal{O}_{X,\eta} \in \mathcal{C}$, then $\mathcal{O}_{X,y} \in \mathcal{D}$.

4.1 Corollary. *Let R be a $\mathbb{Z}^n$-graded noetherian ring, and $P \in \operatorname{Spec} R$. Let $\mathcal{C}$ and $\mathcal{D}$ be classes of noetherian local rings which satisfy 1 and 2 in the theorem. If $R_{P^*} \in \mathcal{C}$, then $R_P \in \mathcal{D}$, where P^* is the prime ideal of R generated by the all homogeneous elements in P.*

The smooth base change holds for F-purity (Lemma 2.7), strong F-regularity (Theorem 3.3), and Cohen–Macaulay F-injectivity (Theorem 2.1). Flat descent also holds, and we have the following.

4.2 Corollary. *Let R be a $\mathbb{Z}^n$-graded noetherian ring of characteristic p. Let P be a prime ideal of R, and P^* the prime ideal generated by the all homogeneous elements of P.*

1. *If R_{P^*} is F-pure, then R_P is F-pure.*
2. *If R_{P^*} is excellent and strongly F-regular, then R_P is strongly F-regular.*
3. *If R_{P^*} is Cohen–Macaulay F-injective, then R_P is Cohen–Macaulay F-injective.*

For applications, see [5].

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