Equivariant Twisted Inverses

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Abstract

An equivariant version of the twisted inverse pseudofunctor is defined, and equivariant versions of some important properties, including the Grothendieck duality of proper morphisms and flat base change are proved. As an application, a generalized version of Watanabe's theorem on the Gorenstein property of the ring of invariants is proved.

Contents

Introduction

Let S be a scheme, G a flat S-group scheme of finite type, X and Y noetherian S-schemes with G-actions, and $f\colon X \to Y$ a finite-type separated G-morphism.

The purpose of these notes is to construct an equivariant version of the twisted inverse functor $f^!$ and study its basic properties.

One of the main motivations of the work is applications to invariant theory. As an example, we give a short proof of a generalized version of Watanabe's theorem on the Gorenstein property of invariant subrings [45]. Also, there might be some meaning in formulating the equivariant duality theorem, of which Serre duality for representations of reductive groups (see [21, (II.4.2)]) is a special case, in a reasonably general form. As a byproduct, we give some foundations for G-equivariant sheaf theory. More generally, we discuss sheaves over diagrams of schemes.

In the case where G is trivial, $f^!$ is defined as follows. For a scheme Z, we denote the category of $\mathcal{O}_Z$-modules by $\operatorname{Mod}(Z)$. By definition, a *plump subcategory* of an abelian category is a non-empty full subcategory which is closed under kernels, cokernels, and extensions [26, (1.9.1)]. We denote the plump subcategory of $\operatorname{Mod}(Z)$ consisting of quasi-coherent $\mathcal{O}_Z$-modules by $\operatorname{Qch}(Z)$.

By Nagata's compactification theorem [34], [27], there exists some factorization

$$X \xrightarrow{i} \bar{X} \xrightarrow{p} Y$$

such that p is proper and i an open immersion. We call such a factorization a *compactification*. We define $f^! : D^+_{\operatorname{Qch}(Y)}(\operatorname{Mod}(Y)) \to D^+_{\operatorname{Qch}(X)}(\operatorname{Mod}(X))$ to be the composite $i^*p^\times$, where $p^\times : D^+_{\operatorname{Qch}(Y)}(\operatorname{Mod}(Y)) \to D^+_{\operatorname{Qch}(\bar{X})}(\operatorname{Mod}(\bar{X}))$ is the right adjoint of Rp_*, and i^* is the restriction. This definition of $f^!$ is independent of the choice of compactification.

In order to consider a non-trivial G, we need to replace $\operatorname{Qch}(X)$ and $\operatorname{Mod}(X)$ by some appropriate categories which respect G-actions. The category $\operatorname{Qch}(G, X)$ which corresponds to $\operatorname{Qch}(X)$ is fairly well-known. It is the category of G-linearized quasi-coherent $\mathcal{O}_X$-modules defined by Mumford [32]. The category $\operatorname{Qch}(G, X)$ is equivalent to the category of quasi-coherent sheaves over the diagram of schemes

$$B_G^M(X) := \left(G \times_S G \times_S X \;\begin{array}{c}\xrightarrow{1_G \times a} \\ \xrightarrow{\mu \times 1_X} \\ \xrightarrow{p_{23}}\end{array}\; G \times_S X \;\begin{array}{c}\xrightarrow{a} \\ \xrightarrow{p_2}\end{array}\; X \right),$$

where $a : G \times X \to X$ is the action, $\mu : G \times G \to G$ the product, and p_{23} and p_2 are appropriate projections. Thus it is natural to embed the category $\mathrm{Qch}(G, X)$ into the category of all $\mathcal{O}_{B_G^M(X)}$-modules $\mathrm{Mod}(B_G^M(X))$, and $\mathrm{Mod}(B_G^M(X))$ is a good substitute of $\mathrm{Mod}(X)$. As G is flat, $\mathrm{Qch}(G, X)$ is a plump subcategory of $\mathrm{Mod}(B_G^M(X))$, and we may consider the triangulated subcategory $D_{\mathrm{Qch}(G,X)}(\mathrm{Mod}(B_G^M(X)))$. However, our construction utilizes an intermediate category $\mathrm{Lqc}(G, X)$ (the category of locally quasi-coherent sheaves), and is not an obvious interpretation of the non-equivariant case.

Note that there is a natural restriction functor $\mathrm{Mod}(B_G^M(X)) \to \mathrm{Mod}(X)$, which sends $\mathrm{Qch}(G, X)$ to $\mathrm{Qch}(X)$. This functor is regarded as the forgetful functor, forgetting the G-action. The equivariant duality theorem which we are going to establish must be compatible with this restriction functor, otherwise the theory would be something different from the usual scheme theory and probably useless.

Most of the discussion in these notes treats more general diagrams of schemes. This makes the discussion easier, as some of the important properties are proved by induction on the number of objects in the diagram. Our main construction and theorems are only for the class of finite diagrams of schemes of certain type, which contains the diagrams of the form $B_G^M(X)$.

In Chapters 1–3, we review some general facts on homological algebra. In Chapter 1, we give some basic facts on commutativity of various diagrams of functors derived from an adjoint pair of almost-pseudofunctors over closed symmetric monoidal categories. In Chapter 2, we give basics about sheaves on ringed sites. In Chapter 3, we give some basics about unbounded derived categories.

The construction of $f^!$ is divided into five steps. The first is to study the functoriality of sheaves over diagrams of schemes. Chapters 4–7 are devoted to this step. The second is the derived version of the first step. This will be done in Chapters 8, 13, and 14. Note that not only the categories of all module sheaves $\mathrm{Mod}(X_\bullet)$ and the category of quasi-coherent sheaves $\mathrm{Qch}(X_\bullet)$, but also the category of locally quasi-coherent sheaves $\mathrm{Lqc}(X_\bullet)$ plays an important role in our construction.

The third is to prove the existence of the right adjoint $p_\bullet^\times$ of $R(p_\bullet)_*$ for (componentwise) proper morphism $p_\bullet$ of diagrams of schemes. This is not so difficult, and is done in Chapter 17. We use Neeman's existence theorem on the right adjoint of triangulated functors. Not only to utilize Neeman's theorem, but to calculate composites of various left and right derived functors, it is convenient to utilize unbounded derived functors. A short survey on unbounded derived functors is given in Chapter 3.

The fourth step is to prove various commutativities related to the well-definedness of the twisted inverse pseudofunctors, Chapters 16, 18, and 19. Among them, the compatibility with restrictions (Proposition 18.14) is the key to our construction. Given a separated G-morphism of finite type $f : X \to Y$ between noetherian G-schemes, the associated morphism $B_G^M(f) : B_G^M(X) \to B_G^M(Y)$ is cartesian, see (4.2) for the definition. If we could find a compactification

$$B_G^M(X) \xrightarrow{i_\bullet} Z_\bullet \xrightarrow{p_\bullet} B_G^M(Y)$$

such that $p_\bullet$ is proper and cartesian, and $i_\bullet$ an image-dense open immersion, then the construction of $f^!$ and the proof of commutativity of various diagrams would be very easy. However, it seems that this is almost the same as the problem of equivariant compactifications. Equivariant compactifications are known to exist only in very restricted cases, see [40]. We avoid this difficult open problem, and prove the commutativity of various diagrams without assuming that $p_\bullet$ is cartesian.

The fifth part is the existence of a factorization $f_\bullet = p_\bullet i_\bullet$, where $p_\bullet$ is proper and $i_\bullet$ an image-dense open immersion. This is easily done utilizing Nagata's compactification theorem, and is done in Chapter 20. This completes the basic construction of the equivariant twisted inverse pseudofunctor. Theorem 20.4 is our main theorem.

In Chapters 21–28, we prove equivariant versions of most of the known results on twisted inverses including equivariant Grothendieck duality and the flat base change, except that equivariant dualizing complexes are treated later. We also prove that the twisted inverse functor preserves quasi-coherence of cohomology groups. As we already know the corresponding results on single schemes and the commutativity with restrictions, this consists in straightforward (but not easy) checking of commutativity of various diagrams of functors.

Almost all results above are valid for any diagram of noetherian schemes with flat arrows over a finite ordered category. Although our construction can be done using the diagram $B_G^M(X)$, some readers might ask why this were not done over the simplicial scheme $B_G(X)$ associated with the action of G on X. We explain the simplicial method and the related descent theory in Chapters 9 and 10. In the literature, it seems that equivariant sheaves with respect to the action of G on X has been regarded as equivariant sheaves on the diagram $B_G(X)$, see for example, [6, Appendix B]. The relation between $X_\bullet := B_G^M(X)$ and $Y_\bullet := B_G(X)$ is subtle. The category $\mathrm{Qch}(X_\bullet)$ and $\mathrm{Qch}(Y_\bullet)$ are equivalent, and the category of equivariant modules $\mathrm{EM}(X_\bullet)$ and $\mathrm{EM}(Y_\bullet)$ are equivalent. However, I do not know anything about the relationship between $\mathrm{Mod}(X_\bullet)$ and $\mathrm{Mod}(Y_\bullet)$. We use $B_G^M(X)$ only because it is a diagram over a finite ordered category.

In Chapter 11, we prove that if $X_\bullet$ is a simplicial groupoid of schemes, $d_0(1)$ and $d_1(1)$ are concentrated, and X_0 is concentrated, then $\mathrm{Qch}(X_\bullet)$ is Grothendieck. If, moreover, X_0 is noetherian, then $\mathrm{Qch}(X_\bullet)$ is locally noetherian, and $\mathcal{M} \in \mathrm{Qch}(X_\bullet)$ is a noetherian object if and only if $\mathcal{M}_0$ is coherent. In Chapter 12, we study groupoids of schemes and their relations with simplicial groupoids of schemes. In Chapter 15, we compare the two derived functors of $\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathcal{M},\mathcal{N})$ for $\mathcal{M} \in D^-_{\mathrm{Coh}}(\mathrm{Qch}(X_\bullet))$ and $\mathcal{N} \in D^+(\mathrm{Qch}(X_\bullet))$. One is the derived functor taken in $D(\mathrm{Mod}(X_\bullet))$ and the other is the derived functor taken in $D(\mathrm{Qch}(X_\bullet))$. They coincide under mild noetherian hypothesis.

Finally, we consider the group actions on schemes. In Chapter 29, we give a groupoid of schemes associated with a group action. The equivariant duality theorem for group actions is established. In Chapter 30, we prove that the equivariant twisted inverse is compatible with the derived G-invariance. In Chapter 31, we give a definition of the equivariant dualizing complexes. As an application, we give a short proof of a generalized version of Watanabe's theorem on the Gorenstein property of invariant subrings in Chapter 32. In Chapter 33, we give some other examples of diagrams of schemes.

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1 Commutativity of diagrams constructed from a monoidal pair of pseudofunctors

(1.1) Let $\mathcal{S}$ be a category. A (covariant) *almost-pseudofunctor* $\#$ on $\mathcal{S}$ assigns to each object $X \in \mathcal{S}$ a category $X_\#$, to each morphism $f\colon X \to Y$ in $\mathcal{S}$ a functor $f_\#\colon X_\# \to Y_\#$, and for each $X \in \mathcal{S}$, a natural isomorphism $\mathfrak{e}_X\colon \mathrm{Id}_{X_\#} \to (\mathrm{id}_X)_\#$ is assigned, and for each composable pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, a natural isomorphism

$$c = c_{f,g}\colon (gf)_\# \xrightarrow{\cong} g_\# f_\#$$

is given, and the following conditions are satisfied.

1. For any $f\colon X \to Y$, the map $f_\# \mathrm{Id}_{X_\#} = f_\# = (f\mathrm{id}_X)_\# \xrightarrow{c_{f,\mathrm{id}}} f_\#(\mathrm{id}_X)_\#$ agrees with $f_\#\mathfrak{e}_X$.

2. For any $f\colon X \to Y$, the map $\mathrm{Id}_{Y_\#} f_\# = f_\# = (\mathrm{id}_Y f)_\# \xrightarrow{c_{\mathrm{id},f}} (\mathrm{id}_Y)_\# f_\#$ agrees with $\mathfrak{e}_Y f_\#$.

3. For any composable triple of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, the diagram
$$\begin{array}{ccc} (hgf)_\# & \xrightarrow{c_{f,hg}} & (hg)_\# f_\# \\ \downarrow c_{gf,h} & & \downarrow c_{g,h} \\ h_\#(gf)_\# & \xrightarrow{c_{f,g}} & h_\# g_\# f_\# \end{array}$$
commutes.

If $(?)_\#$ is an almost-pseudofunctor on $\mathcal{S}$, then $(?)^{\mathrm{op}}_\#$ given by $(X)^{\mathrm{op}}_\# = X^{\mathrm{op}}_\#$ for $X \in \mathcal{S}$ and $(f)^{\mathrm{op}}_\# = f^{\mathrm{op}}_\# : X^{\mathrm{op}}_\# \to Y^{\mathrm{op}}_\#$ for a morphism $f : X \to Y$ of $\mathcal{S}$ together with $\mathfrak{e}^{-1}$ and c^{-1} is again an almost-pseudofunctor on $\mathcal{S}$. Letting $\mathfrak{e}_X = \mathrm{id}$ for each X, a pseudofunctor [26, (3.6.5)] is an almost-pseudofunctor.

(1.2) Let $*$ be an almost-pseudofunctor on $\mathcal{S}$. Let $gf = f'g'$ be a commutative diagram in $\mathcal{S}$. The composite isomorphism

$$g_* f_* \xrightarrow{c^{-1}} (gf)_* = (f'g')_* \xrightarrow{c} f'_* g'_*$$

is also denoted by $c = c(gf = f'g')$, by abuse of notation.

1.3 Lemma. *Let $fg' = gf'$ and $hg'' = g'h'$ be commutative squares in $\mathcal{S}$. Then the diagram*

$$\begin{array}{ccccc}
(fh)_*g''_* & \xrightarrow{c} & f_*h_*g''_* & \xrightarrow{c} & f_*g'_*h'_* \\
\downarrow c & & & & \downarrow c \\
g_*(f'h')_* & & \xrightarrow{c} & & g_*f'_*h'_*
\end{array}$$

is commutative.

Proof. Easy. □

1.4 Lemma. *Let $fg' = gf'$ and $f'h' = hf''$ be commutative squares in $\mathcal{S}$. Then the diagram*

$$\begin{array}{ccccc}
f_*(g'h')_* & \xrightarrow{c} & f_*g'_*h'_* & \xrightarrow{c} & g_*f'_*h'_* \\
\downarrow c & & & & \downarrow c \\
(gh)_*f''_* & & \xrightarrow{c} & & g_*h_*f''_*
\end{array}$$

is commutative.

Proof. Follows from Lemma 1.3. □

(1.5) A *contravariant almost-pseudofunctor* $(?)^\#$ is defined similarly. For $X \in \mathcal{S}$, a category $X^\#$ is assigned, and for a morphism $f\colon X \to Y$ in $\mathcal{S}$, a functor $f^\#\colon Y^\# \to X^\#$ is assigned, and for $X \in \mathcal{S}$, a natural isomorphism $\mathfrak{f} = \mathfrak{f}_X\colon \mathrm{id}_X^\# \to \mathrm{Id}_{X^\#}$ is assigned, and for a composable pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, a natural isomorphism

$$d_{f,g}\colon f^\# g^\# \to (gf)^\#$$

is given, and $((?)^\#)^{\mathrm{op}}$ together with $(\mathfrak{f}_X)_{X\in\mathcal{S}}$ and $(d_{g,f})$ is a covariant almost-pseudofunctor on $\mathcal{S}^{\mathrm{op}}$. If $(?)^\#$ is a contravariant almost-pseudofunctor on $\mathcal{S}$, then $((?)^\#)^{\mathrm{op}}$ together with $\mathfrak{f}^{-1}$ and $(d_{f,g}^{-1})$ is again a contravariant almost-pseudofunctor on $\mathcal{S}$.

(1.6) Let $(?)^*$ be a contravariant almost-pseudofunctor on $\mathcal{S}$. For a commutative diagram $gf = f'g'$ in $\mathcal{S}$, the composite map

$$(g')^*(f')^* \xrightarrow{d} (f'g')^* = (gf)^* \xrightarrow{d^{-1}} f^*g^*$$

is also denoted by $d = d(gf = f'g')$, by abuse of notation.

(1.7) Let $*$ and $\#$ be almost-pseudofunctors on $\mathcal{S}$ such that $X_* = X_\#$. A *morphism of almost-pseudofunctors* $\upsilon\colon * \to \#$ is a family of natural maps $\upsilon_f\colon f_* \to f_\#$ (one for each $f \in \mathrm{Mor}(\mathcal{S})$) such that for any $X \in \mathcal{S}$ the diagram

$$\begin{array}{ccc} \mathrm{Id}_{X_*} & \xrightarrow{\mathrm{id}} & \mathrm{Id}_{X_\#} \\ \downarrow \mathfrak{e} & & \downarrow \mathfrak{e} \\ (\mathrm{id}_X)_* & \xrightarrow{\upsilon} & (\mathrm{id}_X)_\# \end{array}$$

commutes, and for any composable pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, the diagram

$$\begin{tikzcd} (gf)_* \arrow[rr, "\upsilon"] \arrow[d, "c"] & & (gf)_\# \arrow[d, "c"] \\ g_*f_* \arrow[r, "g_*\upsilon"] & g_*f_\# \arrow[r, "\upsilon"] & g_\# f_\# \end{tikzcd}$$

commutes.

If υ is a morphism of almost-pseudofunctors, and υ_f is a natural isomorphism for each f, then we say that υ is an isomorphism of almost-pseudofunctors.

(1.8) Let $\#$ be an almost-pseudofunctor on $\mathcal{S}$. We define $*$ by $X_* = X_\#$ for $X \in \mathcal{S}$, $(\mathrm{id}_X)_* = \mathrm{Id}_{X_\#}$ for $X \in \mathcal{S}$, and $f_* = f_\#$ if $f \neq \mathrm{id}_X$ for any X. For a composable pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, we define $c_{f,g}\colon (gf)_* \to g_*f_*$ to be the identity map if $f = \mathrm{id}_X$ or $g = \mathrm{id}_Y$. If $Z = X$, $g = f^{-1}$ and $f \neq \mathrm{id}_X$, then $c_{f,f^{-1}} : (\mathrm{id}_X)_* \to f^{-1}_* f_*$ is defined to be the composite

$$\mathrm{Id}_{X_\#} \xrightarrow{\mathfrak{e}_X} (\mathrm{id}_X)_\# \xrightarrow{c_{f,f^{-1}}} f^{-1}_\# f_\#.$$

Otherwise, we define $c_{f,g}$ to be the original $c_{f,g}$ of $\#$. It is easy to see that $*$ is a pseudofunctor on $\mathcal{S}$.

We define $\upsilon\colon f_* \to f_\#$ to be $\mathfrak{e}_X$ if $f = \mathrm{id}_X$, and the identity of $f_\#$ otherwise. Then υ is an isomorphism of almost-pseudofunctors. Thus any almost-pseudofunctor is isomorphic to a pseudofunctor. We call $*$ the *associated pseudofunctor* of the almost-pseudofunctor $\#$.

Similarly, any contravariant almost-pseudofunctor is isomorphic to a contravariant pseudofunctor, and the associated contravariant pseudofunctor of a contravariant almost-pseudofunctor is defined.

(1.9) In this paper, various (different) adjoint pairs appears almost everywhere. By abuse of notation, the unit (resp. the counit) of adjunction is usually simply denoted by the same symbol u (resp. ε). When we mention an adjunction of functors, we implicitly (or occasionally explicitly) fix the unit u and the counit ε.

(1.10) We need the notion of the conjugation from [29, (IV.7)] and [26, (3.3.5)]. Let X and Y be categories, and f_* and g_* functors $X \to Y$ with respective left adjoints f^* and g^*. By Hom, we denote the set of natural transformations. Then $\Phi : \operatorname{Hom}(f_*, g_*) \to \operatorname{Hom}(g^*, f^*)$ given by

$$\Phi(\alpha) : g^* \xrightarrow{u} g^* f_* f^* \xrightarrow{\alpha} g^* g_* f^* \xrightarrow{\varepsilon} f^*$$

and $\Psi : \operatorname{Hom}(g^*, f^*) \to \operatorname{Hom}(f_*, g_*)$ given by

$$\Psi(\beta) : f_* \xrightarrow{u} g_* g^* f_* \xrightarrow{\beta} g_* f^* f_* \xrightarrow{\varepsilon} g_*$$

are inverse each other. We say that $\Phi(\alpha)$ is *left conjugate* to α, $\Psi(\beta)$ is *right conjugate* to β, and α and $\Phi(\alpha)$ are *conjugate*. The identity map in $\operatorname{Hom}(f_*, f_*)$ is conjugate to the identity in $\operatorname{Hom}(f^*, f^*)$. Let $h_* : X \to Y$ be a functor with the left adjoint h^*. If $\alpha \in \operatorname{Hom}(f_*, g_*)$ and $\alpha' \in \operatorname{Hom}(g_*, h_*)$, and $\beta \in \operatorname{Hom}(g^*, f^*)$ and $\beta' \in \operatorname{Hom}(h^*, g^*)$ are their respective conjugates, then $\alpha' \circ \alpha$ and $\beta \circ \beta'$ are conjugate. In particular, $\alpha \in \operatorname{Hom}(f_*, g_*)$ is an isomorphism if and only if its conjugate $\beta \in \operatorname{Hom}(g^*, f^*)$ is an isomorphism, and if this is the case, α^{-1} and β^{-1} are conjugate.

1.11 Lemma. *Let X, Y and f_*, g_*, f^*, g^* be as above. Let $\alpha \in \operatorname{Hom}(f_*, g_*)$, and $\beta \in \operatorname{Hom}(g^*, f^*)$. Then the following are equivalent.*

1 *α and β are conjugate.*

2 *One of the following diagrams commutes.*

$$\begin{tikzcd} g^* f_* \arrow[r, "\alpha"] \arrow[d, "\beta"] & g^* g_* \arrow[d, "\varepsilon"] \\ f^* f_* \arrow[r, "\varepsilon"] & 1 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "u"] \arrow[d, "u"] & f_* f^* \arrow[d, "\alpha"] \\ g_* g^* \arrow[r, "\beta"] & g_* f^* \end{tikzcd}$$

Proof. See [29, (IV.7), Theorem 2]. □

(1.12) Let $\mathcal{S}$ be a category, $(?)_*$ be an almost-pseudofunctor on $\mathcal{S}$. Let $(?)^*$ be a left adjoint of $(?)_*$. Namely, for each morphism f of $\mathcal{S}$, we have a left adjoint f^* of f_* (and the explicitly given unit $u : 1 \rightarrow f_*f^*$ and counit $\varepsilon : f^*f_* \rightarrow 1$).

For $X \in \mathcal{S}$, X^* is defined to be X_*.

For composable two morphisms f and g in $\mathcal{S}$, we denote the map $f^*g^* \rightarrow (gf)^*$ conjugate to $c : (gf)_* \rightarrow g_*f_*$ by $d = d_{f,g}$. Thus $d_{f,g}$ is the composite

$$f^*g^* \xrightarrow{u} f^*g^*(gf)_*(gf)^* \xrightarrow{c} f^*g^*g_*f_*(gf)^* \xrightarrow{\varepsilon} f^*f_*(gf)^* \xrightarrow{\varepsilon} (gf)^*.$$

Being the conjugate of an isomorphism, d is an isomorphism. As d^{-1} is conjugate to c^{-1}, it is the composite

$$(gf)^* \xrightarrow{u} (gf)^*g_*g^* \xrightarrow{u} (gf)^*g_*f_*f^*g^* \xrightarrow{c^{-1}} (gf)^*(gf)_*f^*g^* \xrightarrow{\varepsilon} f^*g^*.$$

1.13 Lemma. *Let f and g be morphisms in $\mathcal{S}$, and assume that gf is defined. Then the composite*

$$1 \xrightarrow{u} g_*g^* \xrightarrow{u} g_*f_*f^*g^* \xrightarrow{c^{-1}} (gf)_*f^*g^* \xrightarrow{d} (gf)_*(gf)^*$$

is u.

Proof. Follows immediately from Lemma 1.11. □

1.14 Lemma. *Let f and g be morphisms in $\mathcal{S}$, and assume that gf is defined. Then the composite*

$$(gf)^*(gf)_* \xrightarrow{c} (gf)^*g_*f_* \xrightarrow{d^{-1}} f^*g^*g_*f_* \xrightarrow{\varepsilon} f^*f_* \xrightarrow{\varepsilon} 1$$

is ε.

Proof. Follows immediately from Lemma 1.11. □

(1.15) For $X \in \mathcal{S}$, Id_{X^*} is left adjoint to Id_{X_*} (with $u = \mathrm{id}$ and $\varepsilon = \mathrm{id}$). The morphism left conjugate to $\mathfrak{e}_X \colon \mathrm{Id}_{X_*} \rightarrow (\mathrm{id}_X)_*$ is denoted by $\mathfrak{f}_X \colon (\mathrm{id}_X)^* \rightarrow \mathrm{Id}_{X^*}$. Namely, $\mathfrak{f}_X$ is the composite

$$(\mathrm{id}_X)^* \xrightarrow{\mathrm{id}} (\mathrm{id}_X)^* \mathrm{Id}_{X_*} \mathrm{Id}_{X^*} \xrightarrow{\mathfrak{e}} (\mathrm{id}_X)^*(\mathrm{id}_X)_* \mathrm{Id}_{X^*} \xrightarrow{\varepsilon} \mathrm{Id}_{X^*}.$$

(1.16) Let $\mathcal{S}$, $(?)_*$, and $(?)^*$ be as above. Then it is easy to see that $(?)^*$ together with d and $\mathfrak{f}$ defined above forms a contravariant almost-pseudofunctor. We say that $((?)^*, (?)_*)$ is an *adjoint pair* of almost-pseudofunctors on $\mathcal{S}$, with this situation. For a commutative diagram $gf = f'g'$ in $\mathcal{S}$, the composite maps $c(gf = f'g')$ and $d(gf = f'g')$ are conjugate.

The opposite $((?)_*^{\mathrm{op}}, ((?)^*)^{\mathrm{op}})$ of $((?)^*, (?)_*)$ is an adjoint pair of almost-pseudofunctors on $\mathcal{S}^{\mathrm{op}}$. $c_{f,g}$, $d_{f,g}$, $u : 1 \to f_* f^*$, and $\varepsilon : f^* f_* \to 1$ of $((?)^*, (?)_*)$ correspond to $d_{g,f}$, $c_{g,f}$, ε, and u of $((?)_*^{\mathrm{op}}, ((?)^*)^{\mathrm{op}})$, respectively.

(1.17) Let $\mathcal{S}$ be as above, and $(?)^*$ a given contravariant almost-pseudofunctor, and $(?)_*$ its right adjoint. Then $((?)^*)^{\mathrm{op}}$ is a covariant almost-pseudofunctor on $\mathcal{S}^{\mathrm{op}}$ as in (1.5). Then $((?)_*^{\mathrm{op}}, ((?)^*)^{\mathrm{op}})$ is an adjoint pair of almost-pseudofunctors. So $((?)^*, (?)_*) = (((?)^*)^{\mathrm{opop}}, (?)_*^{\mathrm{opop}})$ is also an adjoint pair of almost-pseudofunctors.

(1.18) Let $\mathcal{S}$ be as above, and $(?)_*$ a given covariant almost-pseudofunctor, and $(?)^!$ its *right* adjoint. For composable morphisms f and g, define $d_{f,g} : f^! g^! \to (gf)^!$ to be the map right conjugate to $c_{f,g} : (gf)_* \to g_* f_*$. For $X \in S$, define $\mathfrak{f}_X : (\mathrm{id}_X)^! \to \mathrm{Id}_{X^!}$ to be the map right conjugate to $\mathfrak{e}_X : \mathrm{Id}_{X_*} \to (\mathrm{id}_X)_*$. Then it is straightforward to check that $(?)^!$ is a contravariant almost-pseudofunctor on $\mathcal{S}$. We say that $((?)_*, (?)^!)$ is an *opposite adjoint pair* of almost-pseudofunctors on $\mathcal{S}$. Opposite adjoint pair is also obtained from a given contravariant almost-pseudofunctor $(?)^!$ and its left adjoint $(?)_*$. Note that $((?)^\#, (?)_\#)$ is an adjoint pair of almost-pseudofunctors on $\mathcal{S}$ if and only if $((?)_\#^{\mathrm{op}}, ((?)^\#)^{\mathrm{op}})$ is an opposite adjoint pair of almost-pseudofunctors on $\mathcal{S}$.

(1.19) Let $((?)^*, (?)_*)$ be an adjoint pair of pseudofunctors on $\mathcal{S}$. Let $\sigma = (fg' = gf')$ be a commutative diagram in $\mathcal{S}$.

1.20 Lemma. *The following composite maps agree:*

1 $g^* f_* \xrightarrow{u} g^* f_* g'_* (g')^* \xrightarrow{c} g^* g_* f'_* (g')^* \xrightarrow{\varepsilon} f'_* (g')^*$;

2 $g^* f_* \xrightarrow{u} f'_* (f')^* g^* f_* \xrightarrow{d} f'_* (g')^* f^* f_* \xrightarrow{\varepsilon} f'_* (g')^*$.

For the proof and more information, see [26, (3.7.2)].

(1.21) We denote the composite map in the lemma by $\theta(\sigma)$ or θ, and call it *Lipman's theta*. Note that $\theta(fg' = gf')$ of $((?)^*, (?)_*)$ is $\theta(g'f = f'g)$ in the opposite $((?)_*^{\mathrm{op}}, ((?)^*)^{\mathrm{op}})$.

1.22 Lemma. *Let $fg' = gf'$ and $hg'' = g'h'$ be commutative squares in $\mathcal{S}$. Then the diagram*

$$\begin{tikzcd}
g^*(fh)_* \arrow[r, "c"] \arrow[d, "\theta"] & g^*f_*h_* \arrow[r, "\theta"] & f'_*(g')^*h_* \arrow[d, "\theta"] \\
(f'h')_*(g'')^* \arrow[rr, "c"] & & f'_*h'_*(g'')^*
\end{tikzcd}$$

is commutative.

Proof. See [26, (3.7.2)]. □

1.23 Lemma. *Let $\sigma = (fg' = gf')$ and $\tau = (f'h' = hf'')$ be commutative diagrams in $\mathcal{S}$. Then the composite*

$$(gh)^*f_* \xrightarrow{d^{-1}} h^*g^*f_* \xrightarrow{\theta} h^*f'_*(g')^* \xrightarrow{\theta} f''_*(h')^*(g')^* \xrightarrow{d} f''_*(g'h')^*$$

agrees with θ for $f(g'h') = (gh)f''$.

Proof. This is the 'opposite assertion' of Lemma 1.22. Namely, Lemma 1.22 applied to the opposite pair $((?)_*^{\mathrm{op}}, ((?)^*)^{\mathrm{op}})$ is this lemma. □

1.24 Lemma. *Let $fg' = gf'$ be a commutative diagram in $\mathcal{S}$. Then the composite*

$$f_* \xrightarrow{u} g_*g^*f_* \xrightarrow{\theta} g_*f'_*(g')^* \xrightarrow{c} f_*g'_*(g')^*$$

is u.

Proof. Obvious by the commutativity of the diagram

$$\begin{tikzcd}
 & g_*g^*f_* \arrow[r, "u"] & g_*g^*f_*g'_*(g')^* \arrow[r, "c"] & g_*g^*g_*f'_*(g')^* \arrow[d, "\varepsilon"] \\
f_* \arrow[ur, "u"] \arrow[r, "u"'] & f_*g'_*(g')^* \arrow[r, "c"] \arrow[drr, "\mathrm{id}"'] & g_*f'_*(g')^* \arrow[ur, "u"] \arrow[r, "\mathrm{id}"] & g_*f'_*(g')^* \arrow[d, "c"] \\
 & & & f_*g'_*(g')^*
\end{tikzcd} .$$ □

1.25 Lemma. *Let $fg' = gf'$ be a commutative diagram in $\mathcal{S}$. Then the composite*

$$g^*g_*f'_* \xrightarrow{c} g^*f_*g'_* \xrightarrow{\theta} f'_*(g')^*g'_* \xrightarrow{\varepsilon} f'_*$$

is ε.

Proof. Obvious by the commutativity of the diagram

$$\begin{tikzcd}
g^*g_*f'_* \arrow[r, "c"] \arrow[ddrr, "\mathrm{id}"'] & g^*f_*g'_* \arrow[r, "u"] \arrow[dr, "\mathrm{id}"'] & g^*f_*g'_*(g')^*g'_* \arrow[r, "c"] \arrow[d, "\varepsilon"] & g^*g_*f'_*(g')^*g'_* \arrow[d, "\varepsilon"] \\
 & & g^*f_*g'_* \arrow[d, "c"] & f'_*(g')^*g'_* \arrow[d, "\varepsilon"] \\
 & & g^*g_*f'_* \arrow[r, "\varepsilon"] & f'_*
\end{tikzcd} \qquad \square$$

1.26 Lemma. *Let $fg' = gf'$ be a commutative diagram in $\mathcal{S}$. Then the composite*

$$g^* \xrightarrow{u} g^*f_*f^* \xrightarrow{\theta} f'_*(g')^*f^* \xrightarrow{d} f'_*(f')^*g^*$$

is u.

Proof. This is the opposite version of Lemma 1.25. $\square$

1.27 Lemma. *Let $fg' = gf'$ be a commutative diagram in $\mathcal{S}$. Then the composite*

$$(g')^*f^*f_* \xrightarrow{d} (f')^*g^*f_* \xrightarrow{\theta} (f')^*f'_*(g')^* \xrightarrow{\varepsilon} (g')^*$$

is ε.

Proof. This is the opposite assertion of Lemma 1.24. $\square$

(1.28) We say that $(?)_*$ is a covariant symmetric monoidal almost-pseudofunctor on a category $\mathcal{S}$, if $(?)_*$ is an almost-pseudofunctor on $\mathcal{S}$, and the following conditions are satisfied. For each $X \in \mathcal{S}$,

$$X_* = (X_*, \otimes, \mathcal{O}_X, \alpha, \lambda, \gamma, [?, *], \pi)$$

is a (symmetric monoidal) closed category (see e.g., [26, (3.5.1)]), where X_* on the right-hand side is the underlying category, $\otimes : X_* \times X_* \to X_*$ the product structure, $\mathcal{O}_X \in X_*$ the unit object, $\alpha : (a \otimes b) \otimes c \cong a \otimes (b \otimes c)$ the associativity isomorphism, $\lambda : \mathcal{O}_X \otimes a \cong a$ the left unit isomorphism, $\gamma : a \otimes b \cong b \otimes a$ the twisting (symmetry) isomorphism, $[?, -] : X_*^{\mathrm{op}} \times X_* \to X_*$ the internal hom, and

$$\pi : X_*(a \otimes b, c) \cong X_*(a, [b, c]) \qquad (1.29)$$

the associative adjunction isomorphism of X, respectively. For a morphism $f : X \to Y$ in $\mathcal{S}$, $f_* : X_* \to Y_*$ is a symmetric monoidal functor [26, (3.4.2)], and $\mathfrak{e}_X : \mathrm{Id}_{X_*} \to (\mathrm{id}_X)_*$ and $c_{f,g}$ are morphisms of symmetric monoidal functors, see [26, (3.6.7)].

(1.30) The unit map and the counit map arising from the adjunction (1.29) are denoted (by less worse abuse of notation, not using u or ε) by (the same symbol)

$$\mathrm{tr} : a \to [b, a \otimes b] \qquad \text{(the trace map)}$$

and

$$\mathrm{ev} : [b, c] \otimes b \to c \qquad \text{(the evaluation map)},$$

respectively.

(1.31) By the definition of closed categories, the associative adjunction isomorphism (1.29) is natural on a, b and c. So not only that tr and ev are natural transformations, we have the following.

1.32 Lemma. *For $a, b, b', c \in X_*$ and a morphism $\varphi\colon b \to b'$, the diagrams*

$$\begin{tikzcd} a \arrow[r, "\mathrm{tr}"] \arrow[d, "\mathrm{tr}"] & {[b, a \otimes b]} \arrow[d, "{[1_b, 1_a \otimes \varphi]}"] \\ {[b', a \otimes b']} \arrow[r, "{[\varphi, 1_a \otimes 1_b]}"] & {[b, a \otimes b']} \end{tikzcd} \qquad \begin{tikzcd} {[b', c] \otimes b} \arrow[r, "{[\varphi, 1_c] \otimes 1_b}"] \arrow[d, "{[1_{b'}, 1_c] \otimes \varphi}"] & {[b, c] \otimes b} \arrow[d, "\mathrm{ev}"] \\ {[b', c] \otimes b'} \arrow[r, "\mathrm{ev}"] & c \end{tikzcd}$$

are commutative.

Proof. We only prove the commutativity of the first diagram. By the naturality of π, the diagram

$$\begin{array}{ccccc} X_*(a \otimes b, a \otimes b) & \xrightarrow{(1_a \otimes \varphi)_*} & X_*(a \otimes b, a \otimes b') & \xleftarrow{(1_a \otimes \varphi)^*} & X_*(a \otimes b', a \otimes b') \\ \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\ X_*(a, [b, a \otimes b]) & \xrightarrow{[1_b, 1_a \otimes \varphi]_*} & X_*(a, [b, a \otimes b']) & \xleftarrow{[\varphi, 1_a \otimes 1_{b'}]_*} & X_*(a, [b', a \otimes b']) \end{array}$$

is commutative. Considering the image of $1_{a \otimes b} \in X_*(a \otimes b, a \otimes b)$ and $1_{a \otimes b'} \in X_*(a \otimes b', a \otimes b')$ in $X_*(a, [b, a \otimes b'])$, we have

$$[1_b, 1_a \otimes \varphi] \circ \mathrm{tr} = \pi(1_a \otimes \varphi) = [\varphi, 1_a \otimes 1_b] \circ \mathrm{tr}\,.$$

This is what we wanted to prove. $\square$

(1.33) Let $f : X \to Y$ be a morphism. Then f_* is a symmetric monoidal functor. The natural map

$$f_*a \otimes f_*b \to f_*(a \otimes b)$$

is denoted by $m = m(f)$, and the map

$$\mathcal{O}_Y \to f_*\mathcal{O}_X$$

is denoted by $\eta = \eta(f)$.

A covariant symmetric monoidal almost-pseudofunctor which is a pseudofunctor is called a covariant symmetric monoidal pseudofunctor. Let $\star$ be the associated pseudofunctor of the symmetric monoidal almost-pseudofunctor $*$. Then letting the closed structure of $X_\star$ be the same as that of X_*, and letting

$$m\colon (\mathrm{id}_X)_\star a \otimes (\mathrm{id}_X)_\star b \to (\mathrm{id}_X)_\star(a \otimes b)$$

and

$$\eta\colon \mathcal{O}_X \to (\mathrm{id}_X)_\star \mathcal{O}_X$$

to be the identity morphisms, $\star$ is a symmetric monoidal pseudofunctor which is isomorphic to $*$ as a symmetric monoidal almost-pseudofunctor.

(1.34) Let $f : X \to Y$ be a morphism in $\mathcal{S}$. The composite natural map

$$f_*[a,b] \xrightarrow{\mathrm{tr}} [f_*a, f_*[a,b] \otimes f_*a] \xrightarrow{\text{via } m} [f_*a, f_*([a,b] \otimes a)] \xrightarrow{\text{via ev}} [f_*a, f_*b]$$

is denoted by H.

(1.35) G. Lewis proved a theorem which guarantee that some diagrams involving two symmetric monoidal closed categories and one symmetric monoidal functor commute [25].

By Lewis's result, we have that the following diagrams are commutative for any morphism $f : X \to Y$ (also checked by a direct computation).

$$\begin{array}{ccc} f_*[a,b] \otimes f_*a & \xrightarrow{H \otimes 1} & [f_*a, f_*b] \otimes f_*a \\ \downarrow m & & \downarrow \mathrm{ev} \\ f_*([a,b] \otimes a) & \xrightarrow{f_*\,\mathrm{ev}} & f_*b \end{array} \tag{1.36}$$

$$\begin{array}{ccc} f_*a & \xrightarrow{f_*\,\mathrm{tr}} & f_*[b, a \otimes b] \\ \downarrow \mathrm{tr} & & \downarrow H \\ [f_*b, f_*a \otimes f_*b] & \xrightarrow{m} & [f_*b, f_*(a \otimes b)] \end{array} \tag{1.37}$$

$$\begin{array}{ccccccc}
f_*a & \xrightarrow{\lambda^{-1}} & \mathcal{O}_Y \otimes f_*a & \xrightarrow{\eta} & f_*\mathcal{O}_X \otimes f_*a & \xrightarrow{\text{via tr}} & f_*[a, \mathcal{O}_X \otimes a] \otimes f_*a \\
\downarrow{\scriptstyle 1} & & & & & & \downarrow{\scriptstyle \text{via } \lambda} \\
f_*a & & \xleftarrow{\qquad\qquad \text{ev} \qquad\qquad} & & [f_*a, f_*a] \otimes f_*a & \xleftarrow{H\otimes 1} & f_*[a,a] \otimes f_*a
\end{array} \tag{1.38}$$

1.39 Lemma. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathcal{S}$. Then the diagram*

$$\begin{array}{ccccc}
g_*f_*[a,b] & \xrightarrow{H} & g_*[f_*a, f_*b] & \xrightarrow{H} & [g_*f_*a, g_*f_*b] \\
\uparrow{\scriptstyle c} & & & & \uparrow{\scriptstyle [c^{-1}, c]} \\
(gf)_*[a,b] & & \xrightarrow{\qquad\qquad H \qquad\qquad} & & [(gf)_*a, (gf)_*b]
\end{array}$$

is commutative.

Proof. Consider the diagram

$$
\begin{tikzcd}
 & (gf)_*[a,b] \otimes g_*f_*a \arrow[r, "H"] \arrow[dl, "c"'] \arrow[d, "c^{-1}"] \arrow[dr, phantom, "(b)"] & {[(gf)_*a, (gf)_*b] \otimes g_*f_*a} \arrow[r, "c"] \arrow[d, "c^{-1}"] \arrow[dr, phantom, "(c)"] & {[(gf)_*a, g_*f_*b] \otimes g_*f_*a} \arrow[d, "c^{-1}"] \arrow[dr, "c^{-1}"] & \\
g_*f_*[a,b] \otimes g_*f_*a \arrow[dd, "H"] \arrow[dr, "m"] \arrow[ur, phantom, "(a)"] & (gf)_*[a,b] \otimes (gf)_*a \arrow[l, "c \otimes c"'] \arrow[r, "H"] \arrow[dr, "m"] \arrow[d, phantom, "(e)"] & {[(gf)_*a, (gf)_*b] \otimes (gf)_*a} \arrow[r, "c"] \arrow[dr, "\mathrm{ev}"] \arrow[d, phantom, "(f)"] & {[(gf)_*a, g_*f_*b] \otimes (gf)_*a} \arrow[dr, "\mathrm{ev}"] \arrow[d, phantom, "(g)"] \arrow[r, phantom, "(d)"] & {[g_*f_*a, g_*f_*b] \otimes g_*f_*a} \arrow[d, "\mathrm{ev}"] \\
 \arrow[dr, phantom, "(i)"] & g_*(f_*[a,b] \otimes f_*a) \arrow[dr, "m"] \arrow[ddr, "H"] & (gf)_*([a,b] \otimes a) \arrow[r, "\mathrm{ev}"] \arrow[d, "c"] \arrow[dr, phantom, "(h)"] & (gf)_*b \arrow[r, "c"] & g_*f_*b \\
g_*[f_*a, f_*b] \otimes g_*f_*a \arrow[drr, dashed, "m"] \arrow[uurrrr, rounded corners, bend right=90, "H"'] & & g_*f_*([a,b] \otimes a) \arrow[urr, dashed, "\mathrm{ev}"] \arrow[d, phantom, "(j)"] & & \\
 & & g_*([f_*a, f_*b] \otimes f_*a) \arrow[uurr, "\mathrm{ev}"'] \arrow[d, phantom, "(k)"] & & \\
 & & & &
\end{tikzcd}
$$

.

The commutativity of (a), (b), (c), (g), (h) and (i) is trivial. The commutativity of (d) is Lemma 1.32. (e) is commutative, since c is assumed to be a morphism of symmetric monoidal functors, see [26, (3.6.7.2)]. The commutativity of (f), (j), and (k) is the commutativity of (1.36). Thus the whole diagram is commutative. Taking the adjoint, we get the commutativity of the diagram in the lemma. $\square$

(1.40) Let $(?)_*$ be a covariant symmetric monoidal almost-pseudofunctor on $\mathcal{S}$. Let $(?)^*$ be its left adjoint. Namely, for each morphism f of $\mathcal{S}$, a left adjoint f^* of f_* (and the unit map $1 \to f_*f^*$ and the counit map $f^*f_* \to 1$) is given. For a morphism $f : X \to Y$, the map $f^*\mathcal{O}_Y \to \mathcal{O}_X$ adjoint to $\eta : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is denoted by C. The composite map

$$f^*(a \otimes b) \xrightarrow{u\otimes u} f^*(f_*f^*a \otimes f_*f^*b) \xrightarrow{m} f^*f_*(f^*a \otimes f^*b) \xrightarrow{\varepsilon} f^*a \otimes f^*b \qquad (1.41)$$

is denoted by Δ.

Almost by definition, the diagrams

$$\begin{array}{ccc} a \otimes b & \xrightarrow{u\otimes u} & f_*f^*a \otimes f_*f^*b \\ \downarrow u & & \downarrow m \\ f_*f^*(a \otimes b) & \xrightarrow{\Delta} & f_*(f^*a \otimes f^*b) \end{array} \qquad (1.42)$$

and

$$\begin{array}{ccc} f^*(f_*a \otimes f_*b) & \xrightarrow{m} & f^*f_*(a \otimes b) \\ \downarrow \Delta & & \downarrow \varepsilon \\ f^*f_*a \otimes f^*f_*b & \xrightarrow{\varepsilon\otimes\varepsilon} & a \otimes b \end{array} \qquad (1.43)$$

are commutative.

If $(?)_\star$ is the associated pseudofunctor of $(?)_*$ and $(?)^\star$ is the associated contravariant pseudofunctor of $(?)^*$, then $((?)_\star, (?)^\star)$ is a monoidal adjoint pair. Note that $u\colon \mathrm{Id} \to f_\star f^\star$ is the identity if $f = \mathrm{id}_X$ for some X, and u agrees with the unit map for the original adjoint pair $((?)_*, (?)^*)$ otherwise. Similarly for the counit map ε.

1.44 Lemma. *Let $((?)^*, (?)_*)$ be a monoidal adjoint pair on $\mathcal{S}$. Let $\sigma = (fg' = gf')$ be a commutative diagram in $\mathcal{S}$. Then the diagram*

$$\begin{array}{ccccc} f^*(g_*a \otimes g_*b) & \xrightarrow{m} & f^*g_*(a \otimes b) & \xrightarrow{\theta} & g'_*(f')^*(a \otimes b) \\ \downarrow \Delta & & & & \downarrow \Delta \\ f^*g_*a \otimes f^*g_*b & \xrightarrow{\theta\otimes\theta} & g'_*(f')^*a \otimes g'_*(f')^*b & \xrightarrow{m} & g'_*((f')^*a \otimes (f')^*b) \end{array}$$

is commutative.

Proof. Utilize the commutativity of (1.42) and (1.43). $\square$

(1.45) For a symmetric monoidal category X_*, the composite

$$a \otimes \mathcal{O}_X \xrightarrow{\gamma} \mathcal{O}_X \otimes a \xrightarrow{\lambda} a$$

is called the *right unit isomorphism*, and is denoted by ρ. Let $f_* : X_* \to Y_*$ be a symmetric monoidal functor. Then it is easy to see that the diagram

$$\begin{tikzcd}
f_*a \otimes \mathcal{O}_Y \arrow[r, "\rho"] \arrow[d, "1\otimes\eta"] & f_*a \\
f_*a \otimes f_*\mathcal{O}_X \arrow[r, "m"] & f_*(a \otimes \mathcal{O}_X) \arrow[u, "\rho"']
\end{tikzcd} \tag{1.46}$$

is commutative for $a \in X_*$.

1.47 Lemma. *Let $f_* : X_* \to Y_*$ be a symmetric monoidal functor between closed categories. Then the diagram*

$$\begin{tikzcd}
f_*[\mathcal{O}_X, c \otimes \mathcal{O}_X] \arrow[r, "\rho"] & f_*[\mathcal{O}_X, c] \arrow[r, "H"] & {[f_*\mathcal{O}_X, f_*c]} \arrow[d, "\eta"] \\
f_*c \arrow[u, "\mathrm{tr}"] \arrow[r, "\mathrm{tr}"] & {[\mathcal{O}_Y, f_*c \otimes \mathcal{O}_Y]} \arrow[r, "\rho"] & {[\mathcal{O}_Y, f_*c]}
\end{tikzcd}$$

is commutative.

Proof. Consider the diagram

$$\begin{tikzcd}
f_*c \arrow[r, "\mathrm{tr}"] \arrow[d, "\mathrm{tr}"] \arrow[dddr, phantom, ""] \arrow[ddd, bend right=80, "\mathrm{tr}"'] \arrow[dr, phantom, "\text{(a)}"] & f_*[\mathcal{O}_X, c \otimes \mathcal{O}_X] \arrow[r, "\rho"] \arrow[d, "H"] \arrow[dr, phantom, "\text{(b)}"] & f_*[\mathcal{O}_X, c] \arrow[d, "H"] \\
{[f_*\mathcal{O}_X, f_*c \otimes f_*\mathcal{O}_X]} \arrow[r, "m"] \arrow[d, "\eta"] \arrow[dr, phantom, "\text{(d)}"] & {[f_*\mathcal{O}_X, f_*(c \otimes \mathcal{O}_X)]} \arrow[r, "\rho"] \arrow[d, "\eta"] \arrow[dr, phantom, "\text{(e)}"] & {[f_*\mathcal{O}_X, f_*c]} \arrow[d, "\eta"] \\
{[\mathcal{O}_Y, f_*c \otimes f_*\mathcal{O}_X]} \arrow[r, "m"] \arrow[dr, phantom, "\text{(f)}"] & {[\mathcal{O}_Y, f_*(c \otimes \mathcal{O}_X)]} \arrow[r, "\rho"] & {[\mathcal{O}_Y, f_*c]} \\
{[\mathcal{O}_Y, f_*c \otimes \mathcal{O}_Y]} \arrow[u, "\eta"] \arrow[urr, dashed, "\rho"] & &
\end{tikzcd} .$$

By the commutativity of (1.37), (a) commutes. Clearly, (b), (d), and (e) commute. By Lemma 1.32, (c) commutes. (f) is nothing but (1.46), and commutes. So the whole diagram commutes, and this is what we want to prove. $\square$

1.48 Definition. A monoidal adjoint pair $((?)^*, (?)_*)$ is said to be *Lipman* if $\Delta : f^*(a \otimes b) \to f^*a \otimes f^*b$ and $C : f^*\mathcal{O}_Y \to \mathcal{O}_X$ are isomorphisms for any morphism $f : X \to Y$ in $\mathcal{S}$ and any $a, b \in Y_*$.

(1.49) Note that $\Delta : f^*(a \otimes b) \to f^*a \otimes f^*b$ is a natural isomorphism if and only if its right conjugate (see (1.10)) is an isomorphism. The right conjugate is the composite

$$f_*[f^*b, c] \xrightarrow{H} [f_*f^*b, f_*c] \xrightarrow{u} [b, f_*c].$$

Let $((?)^*, (?)_*)$ be a Lipman adjoint pair of monoidal almost-pseudofunctors over $\mathcal{S}$. Then $(?)^*$ together with Δ^{-1} and C^{-1} form a *covariant* symmetric monoidal almost-pseudofunctor on $\mathcal{S}^{\mathrm{op}}$.

(1.50) For a morphism $f : X \to Y$ in $\mathcal{S}$, the composite map

$$f^*[a, b] \xrightarrow{\text{via tr}} [f^*a, f^*[a, b] \otimes f^*a] \xrightarrow{\text{via } \Delta^{-1}} [f^*a, f^*([a, b] \otimes a)] \xrightarrow{\text{via ev}} [f^*a, f^*b]$$

is denoted by P. We can apply Lewis's theorem to f^*. In particular, the following diagrams are commutative by (1.35) for a morphism $f : X \to Y$.

$$\begin{array}{ccc} f^*[a, b] \otimes f^*a & \xrightarrow{P \otimes 1} & [f^*a, f^*b] \otimes f^*a \\ \downarrow \Delta^{-1} & & \downarrow \text{ev} \\ f^*([a, b] \otimes a) & \xrightarrow{f^* \text{ev}} & f_*b \end{array} \tag{1.51}$$

$$\begin{array}{ccc} f^*a & \xrightarrow{f^* \text{tr}} & f^*[b, a \otimes b] \\ \downarrow \text{tr} & & \downarrow P \\ [f^*b, f^*a \otimes f^*b] & \xrightarrow{\Delta^{-1}} & [f^*b, f^*(a \otimes b)] \end{array} \tag{1.52}$$

$$\begin{array}{ccccccc} f^*a & \xrightarrow{\lambda^{-1}} & \mathcal{O}_X \otimes f^*a & \xrightarrow{C^{-1}} & f^*\mathcal{O}_Y \otimes f^*a & \xrightarrow{\text{via tr}} & f^*[a, \mathcal{O}_Y \otimes a] \otimes f^*a \\ \downarrow 1 & & & & & & \downarrow \text{via } \lambda \\ f^*a & & \xleftarrow{\qquad\text{ev}\qquad} & & [f^*a, f^*a] \otimes f^*a & \xleftarrow{P \otimes 1} & f^*[a, a] \otimes f^*a \end{array} \tag{1.53}$$

1.54 Lemma. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathcal{S}$. Then the diagram*

$$\begin{array}{ccccc} f^*g^*[a, b] & \xrightarrow{P} & f^*[g^*a, g^*b] & \xrightarrow{P} & [f^*g^*a, f^*g^*b] \\ \uparrow d^{-1} & & & & \uparrow [d, d^{-1}] \\ (gf)^*[a, b] & & \xrightarrow{\qquad\qquad P \qquad\qquad} & & [(gf)^*a, (gf)^*b] \end{array}$$

is commutative.

Proof. Follows instantly by Lemma 1.39. □

1.55 Lemma. *The diagram*

$$
\begin{tikzcd}
{[a,b]\otimes a} \arrow[rr, "\mathrm{ev}"] \arrow[d, "u\otimes u"] & & b \arrow[d, "u"] \\
f_*f^*[a,b]\otimes f_*f^*a \arrow[r, "HP\otimes 1"] & {[f_*f^*a, f_*f^*b]\otimes f_*f^*a} \arrow[r, "\mathrm{ev}"] & f_*f^*b
\end{tikzcd}
$$

is commutative.

Proof. Consider the diagram

$$
\begin{tikzcd}
{[a,b]\otimes a} \arrow[d, "u\otimes u"] \arrow[drr, squiggly, "u"] \arrow[dddrrr, rounded corners, "\mathrm{ev}"] \arrow[dr, phantom, "\text{(a)}"] & & & \\
f_*f^*[a,b]\otimes f_*f^*a \arrow[r, "m"] \arrow[d, "P"] \arrow[dr, phantom, "\text{(b)}"] & f_*(f^*[a,b]\otimes f^*a) \arrow[d, "P"] \arrow[dr, phantom, "\text{(d)}"] & f_*f^*([a,b]\otimes a) \arrow[l, "\Delta"'] \arrow[ddl, "\mathrm{ev}"] & \\
f_*[f^*a, f^*b]\otimes f_*f^*a \arrow[r, "m"] \arrow[d, "H"] \arrow[dr, phantom, "\text{(c)}"] & f_*([f^*a, f^*b]\otimes f^*a) \arrow[d, "\mathrm{ev}"] & \arrow[r, phantom, "\text{(e)}"] & \\
{[f_*f^*a, f_*f^*b]\otimes f_*f^*a} \arrow[r, "\mathrm{ev}"] & f_*f^*b & & b. \arrow[ll, "u"']
\end{tikzcd}
$$

Then (a), (c), and (d) are commutative by the commutativity of (1.42), (1.36), and (1.51), respectively. The commutativity of (b) and (e) is trivial. Thus the whole diagram is commutative, and the lemma follows. □

1.56 Lemma. *The following diagrams are commutative.*

$$
\begin{tikzcd}
{[a,b]} \arrow[r, "u"] \arrow[d, "u"] & {[a, f_*f^*b]} \\
f_*f^*[a,b] \arrow[r, "HP"] & {[f_*f^*a, f_*f^*b]} \arrow[u, "u"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
f^*f_*[a,b] \arrow[r, "PH"] \arrow[d, "\varepsilon"] & {[f^*f_*a, f^*f_*b]} \arrow[d, "\varepsilon"] \\
{[a,b]} \arrow[r, "\varepsilon"] & {[f^*f_*a, b]}
\end{tikzcd}
$$

Proof. We prove the commutativity of the first diagram. Consider the diagram

$$
\begin{tikzcd}
{[a,b]} \arrow[r, "\mathrm{tr}"] \arrow[d, "u"] \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, phantom, "\text{(b)}"] \arrow[rr, phantom, shift left=4, "\text{(a)}"] \arrow[ddrr, phantom, "\text{(c)}"] & {[a,[a,b]\otimes a]} \arrow[r, "\mathrm{ev}"] \arrow[d, "u"] & {[a,b]} \arrow[dddd, bend left, "u"] \\
f_*f^*[a,b] \arrow[r, "\mathrm{tr}"] \arrow[d, "HP"] \arrow[dr, phantom, "\text{(d)}"] & {[a,f_*f^*[a,b]\otimes a]} \arrow[d, "HP"] \arrow[dr, "u"] & \\
{[f_*f^*a,f_*f^*b]} \arrow[r, "\mathrm{tr}"] \arrow[dr, "\mathrm{tr}"] \arrow[ddr, bend right, "\mathrm{id}"'] \arrow[ddr, phantom, "\text{(g)}"] & {[a,[f_*f^*a,f_*f^*b]\otimes a]} \arrow[dr, "u"] \arrow[d, phantom, "\text{(f)}"] \arrow[r, phantom, "\text{(e)}"] & {[a,f_*f^*[a,b]\otimes f_*f^*a]} \arrow[d, "HP"] \\
& {[f_*f^*a,[f_*f^*a,f_*f^*b]\otimes f_*f^*a]} \arrow[r, "u"] \arrow[d, "\mathrm{ev}"] \arrow[dr, phantom, "\text{(h)}"] & {[a,[f_*f^*a,f_*f^*b]\otimes f_*f^*a]} \arrow[d, "\mathrm{ev}"] \\
& {[f_*f^*a,f_*f^*b]} \arrow[r, "u"] & {[a,f_*f^*b]}
\end{tikzcd}
$$

.

The commutativity of (a), (b), (d), (e), (g), and (h) is trivial. The commutativity of (c) is Lemma 1.55. The commutativity of (f) is Lemma 1.32. Thus the whole diagram is commutative, and the first part of the lemma follows.

The commutativity of the second diagram is proved by a similar diagram drawing. The details are left to the reader. □

(1.57) Let X be an object of $\mathcal{S}$. We denote the composite isomorphism

$$X_*(a,b) \xrightarrow{\text{via } \lambda} X_*(\mathcal{O}_X \otimes a, b) \xrightarrow{\pi} X_*(\mathcal{O}_X, [a,b])$$

by h_X.

1.58 Lemma. *Let $f : X \to Y$ be a morphism in $\mathcal{S}$. Then the composite map*

$$X_*(a,b) \xrightarrow{h_X} X_*(\mathcal{O}_X, [a,b]) \xrightarrow{C} X_*(f^*\mathcal{O}_Y, [a,b]) \cong Y_*(\mathcal{O}_Y, f_*[a,b]) \xrightarrow{H} Y_*(\mathcal{O}_Y, [f_*a, f_*b]) \xrightarrow{h_Y^{-1}} Y_*(f_*a, f_*b)$$

agrees with the map given by $\varphi \mapsto f_*\varphi$. The composite map

$$Y_*(a',b') \xrightarrow{h_Y} Y_*(\mathcal{O}_Y, [a',b']) \xrightarrow{f^*} X_*(f^*\mathcal{O}_Y, f^*[a',b']) \xrightarrow{X_*(C^{-1},P)} X_*(\mathcal{O}_X, [f^*a', f^*b']) \xrightarrow{h_X^{-1}} X_*(f^*a', f^*b')$$

agrees with f^*.

Proof. We prove the first assertion. The all maps are natural on a. By Yoneda's lemma, we may assume that $a = b$ and it suffices to show that the identity map 1_b is mapped to 1_{f_*b} by the map. It is straightforward to check that 1_b goes to the composite map

$$f_*b \xrightarrow{\lambda^{-1}} \mathcal{O}_Y \otimes f_*b \xrightarrow{u} f_*f^*\mathcal{O}_Y \otimes f_*b \xrightarrow{C} f_*\mathcal{O}_X \otimes f_*b \xrightarrow{\text{via tr}} f_*[b, \mathcal{O}_X \otimes b] \otimes f_*b \xrightarrow{\lambda} f_*[b,b] \otimes f_*b \xrightarrow{\text{ev}} f_*b.$$

By the commutativity of (1.38), we are done.

The second assertion is proved similarly, utilizing the commutativity of (1.53). □

1.59 Lemma. *Let $\sigma = (fg' = gf')$ be a commutative diagram in $\mathcal{S}$. Then the diagram*

$$\begin{array}{ccccc} g^*f_*[a,b] & & \xrightarrow{\qquad\qquad\theta\qquad\qquad} & & f'_*(g')^*[a,b] \\ \downarrow PH & & & & \downarrow HP \\ [g^*f_*a, g^*f_*b] & \xrightarrow{\theta} & [g^*f_*a, f'_*(g')^*b] & \xleftarrow{\theta} & [f'_*(g')^*a, f'_*(g')^*b] \end{array}$$

is commutative.

Proof. Follows from Lemma 1.56. □

2 Sheaves on ringed sites

(2.1) We fix a universe [16] $\mathcal{U}$ and a universe $\mathcal{V}$ such that $\mathcal{U} \in \mathcal{V}$ or $\mathcal{U} = \mathcal{V}$. A set is said to be *small* or *$\mathcal{U}$-small* if it is an element of $\mathcal{V}$, and is bijective to an element of $\mathcal{U}$. A category is said to be small if both the object set and the set of morphisms are small. Ringed spaces (including schemes) are required to be small, unless otherwise specified.

The categories of small sets and small abelian groups are respectively denoted by $\underline{\mathrm{Set}}$ and $\underline{\mathrm{Ab}}$. For example, M is an object of $\underline{\mathrm{Ab}}$ if and only if M is a group whose underlying set is in $\mathcal{V}$, and M is bijective to some set in $\mathcal{U}$.

A category $\mathcal{C}$ is said to be a *$\mathcal{U}$-category* if for any objects x, y of $\mathcal{C}$, the set $\mathcal{C}(x, y)$ is small. Note that $\underline{\mathrm{Set}}$ and $\underline{\mathrm{Ab}}$ are $\mathcal{U}$-categories.

(2.2) For categories I and $\mathcal{C}$, we denote the functor category $\mathrm{Func}(I^{\mathrm{op}}, \mathcal{C})$ by $\mathcal{P}(I, \mathcal{C})$. An object of $\mathcal{P}(I, \mathcal{C})$ is sometimes referred as a *presheaf* over I with values in $\mathcal{C}$.

If $\mathcal{C}$ is a $\mathcal{U}$-category and I is small, then $\mathcal{P}(I, \mathcal{C})$ is a $\mathcal{U}$-category. For a small category $\mathbb{X}$, we denote $\mathcal{P}(\mathbb{X}, \underline{\mathrm{Ab}})$ by $\mathrm{PA}(\mathbb{X})$.

In these notes, a site (i.e., a category with a Grothendieck topology, in the sense of [42]) is required to be a small category whose topology is defined by a pretopology (see [42]).

Let $\mathcal{C}$ be a category with small products. If $\mathbb{X}$ is a site, then the category of sheaves on $\mathbb{X}$ with values in $\mathcal{C}$ is denoted by $\mathcal{S}(\mathbb{X}, \mathcal{C})$. The inclusion $q : \mathcal{S}(\mathbb{X}, \mathcal{C}) \to \mathcal{P}(\mathbb{X}, \mathcal{C})$ is fully faithful. If $\mathcal{C}$ is a $\mathcal{U}$-category, then $\mathcal{S}(\mathbb{X}, \mathcal{C})$ is a $\mathcal{U}$-category. The category $\mathcal{S}(\mathbb{X}, \underline{\mathrm{Ab}})$ is denoted by $\mathrm{AB}(\mathbb{X})$.

Let $\mathbb{X}$ be a site. The inclusion $\mathrm{AB}(\mathbb{X}) \to \mathrm{PA}(\mathbb{X})$ is denoted by $q(\mathbb{X}, \mathrm{AB})$. We denote the *sheafification* functor $\mathrm{PA}(\mathbb{X}) \to \mathrm{AB}(\mathbb{X})$ by $a(\mathbb{X}, \mathrm{AB})$. Namely, $a = a(\mathbb{X}, \mathrm{AB})$ is the left adjoint of $q(\mathbb{X}, \mathrm{AB})$. Note that a is exact.

We review the construction of the sheafification described in [2, (II.1)].

For $\mathcal{M} \in \mathrm{PA}(\mathbb{X})$, $x \in \mathbb{X}$, and a covering $\mathcal{U} = (x_i \to x)_{i \in I}$ of x, we denote the kernel of the map

$$\prod_{i \in I} \Gamma(x_i, \mathcal{M}) \xrightarrow{\varphi} \prod_{(i,j) \in I \times I} \Gamma(x_i \times_x x_j, \mathcal{M})$$

by $\check{H}^0(\mathcal{U}, \mathcal{M})$, where $\varphi((m_i)_{i \in I}) = (\mathrm{res}_{x_i \times_x x_j, x_j} m_j - \mathrm{res}_{x_i \times_x x_j, x_i} m_i)_{(i,j) \in I \times I}$. Note that $\underline{\check{H}}^0(\mathcal{U}, ?)$ is a functor from $\mathrm{PA}(\mathbb{X})$ to $\underline{\mathrm{Ab}}$, which is compatible with arbitrary limits.

The set of all coverings of x is a directed set. Let $\mathcal{U} = (\phi_i \colon x_i \to x)_{i \in I}$ and $\mathcal{V} = (\psi_j \colon y_j \to x)_{j \in J}$ be coverings of x. We say that $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if there are a map $\tau \colon J \to I$ and a collection of morphisms $(\eta_j \colon y_j \to x_{\tau j})_{j \in J}$ such that $\phi_{\tau j} \circ \eta_j = \psi_j$ for $j \in J$. If $(\tau, (\eta_j))$ makes $\mathcal{V}$ a refinement of $\mathcal{U}$, then we define

$$\mathbb{L} = \mathbb{L}(\mathcal{V}, \mathcal{U}; (\tau, (\eta_j)))(\mathcal{M}) \colon \check{H}^0(\mathcal{U}, \mathcal{M}) \to \check{H}^0(\mathcal{V}, \mathcal{M})$$

by $\mathbb{L}((m_i)_{i \in I}) = (\mathrm{res}_{\eta_j}(m_{\tau j}))_{j \in J}$. It is easy to see that $\mathbb{L}$ is independent of the choice of τ or η_j, and depends only on $\mathcal{U}$ and $\mathcal{V}$, see [31, Lemma III.2.1]. If $\mathcal{W}$ is a refinement of $\mathcal{V}$, then $\mathbb{L}(\mathcal{W}, \mathcal{U}) = \mathbb{L}(\mathcal{W}, \mathcal{V}) \circ \mathbb{L}(\mathcal{V}, \mathcal{U})$. Thus we get an inductive system $(\check{H}^0(\mathcal{U}, \mathcal{M}))_{\mathcal{U}}$, where $\mathcal{U}$ runs through the all coverings of x. We denote $\varinjlim \check{H}^0(\mathcal{U}, \mathcal{M})$ by $\check{H}^0(x, \mathcal{M})$. This is a small abelian group, and $\check{H}^0(x, ?)$ is a left exact functor from $\mathrm{PA}(\mathbb{X})$ to $\underline{\mathrm{Ab}}$.

Let $x' \to x$ be a morphism. Then a covering $\mathcal{U} = (x_i \to x)_{i \in I}$ gives a covering $x' \times_x \mathcal{U} = (x' \times_x x_i \to x')_{i \in I}$ in a natural way. This correspondence induces a map $\check{H}^0(\mathcal{U}, \mathcal{M}) \to \check{H}^0(x' \times_x \mathcal{U}, \mathcal{M})$. So we have a canonical map $\check{H}^0(x, \mathcal{M}) \to \check{H}^0(x', \mathcal{M})$. So we have a presheaf of abelian groups $\underline{\check{H}}^0(\mathcal{M})$ such that $\Gamma(x, \underline{\check{H}}^0(\mathcal{M})) = \check{H}^0(x, \mathcal{M})$. Note that $\underline{\check{H}}^0$ is an endofunctor of $\mathrm{PA}(\mathbb{X})$.

Note that there is a natural map

$$Y = Y(\mathcal{M}) \colon \mathcal{M} \to \underline{\check{H}}^0(\mathcal{M}).$$

The map Y at the object x

$$Y(x) \colon \Gamma(x, \mathcal{M}) \to \Gamma(x, \underline{\check{H}}^0(\mathcal{M})) = \check{H}^0(x, \mathcal{M})$$

is given by $Y(x)(m) = m \in \check{\underline{H}}^0(\mathrm{id}_x, \mathcal{M}) \to \check{\underline{H}}^0(x, \mathcal{M})$ for $m \in \Gamma(x, \mathcal{M})$, where id_x is the covering $(x \to x)$ consisting of the one morphism id_x. $Y = Y(\mathcal{M})$ is an isomorphism if and only if $\mathcal{M}$ is a sheaf.

It is known that $\check{\underline{H}}^0(\check{\underline{H}}^0(\mathcal{M}))$ is a sheaf, and it is the sheafification $a\mathcal{M}$. The composite map

$$u\colon \mathcal{M} \xrightarrow{Y(\mathcal{M})} \check{\underline{H}}^0(\mathcal{M}) \xrightarrow{Y(\check{\underline{H}}^0(\mathcal{M}))} \check{\underline{H}}^0(\check{\underline{H}}^0(\mathcal{M})) = a\mathcal{M} = qa\mathcal{M}$$

is the unit of adjunction. By the naturality of Y, u also agrees with the composite map

$$\mathcal{M} \xrightarrow{Y(\mathcal{M})} \check{\underline{H}}^0(\mathcal{M}) \xrightarrow{\check{\underline{H}}^0(Y(\mathcal{M}))} \check{\underline{H}}^0(\check{\underline{H}}^0(\mathcal{M})).$$

Note that the counit of adjunction $\varepsilon : aq \to \mathrm{Id}$ is given as the unique natural map such that $q\varepsilon : qaq \to q$ is the inverse of uq.

(2.3) Let $\mathbb{X} = (\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be a ringed site. Namely, let $\mathbb{X}$ be a site and $\mathcal{O}_{\mathbb{X}}$ a sheaf of commutative rings on $\mathbb{X}$. We denote the category of presheaves (resp. sheaves) of $\mathcal{O}_{\mathbb{X}}$-modules by $\mathrm{PM}(\mathbb{X})$ (resp. $\mathrm{Mod}(\mathbb{X})$). The inclusion $\mathrm{Mod}(\mathbb{X}) \to \mathrm{PM}(\mathbb{X})$ is denoted by $q(\mathbb{X}, \mathrm{Mod})$. The sheafification $\mathrm{PM}(\mathbb{X}) \to \mathrm{Mod}(\mathbb{X})$ is denoted by $a(\mathbb{X}, \mathrm{Mod})$. Note that $a(\mathbb{X}, \mathrm{Mod})$ is constructed in the same way as in (2.2), since $\check{\underline{H}}^0(\mathcal{M})$ is in $\mathrm{PM}(\mathbb{X})$ in a natural way for $\mathcal{M} \in \mathrm{PM}(\mathbb{X})$. Since q is fully faithful, $\varepsilon : aq \to \mathrm{Id}$ is an isomorphism.

The forgetful functor $\mathrm{Mod}(\mathbb{X}) \to \mathrm{AB}(\mathbb{X})$ is denoted by $F(\mathbb{X})$. The forgetful functor $\mathrm{PM}(\mathbb{X}) \to \mathrm{PA}(\mathbb{X})$ is denoted by $F'(\mathbb{X})$. Thus $F'(\mathbb{X}) \circ q(\mathbb{X}, \mathrm{Mod}) = q(\mathbb{X}, \mathrm{AB}) \circ F(\mathbb{X})$ and $a(\mathbb{X}, \mathrm{AB}) \circ F'(\mathbb{X}) = F(\mathbb{X}) \circ a(\mathbb{X}, \mathrm{Mod})$.

We say that a category $\mathcal{A}$ is *Grothendieck* if it is an abelian $\mathcal{U}$-category with a generator which satisfies the (AB5) condition in [13] (the existence of arbitrary small coproducts, and the exactness of small filtered inductive limits), see [37]. The categories $\mathrm{AB}(\mathbb{X})$ and $\mathrm{Mod}(\mathbb{X})$ are Grothendieck. In general, a Grothendieck category satisfies (AB3*), see [37, Corollary 7.10]. A ringed category $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ is a pair such that $\mathbb{X}$ is a small category, and $\mathcal{O}_{\mathbb{X}}$ is a presheaf of commutative rings on $\mathbb{X}$. If $\mathbb{X}$ is a ringed category, then $\mathrm{PA}(\mathbb{X})$ and $\mathrm{PM}(\mathbb{X})$ are Grothendieck with (AB4*).

(2.4) Let $f : \mathbb{Y} \to \mathbb{X}$ be a functor between small categories. Then the pull-back $\mathrm{PA}(\mathbb{X}) \to \mathrm{PA}(\mathbb{Y})$ is denoted by $f^{\#}_{\mathrm{PA}}$. Note that $f^{\#}_{\mathrm{PA}}(\mathcal{F}) := \mathcal{F} \circ f^{\mathrm{op}}$. In general, the pull-back $\mathcal{P}(\mathbb{X}, \mathcal{C}) \to \mathcal{P}(\mathbb{Y}, \mathcal{C})$ is defined in a similar way, and is denoted by $f^{\#}$. If f is a continuous functor (i.e., $f^{\#}_{\underline{\mathrm{Set}}}$ carries sheaves to sheaves) between sites, then $f^{\#}_{\mathrm{AB}} : \mathrm{AB}(\mathbb{X}) \to \mathrm{AB}(\mathbb{Y})$ is defined to be the restriction of $f^{\#}_{\mathrm{PA}}$. Throughout these notes, we require that a continuous functor $f : \mathbb{Y} \to \mathbb{X}$ between sites satisfies the following condition. For $y \in \mathbb{Y}$, a covering $(y_i \to y)_{i \in I}$, and any $i, j \in I$, the morphisms $f(y_i \times_y y_j) \to f(y_i)$ and $f(y_i \times_y y_j) \to f(y_j)$ make $f(y_i \times_y y_j)$ the fiber product $f(y_i) \times_{f(y)} f(y_j)$. The identity functor is continuous. A composite of continuous functors is again continuous.

Thanks to the re-definition of sites and continuous functors, we have the following.

2.5 Lemma. *Let $f : \mathbb{Y} \to \mathbb{X}$ be a functor between sites. Then f is continuous if and only if the following holds.*

If $(\varphi_i : y_i \to y)_{i \in I}$ is a covering, then $(f\varphi_i : fy_i \to fy)_{i \in I}$ is a covering, and for any $i, j \in I$, the morphisms $f(y_i \times_y y_j) \to f(y_i)$ and $f(y_i \times_y y_j) \to f(y_j)$ make $f(y_i \times_y y_j)$ the fiber product $f(y_i) \times_{f(y)} f(y_j)$.

For the proof, see [43, (1.6)].

(2.6) Let $f : \mathbb{Y} \to \mathbb{X}$ be a functor between small categories. The left adjoint of $f^{\#}_{\mathrm{PA}}$, which exists by Kan's lemma (see e.g., [2, Theorem I.2.1]), is denoted by $f^{\mathrm{PA}}_{\#}$.

For $x \in \mathbb{X}$, we define the small category I^f_x as follows. An object of I^f_x is a pair (y, ϕ) with $y \in \mathbb{Y}$ and $\phi \in \mathbb{X}(x, f(y))$. A morphism $h : (y, \phi) \to (y', \phi')$ is a morphism $h \in \mathbb{Y}(y, y')$ such that $f(h) \circ \phi = \phi'$. Note that $\Gamma(x, f^{\mathrm{PA}}_{\#}(\mathcal{F})) = \varinjlim \Gamma(y, \mathcal{F})$, where the colimit is taken over $(I^f_x)^{\mathrm{op}}$.

The left adjoint of $f^{\#}_{\mathcal{C}} : \mathcal{P}(\mathbb{X}, \mathcal{C}) \to \mathcal{P}(\mathbb{Y}, \mathcal{C})$ is constructed similarly, provided $\mathcal{C}$ has arbitrary small colimits. The left adjoint is denoted by $f^{\mathcal{C}}_{\#}$ or simply by $f_{\#}$.

For a continuous functor $f \colon \mathbb{Y} \to \mathbb{X}$ between sites, the left adjoint $f^{\mathrm{AB}}_{\#}$ of $f^{\#}_{\mathrm{AB}}$ is given by $f^{\mathrm{AB}}_{\#} = a(\mathbb{X}, \mathrm{AB}) \circ f^{\mathrm{PA}}_{\#} \circ q(\mathbb{Y}, \mathrm{AB})$.

2.7 Lemma. *If $(I^f_x)^{\mathrm{op}}$ is pseudofiltered (see e.g., [16, 31]) for each $x \in \mathbb{X}$, then $f^{\mathrm{PA}}_{\#}$ is exact.*

Proof. This is a consequence of [16, Corollaire 2.10]. $\square$

(2.8) We say that $f : \mathbb{Y} \to \mathbb{X}$ is *admissible* if f is continuous and the functor $f^{\mathrm{PA}}_{\#}$ is exact.

(2.9) Let $f : \mathbb{Y} \to \mathbb{X}$ be an admissible functor. Then $f_{\#}^{\mathrm{AB}}$ is exact. Indeed, $f_{\#}^{\mathrm{AB}}$ is right exact, since it is a left adjoint of $f_{\mathrm{AB}}^{\#}$. On the other hand, being a composite of left exact functors, $f_{\#}^{\mathrm{AB}} = af_{\#}^{\mathrm{PA}}q$ is left exact.

(2.10) If $\mathbb{Y}$ has finite limits and f preserves finite limits, then $f_{\#}^{\mathrm{PA}}$ is exact by Lemma 2.7. It follows that a continuous map between topological spaces induces an admissible continuous functor between the corresponding sites.

(2.11) The right adjoint functor of $f_{\mathrm{PA}}^{\#}$, which we denote by $f_{\flat}^{\mathrm{PA}}$ also exists, as $\underline{\mathrm{Ab}}^{\mathrm{op}}$ has arbitrary small colimits (i.e., $\underline{\mathrm{Ab}}$ has small limits). The functor $f_{\flat}^{\mathrm{PA}}$ is the composite

$$\mathrm{Func}(\mathbb{Y}^{\mathrm{op}}, \underline{\mathrm{Ab}}) \xrightarrow{\mathrm{op}} \mathrm{Func}(\mathbb{Y}, \underline{\mathrm{Ab}}^{\mathrm{op}}) \xrightarrow{(f^{\mathrm{op}})_{\#}} \mathrm{Func}(\mathbb{X}, \underline{\mathrm{Ab}}^{\mathrm{op}}) \xrightarrow{\mathrm{op}} \mathrm{Func}(\mathbb{X}^{\mathrm{op}}, \underline{\mathrm{Ab}}),$$

where $(f^{\mathrm{op}})_{\#}$ is the left adjoint of

$$(f^{\mathrm{op}})^{\#} : \mathrm{Func}(\mathbb{X}, \underline{\mathrm{Ab}}^{\mathrm{op}}) \to \mathrm{Func}(\mathbb{Y}, \underline{\mathrm{Ab}}^{\mathrm{op}}),$$

where $f^{\mathrm{op}} = f$ is the opposite of f, namely, f viewed as a functor $\mathbb{Y}^{\mathrm{op}} \to \mathbb{X}^{\mathrm{op}}$.

(2.12) For $\mathcal{M}, \mathcal{N} \in \mathrm{PM}(\mathbb{X})$, the *presheaf tensor product* is denoted by $\otimes_{\mathcal{O}_{\mathbb{X}}}^{p}$. It is defined by

$$\Gamma(x, \mathcal{M} \otimes_{\mathcal{O}_{\mathbb{X}}}^{p} \mathcal{N}) := \Gamma(x, \mathcal{M}) \otimes_{\Gamma(x, \mathcal{O}_{\mathbb{X}})} \Gamma(x, \mathcal{N})$$

for $x \in \mathbb{X}$.

The *sheaf tensor product* $a(q\mathcal{M} \otimes_{\mathcal{O}_{\mathbb{X}}}^{p} q\mathcal{N})$ of $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\mathbb{X})$ is denoted by $\mathcal{M} \otimes_{\mathcal{O}_{\mathbb{X}}} \mathcal{N}$.

Let $\mathcal{M}, \mathcal{N} \in \mathrm{PM}(\mathbb{X})$, $x \in \mathbb{X}$, and $\mathcal{U} = (x_i \to x)_{i \in I}$ and $\mathcal{V} = (x'_j \to x)_{j \in J}$ be coverings of x. We define a map

$$Z = Z(\mathcal{U}, \mathcal{V}; \mathcal{M}, \mathcal{N})\colon\ \check{H}^0(\mathcal{U}, \mathcal{M}) \otimes_{\Gamma(x, \mathcal{O}_{\mathbb{X}})} \check{H}^0(\mathcal{V}, \mathcal{N}) \to \check{H}^0(\mathcal{U} \times \mathcal{V}, \mathcal{M} \otimes^p \mathcal{N})$$

by $Z((m_i)_{i \in I} \otimes (n_j)_{j \in J}) = (m_i \otimes n_j)_{(i,j) \in I \times J}$, where $\mathcal{U} \times \mathcal{V}$ denotes the covering $(x_i \times_x x'_j \to x)_{(i,j) \in I \times J}$ of x. Note that Z induces

$$Z = Z(\mathcal{M}, \mathcal{N})\colon\ \underline{\check{H}}^0(\mathcal{M}) \otimes^p \underline{\check{H}}^0(\mathcal{N}) \to \underline{\check{H}}^0(\mathcal{M} \otimes^p \mathcal{N}).$$

2.13 Lemma. *The composite*

$$\mathcal{M} \otimes^p \mathcal{N} \xrightarrow{Y(\mathcal{M}) \otimes Y(\mathcal{N})} \underline{\check{H}}^0 \mathcal{M} \otimes^p \underline{\check{H}}^0 \mathcal{N} \xrightarrow{Z} \underline{\check{H}}^0(\mathcal{M} \otimes^p \mathcal{N})$$

agrees with $Y(\mathcal{M} \otimes^p \mathcal{N})$.

Proof. This is straightforward, and we omit it. □

2.14 Lemma. *The composite*

$$\underline{\check{H}}^0\,\mathcal{M} \otimes^p \underline{\check{H}}^0\,\mathcal{N} \xrightarrow{Z} \underline{\check{H}}^0(\mathcal{M} \otimes^p \mathcal{N}) \xrightarrow{\underline{\check{H}}^0(Y\otimes Y)} \underline{\check{H}}^0(\underline{\check{H}}^0\,\mathcal{M} \otimes^p \underline{\check{H}}^0\,\mathcal{N})$$

agrees with $Y = Y(\underline{\check{H}}^0\,\mathcal{M} \otimes^p \underline{\check{H}}^0\,\mathcal{N})$.

Proof. Note that a section of $\underline{\check{H}}^0(\underline{\check{H}}^0\,\mathcal{M} \otimes^p \underline{\check{H}}^0\,\mathcal{N})$ at x is represented by data as follows. A covering $\mathcal{V} = (x_i \to x)_{i\in I}$, a collection of coverings $\mathcal{V}_i = (y^i_j \to x_i)_{j\in J_i}$ $(i \in I)$, and a collection of elements $(\sum_l (m^{i,l}_j)_{j\in J_i} \otimes (n^{i,l}_j)_{j\in J_i})_{i\in I}$ subject to the patching conditions, where $m^{i,l}_j \in \Gamma(y^i_j, \mathcal{M})$ and $n^{i,l}_j \in \Gamma(y^i_j, \mathcal{N})$.

Let $\mathcal{U} = (z_l \to x)_{l\in L}$ and $\mathcal{U}' = (z'_{l'} \to x)_{l'\in L'}$ be coverings of x, and $(m_l)_{l\in L}$ and $(n_{l'})_{l'\in L'}$ elements of $\check{H}^0(\mathcal{U}, \mathcal{M})$ and $\check{H}^0(\mathcal{U}', \mathcal{N})$, respectively. Then $Y((m_l) \otimes (n_{l'}))$ is represented by the collection $I = \{\mathrm{id}_x\}$, $\mathcal{V} = (\mathrm{id}_x)$, $J_{\mathrm{id}_x} = L \times L'$, $\mathcal{V}_{\mathrm{id}_x} = (z_l \times_x z'_{l'} \to x)_{(l,l')\in L\times L'}$, and $((\mathrm{res}_{z_l\times_x z'_{l'}, z_l}\, m_l) \otimes (\mathrm{res}_{z_l\times_x z'_{l'}, z'_{l'}}\, n_{l'})) \in \check{H}^0(\mathcal{V}, \underline{\check{H}}^0\,\mathcal{M} \otimes^p \underline{\check{H}}^0\,\mathcal{N})$. As an element of $\check{H}^0(x, \underline{\check{H}}^0\,\mathcal{M} \otimes^p \underline{\check{H}}^0\,\mathcal{N})$, this element is the same as the element represented by the collection $I = L \times L'$, $\mathcal{V} = \mathcal{U} \times \mathcal{U}'$, $J_{l_1,l'_1} = L \times L'$ for any $(l_1, l'_1) \in L \times L'$, $\mathcal{V}_{l_1,l'_1} = z_{l_1} \times_x \mathcal{U} \times_x z'_{l'_1} \times_x \mathcal{U}'$ for $(l_1, l'_1) \in L \times L'$, and $((\mathrm{res}_{z_{l_1,l,l'_1,l'}, z_l}\, m_l) \otimes (\mathrm{res}_{z_{l_1,l,l'_1,l'}, z'_{l'}}\, n_{l'}))_{(l_1,l'_1)\in L\times L'}$, where $z_{l_1,l,l'_1,l'} := z_{l_1} \times_x z_l \times_x z'_{l'_1} \times_x z'_{l'}$. Since $\mathrm{res}_{z_{l_1}\times_x z_l, z_l}\, m_l = \mathrm{res}_{z_{l_1}\times_x z_l, z_{l_1}}\, m_{l_1}$ and $\mathrm{res}_{z'_{l'_1}\times_x z'_{l'}, z'_{l'}}\, n_{l'} = \mathrm{res}_{z'_{l'_1}\times_x z'_{l'}, z'_{l'_1}}\, n_{l'_1}$, this element agrees with the element represented by the collection $I = L \times L'$, $\mathcal{V} = \mathcal{U} \times \mathcal{U}'$, $J_{l_1,l'_1} = L \times L'$ for $(l_1, l'_1) \in L \times L'$, $\mathcal{V}_{l_1,l'_1} = z_{l_1} \times_x \mathcal{U} \times_x z'_{l'_1} \times_x \mathcal{U}'$ for $(l_1, l'_1) \in L \times L'$, and $((\mathrm{res}_{z_{l_1,l,l'_1,l'}, z_{l_1}}\, m_{l_1}) \otimes (\mathrm{res}_{z_{l_1,l,l'_1,l'}, z'_{l'_1}}\, n_{l'_1}))_{(l_1,l'_1)\in L\times L'}$. It also agrees with the element represented by the collection $I = L \times L'$, $\mathcal{V} = \mathcal{U} \times \mathcal{U}'$, J_{l_1,l'_1} is the singleton $\{\mathrm{id}_{z_{l_1}\times_x z'_{l'_1}}\}$ for $(l_1, l'_1) \in L \times L'$, $\mathcal{V}_{l_1,l'_1} = (\mathrm{id}_{z_{l_1}\times_x z'_{l'_1}})$ for $(l_1, l'_1) \in L \times L'$, and $((\mathrm{res}_{z_{l_1}\times_x z'_{l'_1}, z_{l_1}}\, m_{l_1}) \otimes (\mathrm{res}_{z_{l_1}\times_x z'_{l'_1}, z'_{l'_1}}\, n_{l'_1}))_{(l_1,l'_1)\in L\times L'}$, which agrees with the image of $(m_l) \otimes (n_{l'})$ by $\underline{\check{H}}^0(Y \otimes Y) \circ Z$. This shows that $Y = \underline{\check{H}}^0(Y \otimes Y) \circ Z$. □

(2.15) We define a natural map

$$m' \colon qa\mathcal{M} \otimes^p qa\mathcal{N} \to qa(\mathcal{M} \otimes^p \mathcal{N})$$

as the composite

$$\begin{aligned} qa\mathcal{M} \otimes^p qa\mathcal{N} = \underline{\check{H}}^0\, \underline{\check{H}}^0\, \mathcal{M} \otimes^p \underline{\check{H}}^0\, \underline{\check{H}}^0\, \mathcal{N} &\xrightarrow{Z} \underline{\check{H}}^0 (\underline{\check{H}}^0\, \mathcal{M} \otimes^p \underline{\check{H}}^0\, \mathcal{N}) \\ &\xrightarrow{\underline{\check{H}}^0\, Z} \underline{\check{H}}^0\, \underline{\check{H}}^0 (\mathcal{M} \otimes^p \mathcal{N}) = qa(\mathcal{M} \otimes^p \mathcal{N}). \end{aligned}$$

2.16 Lemma. *The composite*

$$\mathcal{M} \otimes^p \mathcal{N} \xrightarrow{u \otimes u} qa\mathcal{M} \otimes^p qa\mathcal{N} \xrightarrow{m'} qa(\mathcal{M} \otimes^p \mathcal{N})$$

agrees with the unit map u.

Proof. Consider the diagram

$$\begin{tikzcd}
\mathcal{M} \otimes^p \mathcal{N} \arrow[r, "Y \otimes Y"] \arrow[dr, "Y"] & h\mathcal{M} \otimes^p h\mathcal{N} \arrow[r, "Y \otimes Y"] \arrow[d, "Z"] \arrow[dr, "Y"] & hh\mathcal{M} \otimes^p hh\mathcal{N} \arrow[d, "Z"] \\
 & h(\mathcal{M} \otimes^p \mathcal{N}) \arrow[r, "h(Y \otimes Y)"] \arrow[dr, "hY"] & h(h\mathcal{M} \otimes^p h\mathcal{N}) \arrow[d, "hZ"] \\
 & & hh(\mathcal{M} \otimes^p \mathcal{N})
\end{tikzcd} ,$$

where $h = \underline{\check{H}}^0$. Then the four triangles in the diagram commutes by Lemma 2.13 and Lemma 2.14. So the whole diagram commutes, and the lemma follows. ☐

2.17 Lemma. *The composite*

$$qa\mathcal{M} \otimes^p qa\mathcal{N} \xrightarrow{m'} qa(\mathcal{M} \otimes^p \mathcal{N}) \xrightarrow{qa(u \otimes^p u)} qa(qa\mathcal{M} \otimes^p qa\mathcal{N})$$

agrees with u.

Proof. Consider the diagram

$$\begin{tikzcd}
hh\mathcal{M} \otimes^p hh\mathcal{N} \arrow[r, "Z"] \arrow[dr, "Y"] & h(h\mathcal{M} \otimes^p h\mathcal{N}) \arrow[r, "hZ"] \arrow[d, "h(Y \otimes^p Y)"] \arrow[dr, "hY"] & hh(\mathcal{M} \otimes^p \mathcal{N}) \arrow[d, "hh(Y \otimes^p Y)"] \\
 & h(hh\mathcal{M} \otimes^p hh\mathcal{N}) \arrow[r, "hZ"] \arrow[dr, "hY"] & hh(h\mathcal{M} \otimes^p h\mathcal{N}) \arrow[d, "hh(Y \otimes^p Y)"] \\
 & & hh(hh\mathcal{M} \otimes^p hh\mathcal{N})
\end{tikzcd},$$

where $h = \underline{\check{H}}^0$. The four triangles in the diagram are commutative by Lemma 2.13 and Lemma 2.14. So the whole diagram is commutative, and the lemma follows. $\square$

2.18 Lemma. *For $\mathcal{M}, \mathcal{N} \in \mathrm{PM}(\mathbb{X})$, the natural map*

$$\bar{\Delta} := a(u \otimes^p u) \colon a(\mathcal{M} \otimes^p \mathcal{N}) \to a(qa\mathcal{M} \otimes^p qa\mathcal{N})$$

is an isomorphism.

Proof. Consider the diagram

$$\begin{tikzcd}
\mathcal{M} \otimes^p \mathcal{N} \arrow[r, "\varphi"] \arrow[dr, "\tau"] & qa\mathcal{M} \otimes^p qa\mathcal{N} \arrow[d, "m'"] \arrow[dr, "\beta"] & \\
& qa(\mathcal{M} \otimes^p \mathcal{N}) \arrow[r, shift left, "q(\bar{\Delta})"] & qa(qa\mathcal{M} \otimes^p qa\mathcal{N}), \arrow[l, shift left, "q(\psi)"]
\end{tikzcd}$$

where $\varphi = u \otimes^p u$, $\tau = u$, $\beta = u$, $\bar{\Delta} = a(u \otimes^p u)$, and $\psi \colon a(qa\mathcal{M} \otimes^p qa\mathcal{N}) \to a(\mathcal{M} \otimes^p \mathcal{N})$ is the unique map of sheaves such that $m' = q(\psi)\beta$ (this map exists by the universality of the sheafification). By Lemma 2.16, $\tau = m'\varphi$. By Lemma 2.17, $\beta = q(\bar{\Delta})m'$.

So

$$q(\psi\bar{\Delta})\tau = q(\psi)q(\bar{\Delta})m'\varphi = q(\psi)\beta\varphi = m'\varphi = \tau = q(\mathrm{id})\tau.$$

By the universality of the sheafification τ, we have that $\psi\bar{\Delta} = \mathrm{id}$. Moreover,

$$q(\bar{\Delta}\psi)\beta = q(\bar{\Delta})q(\psi)\beta = q(\bar{\Delta})m' = \beta = q(\mathrm{id})\beta.$$

By the universality of the sheafification β, we have that $\bar{\Delta}\psi = \mathrm{id}$. This shows that $\bar{\Delta}$ is an isomorphism. $\square$

(2.19) Let $(\mathbb{Y}, \mathcal{O}_{\mathbb{Y}})$ and $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be ringed categories. We say that $f : (\mathbb{Y}, \mathcal{O}_{\mathbb{Y}}) \to (\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ is a *ringed functor* if $f : \mathbb{Y} \to \mathbb{X}$ is a functor, and a morphism of presheaves of rings $\eta : \mathcal{O}_{\mathbb{Y}} \to f^{\#}\mathcal{O}_{\mathbb{X}}$ is given.

If, moreover, both $(\mathbb{Y}, \mathcal{O}_{\mathbb{Y}})$ and $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ are ringed sites and f is continuous, then we call f a *ringed continuous functor*.

The pull-back $\mathrm{PM}(\mathbb{X}) \to \mathrm{PM}(\mathbb{Y})$ is denoted by $f^{\#}_{\mathrm{PM}}$, and its left adjoint is denoted by $f_{\#}^{\mathrm{PM}}$. The left adjoint $f_{\#}^{\mathrm{PM}}$ is defined by

$$\Gamma(x, f_{\#}^{\mathrm{PM}}\mathcal{M}) := \varinjlim \Gamma(x, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y, \mathcal{O}_{\mathbb{Y}})} \Gamma(y, \mathcal{M})$$

for $x \in \mathbb{X}$ and $\mathcal{M} \in \mathrm{PM}(\mathbb{Y})$, where the colimit is taken over the category $(I_x^f)^{\mathrm{op}}$. Similarly, $f^\#_{\mathrm{Mod}} : \mathrm{Mod}(\mathbb{X}) \to \mathrm{Mod}(\mathbb{Y})$ and its left adjoint $f_\#^{\mathrm{Mod}} = af^\#_{\mathrm{PM}}q$ is defined. Note that $qf^\#_{\mathrm{Mod}} = f^\#_{\mathrm{PM}}q$ and $qf^\#_{\mathrm{AB}} = f^\#_{\mathrm{PA}}q$. We sometimes denote the identity map $qf^\#_{\mathrm{Mod}} = f^\#_{\mathrm{PM}}q$ and $qf^\#_{\mathrm{AB}} = f^\#_{\mathrm{PA}}q$ and their inverses by $c = c(f)$. If $(I_x^f)^{\mathrm{op}}$ is filtered for any $x \in \mathbb{X}$, then $f_\#^{\mathrm{PA}}\mathcal{O}_{\mathbb{Y}}$ has a structure of a presheaf of rings in a natural way, and there is a canonical isomorphism $f_\#^{\mathrm{PM}}\mathcal{M} \cong \mathcal{O}_{\mathbb{X}} \otimes^p_{f_\#^{\mathrm{PA}}\mathcal{O}_{\mathbb{Y}}} f_\#^{\mathrm{PA}}\mathcal{M}$. The right adjoint of $f^\#_{\mathrm{PM}}$, which exists as in (2.11), is denoted by $f_\flat^{\mathrm{PM}}$.

(2.20) Let $f : (\mathbb{Y}, \mathcal{O}_{\mathbb{Y}}) \to (\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be a ringed continuous functor. For later use, we need the explicit description of the unit $u : \mathrm{Id} \to f^\#_\heartsuit f^\heartsuit_\#$ and the counit $\varepsilon : f^\heartsuit_\# f^\#_\heartsuit \to \mathrm{Id}$, where $\heartsuit$ denotes either PM or Mod. The unit u for the case $\heartsuit = \mathrm{PM}$ is induced by the map

$$\begin{aligned}\Gamma(y, \mathcal{M}) \to \Gamma(fy, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y,\mathcal{O}_{\mathbb{Y}})} \Gamma(y, \mathcal{M}) \to \\ \varinjlim \Gamma(fy, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y',\mathcal{O}_{\mathbb{Y}})} \Gamma(y', \mathcal{M}) = \Gamma(fy, f_\#\mathcal{M}) = \Gamma(y, f^\# f_\# \mathcal{M}),\end{aligned}$$

where the first map sends m to $1 \otimes m$, and the second map is the obvious map. The counit ε for the case $\heartsuit = \mathrm{PM}$ is induced by the map

$$\Gamma(x, f_\# f^\# \mathcal{M}) = \varinjlim \Gamma(x, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y,\mathcal{O}_{\mathbb{Y}})} \Gamma(fy, \mathcal{M}) \to \Gamma(x, \mathcal{M}),$$

where the colimit is taken over $(I_x^f)^{\mathrm{op}}$, and the last map is given by $a \otimes m \mapsto a\,\mathrm{res}_{x,fy}(m)$. It is easy to verify that the composite

$$f_\# \xrightarrow{u} f_\# f^\# f_\# \xrightarrow{\varepsilon} f_\#$$

is the identity, and the composite

$$f^\# \xrightarrow{u} f^\# f_\# f^\# \xrightarrow{\varepsilon} f^\#$$

is the identity, and thus certainly $(f_\#, f^\#)$ is an adjoint pair.

The unit $u : \mathrm{Id} \to f^\#_{\mathrm{Mod}} f_\#^{\mathrm{Mod}}$ is the composite

$$\mathrm{Id} \xrightarrow{\varepsilon^{-1}} aq \xrightarrow{u} af^\#_{\mathrm{PM}} f_\#^{\mathrm{PM}} q \xrightarrow{\theta} f^\#_{\mathrm{Mod}} a f_\#^{\mathrm{PM}} q = f^\#_{\mathrm{Mod}} f_\#^{\mathrm{Mod}},$$

where θ is the composite

$$af^\# \xrightarrow{u} af^\# qa \xrightarrow{c} aqf^\# a \xrightarrow{\varepsilon} f^\# a,$$

see Lemma 2.32 below.

The counit $\varepsilon : f_\#^{\mathrm{Mod}} f^\#_{\mathrm{Mod}} \to \mathrm{Id}$ is the composite

$$f_\#^{\mathrm{Mod}} f^\#_{\mathrm{Mod}} = af_\#^{\mathrm{PM}} q f^\#_{\mathrm{Mod}} \xrightarrow{c} af_\#^{\mathrm{PM}} f^\#_{\mathrm{PM}} q \xrightarrow{\varepsilon} aq \xrightarrow{\varepsilon} \mathrm{Id}\,.$$

(2.21) If there is no confusion, the $\heartsuit$ attached to the functors of sheaves defined above are omitted. For example, $f^{\#}$ stands for $f^{\#}_{\heartsuit}$. Note that $f^{\#}_{\heartsuit}(\mathcal{F})$ viewed as a presheaf of abelian groups is independent of $\heartsuit$.

(2.22) Let $\mathbb{X}$ be a ringed site, and $x \in \mathbb{X}$. The category $\mathbb{X}/x$ is a site with the same topology as that of $\mathbb{X}$. The canonical functor $\mathfrak{R}_x : \mathbb{X}/x \to \mathbb{X}$ is a ringed continuous functor, and yields the pull-backs $(\mathfrak{R}_x)^{\#}_{\mathrm{AB}}$ and $(\mathfrak{R}_x)^{\#}_{\mathrm{PA}}$, which we denote by $(?)|_x^{\mathrm{AB}}$ and $(?)|_x^{\mathrm{PA}}$, respectively. Their left adjoints are denoted by L_x^{AB} and L_x^{PA}, respectively. Note that $\mathfrak{R}_x$ is admissible, see [31, p.78].

(2.23) Note that $\mathbb{X}/x$ is a ringed site with the structure sheaf $\mathcal{O}_{\mathbb{X}}|_x$. Thus, $(?)|_x^{\mathrm{Mod}}$ and $(?)|_x^{\mathrm{PM}}$ are defined in an obvious way, and their left adjoints L_x^{Mod} and L_x^{PM} are also defined. Note that L_x^{Mod} and L_x^{PM} are faithful and exact.

(2.24) For a morphism $\phi : x \to y$, we have an obvious admissible ringed continuous functor $\mathfrak{R}_\phi : \mathbb{X}/x \to \mathbb{X}/y$. The corresponding pull-back is denoted by $\phi^{\star}_{\heartsuit}$, and its left adjoint is denoted by $\phi_{\star}^{\heartsuit}$, where $\heartsuit$ is AB, PA, Mod or PM.

For $\mathcal{M}, \mathcal{N} \in \heartsuit(\mathbb{X})$, we define $\underline{\mathrm{Hom}}_{\heartsuit(\mathbb{X})}(\mathcal{M}, \mathcal{N})$ to be the object of $\heartsuit(\mathbb{X})$ given by

$$\Gamma(x, \underline{\mathrm{Hom}}_{\heartsuit(\mathbb{X})}(\mathcal{M}, \mathcal{N})) := \mathrm{Hom}_{\heartsuit(\mathbb{X}/x)}(\mathcal{M}|_x^{\heartsuit}, \mathcal{N}|_x^{\heartsuit}),$$

where $\heartsuit = \mathrm{PA}, \mathrm{AB}, \mathrm{PM}$, or Mod. For $\phi : x \to y$, the restriction map

$$\mathrm{Hom}_{\heartsuit(\mathbb{X}/y)}(\mathcal{M}|_y^{\heartsuit}, \mathcal{N}|_y^{\heartsuit}) \to \mathrm{Hom}_{\heartsuit(\mathbb{X}/x)}(\mathcal{M}|_x^{\heartsuit}, \mathcal{N}|_x^{\heartsuit})$$

is given by $\phi^{\star}_{\heartsuit}$. It is easy to see that if $\mathcal{N}$ is a sheaf, then $\underline{\mathrm{Hom}}_{\heartsuit(\mathbb{X})}(\mathcal{M}, \mathcal{N})$ is a sheaf. Note that $\underline{\mathrm{Hom}}_{\heartsuit(\mathbb{X})}(\mathcal{M}, \mathcal{N})$ is a functor from $\heartsuit(\mathbb{X})^{\mathrm{op}} \times \heartsuit(\mathbb{X})$ to $\heartsuit(\mathbb{X})$.

(2.25) Let $f \colon \mathbb{Y} \to \mathbb{X}$ be a ringed continuous functor, $y \in \mathbb{Y}$, and $\mathcal{U} = (y_i \to y)_{i \in I}$ a covering of y. Then $f\mathcal{U} = (fy_i \to fy)_{i \in I}$ is a covering of fy, since f is continuous, see Lemma 2.5. Let $\mathcal{M} \in \mathrm{PM}(\mathbb{X})$. Then we have a canonical isomorphism

$$\nu \colon \check{H}^0(\mathcal{U}, f^{\#}\mathcal{M}) \cong \check{H}^0(f\mathcal{U}, \mathcal{M}),$$

since the canonical map $f(y_i \times_y y_j) \to fy_i \times_{fy} fy_j$ is an isomorphism. This induces a natural map

$$\nu \colon \underline{\check{H}}^0 f^{\#}\mathcal{M} \to f^{\#}\, \underline{\check{H}}^0\, \mathcal{M}.$$

2.26 Lemma. *Let f and $\mathcal{M}$ be as above. Then the composite*

$$f^{\#}\mathcal{M} \xrightarrow{Y} \underline{\check{H}}^0\, f^{\#}\mathcal{M} \xrightarrow{\nu} f^{\#}\, \underline{\check{H}}^0\, \mathcal{M}$$

is $f^{\#}Y$.

Proof. This is straightforward, and left to the reader. □

2.27 Lemma. *Let f and $\mathcal{M}$ be as above. Then the composite*

$$\underline{\check{H}}^0\, f^{\#}\mathcal{M} \xrightarrow{\nu} f^{\#}\, \underline{\check{H}}^0\, \mathcal{M} \xrightarrow{Y} \underline{\check{H}}^0\, f^{\#}\, \underline{\check{H}}^0\, \mathcal{M}$$

agrees with $\underline{\check{H}}^0\, f^{\#}Y$.

Proof. This is proved quite similarly to Lemma 2.14, and we omit the proof. □

(2.28) Let $f\colon \mathbb{Y} \to \mathbb{X}$ be a ringed continuous functor, and $\mathcal{M} \in \mathrm{PM}(\mathbb{X})$. Then we define the natural map $\bar{\theta}\colon af^{\#}\mathcal{M} \to f^{\#}a\mathcal{M}$ to be the unique map such that $q\bar{\theta}$ is the composite

$$\begin{aligned} q\bar{\theta}\colon qaf^{\#}\mathcal{M} = \underline{\check{H}}^0\, \underline{\check{H}}^0\, f^{\#}\mathcal{M} &\xrightarrow{\underline{\check{H}}^0\,\nu} \underline{\check{H}}^0\, f^{\#}\, \underline{\check{H}}^0\, \mathcal{M} \\ &\xrightarrow{\nu} f^{\#}\, \underline{\check{H}}^0\, \underline{\check{H}}^0\, \mathcal{M} = f^{\#}qa\mathcal{M} \xrightarrow{c} qf^{\#}a\mathcal{M}. \end{aligned}$$

2.29 Lemma. *Let f and $\mathcal{M}$ be as above. Then the composite*

$$f^{\#}\mathcal{M} \xrightarrow{u} qaf^{\#}\mathcal{M} \xrightarrow{q\bar{\theta}} qf^{\#}a\mathcal{M} \xrightarrow{c} f^{\#}qa\mathcal{M}$$

is $f^{\#}u$.

Proof. Consider the diagram

$$\begin{tikzcd}
f^{\#}\mathcal{M} \arrow[r, "Y"] \arrow[dr, "f^{\#}Y"'] & hf^{\#}\mathcal{M} \arrow[r, "hY"] \arrow[d, "\nu"] \arrow[dr, "hf^{\#}Y"] & hhf^{\#}\mathcal{M} \arrow[d, "\nu"] \\
& f^{\#}h\mathcal{M} \arrow[r, "Y"] \arrow[dr, "f^{\#}Y"] & hf^{\#}h\mathcal{M} \arrow[d, "\nu"] \\
& & f^{\#}hh\mathcal{M},
\end{tikzcd}$$

where $h = \underline{\check{H}}^0$. The four triangles in the diagram commutes by Lemma 2.26 and Lemma 2.27. So the whole diagram commutes, and the lemma follows. □

(2.30) Let $f : \mathbb{Y} \to \mathbb{X}$ be a ringed continuous functor between ringed sites. The following is a restricted version of the results on cocontinuous functors in [43]. We give a proof for convenience of readers.

2.31 Lemma. *Assume that for any $y \in \mathbb{Y}$ and any covering $(x_\lambda \to fy)_{\lambda\in\Lambda}$ of fy, there is a covering $(y_\mu \to y)_{\mu\in M}$ of y such that there is a map $\phi : M \to \Lambda$ such that $fy_\mu \to fy$ factors through $x_{\phi(\mu)} \to fy$ for each μ. Then the pull-back $f^\#$ is compatible with the sheafification in the sense that the canonical natural transformation*

$$\bar{\theta} : a(\mathbb{Y}, \mathrm{Mod})f^\#_{\mathrm{PM}} \to f^\#_{\mathrm{Mod}}a(\mathbb{X}, \mathrm{Mod})$$

is a natural isomorphism. If this is the case, $f^\#_{\mathrm{Mod}}$ has the right adjoint $f^{\mathrm{Mod}}_\flat$, and in particular, it preserves arbitrary limits and arbitrary colimits.

Proof. Let $\mathcal{M} \in \mathrm{PM}(\mathbb{X})$ and $y \in \mathbb{Y}$. Recall that

$$\nu\colon\ \check{H}^0((y_i \to y)_{i\in I}, f^\#\mathcal{M}) \to \check{H}^0((fy_i \to fy)_{i\in I}, \mathcal{M})$$

is an isomorphism, and induces

$$\nu\colon\ \check{H}^0(y, f^\#\mathcal{M}) = \varinjlim \check{H}^0((y_i \to y)_{i\in I}, f^\#\mathcal{M}) \xrightarrow{\nu} \\ \varinjlim \check{H}^0((fy_i \to fy)_{i\in I}, \mathcal{M}) \to \varinjlim \check{H}^0((x_j \to fy)_{j\in J}, \mathcal{M}) = \check{H}^0(fy, \mathcal{M}).$$

This is also an isomorphism, since coverings of the form $(fy_i \to fy)$ is final in the category of all coverings of fy. By the definition of $\bar{\theta}$, $\bar{\theta}$ is an isomorphism.

Next we show that $f^\#_{\mathrm{Mod}}$ has a right adjoint. To prove this, it suffices to show that $f^{\mathrm{PM}}_\flat(\mathcal{M})$ is a sheaf if so is $\mathcal{M}$ for $\mathcal{M} \in \mathrm{PM}(\mathbb{X})$. Let $u : \mathrm{Id}_{\mathrm{PM}(\mathbb{X})} \to f^{\mathrm{PM}}_\flat f^\#_{\mathrm{PM}}$ be the unit of adjunction, $\varepsilon : f^\#_{\mathrm{PM}} f^{\mathrm{PM}}_\flat \to \mathrm{Id}_{\mathrm{PM}(\mathbb{Y})}$ the counit of adjunction, $v(\mathbb{X}) : \mathrm{Id}_{\mathrm{PM}(\mathbb{X})} \to q(\mathbb{X}, \mathrm{Mod})a(\mathbb{X}, \mathrm{Mod})$ the unit of adjunction, and $v(\mathbb{Y}) : \mathrm{Id}_{\mathrm{PM}(\mathbb{Y})} \to q(\mathbb{Y}, \mathrm{Mod})a(\mathbb{Y}, \mathrm{Mod})$ the unit of adjunction. Then the diagram of functors

$$\begin{tikzcd}
f^{\mathrm{PM}}_\flat \arrow[r, "uf^{\mathrm{PM}}_\flat"] \arrow[d, "v(\mathbb{X})f^{\mathrm{PM}}_\flat"] & f^{\mathrm{PM}}_\flat f^\#_{\mathrm{PM}} f^{\mathrm{PM}}_\flat \arrow[r, "\mathrm{id}"] \arrow[d, "f^{\mathrm{PM}}_\flat f^\#_{\mathrm{PM}} v(\mathbb{X}) f^{\mathrm{PM}}_\flat"] & f^{\mathrm{PM}}_\flat f^\#_{\mathrm{PM}} f^{\mathrm{PM}}_\flat \arrow[r, "f^{\mathrm{PM}}_\flat \varepsilon"] \arrow[d, "f^{\mathrm{PM}}_\flat v(\mathbb{Y}) f^\#_{\mathrm{PM}} f^{\mathrm{PM}}_\flat"] & f^{\mathrm{PM}}_\flat \arrow[d, "f^{\mathrm{PM}}_\flat v(\mathbb{Y})"] \\
qaf^{\mathrm{PM}}_\flat \arrow[r, "uqaf^{\mathrm{PM}}_\flat"] & f^{\mathrm{PM}}_\flat f^\#_{\mathrm{PM}} qaf^{\mathrm{PM}}_\flat \arrow[r, "\cong"] & f^{\mathrm{PM}}_\flat qaf^\#_{\mathrm{PM}} f^{\mathrm{PM}}_\flat \arrow[r, "f^{\mathrm{PM}}_\flat qa\varepsilon"] & f^{\mathrm{PM}}_\flat qa
\end{tikzcd}$$

is commutative, where $\cong$ is the *inverse* of the canonical map caused by $qaf^\#_{\mathrm{PM}} \xrightarrow{q\bar{\theta}} qf^\#_{\mathrm{Mod}}a \xrightarrow{c} f^\#_{\mathrm{PM}}qa$, which exists by the first part. As $(f^\#_{\mathrm{PM}}, f^{\mathrm{PM}}_\flat)$ is an adjoint pair, the composite of the first row of the diagram is the identity. As $v(\mathbb{Y})(\mathcal{M})$ is an isomorphism, the right-most vertical arrow evaluated at $\mathcal{M}$ is an isomorphism. Hence, $v(\mathbb{X})f_\flat^{\mathrm{PM}}(\mathcal{M})$, which is the left-most vertical arrow evaluated at $\mathcal{M}$, is a split monomorphism. As it is a direct summand of a sheaf, $f_\flat^{\mathrm{PM}}(\mathcal{M})$ is a sheaf, as desired.

As it is a right adjoint of $f_\#^{\mathrm{Mod}}$, the functor $f^\#_{\mathrm{Mod}}$ preserves arbitrary limits. As it is a left adjoint of $f_\flat^{\mathrm{Mod}}$, the functor $f^\#_{\mathrm{Mod}}$ preserves arbitrary colimits. □

2.32 Lemma. *Let $f\colon \mathbb{Y} \to \mathbb{X}$ be a ringed continuous functor. Then $\bar\theta\colon af^\#_{\mathrm{PM}} \to f^\#_{\mathrm{Mod}}a$ agrees with the composite*

$$\theta\colon af^\# \xrightarrow{u} af^\# qa \xrightarrow{c} aqf^\# a \xrightarrow{\varepsilon} f^\# a.$$

Proof. Consider the diagram

$$\begin{tikzcd}
f^\# \arrow[r, "u"] \arrow[d, "u"] \arrow[dr, phantom, "\text{(a)}"] & f^\# qa \arrow[r, "c"] \arrow[d, "u"] \arrow[dr, phantom, "\text{(b)}"] & qf^\# a \arrow[d, "u"] \arrow[dr, "\mathrm{id}"] \arrow[d, phantom, shift left=4, "\text{(c)}"] & \\
qaf^\# \arrow[r, "u"] & qaf^\# qa \arrow[r, "c"] & qaqf^\# a \arrow[r, "\varepsilon"] & qf^\# a\ .
\end{tikzcd}$$

(a) and (b) are commutative by the naturality of u. The commutativity of (c) is basics on adjunction. So the adjoint $q\theta \circ u$ of θ agrees with the composite

$$f^\# \xrightarrow{u} f^\# qa \xrightarrow{c} qf^\# a.$$

By Lemma 2.29, this agrees with the adjoint $q\bar\theta \circ u$ of $\bar\theta$. Since the adjoint maps agree, we have $\theta = \bar\theta$. □

2.33 Lemma. *Let $f : \mathbb{Y} \to \mathbb{X}$ be a ringed continuous functor. Then the composite*

$$f^\#_{\mathrm{PM}} \xrightarrow{u} qaf^\#_{\mathrm{PM}} \xrightarrow{\theta} qf^\#_{\mathrm{Mod}}a \xrightarrow{c} f^\#_{\mathrm{PM}}qa$$

agrees with u.

Proof. Follows from Lemma 2.29 and Lemma 2.32. □

2.34 Lemma. *Let $f : \mathbb{Y} \to \mathbb{X}$ be a ringed continuous functor. Then the conjugate of $c : qf^\#_{\mathrm{Mod}} \to f^\#_{\mathrm{PM}}q$ agrees with*

$$af_\#^{\mathrm{PM}} \xrightarrow{u} af_\#^{\mathrm{PM}}qa = f_\#^{\mathrm{Mod}}a. \tag{2.35}$$

In particular, being a conjugate of an isomorphism, (2.35) *is an isomorphism.*

Proof. Straightforward. □

(2.36) Let $\mathbb{X}$ be a ringed site, and $x \in \mathbb{X}$. It is easy to see that $\mathfrak{R}_x : \mathbb{X}/x \to \mathbb{X}$ satisfies the condition in Lemma 2.31. So $(?)|_x^{\mathrm{Mod}}$ preserves arbitrary limits and colimits. In particular, $(?)|_x^{\mathrm{Mod}}$ is exact. Similarly, for a morphism $\phi : x \to y$ in $\mathbb{X}$, $\phi^\star_{\mathrm{Mod}}$ preserves arbitrary limits and colimits.

(2.37) Let $\mathbb{X}$ be a ringed site, $\mathcal{M} \in \mathrm{PM}(\mathbb{X})$, and $\mathcal{N} \in \mathrm{Mod}(\mathbb{X})$. We define an isomorphism

$$\mathbb{V}\colon q\,\underline{\mathrm{Hom}}_{\mathrm{Mod}(\mathbb{X})}(a\mathcal{M},\mathcal{N}) \to \underline{\mathrm{Hom}}_{\mathrm{PM}(\mathbb{X})}(\mathcal{M}, q\mathcal{N})$$

as follows. For $x \in \mathbb{X}$, the map $\mathbb{V}$ at x is the composite

$$\begin{aligned}\mathbb{V}(x)\colon\ \mathrm{Hom}_{\mathrm{Mod}(\mathbb{X}/x)}((a\mathcal{M})|_x,\mathcal{N}|_x) &\xrightarrow{\bar\theta} \mathrm{Hom}_{\mathrm{Mod}(\mathbb{X}/x)}(a(\mathcal{M}|_x),\mathcal{N}|_x)\\ &\cong \mathrm{Hom}_{\mathrm{PM}(\mathbb{X}/x)}(\mathcal{M}|_x, q(\mathcal{N}|_x)) \xrightarrow{c} \mathrm{Hom}_{\mathrm{PM}(\mathbb{X}/x)}(\mathcal{M}|_x, (q\mathcal{N})|_x).\end{aligned}$$

The $\cong$ is an isomorphism coming from the adjunction. Note that $\bar\theta\colon a(\mathcal{M}|_x) \to (a\mathcal{M})|_x$ is an isomorphism by Lemma 2.31.

A morphism $\varphi\colon (a\mathcal{M})|_x \to \mathcal{N}|_x$ is mapped to the composite

$$\mathcal{M}|_x \xrightarrow{u|_x} (qa\mathcal{M})|_x \xrightarrow{c} q((a\mathcal{M})|_x) \xrightarrow{q\varphi} q(\mathcal{N}|_x) \xrightarrow{c} (q\mathcal{N})|_x.$$

Note that $\mathbb{V}$ is a map of presheaves, that is, $\mathbb{V}$ is compatible with the restriction. Indeed, for $\varphi \in \mathrm{Hom}_{\mathrm{Mod}(\mathbb{X}/x)}((a\mathcal{M})|_x,\mathcal{N}|_x)$ and $\phi : y \to x$, $\phi^\star\mathbb{V}\varphi$ is the composite

$$\begin{aligned}&\phi^\star(\mathcal{M}|_x) \xrightarrow{\phi^\star(u|_x)} \phi^\star((qa\mathcal{M})|_x) \xrightarrow{c}\\ &\qquad\qquad \phi^\star(q((a\mathcal{M})|_x)) \xrightarrow{\phi^\star(q\varphi)} \phi^\star(q(\mathcal{N}|_x)) \xrightarrow{c} \phi^\star((q\mathcal{N})|_x),\end{aligned}$$

which can be identified with the composite

$$\mathcal{M}|_y \xrightarrow{u|_y} (qa\mathcal{M})|_y \xrightarrow{c} q((a\mathcal{M})|_y) \xrightarrow{q(\phi^\star\varphi)} q(\mathcal{N}|_y) \xrightarrow{c} (q\mathcal{N})|_y.$$

This map is $\mathbb{V}\phi^\star\varphi$, and $\mathbb{V}$ is compatible with the restriction maps.

2.38 Lemma. *Let $\mathbb{X}$ be a ringed site, and $\mathcal{M},\mathcal{N} \in \mathrm{Mod}(\mathbb{X})$. Then the composite*

$$\bar H\colon q\,\underline{\mathrm{Hom}}_{\mathrm{Mod}(\mathbb{X})}(\mathcal{M},\mathcal{N}) \xrightarrow{\varepsilon} q\,\underline{\mathrm{Hom}}_{\mathrm{Mod}(\mathbb{X})}(aq\mathcal{M},\mathcal{N}) \xrightarrow{\mathbb{V}} \underline{\mathrm{Hom}}_{\mathrm{PM}(\mathbb{X})}(q\mathcal{M}, q\mathcal{N})$$

is given as follows. For $x \in \mathbb{X}$ and $\varphi \in \mathrm{Hom}_{\mathrm{Mod}(\mathbb{X}/x)}(\mathcal{M}|_x,\mathcal{N}|_x)$, $\bar H(x)(\varphi) \in \mathrm{Hom}_{\mathrm{PM}(\mathbb{X}/x)}((q\mathcal{M})|_x, (q\mathcal{N})|_x)$ is the composite

$$(q\mathcal{M})|_x \xrightarrow{c} q(\mathcal{M}|_x) \xrightarrow{q\varphi} q(\mathcal{N}|_x) \xrightarrow{c} (q\mathcal{N})|_x.$$

Proof. By the definition of $\mathbb{V}$, the map in question is the composite

$$(q\mathcal{M})|_x \xrightarrow{u} (qaq\mathcal{M})|_x \xrightarrow{c} q((aq\mathcal{M})|_x) \xrightarrow{\varepsilon} q(\mathcal{M}|_x) \xrightarrow{q\varphi} q(\mathcal{N}|_x) \xrightarrow{c} (q\mathcal{N})|_x.$$

It agrees with the composite

$$(q\mathcal{M})|_x \xrightarrow{u} (qaq\mathcal{M})|_x \xrightarrow{\varepsilon} (q\mathcal{M})|_x \xrightarrow{c} q(\mathcal{M}|_x) \xrightarrow{q\varphi} q(\mathcal{N}|_x) \xrightarrow{c} (q\mathcal{N})|_x.$$

Since $\varepsilon u = \mathrm{id}$, the assertion follows. □

(2.39) Let $\mathcal{M}, \mathcal{N} \in \mathrm{PM}(\mathbb{X})$. The composite

$$a \underline{\mathrm{Hom}}_{\mathrm{PM}(\mathbb{X})}(\mathcal{M}, \mathcal{N}) \xrightarrow{u} a \underline{\mathrm{Hom}}_{\mathrm{PM}(\mathbb{X})}(\mathcal{M}, qa\mathcal{N}) \xrightarrow{\mathbb{V}^{-1}} \\ aq \underline{\mathrm{Hom}}_{\mathrm{Mod}(\mathbb{X})}(a\mathcal{M}, a\mathcal{N}) \xrightarrow{\varepsilon} \underline{\mathrm{Hom}}_{\mathrm{Mod}(\mathbb{X})}(a\mathcal{M}, a\mathcal{N})$$

is denoted by $\bar{P}$.

2.40 Lemma. *Let $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be a ringed site. The category $\mathrm{PM}(\mathbb{X})$ is a closed symmetric monoidal category (see [29, (VII.7)]) with $\otimes^p_{\mathcal{O}_{\mathbb{X}}}$ the multiplication, $q\mathcal{O}_{\mathbb{X}}$ the unit object, $\underline{\mathrm{Hom}}_{\mathrm{PM}(\mathbb{X})}(?, ?)$ the internal hom, etc., etc.*

The proof of the lemma (including the precise statement) is straightforward, but we give some remarks on non-trivial natural maps.

(2.41) The evaluation map

$$\mathrm{ev}\colon \underline{\mathrm{Hom}}_{\mathrm{PM}(\mathbb{X})}(\mathcal{M}, \mathcal{N}) \otimes^p \mathcal{M} \to \mathcal{N}$$

at the section $\Gamma(x, ?)$,

$$\Gamma(x, \underline{\mathrm{Hom}}_{\mathrm{PM}(\mathbb{X})}(\mathcal{M}, \mathcal{N}) \otimes^p \mathcal{M}) = \\ \mathrm{Hom}_{\mathrm{Mod}(\mathbb{X}/x)}(\mathcal{M}|_x, \mathcal{N}|_x) \otimes_{\Gamma(x, \mathcal{O}_{\mathbb{X}})} \Gamma(x, \mathcal{M}) \to \Gamma(x, \mathcal{N}),$$

is given by $\varphi \otimes a \mapsto \varphi(\mathrm{id}_x)(a)$ for $\varphi \in \mathrm{Hom}_{\mathrm{Mod}(\mathbb{X}/x)}(\mathcal{M}|_x, \mathcal{N}|_x)$ and $a \in \Gamma(x, \mathcal{M})$.

(2.42) The trace map

$$\mathrm{tr}\colon \mathcal{M} \to \underline{\mathrm{Hom}}_{\mathrm{PM}(\mathbb{X})}(\mathcal{N}, \mathcal{M} \otimes^p_{\mathcal{O}_{\mathbb{X}}} \mathcal{N})$$

at the section $\Gamma(x, ?)$,

$$\Gamma(x, \mathcal{M}) \to \mathrm{Hom}_{\mathrm{PM}(\mathbb{X}/x)}(\mathcal{N}|_x, (\mathcal{M} \otimes^p_{\mathcal{O}_{\mathbb{X}}} \mathcal{N})|_x),$$

maps $\alpha \in \Gamma(x, \mathcal{M})$ to the map

$$\mathrm{tr}(x)(\alpha) \in \mathrm{Hom}_{\mathrm{PM}(\mathbb{X}/x)}(\mathcal{N}|_x, (\mathcal{M} \otimes^p_{\mathcal{O}_\mathbb{X}} \mathcal{N})|_x)$$

as follows. For $\phi\colon x' \to x$, $\beta \in \Gamma(\phi, \mathcal{N}|_x) = \Gamma(x', \mathcal{N})$ is mapped to

$$\mathrm{res}_{x',x}(\alpha) \otimes \beta \in \Gamma(x', \mathcal{M}) \otimes_{\Gamma(x', \mathcal{O}_\mathbb{X})} \Gamma(x', \mathcal{N}) = \Gamma(\phi, (\mathcal{M} \otimes^p_{\mathcal{O}_\mathbb{X}} \mathcal{N})|_x)$$

by $\mathrm{tr}(x)(\alpha)$.

2.43 Lemma. *The category* $\mathrm{Mod}(\mathbb{X})$ *is a closed symmetric monoidal category with* $\otimes_{\mathcal{O}_\mathbb{X}}$ *the multiplication,* $\mathcal{O}_\mathbb{X}$ *the unit object,* $\underline{\mathrm{Hom}}_{\mathrm{Mod}(\mathbb{X})}(?, ?)$ *the internal hom, etc., etc.*

The precise statement and the proof is left to the reader. We only remark the following.

(2.44) Let $\mathcal{M}, \mathcal{N}, \mathcal{P} \in \mathrm{Mod}(\mathbb{X})$. Then the associativity morphism

$$\alpha\colon (\mathcal{M} \otimes \mathcal{N}) \otimes \mathcal{P} \to \mathcal{M} \otimes (\mathcal{N} \otimes \mathcal{P})$$

is the composite

$$\begin{aligned}(\mathcal{M} \otimes \mathcal{N}) \otimes \mathcal{P} = a(qa(q\mathcal{M} \otimes^p q\mathcal{N}) \otimes^p q\mathcal{P}) &\xrightarrow{u} a(qa(q\mathcal{M} \otimes^p q\mathcal{N}) \otimes^p qaq\mathcal{P}) \\ \xrightarrow{\bar{\Delta}^{-1}} a((q\mathcal{M} \otimes^p q\mathcal{N}) \otimes^p q\mathcal{P}) &\xrightarrow{a\alpha'} a(q\mathcal{M} \otimes^p (q\mathcal{N} \otimes^p q\mathcal{P})) \\ &\xrightarrow{u} a(q\mathcal{M} \otimes^p qa(q\mathcal{N} \otimes^p q\mathcal{P})) = \mathcal{M} \otimes (\mathcal{N} \otimes \mathcal{P}),\end{aligned}$$

where $\bar{\Delta}^{-1}$ is the inverse of the map $\bar{\Delta}$, see Lemma 2.18, and α' is the associativity morphism for presheaves.

(2.45) The left unit isomorphism $\lambda\colon \mathcal{O}_\mathbb{X} \otimes \mathcal{M} \to \mathcal{M}$ is defined to be the composite

$$\mathcal{O}_\mathbb{X} \otimes \mathcal{M} = a(q\mathcal{O}_\mathbb{X} \otimes^p q\mathcal{M}) \xrightarrow{a\lambda'} aq\mathcal{M} \xrightarrow{\varepsilon} \mathcal{M},$$

where λ' is the left unit isomorphism for presheaves.

(2.46) The twisting isomorphism

$$\gamma\colon \mathcal{M} \otimes \mathcal{N} \to \mathcal{N} \otimes \mathcal{M}$$

is nothing but

$$\mathcal{M} \otimes \mathcal{N} = a(q\mathcal{M} \otimes^p q\mathcal{N}) \xrightarrow{a\gamma'} a(q\mathcal{N} \otimes^p q\mathcal{M}) = \mathcal{N} \otimes \mathcal{M},$$

where γ' is the twisting map for presheaves.

(2.47) The natural map ev is the composite

$$\underline{\mathrm{Hom}}_{\mathrm{Mod}(\mathbb{X})}(\mathcal{M},\mathcal{N}) \otimes_{\mathcal{O}_{\mathbb{X}}} \mathcal{M} = a(q\,\underline{\mathrm{Hom}}_{\mathrm{Mod}(\mathbb{X})}(\mathcal{M},\mathcal{N}) \otimes^p q\mathcal{M}) \xrightarrow{\bar{H}} a(\underline{\mathrm{Hom}}_{\mathrm{PM}(\mathbb{X})}(q\mathcal{M}, q\mathcal{N}) \otimes^p q\mathcal{M}) \xrightarrow{a\,\mathrm{ev}'} aq\mathcal{M} \xrightarrow{\varepsilon} \mathcal{M},$$

where ev' is the evaluation map for presheaves.

(2.48) The natural map tr is the composite

$$\begin{aligned}\mathcal{M} \xrightarrow{\varepsilon^{-1}} aq\mathcal{M} \xrightarrow{\mathrm{tr}'} a\,\underline{\mathrm{Hom}}_{\mathrm{PM}(\mathbb{X})}(q\mathcal{N}, q\mathcal{M} \otimes^p q\mathcal{N}) \\ \xrightarrow{\bar{P}} \underline{\mathrm{Hom}}_{\mathrm{Mod}(\mathbb{X})}(aq\mathcal{N}, a(q\mathcal{M} \otimes^p q\mathcal{N})) \\ = \underline{\mathrm{Hom}}_{\mathrm{Mod}(\mathbb{X})}(aq\mathcal{N}, \mathcal{M} \otimes \mathcal{N}) \xrightarrow{\varepsilon^{-1}} \underline{\mathrm{Hom}}_{\mathrm{Mod}(\mathbb{X})}(\mathcal{N}, \mathcal{M} \otimes \mathcal{N}),\end{aligned}$$

where tr' is the trace map for presheaves.

2.49 Lemma. *The inclusion* $q : \mathrm{Mod}(\mathbb{X}) \to \mathrm{PM}(\mathbb{X})$ *and the natural transformations*

$$m \colon q\mathcal{M} \otimes^p_{\mathcal{O}_{\mathbb{X}}} q\mathcal{N} \xrightarrow{u} qa(q\mathcal{M} \otimes^p_{\mathcal{O}_{\mathbb{X}}} q\mathcal{N}) = q(\mathcal{M} \otimes_{\mathcal{O}_{\mathbb{X}}} \mathcal{N})$$

and

$$\eta \colon q\mathcal{O}_{\mathbb{X}} \xrightarrow{\mathrm{id}} q\mathcal{O}_{\mathbb{X}}$$

form a symmetric monoidal functor, see [26, (3.4.2)].

Letting $\mathcal{S}$ be the connected category with two objects and one non-trivial morphism, (a, q) forms a Lipman monoidal adjoint pair. The map H (see for the definition, (1.34)*) agrees with $\bar{H}$ (see* Lemma 2.38*). The map P (see for the definition,* (1.50)*) agrees with $\bar{P}$ (see* (2.39)*). The map Δ (see* (1.40)*) agrees with $\bar{\Delta}$ (see* Lemma 2.18*).*

Proof. We prove the first assertion. First we prove that the diagram

$$\begin{array}{ccc} q\mathcal{O}_{\mathbb{X}} \otimes^p q\mathcal{M} & \xrightarrow{m} & q(\mathcal{O}_{\mathbb{X}} \otimes \mathcal{M}) \\ \uparrow {\scriptstyle \eta \otimes^p 1} & & \downarrow {\scriptstyle q\lambda} \\ q\mathcal{O}_{\mathbb{X}} \otimes^p q\mathcal{M} & \xrightarrow{\lambda'} & q\mathcal{M} \end{array}$$

is commutative for $\mathcal{M} \in \mathrm{Mod}(\mathbb{X})$. This is trivial from the definition of the sheaf tensor product (2.12), the definition of λ (2.45), and the commutativity of the diagram

$$\begin{tikzcd}
q\mathcal{O}_{\mathbb{X}} \otimes^p q\mathcal{M} \arrow[r, "u"] \arrow[d, "\lambda'"] & qa(q\mathcal{O}_{\mathbb{X}} \otimes q\mathcal{M}) \arrow[d, "qa\lambda'"] \\
q\mathcal{M} \arrow[r, "u"] \arrow[dr, "\mathrm{id}"] & qaq\mathcal{M} \arrow[d, "\varepsilon"] \\
& q\mathcal{M} \ .
\end{tikzcd}$$

Next we prove that the diagram

$$\begin{array}{ccc}
q\mathcal{M} \otimes^p q\mathcal{N} & \xrightarrow{m} & q(\mathcal{M} \otimes \mathcal{N}) \\
\downarrow \gamma' & & \downarrow q\gamma \\
q\mathcal{N} \otimes^p q\mathcal{M} & \xrightarrow{m} & q(\mathcal{N} \otimes \mathcal{M})
\end{array}$$

is commutative for $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\mathbb{X})$. By the definition of m and γ (2.46), the diagram is nothing but

$$\begin{array}{ccc}
q\mathcal{M} \otimes^p q\mathcal{N} & \xrightarrow{u} & qa(q\mathcal{M} \otimes^p q\mathcal{N}) \\
\downarrow \gamma' & & \downarrow qa\gamma' \\
q\mathcal{N} \otimes^p q\mathcal{M} & \xrightarrow{u} & qa(q\mathcal{N} \otimes^p q\mathcal{M})
\end{array} ,$$

which is commutative by the naturality of u.

To prove that q is a symmetric monoidal functor, it remains to prove that

$$\begin{tikzcd}
(q\mathcal{M} \otimes^p q\mathcal{N}) \otimes^p q\mathcal{P} \arrow[r, "\alpha'"] \arrow[d, "m"] & q\mathcal{M} \otimes^p (q\mathcal{N} \otimes^p q\mathcal{P}) \arrow[d, "m"] \\
q(\mathcal{M} \otimes \mathcal{N}) \otimes^p q\mathcal{P} \arrow[d, "m"] & q\mathcal{M} \otimes^p q(\mathcal{N} \otimes \mathcal{P}) \arrow[d, "m"] \\
q((\mathcal{M} \otimes \mathcal{N}) \otimes \mathcal{P}) \arrow[r, "q\alpha"] & q(\mathcal{M} \otimes (\mathcal{N} \otimes \mathcal{P}))
\end{tikzcd}$$

is commutative, where α' is the associativity map for presheaves. By the definition of m, the diagram equals

$$\begin{tikzcd}
(q\mathcal{M} \otimes^p q\mathcal{N}) \otimes^p q\mathcal{P} \arrow[r, "\alpha'"] \arrow[d, "u"] & q\mathcal{M} \otimes^p (q\mathcal{N} \otimes^p q\mathcal{P}) \arrow[d, "u"] \\
qa(q\mathcal{M} \otimes^p q\mathcal{N}) \otimes^p q\mathcal{P} \arrow[d, "u"] & q\mathcal{M} \otimes^p qa(q\mathcal{N} \otimes^p q\mathcal{P}) \arrow[d, "u"] \\
qa(qa(q\mathcal{M} \otimes^p q\mathcal{N}) \otimes^p q\mathcal{P}) \arrow[r, "q\alpha"] & qa(q\mathcal{M} \otimes^p qa(q\mathcal{N} \otimes^p q\mathcal{P})) \ .
\end{tikzcd}$$

By the naturality of u, the commutativity of this diagram is reduced to the commutativity of

$$\begin{tikzcd}
(q\mathcal{M} \otimes^p q\mathcal{N}) \otimes^p q\mathcal{P} \arrow[r, "\alpha'"] \arrow[d, "u"] \arrow[dr, phantom, "\text{(a)}"] & q\mathcal{M} \otimes^p (q\mathcal{N} \otimes^p q\mathcal{P}) \arrow[d, "u"] \\
qa((q\mathcal{M} \otimes^p q\mathcal{N}) \otimes^p q\mathcal{P}) \arrow[r, "qa\alpha'"] \arrow[d, "u"] \arrow[dr, phantom, "\text{(b)}"] & qa(q\mathcal{M} \otimes^p (q\mathcal{N} \otimes^p q\mathcal{P})) \arrow[d, "u"] \\
qa(qa(q\mathcal{M} \otimes^p q\mathcal{N}) \otimes^p q\mathcal{P}) \arrow[r, "q\alpha"] & qa(q\mathcal{M} \otimes^p qa(q\mathcal{N} \otimes^p q\mathcal{P}))\ .
\end{tikzcd}$$

The commutativity of (a) is obvious by the naturality of u. The commutativity of (b) follows from the definition of α (2.44). We have proved that q is a symmetric monoidal functor.

Next we prove that Δ agrees with $\bar{\Delta}$. By definition (1.40), Δ is the composite

$$a(\mathcal{M} \otimes^p \mathcal{N}) \xrightarrow{\bar{\Delta}} a(qa\mathcal{M} \otimes^p qa\mathcal{N}) \xrightarrow{u} aqa(qa\mathcal{M} \otimes^p qa\mathcal{N}) \xrightarrow{\varepsilon} a(qa\mathcal{M} \otimes^p qa\mathcal{N}),$$

which agrees with $\bar{\Delta}$.

Next we prove that $H = \bar{H}$. For $b, c \in \mathrm{Mod}(\mathbb{X})$, the diagram

$$\begin{tikzcd}
q[b,c] \arrow[r, "\mathrm{tr}'"] \arrow[d, "\bar{H}"] \arrow[dr, phantom, "\text{(a)}"] & {[qb, q[b,c] \otimes^p qb]} \arrow[r, "u"] \arrow[d, "\bar{H}"] \arrow[dr, phantom, "\text{(b)}"] & {[qb, qa(q[b,c] \otimes^p qb)]} \arrow[d, "\bar{H}"] \\
{[qb, qc]} \arrow[r, "\mathrm{tr}'"] \arrow[dr, "\mathrm{id}"] & {[qb, [qb,qc] \otimes^p qb]} \arrow[r, "u"] \arrow[d, "\mathrm{ev}'"] \arrow[dr, phantom, "\text{(c)}"] & {[qb, qa([qb,qc] \otimes^p qb)]} \arrow[d, "\mathrm{ev}'"] \\
& {[qb, qc]} \arrow[r, "u"] \arrow[dr, "\mathrm{id}"] & {[qb, qaqc]} \arrow[d, "\varepsilon"] \\
& & {[qb, qc]}
\end{tikzcd}$$

is commutative, where $[d, e]$ stands for $\underline{\mathrm{Hom}}(d, e)$. Indeed, (a) is commutative by the naturality of tr', and (b) and (c) are commutative by the naturality of u. The commutativity of the two triangles are obvious.

By the definition of H (1.34) and ev (2.47), the composite $\varepsilon\, \mathrm{ev}'\, \bar{H} u\, \mathrm{tr}'$ agrees with H. By the commutativity of the diagram, we have that $H = \bar{H}$.

Now we prove that (a, q) forms a Lipman adjoint pair. That is, Δ and C are isomorphisms. Since $\bar{\Delta}$ is an isomorphism by Lemma 2.18 and $\Delta = \bar{\Delta}$, Δ is an isomorphism. Note that $C\colon aq\mathcal{O}_{\mathbb{X}} \to \mathcal{O}_{\mathbb{X}}$ is nothing but ε by definition. Since q is fully faithful, ε is an isomorphism (apply [19, Lemma I.1.2.6, **4**] for the adjoint pair $(q^{\mathrm{op}}, a^{\mathrm{op}})$), and we are done. So the definition of P makes sense.

We prove that $P = \bar{P}$. For $b, c \in \mathrm{PM}(\mathbb{X})$, the diagram

$$
\begin{tikzcd}
 & & & a[b,c] \arrow[rr, "\mathrm{id}"] \arrow[drr, phantom, "(a)"] & & a[b,c] \arrow[d, "u=\varepsilon^{-1}"] \arrow[dddddll, bend left=20, "\mathrm{tr}"] \\
 & a[b,c] \arrow[urr, "\mathrm{id}"] \arrow[rr, "\mathrm{tr}'"] \arrow[dl, swap, "\mathrm{id}"] \arrow[d, "\mathrm{tr}'"] \arrow[drr, phantom, "(c)"] & & a[qab,[b,c]\otimes^p qab] \arrow[d, "u"] \arrow[drr, "u"] & & aqa[b,c] \arrow[d, "\mathrm{tr}'"] & (d) \\
a[b,c] \arrow[d, "\bar{P}"] \arrow[dr, phantom, "(e)"] & a[b,[b,c]\otimes^p b] \arrow[l, swap, "\mathrm{ev}'"] \arrow[ul, phantom, "(b)"] \arrow[rr, "u"] \arrow[drr, swap, "u\otimes^p u"] & & a[b,[b,c]\otimes^p qab] \arrow[d, "u"] \arrow[dl, phantom, "(f)"] & (g) & a[qab,qa[b,c]\otimes^p qab] \arrow[dll, swap, "u"] \arrow[d, "\bar{P}"] \\
[ab,ac] & {[ab,a([b,c]\otimes^p b)]} \arrow[l, swap, "\mathrm{ev}'"] \arrow[drr, "u\otimes^p u"] \arrow[d, phantom, "(j)"] & (h) & a[b,qa[b,c]\otimes^p qab] \arrow[d, "\bar{P}"] & (i) & {[aqab,a(qa[b,c]\otimes^p qab)]} \arrow[dll, "u=\varepsilon^{-1}"] \\
 & {[ab,a([b,c]\otimes^p b)]} \arrow[ul, "\mathrm{ev}'"] & & {[ab,a(qa[b,c]\otimes^p qab)]} \arrow[ll, swap, "(u\otimes^p u)^{-1}"] \arrow[d, "\mathrm{id}"] \arrow[dll, phantom, "(k)"] & & \\
 & {[ab,a([b,c]\otimes^p b)]} \arrow[u, "\mathrm{id}"] & & {[ab,a[b,c]\otimes ab]} \arrow[ll, swap, "\Delta^{-1}"] & &
\end{tikzcd}
$$

is commutative. Note that $\varepsilon\colon aqa \to a$ has an inverse, which must agree with u. The commutativity of (a) is the naturality of tr'. The commutativity of (b) is the basics on adjunction. The commutativity of (c) is Lemma 1.32. The commutativity of (d) is the definition of tr, see (2.48). The commutativity of (e), (h) and (i) is the naturality of $\bar{P}$. The commutativity of (f), (g) and (j) is trivial. The commutativity of (k) is the definition of $\bar{\Delta}$. Thus the diagram is commutative, and we have $\bar{P} = P$, by the definition of P. $\square$

(2.50) Let $f\colon \mathbb{Y} \to \mathbb{X}$ be a ringed continuous functor. For $\mathcal{M}, \mathcal{N} \in \mathrm{PM}(\mathbb{X})$, we define

$$m = m_{\mathrm{PM}}(f)\colon f^{\#}_{\mathrm{PM}}\mathcal{M} \otimes^p_{\mathcal{O}_{\mathbb{Y}}} f^{\#}_{\mathrm{PM}}\mathcal{N} \to f^{\#}_{\mathrm{PM}}(\mathcal{M} \otimes^p_{\mathcal{O}_{\mathbb{X}}} \mathcal{N})$$

by

$$\begin{aligned}\Gamma(y, f^{\#}_{\mathrm{PM}}\mathcal{M} \otimes^p_{\mathcal{O}_{\mathbb{Y}}} f^{\#}_{\mathrm{PM}}\mathcal{N}) &= \Gamma(fy, \mathcal{M}) \otimes_{\Gamma(y,\mathcal{O}_{\mathbb{Y}})} \Gamma(fy, \mathcal{N}) \\ &\to \Gamma(fy, \mathcal{M}) \otimes_{\Gamma(fy,\mathcal{O}_{\mathbb{X}})} \Gamma(fy, \mathcal{N}) = \Gamma(y, f^{\#}_{\mathrm{PM}}(\mathcal{M} \otimes^p_{\mathcal{O}_{\mathbb{X}}} \mathcal{N})) \\ &\qquad\qquad (m \otimes n \mapsto m \otimes n)\end{aligned}$$

for each $y \in \mathbb{Y}$.

We also define $\eta = \eta_{\mathrm{PM}}(f)$ to be the canonical map

$$q\mathcal{O}_{\mathbb{Y}} \to q f^{\#}_{\mathrm{Mod}} \mathcal{O}_{\mathbb{X}} \xrightarrow{c} f^{\#}_{\mathrm{PM}} q\mathcal{O}_{\mathbb{X}}.$$

2.51 Lemma. *The functor $f^{\#}_{\mathrm{PM}}$ together with m and η above forms a symmetric monoidal functor.*

Proof. Consider the diagram

$$\begin{array}{ccc} f^{\#}q\mathcal{O}_{\mathbb{X}} \otimes^p f^{\#}\mathcal{M} & \xrightarrow{m} & f^{\#}(q\mathcal{O}_{\mathbb{X}} \otimes^p \mathcal{M}) \\ \uparrow {\scriptstyle \eta \otimes^p 1} & & \downarrow {\scriptstyle f^{\#}\lambda} \\ q\mathcal{O}_{\mathbb{Y}} \otimes^p f^{\#}\mathcal{M} & \xrightarrow{\lambda} & f^{\#}\mathcal{M} \end{array},$$

whose commutativity we need to prove. For $y \in \mathbb{Y}$, applying $\Gamma(y, ?)$ to this diagram, we get

$$\begin{array}{ccc} \Gamma(fy, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y,\mathcal{O}_{\mathbb{Y}})} \Gamma(fy, \mathcal{M}) & \to & \Gamma(fy, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(fy,\mathcal{O}_{\mathbb{X}})} \Gamma(fy, \mathcal{M}) \\ \uparrow & & \downarrow \\ \Gamma(y, \mathcal{O}_{\mathbb{Y}}) \otimes_{\Gamma(y,\mathcal{O}_{\mathbb{Y}})} \Gamma(fy, \mathcal{M}) & \to & \Gamma(fy, \mathcal{M}) \end{array}.$$

By the bottom horizontal arrow, $a \otimes m \in \Gamma(y, \mathcal{O}_{\mathbb{Y}}) \otimes_{\Gamma(y,\mathcal{O}_{\mathbb{Y}})} \Gamma(fy, \mathcal{M})$ goes to $\eta(a)m$. We get the same result when we keep track the other path in the diagram. So the diagram in question is commutative.

The rest of the proof is similar, and we leave it to the reader. $\square$

(2.52) Let $f\colon \mathbb{Y}\to\mathbb{X}$ be a ringed continuous functor as above. We define $m = m_{\mathrm{Mod}}(f)$ to be the composite map

$$f^\#_{\mathrm{Mod}}\mathcal{M}\otimes_{\mathcal{O}_{\mathbb{Y}}} f^\#_{\mathrm{Mod}}\mathcal{N} = a(qf^\#_{\mathrm{Mod}}\mathcal{M}\otimes^p_{\mathcal{O}_{\mathbb{Y}}} qf^\#_{\mathrm{Mod}}\mathcal{N}) \xrightarrow{c} a(f^\#_{\mathrm{PM}}q\mathcal{M}\otimes^p_{\mathcal{O}_{\mathbb{Y}}} f^\#_{\mathrm{PM}}q\mathcal{N})$$
$$\xrightarrow{\text{via } m_{\mathrm{PM}}} af^\#_{\mathrm{PM}}(q\mathcal{M}\otimes^p_{\mathcal{O}_{\mathbb{X}}} q\mathcal{N}) \xrightarrow{\text{via } \theta} f^\#_{\mathrm{Mod}}a(q\mathcal{M}\otimes^p_{\mathcal{O}_{\mathbb{X}}} q\mathcal{N}) = f^\#_{\mathrm{Mod}}(\mathcal{M}\otimes_{\mathcal{O}_{\mathbb{X}}}\mathcal{N}),$$

where $\theta : af^\#_{\mathrm{PM}}\to f^\#_{\mathrm{Mod}}a$ is the composite map

$$af^\#_{\mathrm{PM}} \xrightarrow{\text{via } u} af^\#_{\mathrm{PM}}qa \xrightarrow{c} aqf^\#_{\mathrm{Mod}}a \xrightarrow{\text{via } \varepsilon} f^\#_{\mathrm{Mod}}a.$$

We define $\eta = \eta_{\mathrm{Mod}}(f)$ to be the given map of sheaves of rings $\mathcal{O}_{\mathbb{Y}}\to f^\#_{\mathrm{Mod}}\mathcal{O}_{\mathbb{X}}$.

2.53 Lemma. *Let the notation be as above. Then the functor $f^\#_{\mathrm{Mod}}\colon$ Mod$(\mathbb{X})\to$ Mod$(\mathbb{Y})$, together with m and η, is a symmetric monoidal functor.*

Proof. The diagram

$$\begin{array}{ccccccc} af^\#q & \xrightarrow{u} & af^\#qaq & \xrightarrow{c} & aqf^\#aq & \xrightarrow{\varepsilon} & f^\#aq \\ & \searrow \mathrm{id} & \downarrow \varepsilon & & \downarrow \varepsilon & & \downarrow\varepsilon \\ & & af^\#q & \xrightarrow{c} & aqf^\# & \xrightarrow{\varepsilon} & f^\# \end{array} \tag{2.54}$$

is commutative. For $\mathcal{M}\in$ Mod$(\mathbb{X})$, consider the diagram

$$\begin{array}{ccccc} a(f^\#q\mathcal{O}_{\mathbb{X}}\otimes^p f^\#q\mathcal{M}) & \xrightarrow{m_{\mathrm{PM}}} & af^\#(q\mathcal{O}_{\mathbb{X}}\otimes^p q\mathcal{M}) & \xrightarrow{\theta} & f^\#a(q\mathcal{O}_{\mathbb{X}}\otimes^p q\mathcal{M}) \\ \uparrow \eta_{\mathrm{PM}} & \text{(a)} & \downarrow\lambda & \text{(b)} & \downarrow\lambda \\ a(q\mathcal{O}_{\mathbb{Y}}\otimes^p f^\#q\mathcal{M}) & \xrightarrow{\lambda} & af^\#q\mathcal{M} & \xrightarrow{\theta} & f^\#aq\mathcal{M} \\ \downarrow c & \text{(c)} & \downarrow c & \text{(d)} & \downarrow\varepsilon \\ a(q\mathcal{O}_{\mathbb{Y}}\otimes^p qf^\#\mathcal{M}) & \xrightarrow{\lambda} & aqf^\#\mathcal{M} & \xrightarrow{\varepsilon} & f^\#\mathcal{M} \end{array}.$$

(a) is commutative by Lemma 2.51. (b) is commutative by the naturality of θ. (c) is commutative by the naturality of λ. (d) is commutative by the commutativity of (2.54). So the whole diagram is commutative.

This shows that the diagram

$$\begin{array}{ccc} f^\#_{\mathrm{Mod}}\mathcal{O}_{\mathbb{X}}\otimes f^\#_{\mathrm{Mod}}\mathcal{M} & \xrightarrow{m_{\mathrm{Mod}}} & f^\#_{\mathrm{Mod}}(\mathcal{O}_{\mathbb{X}}\otimes\mathcal{M}) \\ \uparrow\eta & & \downarrow\lambda \\ \mathcal{O}_{\mathbb{Y}}\otimes f^\#_{\mathrm{Mod}}\mathcal{M} & \xrightarrow{\lambda} & f^\#_{\mathrm{Mod}}\mathcal{M} \end{array}$$

is commutative.

The other axioms are checked similarly. The details are left to the reader. $\square$

Now the following is easy to prove.

2.55 Lemma. *Let $\mathcal{S}'$ denote the category of ringed categories and ringed functors. Then $((?)^{\mathrm{PM}}_{\#}, (?)^{\#}_{\mathrm{PM}})$ is an adjoint pair of monoidal almost-pseudofunctors on $(\mathcal{S}')^{\mathrm{op}}$. Let $\mathcal{S}$ denote the category of ringed sites and ringed continuous functors. Then $((?)^{\mathrm{Mod}}_{\#}, (?)^{\#}_{\mathrm{Mod}})$ is an adjoint pair of monoidal almost-pseudofunctors on $\mathcal{S}^{\mathrm{op}}$, see* (1.40).

(2.56) We give some comments on Lemma 2.55. Let $g : \mathbb{Z} \to \mathbb{Y}$ and $f : \mathbb{Y} \to \mathbb{X}$ be morphisms in $\mathcal{S}$. Then $c : (fg)^{\#}_{\heartsuit} \to g^{\#}_{\heartsuit} f^{\#}_{\heartsuit}$ is the identity map for $\heartsuit = \mathrm{PM}, \mathrm{Mod}$. For $z \in \mathbb{Z}$,

$$\Gamma(z, (fg)^{\#}\mathcal{M}) = \Gamma(fgz, \mathcal{M}) = \Gamma(gz, f^{\#}\mathcal{M}) = \Gamma(z, g^{\#} f^{\#}\mathcal{M}).$$

In particular, the diagram

$$\begin{array}{ccccc} q(fg)^{\#} & & \xrightarrow{\quad c \quad} & & (fg)^{\#}q \\ \downarrow c & & & & \downarrow c \\ qf^{\#}g^{\#} & \xrightarrow{c} & f^{\#}qg^{\#} & \xrightarrow{c} & f^{\#}g^{\#}q \end{array}$$

is commutative.

Note also that $(\mathrm{id}_X)^{\#} : \heartsuit(X) \to \heartsuit(X)$ is the identity functor, and $\mathfrak{e}_X : \mathrm{Id} \to (\mathrm{id}_X)^{\#}$ is the identity.

A straightforward computation shows that $d : f^{\mathrm{PM}}_{\#} g^{\mathrm{PM}}_{\#} \to (fg)^{\mathrm{PM}}_{\#}$ is given by

$$\Gamma(x, f_{\#}g_{\#}\mathcal{M}) = \varinjlim \Gamma(x, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y,\mathcal{O}_{\mathbb{Y}})} \varinjlim \Gamma(y, \mathcal{O}_{\mathbb{Y}}) \otimes_{\Gamma(z,\mathcal{O}_{\mathbb{Z}})} \Gamma(z, \mathcal{M}) \to \varinjlim \Gamma(x, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(z,\mathcal{O}_{\mathbb{Z}})} \Gamma(z, \mathcal{M}) = \Gamma(x, (fg)_{\#}\mathcal{M}),$$

where $\to$ is given by $a \otimes b \otimes m \mapsto ab \otimes m$.

A straightforward diagram chasing (using Lemma 2.33) shows that $d : f^{\mathrm{Mod}}_{\#} g^{\mathrm{Mod}}_{\#} \to (fg)^{\mathrm{Mod}}_{\#}$ agrees with the composite

$$f^{\mathrm{Mod}}_{\#} g^{\mathrm{Mod}}_{\#} = af^{\mathrm{PM}}_{\#} qag^{\mathrm{PM}}_{\#} q \xrightarrow{u^{-1}} af^{\mathrm{PM}}_{\#} g^{\mathrm{PM}}_{\#} q \xrightarrow{d} a(fg)^{\mathrm{PM}}_{\#} q = (fg)_{\#},$$

where u^{-1} is the inverse of the isomorphism $u : af_{\#} \to af_{\#}qa$, see Lemma 2.34.

(2.57) Let

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{g'} & \mathbb{X}' \\ \uparrow f & & \uparrow f' \\ \mathbb{Y} & \xrightarrow{g} & \mathbb{Y}' \end{array} \tag{2.58}$$

be a commutative diagram of ringed sites and ringed (continuous) functors. Then the natural map $\theta : g_\# f^\# \to (f')^\# g'_\#$ is defined, see (1.21). Using the explicit description of the unit map $u : 1 \to (g')^\# g'_\#$ and the counit map $\varepsilon : g_\# g^\# \to 1$ in (2.20) and $c : f^\#(g')^\# \cong g^\#(f')^\#$ in (2.56), it is straightforward to check the following. For $\mathcal{M} \in \mathrm{PM}(\mathbb{X})$ and $y' \in \mathbb{Y}$,

$$\begin{aligned}\Gamma(y', \theta) : \Gamma(y', g_\# f^\# \mathcal{M}) &= \varinjlim_{y' \to gy} \Gamma(y', \mathcal{O}_{\mathbb{Y}'}) \otimes_{\Gamma(y, \mathcal{O}_{\mathbb{Y}})} \Gamma(fy, \mathcal{M}) \\ \to \Gamma(y', (f')^\# g'_\# \mathcal{M}) &= \varinjlim_{f'y' \to g'x} \Gamma(f'y', \mathcal{O}_{\mathbb{X}'}) \otimes_{\Gamma(x, \mathcal{O}_{\mathbb{X}})} \Gamma(x, \mathcal{M})\end{aligned}$$

is induced by the map

$$\Gamma(y', \mathcal{O}_{\mathbb{Y}'}) \otimes_{\Gamma(y, \mathcal{O}_{\mathbb{Y}})} \Gamma(fy, \mathcal{M}) \to \Gamma(f'y', \mathcal{O}_{\mathbb{X}'}) \otimes_{\Gamma(fy, \mathcal{O}_{\mathbb{X}})} \Gamma(fy, \mathcal{M}) \qquad (a \otimes m \mapsto a \otimes m)$$

for $(y' \to gy) \in (I^g_{y'})^{\mathrm{op}}$.

θ_{Mod} is described by θ_{PM} as follows.

2.59 Lemma. *Let* (2.58) *be a commutative diagram of ringed sites and ringed continuous functors. Then* $\theta_{\mathrm{Mod}} : g_\# f^\# \to (f')^\# g'_\#$ *is the composite*

$$g_\# f^\# = a g_\# q f^\# \xrightarrow{c} q g_\# f^\# q \xrightarrow{\theta_{\mathrm{PM}}} a (f')^\# g'_\# q \xrightarrow{\theta} (f')^\# a g'_\# q = (f')^\# g'_\#.$$

Proof. Left to the reader as an exercise (utilize Lemma 2.60). □

2.60 Lemma. *Let* $f\colon \mathbb{Y} \to \mathbb{X}$ *be a ringed continuous functor. Then the diagram of functors* $\mathrm{PM}(\mathbb{X}) \to \mathrm{PM}(\mathbb{Y})$

$$\begin{array}{ccc} f^\# & \xrightarrow{u} & f^\# q a \\ \downarrow u & & \downarrow c(f) \\ q a f^\# & \xrightarrow{\theta} & q f^\# a \end{array}$$

is commutative.

Proof. Left to the reader as an exercise. □

(2.61) Let $\mathcal{S}'$ and $\mathcal{S}$ be as in Lemma 2.55. Then the monoidal adjoint pairs $((?)_{\#}^{\mathrm{PM}}, (?)^{\#}_{\mathrm{PM}})$ and $((?)_{\#}^{\mathrm{Mod}}, (?)^{\#}_{\mathrm{Mod}})$ are not Lipman, see (6.9).

2.62 Lemma. *Let $\mathcal{A}$ be an abelian category which satisfies the* (AB3) *condition, I a small category, and $((a_\lambda)_{\lambda\in I}, (\varphi_f)_{f\in \mathrm{Mor}(I)})$ a direct system in $\mathcal{A}$. Assume that I has an initial object λ_0, and φ_f is an isomorphism for any $f \in \mathrm{Mor}(I)$. Then $a_\lambda \to \varinjlim a_\lambda$ is an isomorphism for any λ.*

Proof. It suffices to show that $a_{\lambda_0} \to \varinjlim a_\lambda$ has an inverse. For each λ, consider $\varphi_{f(\lambda)}^{-1} : a_\lambda \to a_{\lambda_0}$, where $f(\lambda)$ is the unique map $\lambda_0 \to \lambda$. Then the collection $(\varphi_{f(\lambda)}^{-1})$ induces a morphism $\varinjlim a_\lambda \to a_{\lambda_0}$. This gives the desired inverse. □

2.63 Lemma. *Let $\mathbb{Y}$ and $\mathbb{X}$ be ringed categories, and $f : \mathbb{Y} \to \mathbb{X}$ be a ringed functor. Assume that for each $x \in \mathbb{X}$, I_x^f has a terminal object. Then $C : f_{\#}\mathcal{O}_{\mathbb{Y}} \to \mathcal{O}_{\mathbb{X}}$ is an isomorphism.*

Proof. Let $x \in \mathbb{X}$. Then $\Gamma(x, f_{\#}\mathcal{O}_{\mathbb{Y}}) = \varinjlim \Gamma(x, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y,\mathcal{O}_{\mathbb{Y}})} \Gamma(y, \mathcal{O}_{\mathbb{Y}})$, where the colimit is taken over $(I_x^f)^{\mathrm{op}}$, which has an initial object $(y_0, (h : x \to fy_0))$ by assumption. By Lemma 2.62, the canonical maps

$$\Gamma(x, \mathcal{O}_{\mathbb{X}}) \cong \Gamma(x, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y_0,\mathcal{O}_{\mathbb{Y}})} \Gamma(y_0, \mathcal{O}_{\mathbb{Y}}) \to \Gamma(x, f_{\#}\mathcal{O}_{\mathbb{Y}})$$

are isomorphisms. So C is an isomorphism, as can be seen easily. □

2.64 Corollary. *Let $f : \mathbb{Y} \to \mathbb{X}$ be a morphism of ringed sites. If I_x^f has a terminal object for $x \in \mathbb{X}$, then $C : f_{\#}^{\mathrm{Mod}}\mathcal{O}_{\mathbb{Y}} \to \mathcal{O}_{\mathbb{X}}$ is an isomorphism.*

Proof. Note that C is the composite

$$f_{\#}^{\mathrm{Mod}}\mathcal{O}_{\mathbb{Y}} = af_{\#}q\mathcal{O}_{\mathbb{Y}} \xrightarrow{C'} aq\mathcal{O}_{\mathbb{X}} \xrightarrow{\varepsilon} \mathcal{O}_{\mathbb{X}},$$

where $C' : f_{\#}q\mathcal{O}_{\mathbb{Y}} \to q\mathcal{O}_{\mathbb{X}}$ is the C associated with the ringed functor $(\mathbb{Y}, q\mathcal{O}_{\mathbb{Y}}) \to (\mathbb{X}, q\mathcal{O}_{\mathbb{X}})$. Note that C' is an isomorphism by the lemma. As $\varepsilon : aq \to 1$ is also an isomorphism, C is also an isomorphism, as desired. □

2.65 Corollary. *Let $f : X \to Y$ be a morphism of ringed spaces. Then $C : f^*\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism.*

Proof. For each open subset U of X, $I_U^{f^{-1}}$ has the terminal object $(Y, (U \hookrightarrow X))$. □

2.66 Corollary. *Let $f : \mathbb{Y} \to \mathbb{X}$ be a morphism of ringed sites, and $y \in \mathbb{Y}$. Then we may consider the induced morphism of ringed sites $f/y : \mathbb{Y}/y \to \mathbb{X}/fy$. The canonical map $C : (f/y)_\#(\mathcal{O}_{\mathbb{Y}}|_y) \to \mathcal{O}_{\mathbb{X}}|_x$ is an isomorphism.*

Proof. For $\varphi : x \to fy$ in $\mathbb{X}/fy$, $I_\varphi^{f/y}$ has a terminal object (id_y, φ). $\square$

3 Derived categories and derived functors of sheaves on ringed sites

We utilize the notation and terminology on triangulated categories in [44]. However, we usually write the suspension (translation) functor of a triangulated category by Σ or $(?)[1]$.

Let $\mathcal{T}$ be a triangulated category.

3.1 Lemma. *Let*

$$(a_\lambda \xrightarrow{f_\lambda} b_\lambda \xrightarrow{g_\lambda} c_\lambda \xrightarrow{h_\lambda} \Sigma a_\lambda)$$

be a small family of distinguished triangles in $\mathcal{T}$. Assume that the coproducts $\bigoplus a_\lambda$, $\bigoplus b_\lambda$, and $\bigoplus c_\lambda$ exist. Then the triangle

$$\bigoplus a_\lambda \xrightarrow{\bigoplus f_\lambda} \bigoplus b_\lambda \xrightarrow{\bigoplus g_\lambda} \bigoplus c_\lambda \xrightarrow{H \circ \bigoplus h_\lambda} \Sigma(\bigoplus a_\lambda)$$

is distinguished, where $H : \bigoplus \Sigma a_\lambda \to \Sigma(\bigoplus a_\lambda)$ is the canonical isomorphism. Similarly, a product of distinguished triangles is a distinguished triangle.

We refer the reader to [36, Proposition 1.2.1] for the proof of the second assertion. The proof for the first assertion is similar [36, Remark 1.2.2].

(3.2) Let $\mathcal{A}$ be an abelian category. The category of unbounded (resp. bounded below, bounded above, bounded) complexes in $\mathcal{A}$ is denoted by $C(\mathcal{A})$ (resp. $C^+(\mathcal{A})$, $C^-(\mathcal{A})$, $C^b(\mathcal{A})$). The corresponding homotopy category and the derived category are denoted by $K^?(\mathcal{A})$ and $D^?(\mathcal{A})$, where ? is either $\emptyset$ (i.e., nothing), $+$, $-$ or b. The localization $K^?(\mathcal{A}) \to D^?(\mathcal{A})$ is denoted by Q. We denote the homotopy category of complexes in $\mathcal{A}$ with unbounded (resp. bounded below, bounded above, bounded) cohomology groups by $\bar{K}^?(\mathcal{A})$. The corresponding derived category is denoted by $\bar{D}^?(\mathcal{A})$.

For a plump subcategory $\mathcal{A}'$ of $\mathcal{A}$, we denote by $K^?_{\mathcal{A}'}(\mathcal{A})$ (resp. $\bar{K}^?_{\mathcal{A}'}(\mathcal{A})$) the full subcategory of $K^?(\mathcal{A})$ (resp. $\bar{K}^?(\mathcal{A})$) consisting of complexes whose cohomology groups are objects of $\mathcal{A}'$. The localization of $K^?_{\mathcal{A}'}(\mathcal{A})$ by the épaisse subcategory (see for the definition, [44, Chapitre 1, §2, (1.1)]) of exact complexes is denoted by $D^?_{\mathcal{A}'}(\mathcal{A})$. The category $\bar{D}^?_{\mathcal{A}'}(\mathcal{A})$ is defined similarly. Note that the canonical functor $D^?_{\mathcal{A}'}(\mathcal{A}) \to D(\mathcal{A})$ is fully faithful, and hence $D^?_{\mathcal{A}'}(\mathcal{A})$ is identified with the full subcategory of $D(\mathcal{A})$ consisting of unbounded (resp. bounded below, bounded above, bounded) complexes whose cohomology groups are in $\mathcal{A}'$. Note also that the canonical functor $D^?_{\mathcal{A}'}(\mathcal{A}) \to \bar{D}^?_{\mathcal{A}'}(\mathcal{A})$ is an equivalence.

(3.3) Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and $F : K^?(\mathcal{A}) \to K^*(\mathcal{B})$ a triangulated functor. Let $\mathcal{C}$ be a triangulated subcategory of $K^?(\mathcal{A})$ such that

1 If $c \in \mathcal{C}$ is exact, then Fc is exact.

2 For any $a \in K^?(\mathcal{A})$, there exists some quasi-isomorphism $a \to c$.

The condition **2** implies that the canonical functor

$$i(\mathcal{C}) : \mathcal{C}/(\mathcal{E} \cap \mathcal{C}) \to K^?(\mathcal{A})/\mathcal{E} = D^?(\mathcal{A})$$

is an equivalence, where $\mathcal{E}$ denotes the épaisse subcategory of exact complexes in $K^?(\mathcal{A})$, see [44, Chapitre 2, §1, (2.3)]. We fix a quasi-inverse $p(\mathcal{C}) : D^?(\mathcal{A}) \to \mathcal{C}/(\mathcal{E} \cap \mathcal{C})$. On the other hand, the composite

$$\mathcal{C} \hookrightarrow K^?(\mathcal{A}) \xrightarrow{F} K^*(\mathcal{B}) \xrightarrow{Q} D^*(\mathcal{B}) \tag{3.4}$$

is factorized as

$$\mathcal{C} \xrightarrow{Q_{\mathcal{C}}} \mathcal{C}/(\mathcal{E} \cap \mathcal{C}) \xrightarrow{\mathfrak{F}} D^*(\mathcal{B}) \tag{3.5}$$

up to a unique natural isomorphism, by the universality of localization and the condition **1** above. Under the setting above, we have the following [17, (I.5.1)].

3.6 Lemma. *The composite functor*

$$RF : D^?(\mathcal{A}) \xrightarrow{p(\mathcal{C})} \mathcal{C}/(\mathcal{E} \cap \mathcal{C}) \xrightarrow{\mathfrak{F}} D^*(\mathcal{B})$$

is a right derived functor of F.

For more about the existence of a derived functor, see [26, (2.2)]. We denote the map $QF \to (RF)Q$ in the definition of RF (see [17, p.51]) by Ξ or $\Xi(F)$.

(3.7) Here we are going to review Spaltenstein's work on unbounded derived categories [39].

A chain complex I of $\mathcal{A}$ is called K-*injective* if for any exact sequence E of $\mathcal{A}$, the complex of abelian groups $\mathrm{Hom}^\bullet_{\mathcal{A}}(E, I)$ is also exact.

A morphism $f : C \to I$ in $K(\mathcal{A})$ is called a K-*injective resolution* of C, if I is K-injective and f is a quasi-isomorphism.

The following is pointed out in [9].

3.8 Lemma. *Let $\mathcal{A}$ be an abelian category, and $\mathbb{I} \in C(\mathcal{A})$. Then the following are equivalent.*

1 *$\mathbb{I}$ is K-injective, and $\mathbb{I}^n$ is an injective object of $\mathcal{A}$ for each $n \in \mathbb{Z}$.*

2 *For an exact sequence $0 \to \mathbb{A} \to \mathbb{B} \to \mathbb{C} \to 0$ in $C(\mathcal{A})$ with $\mathbb{C}$ exact, any chain map $\mathbb{A} \to \mathbb{I}$ lifts to $\mathbb{B}$.*

Proof. **1⇒2**. The sequence of complexes of abelian groups

$$0 \to \mathrm{Hom}^\bullet_{\mathcal{A}}(\mathbb{C}, \mathbb{I}) \to \mathrm{Hom}^\bullet_{\mathcal{A}}(\mathbb{B}, \mathbb{I}) \to \mathrm{Hom}^\bullet_{\mathcal{A}}(\mathbb{A}, \mathbb{I}) \to 0$$

is exact, since each term of $\mathbb{I}$ is injective. So

$$H^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(\mathbb{B}, \mathbb{I})) \to H^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(\mathbb{A}, \mathbb{I})) \to H^1(\mathrm{Hom}^\bullet_{\mathcal{A}}(\mathbb{C}, \mathbb{I}))$$

is exact. But $H^1(\mathrm{Hom}^\bullet_{\mathcal{A}}(\mathbb{C}, \mathbb{I})) = 0$, since $\mathbb{C}$ is exact and $\mathbb{I}$ is K-injective. So $H^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(\mathbb{B}, \mathbb{I})) \to H^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(\mathbb{A}, \mathbb{I}))$ is surjective. Since $\mathrm{Hom}^{-1}_{\mathcal{A}}(\mathbb{B}, \mathbb{I}) \to \mathrm{Hom}^{-1}_{\mathcal{A}}(\mathbb{A}, \mathbb{I})$ is surjective, the commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}^{-1}_{\mathcal{A}}(\mathbb{B}, \mathbb{I}) \arrow[r, "\partial"] \arrow[d] & Z^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(\mathbb{B}, \mathbb{I})) \arrow[r] \arrow[d] & H^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(\mathbb{B}, \mathbb{I})) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}^{-1}_{\mathcal{A}}(\mathbb{A}, \mathbb{I}) \arrow[r, "\partial"] & Z^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(\mathbb{A}, \mathbb{I})) \arrow[r] & H^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(\mathbb{A}, \mathbb{I})) \arrow[r] & 0
\end{tikzcd}$$

shows that $Z^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(\mathbb{B}, \mathbb{I})) \to Z^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(\mathbb{A}, \mathbb{I}))$ is surjective. This is what we wanted to prove.

2⇒1 First we prove that $\mathbb{I}$ is K-injective. It suffices to show that for any exact complex $\mathbb{F}$, any chain map $\varphi : \mathbb{F} \to \mathbb{I}$ is null-homotopic. Let $\mathbb{C} = \mathrm{Cone}(\varphi)$, where Cone denotes the mapping cone. Consider the exact sequence

$$0 \to \mathbb{I} \to \mathbb{C} \to \mathbb{F}[1] \to 0.$$

So the identity map $\mathbb{I} \to \mathbb{I}$ lifts to $\psi : \mathbb{C} \to \mathbb{I}$. Let s be the restriction of ψ to $\mathbb{F}[1] \subset \mathbb{C}$. It is easy to see that $\varphi = sd + ds$. So φ is null-homotopic, as desired.

Next we show that $\mathbb{I}^n$ is injective for any n. To prove this, let $f : A \to B$ be a monomorphism in $\mathcal{A}$, and $\varphi : A \to \mathbb{I}^n$ a morphism. Let C be the cokernel of f. Define a complex $\mathbb{A}$ by $\mathbb{A}^n = \mathbb{A}^{n+1} = A$, $d^n_{\mathbb{A}} = \mathrm{id}$, and $\mathbb{A}^i = 0$ $(i \neq n, n+1)$. Replacing A by B and C, we define the complexes $\mathbb{B}$ and $\mathbb{C}$, respectively. Define $f^\bullet : \mathbb{A} \to \mathbb{B}$ by $f^n = f^{n+1} = f$ and $f^i = 0$ for $i \neq n, n+1$. Obviously, $\operatorname{Coker} f^\bullet \cong \mathbb{C}$ is exact.

Define a chain map $\Phi : \mathbb{A} \to \mathbb{I}$ by $\Phi^n = \varphi$ and $\Phi^{n+1} = d^n_{\mathbb{I}} \circ \varphi$. By assumption, there is a chain map $\Psi : \mathbb{B} \to \mathbb{I}$ such that $\Phi = \Psi f^\bullet$. So $\varphi = \Phi^n = \Psi^n \circ f^n = \Psi^n \circ f$, and Ψ^n lifts φ. $\square$

For $\mathbb{I} \in C(\mathcal{A})$, we say that $\mathbb{I}$ is *strictly injective* if $\mathbb{I}$ satisfies the equivalent conditions in the lemma. A *strictly injective resolution* is a quasi-isomorphism $\mathbb{F} \to \mathbb{I}$ with $\mathbb{I}$ strictly injective. The following is proved in [9]. See also [39] and [1].

3.9 Lemma. *If $\mathcal{A}$ is Grothendieck, then for any chain complex $\mathbb{F} \in C(\mathcal{A})$ admits a strictly injective resolution $\mathbb{F} \to \mathbb{I}$ which is a monomorphism.*

A chain complex I is K-injective if and only if $K(\mathcal{A})(E, I) = 0$ for any exact sequence E. It is easy to see that the K-injective complexes form an épaisse subcategory $I(\mathcal{A})$ of $K(\mathcal{A})$.

(3.10) Let $F : K(\mathcal{A}) \to K(\mathcal{B})$ be a triangulated functor, and assume that $\mathcal{A}$ is Grothendieck. Let $\mathcal{I}$ be the full subcategory of K-injective complexes of $K(\mathcal{A})$. It is easy to see that $\mathcal{I}$ is triangulated, and $\mathcal{I} \cap \mathcal{E} = 0$. By Lemma 3.6, the composite

$$D(\mathcal{A}) \xrightarrow{p(\mathcal{I})} \mathcal{I} \xrightarrow{\mathfrak{F}} D(\mathcal{B})$$

is a right derived functor RF of F. Note that to fix $p(\mathcal{I})$ and the isomorphism $\mathrm{Id}_{D(\mathcal{A})} \to i(\mathcal{I})p(\mathcal{I})$ is nothing but to fix a functorial K-injective resolution $\mathbb{F} \to pQ\mathbb{F} = \mathbb{I}_{\mathbb{F}}$ in $K(\mathcal{A})$.

(3.11) Let $F : K(\mathcal{A}) \to K(\mathcal{B})$ be a triangulated functor. Assume that $\mathcal{A}$ is Grothendieck. For $\mathbb{F} \in K(\mathcal{A})$, $\mathbb{F}$ is (right) *F-acyclic* (more precisely, $Q \circ F$-acyclic, where $Q : K(\mathcal{B}) \to D(\mathcal{B})$ is the localization. See for the definition, [26, (2.2.5)]) if and only if for some K-injective resolution $\mathbb{F} \to \mathbb{I}$, $F(\mathbb{F}) \to F(\mathbb{I})$ is a quasi-isomorphism, if and only if for any K-injective resolution

$\mathbb{F} \to \mathbb{I}$, $F(\mathbb{F}) \to F(\mathbb{I})$ is a quasi-isomorphism. Note that the set of F-acyclic objects in $K(\mathcal{A})$ forms a localizing subcategory of $K(\mathcal{A})$, see [26, (2.2.5.1)].

3.12 Lemma. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and $F : \mathcal{A} \to \mathcal{B}$ an exact functor with the right adjoint G. Assume that $\mathcal{B}$ is Grothendieck. Then $KG : K(\mathcal{B}) \to K(\mathcal{A})$ preserves K-injective complexes. Moreover, $RG : D(\mathcal{B}) \to D(\mathcal{A})$ is the right adjoint of $LF = F$.*

Proof. Let $\mathbb{M} \in K(\mathcal{A})$, and $\mathbb{I}$ a K-injective complex of $K(\mathcal{B})$. Then

$$\begin{aligned}\operatorname{Hom}_{K(\mathcal{A})}(\mathbb{M}, (KG)\mathbb{I}) &\cong H^0(\operatorname{Hom}^\bullet_{\mathcal{A}}(\mathbb{M}, G\mathbb{I})) \\ &\cong H^0(\operatorname{Hom}^\bullet_{\mathcal{B}}(F\mathbb{M}, \mathbb{I})) = \operatorname{Hom}_{K(\mathcal{B})}(F\mathbb{M}, \mathbb{I}).\end{aligned}$$

If $\mathbb{M}$ is exact, then the last group is zero. This shows $(KG)\mathbb{I}$ is K-injective.

Now let $\mathbb{M} \in D(\mathcal{A})$ and $\mathbb{N} \in D(\mathcal{B})$ be arbitrary. Then by the first part, we have a functorial isomorphism

$$\begin{aligned}\operatorname{Hom}_{D(\mathcal{A})}(\mathbb{M}, (RG)\mathbb{N}) &\cong \operatorname{Hom}_{K(\mathcal{A})}(\mathbb{M}, (KG)\mathbb{I}_{\mathbb{N}}) \\ &\cong \operatorname{Hom}_{K(\mathcal{B})}(F\mathbb{M}, \mathbb{I}_{\mathbb{N}}) \cong \operatorname{Hom}_{D(\mathcal{B})}(F\mathbb{M}, \mathbb{N}).\end{aligned}$$

This proves the last assertion. □

3.13 Remark. Note that for an abelian category $\mathcal{A}$, we have $\operatorname{ob}(C(\mathcal{A})) = \operatorname{ob}(K(\mathcal{A})) = \operatorname{ob}(D(\mathcal{A}))$. Thus, an object of one of the three categories is sometimes viewed as an object of another.

(3.14) Let $\mathcal{A}$ be a closed symmetric monoidal abelian category which satisfies the (AB3) and (AB3*) conditions. Let $\otimes$ be the multiplication and $[?, ?]$ be the internal hom. For a fixed $b \in \mathcal{A}$, $(? \otimes b, [b, ?])$ is an adjoint pair. In particular, $? \otimes b$ preserves colimits, and $[b, ?]$ preserves limits. By symmetry, $a \otimes ?$ also preserves colimits. As we have an isomorphism

$$\mathcal{A}(a, [b, c]) \cong \mathcal{A}(a \otimes b, c) \cong \mathcal{A}(b \otimes a, c) \cong \mathcal{A}(b, [a, c]) \cong \mathcal{A}^{\mathrm{op}}([a, c], b),$$

we have $[?, c] : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$ is right adjoint to $[?, c] : \mathcal{A} \to \mathcal{A}^{\mathrm{op}}$. This shows that $[?, c]$ changes colimits to limits.

As in [17], we define the tensor product $\mathbb{F} \otimes^\bullet \mathbb{G}$ of $\mathbb{F}, \mathbb{G} \in C(\mathcal{A})$ by

$$(\mathbb{F} \otimes^\bullet \mathbb{G})^n := \bigoplus_{p+q=n} \mathbb{F}^p \otimes \mathbb{G}^q.$$

The differential d^n on $\mathbb{F}^p \otimes \mathbb{G}^q$ is defined to be

$$d^n = d_{\mathbb{F}} \otimes 1 + (-1)^p 1 \otimes d_{\mathbb{G}}$$

(the sign convention is slightly different from [17], but this is not essential).

We have $\mathbb{F} \otimes^\bullet \mathbb{G} \in C(\mathcal{A})$. Similarly, $[\mathbb{F}, \mathbb{G}]^\bullet \in C(\mathcal{A})$ is defined by

$$[\mathbb{F}, \mathbb{G}]^n := \prod_{p \in \mathbb{Z}} [\mathbb{F}^p, \mathbb{G}^{n+p}]$$

and

$$d^n := [d_{\mathbb{F}}, 1] + (-1)^{n+1}[1, d_{\mathbb{G}}].$$

It is straightforward to prove the following.

3.15 Lemma. *Let $\mathcal{A}$ be as above. Then the category of chain complexes $C(\mathcal{A})$ is closed symmetric monoidal with $\otimes^\bullet$ the multiplication and $[?, ?]^\bullet$ the internal hom. The bi-triangulated functors*

$$\otimes^\bullet : K(\mathcal{A}) \times K(\mathcal{A}) \to K(\mathcal{A})$$

and

$$[?, ?]^\bullet : K(\mathcal{A})^{\mathrm{op}} \times K(\mathcal{A}) \to K(\mathcal{A}),$$

are induced, and $K(\mathcal{A})$ is a closed symmetric monoidal triangulated category (*see* [26, (3.5), (3.6)]).

(3.16) Let $\mathcal{A}$ be an abelian category, and $\mathfrak{P}$ a full subcategory of $C(\mathcal{A})$. An inverse system $(\mathbb{F}_i)_{i \in I}$ in $C(\mathcal{A})$ is said to be $\mathfrak{P}$-*special* if the following conditions are satisfied.

i I is well-ordered.

ii If $i \in I$ has no predecessor, then the canonical map $I_i \to \varprojlim_{j<i} I_j$ is an isomorphism (in particular, $I_{i_0} = 0$ if i_0 is the minimum element of I).

iii If $i \in I$ has a predecessor $i-1$, then the natural chain map $I_i \to I_{i-1}$ is an epimorphism, the kernel C_i is isomorphic to some object of $\mathfrak{P}$, and the exact sequence

$$0 \to C_i \to I_i \to I_{i-1} \to 0$$

is semi-split.

Similarly, $\mathfrak{P}$-special direct systems are also defined, see [39].

The full subcategory of $C(\mathcal{A})$ consisting of inverse (resp. direct) limits of $\mathfrak{P}$-special inverse (resp. direct) systems is denoted by $\underleftarrow{\mathfrak{P}}$ (resp. $\underrightarrow{\mathfrak{P}}$).

(3.17) Let $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be a ringed site. Various definitions and results on unbounded complexes of sheaves over a ringed space by Spaltenstein [39] is generalized to those for ringed sites. However, note that we can not utilize the notion related to closed subsets, points, or stalks of sheaves.

(3.18) We say that a complex $\mathbb{F} \in C(\mathrm{Mod}(\mathbb{X}))$ is *K-flat* if $\mathbb{G} \otimes^\bullet \mathbb{F}$ is exact whenever $\mathbb{G}$ is an exact complex in $\mathrm{Mod}(\mathbb{X})$. We say that $\mathbb{A} \in C(\mathrm{Mod}(\mathbb{X}))$ is *weakly K-injective* if $\mathbb{A}$ is $\mathrm{Hom}^\bullet_{\mathrm{Mod}(\mathbb{X})}(\mathbb{F}, ?)$-acyclic for any K-flat complex $\mathbb{F}$.

(3.19) Let $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be a ringed site. For $x \in \mathbb{X}$, we define $\mathcal{O}_x^p$ to be $L_x^{\mathrm{PM}}((q\mathcal{O}_{\mathbb{X}})|_x)$, and $\mathcal{O}_x := L_x^{\mathrm{Mod}}(\mathcal{O}_{\mathbb{X}}|_x) \cong a\mathcal{O}_x^p$. We denote by $\mathfrak{P}_0 = \mathfrak{P}_0(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ the full subcategory of $C(\mathrm{Mod}(\mathbb{X}))$ consisting of complexes of the form $\mathcal{O}_x[n]$ with $x \in \mathbb{X}$. We define $\mathfrak{P} = \mathfrak{P}(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ to be $\underrightarrow{\mathfrak{P}_0}$. We call an object of $\mathfrak{P}$ a *strongly K-flat* complex. We also define $\mathfrak{Q}$ to be the full subcategory of $C(\mathrm{Mod}(\mathbb{X}))$ consisting of bounded above complexes whose terms are direct sums of copies of $\mathcal{O}_x$.

We say that $\mathbb{A} \in C(\mathrm{Mod}(\mathbb{X}))$ is *K-limp* if $\mathbb{A}$ is $\mathrm{Hom}^\bullet_{\mathrm{Mod}(\mathbb{X})}(\mathbb{F}, ?)$-acyclic for any strongly K-flat complex $\mathbb{F}$.

3.20 Lemma. *Let $f : (\mathbb{Y}, \mathcal{O}_{\mathbb{Y}}) \to (\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be a ringed continuous functor. Then we have an isomorphism*

$$f_\#^{\mathrm{Mod}}(\mathcal{O}_y) \cong \mathcal{O}_{fy}$$

for $y \in \mathbb{Y}$. In particular, if $\mathbb{F} \in \mathfrak{P}(\mathbb{Y})$, then $f_\#\mathbb{F} \in \mathfrak{P}(\mathbb{X})$.

Proof. For $y \in \mathbb{Y}$, we denote the canonical continuous ringed functor

$$(\mathbb{Y}/y, \mathcal{O}_{\mathbb{Y}}|_y) \to (\mathbb{X}/fy, \mathcal{O}_{\mathbb{X}}|_{fy})$$

by f/y. We have $\mathfrak{R}_{fy} \circ (f/y) = f \circ \mathfrak{R}_y$. Hence by Corollary 2.66,

$$f_\#\mathcal{O}_y = f_\# L_y(\mathcal{O}_{\mathbb{Y}}|_y) \cong L_{fy}(f/y)_\#(\mathcal{O}_{\mathbb{Y}}|_y) \cong L_{fy}(\mathcal{O}_{\mathbb{X}}|_{fy}) = \mathcal{O}_{fy}.$$

$\square$

3.21 Lemma. *Let $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be a ringed site, and $\mathbb{F}, \mathbb{G} \in C(\mathrm{Mod}(\mathbb{X}))$. Then the following hold:*

1 *$\mathbb{F}$ is K-flat if and only if $\underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(\mathbb{X})}(\mathbb{F}, \mathbb{I})$ is K-injective for any K-injective complex $\mathbb{I}$.*

2 *If* $\mathbb{F}$ *is* K*-flat exact, then* $\mathbb{G} \otimes^\bullet_{\mathcal{O}_\mathbb{X}} \mathbb{F}$ *is exact.*

3 *The inductive limit of a pseudo-filtered inductive system of* K*-flat complexes is again* K*-flat.*

4 *The tensor product of two* K*-flat complexes is again* K*-flat.*

See [39] for the proof. For **2**, utilize **3** of Lemma 3.25 and Corollary 3.23 below.

3.22 Proposition. *Let* $(\mathbb{X}, \mathcal{O}_\mathbb{X})$ *be a ringed site, and* $x \in \mathbb{X}$*. Then* $\mathcal{O}_x$ *is* K*-flat.*

Proof. It suffices to show that for any exact complex $\mathcal{E}$, $\mathcal{E} \otimes \mathcal{O}_x$ is exact. To verify this, it suffices to show that for any K-injective complex $\mathcal{I}$, the complex $\operatorname{Hom}^\bullet_{\mathcal{O}_\mathbb{X}}(\mathcal{E} \otimes \mathcal{O}_x, \mathcal{I})$ is exact. Indeed, then if we consider the K-injective resolution $\mathcal{E} \otimes \mathcal{O}_x \to \mathcal{I}$, it must be null-homotopic and thus $\mathcal{E} \otimes \mathcal{O}_x$ must be exact.

Note that we have

$$\operatorname{Hom}^\bullet_{\mathcal{O}_\mathbb{X}}(\mathcal{E} \otimes \mathcal{O}_x, \mathcal{I}) \cong \operatorname{Hom}^\bullet_{\mathcal{O}_\mathbb{X}}(\mathcal{O}_x, \underline{\operatorname{Hom}}^\bullet_{\mathcal{O}_\mathbb{X}}(\mathcal{E}, \mathcal{I})) \cong \\ \operatorname{Hom}^\bullet_{\operatorname{Mod}(\mathbb{X}/x)}(\mathcal{E}|_x, \mathcal{I}|_x) \cong \operatorname{Hom}^\bullet_{\mathcal{O}_\mathbb{X}}(L_x(\mathcal{E}|_x), \mathcal{I}).$$

As $(?)|_x$ and L_x are exact (2.36), (2.23), the last complex is exact, and we are done. □

3.23 Corollary. *A strongly* K*-flat complex is* K*-flat.*

Proof. Follows immediately from the proposition. □

(3.24) Let $\mathcal{A}$ be an abelian category. For an object

$$\mathbb{F} : \cdots \to F^n \xrightarrow{d^n} F^{n+1} \xrightarrow{d^{n+1}} \to \cdots$$

in $C(\mathcal{A})$, we denote the truncated complex

$$0 \to F^n / \operatorname{Im} d^{n-1} \xrightarrow{d^n} F^{n+1} \xrightarrow{d^{n+1}} \to \cdots$$

by $\tau_{\geq n}\mathbb{F}$. Similarly, the truncated complex

$$\cdots \to F^{n-2} \xrightarrow{d^{n-2}} F^{n-1} \xrightarrow{d^{n-1}} \operatorname{Ker} d^n \to 0$$

is denoted by $\tau_{\leq n}\mathbb{F}$.

3.25 Lemma. *Let $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be a ringed site and $\mathbb{F} \in C(\mathrm{Mod}(\mathbb{X}))$.*

1 *We have $\mathfrak{Q} \subset \mathfrak{P}$.*

2 *A K-injective complex is weakly K-injective, and a weakly K-injective complex is K-limp.*

3 *For any $\mathbb{H} \in C(\mathrm{Mod}(\mathbb{X}))$, there is a $\mathfrak{Q}$-special direct system $(\mathbb{F}_n)$ and a direct system of chain maps $(f_n : \mathbb{F}_n \to \tau_{\leq n}\mathbb{H})$ such that f_n is a quasi-isomorphism for each $n \in \mathbb{N}$, and $(\mathbb{F}_n)_l = 0$ for $l \geq n+1$. We have $\varinjlim \mathbb{F}_n \to \mathbb{H}$ is a quasi-isomorphism, and $\varinjlim \mathbb{F}_n \in \mathfrak{P}$. Moreover, $Q\mathbb{H}$ is the homotopy colimit of the inductive system $(\tau_{\leq n}Q\mathbb{H})$ in the category $D(\mathrm{Mod}(\mathbb{X}))$.*

4 *The following are equivalent.*

i *$\mathbb{F}$ is K-limp.*

ii *$\mathbb{F}$ is K-limp as a complex of sheaves of abelian groups.*

iii *$\mathbb{F}$ is $\mathrm{Hom}^\bullet_{\mathrm{Mod}(\mathbb{X})}(\mathcal{O}_x, ?)$-acyclic for $x \in \mathbb{X}$.*

iv *$\mathbb{F}$ is $\Gamma(x, ?)$-acyclic for $x \in \mathbb{X}$.*

v *If $\mathbb{G} \in \mathfrak{P}$ and $\mathbb{G}$ is exact, then $\mathrm{Hom}^\bullet_{\mathrm{Mod}(\mathbb{X})}(\mathbb{G}, \mathbb{F})$ is exact.*

5 *The following are equivalent.*

i *$\mathbb{F}$ is weakly K-injective.*

ii *If $\mathbb{G}$ is K-flat exact, then $\mathrm{Hom}^\bullet_{\mathrm{Mod}(\mathbb{X})}(\mathbb{G}, \mathbb{F})$ is exact.*

iii *For any K-flat complex $\mathbb{G}$, $\underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(\mathbb{X})}(\mathbb{G}, \mathbb{F})$ is weakly K-injective.*

Proof. **1** is trivial. **2** follows from the definition and Corollary 3.23. The proof of **3** and **4** are left to the reader, see [39, (3.2), (3.3), (5.16), (5.17), (5.21)]. **5** is similar. □

3.26 Lemma. *Let $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be a ringed site and $\mathbb{F}, \mathbb{G} \in C(\mathrm{Mod}(\mathbb{X}))$. If $\mathbb{F}$ is weakly K-injective and $\mathbb{G}$ is K-flat, then $\mathbb{F}$ is $\underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(\mathbb{X})}(\mathbb{G}, ?)$-acyclic.*

Proof. Let $\mathbb{F} \to \mathbb{I}$ be the K-injective resolution, and $\mathbb{J}$ the mapping cone. Let $\varphi : \mathbb{H} \to \underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(\mathbb{X})}(\mathbb{G}, \mathbb{J})$ be a $\mathfrak{P}$-resolution. As $\mathbb{H} \otimes^\bullet \mathbb{G}$ is K-flat and $\mathbb{J}$ is weakly K-injective exact,

$$\mathrm{Hom}^\bullet_{\mathrm{Mod}(\mathbb{X})}(\mathbb{H}, \underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(\mathbb{X})}(\mathbb{G}, \mathbb{J})) \cong \mathrm{Hom}^\bullet_{\mathrm{Mod}(\mathbb{X})}(\mathbb{H} \otimes^\bullet \mathbb{G}, \mathbb{J})$$

is exact. So φ must be null-homotopic, and hence $\underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(\mathbb{X})}(\mathbb{G}, \mathbb{J})$ is exact. This is what we wanted to prove. □

(3.27) Let $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be a ringed site. For $\mathbb{G} \in C(\mathrm{Mod}(\mathbb{X}))$, it is easy to see that $\mathbb{G}\otimes^{\bullet}_{\mathcal{O}_{\mathbb{X}}}?$ induces a functor from $K(\mathrm{Mod}(\mathbb{X}))$ to itself. By Lemma 3.25, **3** and the dual assertion of [17, Theorem I.5.1], the derived functor $L(\mathbb{G}\otimes^{\bullet}_{\mathcal{O}_{\mathbb{X}}}?)$ is induced, and it is calculated using any K-flat resolution of ?. If we fix ?, then $L(\mathbb{G}\otimes^{\bullet}_{\mathcal{O}_{\mathbb{X}}}?)$ is a functor on $\mathbb{G}$, and it induces a bifunctor

$$*\otimes^{\bullet,L}_{\mathcal{O}_{\mathbb{X}}}? : D(\mathrm{Mod}(\mathbb{X})) \times D(\mathrm{Mod}(\mathbb{X})) \to D(\mathrm{Mod}(\mathbb{X})).$$

$\mathbb{G}\otimes^{\bullet,L}_{\mathcal{O}_{\mathbb{X}}}\mathbb{F}$ is calculated using any K-flat resolution of $\mathbb{F}$ or any K-flat resolution of $\mathbb{G}$. Note that $\otimes^{\bullet,L}_{\mathcal{O}_{\mathbb{X}}}$ is a $\triangle$-functor as in [26, (2.5.7)].

We define the *hyperTor* functor as follows:

$$\underline{\mathrm{Tor}}_i^{\mathcal{O}_{\mathbb{X}}}(\mathbb{F}, \mathbb{G}) := H^{-i}(\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{\mathbb{X}}} \mathbb{G}).$$

(3.28) Let $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be a ringed site. For $\mathbb{F} \in C(\mathrm{Mod}(\mathbb{X}))$, the functor $\underline{\mathrm{Hom}}^{\bullet}_{\mathcal{O}_{\mathbb{X}}}(\mathbb{F}, ?)$ induces a functor from $K(\mathrm{Mod}(\mathbb{X}))$ to itself. As $\mathrm{Mod}(\mathbb{X})$ is Grothendieck, we can take K-injective resolutions, and hence the right derived functor $R\,\underline{\mathrm{Hom}}^{\bullet}_{\mathcal{O}_{\mathbb{X}}}(\mathbb{F}, ?)$ is induced. Thus a bifunctor

$$R\,\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathbb{X}}}(*, ?) : D(\mathrm{Mod}(\mathbb{X}))^{\mathrm{op}} \times D(\mathrm{Mod}(\mathbb{X})) \to D(\mathrm{Mod}(\mathbb{X}))$$

is induced. For $\mathbb{F}, \mathbb{G} \in D(\mathrm{Mod}(\mathbb{X}))$, we define the *hyperExt* sheaf of $\mathbb{F}$ and $\mathbb{G}$ by

$$\underline{\mathrm{Ext}}^i_{\mathcal{O}_{\mathbb{X}}}(\mathbb{F}, \mathbb{G}) := H^i(R\,\underline{\mathrm{Hom}}^{\bullet}_{\mathcal{O}_{\mathbb{X}}}(\mathbb{F}, \mathbb{G})).$$

Similarly, the functor $\mathrm{Hom}^{\bullet}_{\mathcal{O}_{\mathbb{X}}}(*, ?)$ induces

$$R\,\mathrm{Hom}^{\bullet}_{\mathcal{O}_{\mathbb{X}}}(*, ?) : D(\mathrm{Mod}(\mathbb{X}))^{\mathrm{op}} \times D(\mathrm{Mod}(\mathbb{X})) \to D(\underline{\mathrm{Ab}}).$$

Almost by definition, we have

$$H^i(R\,\mathrm{Hom}^{\bullet}_{\mathcal{O}_{\mathbb{X}}}(\mathbb{F}, \mathbb{G})) \cong \mathrm{Hom}_{D(\mathrm{Mod}(\mathbb{X}))}(\mathbb{F}, \mathbb{G}[i]).$$

Sometimes we denote these groups by $\mathrm{Ext}^i_{\mathcal{O}_{\mathbb{X}}}(\mathbb{F}, \mathbb{G})$.

3.29 Lemma. *Let $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be a ringed site. Then $D(\mathrm{Mod}(\mathbb{X}))$ is a closed symmetric monoidal triangulated category with $\otimes^{\bullet,L}_{\mathcal{O}_{\mathbb{X}}}$ its product and $R\,\underline{\mathrm{Hom}}^{\bullet}_{\mathcal{O}_{\mathbb{X}}}(*, ?)$ its internal hom.*

Proof. This is straightforward. $\square$

3.30 Lemma. *Let* $f\colon \mathbb{Y} \to \mathbb{X}$ *be a continuous functor between sites. If* $\mathbb{I} \in C(\mathrm{AB}(\mathbb{X}))$ *is* K*-limp and exact, then* $f^{\#}_{\mathrm{AB}}\mathbb{I}$ *is exact.*

Proof. Let $\xi : \mathbb{F} \to f^{\#}_{\mathrm{AB}}\mathbb{I}$ be a $\mathfrak{P}(\mathrm{AB})$-resolution of $f^{\#}_{\mathrm{AB}}\mathbb{I}$. It suffices to show $\mathrm{Hom}^\bullet_{\mathrm{AB}(\mathbb{Y})}(\mathbb{F}, f^{\#}_{\mathrm{AB}}\mathbb{I})$ is exact (if so, then ξ must be null-homotopic). Since $f_{\#}^{\mathrm{AB}}\mathbb{F} \in \mathfrak{P}(\mathrm{AB})$ and $\mathbb{I}$ is K-limp exact, this is obvious. □

By the lemma, a K-limp complex is $f^{\#}$-acyclic.

3.31 Lemma. *Let* $f : (\mathbb{Y}, \mathcal{O}_{\mathbb{Y}}) \to (\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ *be an admissible ringed continuous functor. Then the following hold:*

1 *If* $\mathbb{I} \in C(\mathrm{AB}(\mathbb{X}))$ *is a* K*-injective* (*resp.* K*-limp*) *complex of sheaves of abelian groups, then so is* $f^{\#}_{\mathrm{AB}}\mathbb{I}$.

2 *If* $\mathbb{F} \in C(\mathrm{Mod}(\mathbb{Y}))$ *is strongly* K*-flat and exact, then* $f_{\#}^{\mathrm{Mod}}\mathbb{F}$ *is strongly* K*-flat and exact.*

Proof. As $f^{\#}_{\mathrm{AB}}$ has an exact left adjoint $f_{\#}^{\mathrm{AB}}$, the assertion for K-injectivity in **1** is obvious.

We prove the assertion for the K-limp property in **1**. Let $\mathbb{P} \in \mathfrak{P}(\mathbb{Y})$ be an exact complex. As $f_{\#}^{\mathrm{AB}}$ is exact, $f_{\#}^{\mathrm{AB}}\mathbb{P}$ is exact and a complex in $\mathfrak{P}(\mathbb{X})$ by Lemma 3.20. Hence,

$$\mathrm{Hom}^\bullet_{\mathrm{AB}(\mathbb{Y})}(\mathbb{P}, f^{\#}_{\mathrm{AB}}\mathbb{I}) \cong \mathrm{Hom}^\bullet_{\mathrm{AB}(\mathbb{X})}(f_{\#}^{\mathrm{AB}}\mathbb{P}, \mathbb{I})$$

is exact. This shows $f^{\#}_{\mathrm{AB}}\mathbb{I}$ is K-limp.

We prove **2**. We already know that $f_{\#}\mathbb{F}$ is strongly K-flat by Lemma 3.20. We prove that $\mathrm{Hom}^\bullet_{\mathrm{Mod}(\mathbb{X})}(f_{\#}^{\mathrm{Mod}}\mathbb{F}, \mathbb{I})$ is exact, where $\eta : f_{\#}^{\mathrm{Mod}}\mathbb{F} \to \mathbb{I}$ is a K-injective resolution. Then η must be null-homotopic, and we have $f_{\#}^{\mathrm{Mod}}\mathbb{F}$ is exact and the proof is complete. Clearly, $\mathbb{I}$ is K-limp, and hence so is $f^{\#}_{\mathrm{Mod}}\mathbb{I}$ by **1** and Lemma 3.25, **4**. The assertion follows immediately by adjunction. □

(3.32) By the lemma, there is a derived functor $Lf_{\#}^{\mathrm{Mod}} : D(\mathrm{Mod}(\mathbb{Y})) \to D(\mathrm{Mod}(\mathbb{X}))$ of $f_{\#}^{\mathrm{Mod}}$ for an admissible ringed continuous functor f. It is calculated via strongly K-flat resolutions.

Now as in [39, section 6] and [26], the following is proved.

3.33 Lemma. *Let* $\mathcal{S}$ *be the category of ringed sites and admissible ringed continuous functors. Then* $(L(?)_{\#}^{\mathrm{Mod}}, R(?)^{\#}_{\mathrm{Mod}})$ *is a monoidal adjoint pair of* Δ*-almost-pseudofunctors* (*defined appropriately as in* [26, (3.6.7)]) *on* $\mathcal{S}^{\mathrm{op}}$.

The proof is basically the same as that in [26, Chapter 1–3], and left to the reader.

3.34 Remark. Later we will treat ringed continuous functor f which may not be admissible. In this case, we may use $Rf^{\#}$ and related functorialities, but not $Lf_{\#}$.

4 Sheaves over a diagram of S-schemes

(4.1) Let S be a (small) scheme, and I a small category. We call an object of $\mathcal{P}(I^{\mathrm{op}}, \underline{\mathrm{Sch}}/S)$ an I-diagram of S-schemes, where $\underline{\mathrm{Sch}}/S$ denotes the category of (small) S-schemes. We denote $\underline{\mathrm{Sch}}/\operatorname{Spec}\mathbb{Z}$ simply by $\underline{\mathrm{Sch}}$. So an object of $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$ is referred as an I^{op}-diagram of S-schemes. Let $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. We denote $X_\bullet(i)$ by X_i for $i \in I$, and $X_\bullet(\phi)$ by X_ϕ for $\phi \in \operatorname{Mor}(I)$. Let $\mathbb{P}$ be a property of schemes (e.g., quasi-compact, locally noetherian, regular). We say that $X_\bullet$ satisfies $\mathbb{P}$ if X_i satisfies $\mathbb{P}$ for any $i \in I$. Let $\mathbb{Q}$ be a property of morphisms of schemes (e.g., quasi-compact, locally of finite type, smooth). We say that $X_\bullet$ is $\mathbb{Q}$ over S if the structure map $X_i \to S$ satisfies $\mathbb{Q}$ for any $i \in I$. We say that $X_\bullet$ has $\mathbb{Q}$ arrows if X_ϕ satisfies $\mathbb{Q}$ for any $\phi \in \operatorname{Mor}(I)$.

(4.2) Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. For $i \in I$, we denote $f_\bullet(i) : X_i \to Y_i$ by f_i. For a property $\mathbb{Q}$ of morphisms of schemes, we say that $f_\bullet$ satisfies $\mathbb{Q}$ if so does f_i for any $i \in I$. We say that $f_\bullet$ is *cartesian* if the canonical map $(f_j, X_\phi) : X_j \to Y_j \times_{Y_i} X_i$ is an isomorphism for any morphism $\phi : i \to j$ of I.

(4.3) Let S, I and $X_\bullet$ be as above. We define the *Zariski site* of $X_\bullet$, denoted by $\operatorname{Zar}(X_\bullet)$, as follows. An object of $\operatorname{Zar}(X_\bullet)$ is a pair (i, U) such that $i \in I$ and U is an open subset of X_i. A morphism $(\phi, h) : (j, V) \to (i, U)$ is a pair (ϕ, h) such that $\phi \in I(i, j)$ and $h : V \to U$ is the restriction of X_ϕ. For a given morphism $\phi : i \to j$, U, and V, such an h exists if and only if $V \subset X_\phi^{-1}(U)$, and it is unique. We denote this h by $h(\phi; U, V)$. The composition of morphisms is defined in an obvious way. Thus $\operatorname{Zar}(X_\bullet)$ is a small category. For $(i, U) \in \operatorname{Zar}(X_\bullet)$, a covering of (i, U) is a family of morphisms of the form

$$((\mathrm{id}_i, h(\mathrm{id}_i; U, U_\lambda)) : (i, U_\lambda) \to (i, U))_{\lambda \in \Lambda}$$

such that $\bigcup_{\lambda\in\Lambda} U_\lambda = U$. This defines a pretopology of $\mathrm{Zar}(X_\bullet)$, and $\mathrm{Zar}(X_\bullet)$ is a site. As we will consider only the Zariski topology, a presheaf or sheaf on $\mathrm{Zar}(X_\bullet)$ will be sometimes referred as a presheaf or sheaf on $X_\bullet$, if there is no danger of confusion. Thus $\mathcal{P}(X_\bullet, \mathcal{C})$ and $\mathcal{S}(X_\bullet, \mathcal{C})$ mean $\mathcal{P}(\mathrm{Zar}(X_\bullet), \mathcal{C})$ and $\mathcal{S}(\mathrm{Zar}(X_\bullet), \mathcal{C})$, respectively.

(4.4) Let S and I be as above, and let $\sigma : J \hookrightarrow I$ be a subcategory of I. Then we have an obvious restriction functor $\sigma^{\#} : \mathcal{P}(I, \underline{\mathrm{Sch}}/S) \to \mathcal{P}(J, \underline{\mathrm{Sch}}/S)$, which we denote by $(?)|_J$. If $\mathrm{ob}(J)$ is finite and $I(j,i)$ is finite for each $j \in J$ and $i \in I$, then $(?)|_J$ has a right adjoint functor $\mathrm{cosk}_J^I = (?)^{\mathrm{op}}\sigma_{\#}^{\mathrm{op}}(?)^{\mathrm{op}}$, because $\underline{\mathrm{Sch}}/S$ has finite limits. Note that $(\mathrm{cosk}_J^I X_\bullet)_i = \varprojlim X_j$, where the limit is taken over $I_i^{\sigma^{\mathrm{op}}}$, where $\sigma^{\mathrm{op}} : J^{\mathrm{op}} \to I^{\mathrm{op}}$ is the opposite of σ. See [10, pp. 9–12].

(4.5) Let $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Then we have an obvious continuous functor $Q(X_\bullet, J) : \mathrm{Zar}((X_\bullet)|_J) \hookrightarrow \mathrm{Zar}(X_\bullet)$. Note that $Q(X_\bullet, J)$ may not be admissible. The restriction functors $Q(X_\bullet, J)^{\#}_{\mathrm{AB}}$ and $Q(X_\bullet, J)^{\#}_{\mathrm{PA}}$ are denoted by $(?)_J^{\mathrm{AB}}$ and $(?)_J^{\mathrm{PA}}$, respectively. For $i \in I$, we consider that i is the subcategory of I whose object set is $\{i\}$ with $\mathrm{Hom}_i(i,i) = \{\mathrm{id}\}$. The restrictions $(?)_i^{\heartsuit}$ for $\heartsuit = \mathrm{AB}, \mathrm{PA}$ are defined.

(4.6) Let $\mathcal{F} \in \mathrm{PA}(X_\bullet)$ and $i \in I$. Then $\mathcal{F}_i \in \mathrm{PA}(X_i)$, and thus we have a family of sheaves $(\mathcal{F}_i)_{i\in I}$. Moreover, for $(i, U) \in \mathrm{Zar}(X_\bullet)$ and $\phi : i \to j$, we have the restriction map

$$\Gamma(U, \mathcal{F}_i) = \Gamma((i,U), \mathcal{F}) \xrightarrow{\mathrm{res}} \Gamma((j, X_\phi^{-1}(U)), \mathcal{F}) = \Gamma(X_\phi^{-1}(U), \mathcal{F}_j) = \Gamma(U, (X_\phi)_*\mathcal{F}_j),$$

which induces

$$\beta_\phi(\mathcal{F}) \in \mathrm{Hom}_{\mathrm{PA}(X_i)}(\mathcal{F}_i, (X_\phi)_*\mathcal{F}_j). \tag{4.7}$$

The corresponding map in $\mathrm{Hom}_{\mathrm{PA}(X_j)}((X_\phi)^*_{\mathrm{PA}}(\mathcal{F}_i), \mathcal{F}_j)$ is denoted by $\alpha_\phi^{\mathrm{PA}}(\mathcal{F})$. If $\mathcal{F}$ is a sheaf, then (4.7) yields

$$\alpha_\phi^{\mathrm{AB}}(\mathcal{F}) \in \mathrm{Hom}_{\mathrm{AB}(X_j)}((X_\phi)^*_{\mathrm{AB}}(\mathcal{F}_i), \mathcal{F}_j).$$

It is straightforward to check the following.

4.8 Lemma ([10]). *Let $\heartsuit$ be either* AB *or* PA*. The following hold:*

1 *For any $i \in I$, we have $\alpha_{\mathrm{id}_i}^{\heartsuit} : (X_{\mathrm{id}_i})^*_{\heartsuit}(\mathcal{F}_i) \to \mathcal{F}_i$ is the canonical identification $\mathfrak{f}_{X_i}$.*

2 *If $\phi \in I(i,j)$ and $\psi \in I(j,k)$, then the composite map*

$$(X_{\psi\phi})^*_\heartsuit(\mathcal{F}_i) \xrightarrow{d^{-1}} (X_\psi)^*_\heartsuit(X_\phi)^*_\heartsuit(\mathcal{F}_i) \xrightarrow{(X_\psi)^*_\heartsuit\alpha^\heartsuit_\phi} (X_\psi)^*_\heartsuit(\mathcal{F}_j) \xrightarrow{\alpha^\heartsuit_\psi} \mathcal{F}_k \qquad (4.9)$$

agrees with $\alpha^\heartsuit_{\psi\phi}$.

3 *Conversely, a family $((\mathcal{G}_i)_{i\in I}, (\alpha_\phi)_{\phi\in \mathrm{Mor}(I)})$ such that $\mathcal{G}_i \in \heartsuit(X_i)$, $\alpha_\phi \in \mathrm{Hom}_{\heartsuit(X_j)}((X_\phi)^*_\heartsuit(\mathcal{G}_i), \mathcal{G}_j)$ for $\phi \in I(i,j)$, and that the conditions corresponding to* **1,2** *are satisfied yields $\mathcal{G} \in \heartsuit(X_\bullet)$, and this correspondence gives an equivalence.*

(4.10) Similarly, a family $((\mathcal{G}_i)_{i\in \mathrm{ob}(I)}, (\beta_\phi)_{\phi\in\mathrm{Mor}(I)})$ with

$$\mathcal{G}_i \in \heartsuit(X_i) \text{ and } \beta_\phi \in \mathrm{Hom}_{\heartsuit(X_i)}(\mathcal{G}_i, (X_\phi)^\heartsuit_* \mathcal{G}_j)$$

satisfying the conditions

1' For $i \in \mathrm{ob}(I)$, $\beta_{\mathrm{id}_i} : \mathcal{G}_i \to (X_{\mathrm{id}_i})_*\mathcal{G}_i$ is the canonical identification $\mathfrak{e}_{X_i}$;

2' For $\phi \in I(i,j)$ and $\psi \in I(j,k)$, the composite

$$\mathcal{F}_i \xrightarrow{\beta_\phi} (X_\phi)_*(\mathcal{F}_j) \xrightarrow{(X_\phi)_*\beta_\psi} (X_\phi)_*(X_\psi)_*(\mathcal{F}_k) \xrightarrow{c^{-1}} (X_{\psi\phi})_*(\mathcal{F}_k)$$

agrees with $\beta_{\psi\phi}$

is in one to one correspondence with $\mathcal{G} \in \heartsuit(X_\bullet)$.

(4.11) Let $\mathcal{F} \in \mathrm{AB}(X_\bullet)$. We say that $\mathcal{F}$ is an *equivariant* abelian sheaf if $\alpha^{\mathrm{AB}}_\phi$ are isomorphisms for all $\phi \in \mathrm{Mor}(I)$. For $\mathcal{F} \in \mathrm{PA}(X_\bullet)$, we say that $\mathcal{F}$ is an equivariant abelian presheaf if $\alpha^{\mathrm{PA}}_\phi$ are isomorphisms for all $\phi \in \mathrm{Mor}(I)$. An equivariant sheaf may not be an equivariant presheaf. However, an equivariant presheaf which is a sheaf is an equivariant sheaf. We denote the category of equivariant sheaves and presheaves by $\mathrm{EqAB}(X_\bullet)$ and $\mathrm{EqPA}(X_\bullet)$, respectively. As $(X_\phi)^*_\heartsuit$ is exact for $\heartsuit = \mathrm{AB}, \mathrm{PA}$ and any ϕ, we have that $\mathrm{EqAB}(X_\bullet)$ is plump in $\mathrm{AB}(X_\bullet)$, and $\mathrm{EqPA}(X_\bullet)$ is plump in $\mathrm{PA}(X_\bullet)$.

(4.12) Let $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. The data

$$((\mathcal{O}_{X_i})_{i\in I}, (\beta_\phi = \eta : \mathcal{O}_{X_i} \to (X_\phi)_*\mathcal{O}_{X_j})_{\phi\in\mathrm{Mor}(I)})$$

gives a sheaf of commutative rings on $X_\bullet$, which we denote by $\mathcal{O}_{X_\bullet}$, and thus $\mathrm{Zar}(X_\bullet)$ is a ringed site. The categories $\mathrm{PM}(\mathrm{Zar}(X_\bullet))$ and $\mathrm{Mod}(\mathrm{Zar}(X_\bullet))$ are denoted by $\mathrm{PM}(X_\bullet)$ and $\mathrm{Mod}(X_\bullet)$, respectively. Let $\heartsuit = \mathrm{PM}, \mathrm{Mod}$. Note that for $\mathcal{M} \in \heartsuit(X_\bullet)$ and $\phi : i \to j$, $\beta_\phi : \mathcal{M}_i \to (X_\phi)_*\mathcal{M}_j$ is a morphism in $\heartsuit(X_i)$, which we denote by $\beta_\phi^\heartsuit$. The adjoint morphism $X_\phi^*\mathcal{M}_i \to \mathcal{M}_j$ is denoted by $\alpha_\phi^\heartsuit$. α is not compatible with the forgetful functors in general.

(4.13) For $J \subset I$, we have $\mathcal{O}_{X_\bullet|_J} = (\mathcal{O}_{X_\bullet})_J$ by definition. The continuous functor

$$Q(X_\bullet, J) : (\mathrm{Zar}(X_\bullet|_J), \mathcal{O}_{X_\bullet|_J}) \to (\mathrm{Zar}(X_\bullet), \mathcal{O}_{X_\bullet})$$

is actually a ringed continuous functor.

The corresponding restriction $Q(X_\bullet, J)^\#_\heartsuit$ is denoted by $(?)^\heartsuit_J$ for $\heartsuit = \mathrm{PM}, \mathrm{Mod}$. For subcategories $J_1 \subset J \subset I$ of I, we denote the restriction $(?)^\heartsuit_{J_1} : \heartsuit(X_\bullet|_J) \to \heartsuit(X_\bullet|_{J_1})$ by $(?)^\heartsuit_{J_1,J}$, to emphasize J.

(4.14) Let $\heartsuit$ be PM or Mod. Note that $\mathcal{M} \in \heartsuit(X_\bullet)$ is nothing but a family

$$\mathrm{Dat}(\mathcal{M}) := ((\mathcal{M}_i)_{i\in I}, (\alpha_\phi^\heartsuit)_{\phi\in\mathrm{Mor}(I)})$$

such that $\mathcal{M}_i \in \heartsuit(X_i)$, $\alpha_\phi^\heartsuit : (X_\phi)^*_\heartsuit(\mathcal{M}_i) \to \mathcal{M}_j$ is a morphism of $\heartsuit(X_\bullet)$ for any $\phi : i \to j$, and the conditions corresponding to **1,2** in Lemma 4.8 are satisfied.

We say that $\mathcal{M} \in \heartsuit(X_\bullet)$ is *equivariant* if $\alpha_\phi^\heartsuit$ is an isomorphism for any $\phi \in \mathrm{Mor}(I)$. Note that equivariance depends on $\heartsuit$, and is not preserved by the forgetful functors in general. We denote the full subcategory of $\mathrm{Mod}(X_\bullet)$ consisting of equivariant objects by $\mathrm{EM}(X_\bullet)$.

(4.15) Let $\heartsuit$ be Mod or AB, and $\mathcal{M} \in \heartsuit(X_\bullet)$. For a morphism $\phi : i \to j$, the diagram

$$\begin{array}{ccccc} q\mathcal{M}_i & & \xrightarrow{\quad\quad\beta\quad\quad} & & q(X_\phi)_*\mathcal{M}_j \\ \downarrow c & & & & \downarrow c \\ (q\mathcal{M})_i & \xrightarrow{\beta} & (X_\phi)_*(q\mathcal{M})_j & \xrightarrow{c} & (X_\phi)_*q\mathcal{M}_j \end{array} \tag{4.16}$$

is commutative. This is checked at the section level directly. Utilizing this fact, we have the following.

4.17 Lemma. *Let $\mathcal{M} \in \mathrm{PM}(X_\bullet)$. For $\phi : i \to j$, the diagram*

$$\begin{array}{ccccc} a\mathcal{M}_i & \xrightarrow{\beta} & a(X_\phi)_*\mathcal{M}_j & \xrightarrow{\theta} & (X_\phi)_*a\mathcal{M}_j \\ \downarrow \theta & & & & \downarrow \theta \\ (a\mathcal{M})_i & & \xrightarrow{\qquad\qquad\beta\qquad\qquad} & & (X_\phi)_*(a\mathcal{M})_j \end{array}$$

is commutative.

Proof. Straightforward diagram drawing. □

(4.18) Let $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$, and $\phi : i \to j$ be a morphism of I. Let $\mathcal{M} \in \mathrm{PM}(X_\bullet)$. Then $\alpha_\phi : X_\phi^*\mathcal{M}_i \to \mathcal{M}_j$ is the composite

$$X_\phi^*\mathcal{M}_i \xrightarrow{\beta} X_\phi^*(X_\phi)_*\mathcal{M}_j \xrightarrow{\varepsilon} \mathcal{M}_j. \tag{4.19}$$

Thus for $U \in \mathrm{Zar}(X_j)$, α_ϕ is given by

$$\Gamma(U, X_\phi^*\mathcal{M}_i) = \varinjlim \Gamma(U, \mathcal{O}_{X_j}) \otimes_{\Gamma(V,\mathcal{O}_{X_i})} \Gamma((i,V),\mathcal{M}) \\ \to \Gamma((j,U),\mathcal{M}) = \Gamma(U,\mathcal{M}_j),$$

where the colimit is taken over the open subsets V of X_i such that $U \subset X_\phi^{-1}(V)$, and the arrow is given by $a \otimes m \mapsto a \,\mathrm{res}_{(j,U),(i,V)}\, m$, see (2.20).

(4.20) Let $X_\bullet$ and $\phi : i \to j$ be as in (4.18). Let $\mathcal{M} \in \mathrm{Mod}(X_\bullet)$. Then $\alpha_\phi : X_\phi^*\mathcal{M}_i \to \mathcal{M}_j$ is also given by the composite (4.19). By (2.20), it is given by the composite

$$X_\phi^*(?)_i = aX_\phi^*q(?)_i \xrightarrow{\beta} aX_\phi^*q(X_\phi)_*(?)_j \xrightarrow{c} aX_\phi^*(X_\phi)_*q(?)_j \xrightarrow{\varepsilon} aq(?)_j \xrightarrow{\varepsilon} (?)_j.$$

By the commutativity of (4.16), it is easy to see that it agrees with

$$X_\phi^*(?)_i = aX_\phi^*q(?)_i \xrightarrow{c} aX_\phi^*(?)_iq \xrightarrow{\alpha_\phi} a(?)_jq \xrightarrow{c} aq(?)_j \xrightarrow{\varepsilon} (?)_j.$$

Thus if $\mathcal{M} \in \mathrm{Mod}(X_\bullet)$ is equivariant as an object of $\mathrm{PM}(X_\bullet)$, then it is equivariant as an object of $\mathrm{Mod}(X_\bullet)$.

5 The left and right inductions and the direct and inverse images

Let I be a small category, S a scheme, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$.

(5.1) Let J be a subcategory of I. The left adjoint $Q(X_\bullet, J)^\heartsuit_\#$ of $(?)^\heartsuit_J$ (see (4.13)) is denoted by $L^\heartsuit_J$ for $\heartsuit = \mathrm{PA}, \mathrm{AB}, \mathrm{PM}, \mathrm{Mod}$. The right adjoint $Q(X_\bullet, J)^\heartsuit_\flat$ of $(?)^\heartsuit_J$, which exists by Lemma 2.31, is denoted by $R^\heartsuit_J$ for $\heartsuit = \mathrm{PA}, \mathrm{AB}, \mathrm{PM}, \mathrm{Mod}$. We call $L^\heartsuit_J$ and $R^\heartsuit_J$ the left and right induction functor, respectively.

Let $J_1 \subset J \subset I$ be subcategories of I. The left and right adjoints of $(?)^\heartsuit_{J_1,J}$ are denoted by $L^\heartsuit_{J,J_1}$ and $R^\heartsuit_{J,J_1}$, respectively. As $(?)^\heartsuit_J$ has both a left adjoint and a right adjoint, we have

5.2 Lemma. *The functor $(?)^\heartsuit_J$ preserves arbitrary limits and colimits (hence is exact) for $\heartsuit = \mathrm{PA}, \mathrm{AB}, \mathrm{PM}, \mathrm{Mod}$.*

The functor $\heartsuit(X_\bullet) \to \prod_{i\in I} \heartsuit(X_i)$ given by $\mathcal{F} \mapsto (\mathcal{F}_i)_{i\in I}$ is faithful for $\heartsuit = \mathrm{PA}, \mathrm{AB}, \mathrm{PM}, \mathrm{Mod}$.

(5.3) Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. This induces an obvious ringed continuous functor

$$f_\bullet^{-1} : (\mathrm{Zar}(Y_\bullet), \mathcal{O}_{Y_\bullet}) \to (\mathrm{Zar}(X_\bullet), \mathcal{O}_{X_\bullet}).$$

We have $\mathrm{id}^{-1} = \mathrm{id}$, and $(g_\bullet \circ f_\bullet)^{-1} = f_\bullet^{-1} \circ g_\bullet^{-1}$ for $g_\bullet : Y_\bullet \to Z_\bullet$.

We define the *direct image* $(f_\bullet)^\heartsuit_*$ to be $(f_\bullet^{-1})^\#_\heartsuit$, and the *inverse image* $(f_\bullet)^*_\heartsuit$ to be $(f_\bullet^{-1})^\heartsuit_\#$ for $\heartsuit = \mathrm{Mod}, \mathrm{PM}, \mathrm{AB}, \mathrm{PA}$.

5.4 Lemma. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$, and $K \subset J \subset I$. Then we have*

1 $Q(X_\bullet, J) \circ Q(X_\bullet|_J, K) = Q(X_\bullet, K)$

2 $f_\bullet^{-1} \circ Q(Y_\bullet, J) = Q(X_\bullet, J) \circ (f_\bullet|_J)^{-1}$.

(5.5) Let us fix I and S. By Lemma 2.43 and Lemma 2.55, we have various natural maps between functors on sheaves arising from the closed structures and the monoidal pairs, involving various J-diagrams of schemes, where J varies subcategories of I. In the sequel, many of the natural maps are referred as 'the canonical maps' or 'the canonical isomorphisms' without any explicit definitions. Many of them are defined in [26] and Chapter 1, and various commutativity theorems are proved there.

5.6 Example. Let I be a small category, S a scheme, and $f_\bullet : X_\bullet \to Y_\bullet$ and $g_\bullet : Y_\bullet \to Z_\bullet$ are morphisms in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Let $K \subset J \subset I$ be subcategories, and $\heartsuit$ denote PM, Mod, PA, or AB.

1 There is a natural isomorphism

$$c^{\heartsuit}_{I,J,K} : (?)^{\heartsuit}_{K,I} \cong (?)^{\heartsuit}_{K,J} \circ (?)^{\heartsuit}_{J,I}.$$

Taking the conjugate,

$$d^{\heartsuit}_{I,J,K} : L^{\heartsuit}_{I,J} \circ L^{\heartsuit}_{J,K} \cong L^{\heartsuit}_{I,K}$$

is induced.

2 There is a natural isomorphism

$$c^{\heartsuit}_{J,f_\bullet} : (?)^{\heartsuit}_{J} \circ (f_\bullet)^{\heartsuit}_{*} \cong (f_\bullet|_J)^{\heartsuit}_{*} \circ (?)^{\heartsuit}_{J}$$

and its conjugate

$$d^{\heartsuit}_{J,f_\bullet} : L^{\heartsuit}_{J} \circ (f_\bullet|_J)^{*}_{\heartsuit} \cong (f_\bullet)^{*}_{\heartsuit} \circ L^{\heartsuit}_{J}.$$

3 We have

$$(c^{\heartsuit}_{K,f_\bullet|_J}(?)^{\heartsuit}_{J}) \circ ((?)^{\heartsuit}_{K,J} c^{\heartsuit}_{J,f_\bullet}) = ((f_\bullet|_K)^{\heartsuit}_{*} c^{\heartsuit}_{I,J,K}) \circ c^{\heartsuit}_{K,f_\bullet} \circ ((c^{\heartsuit}_{I,J,K})^{-1}(f_\bullet)^{\heartsuit}_{*}).$$

4 We have

$$((g_\bullet|_J)^{\heartsuit}_{*} c^{\heartsuit}_{J,f_\bullet}) \circ (c^{\heartsuit}_{J,g_\bullet}(f_\bullet)^{\heartsuit}_{*}) = (c^{\heartsuit}_{f_\bullet|_J,g_\bullet|_J}(?)^{\heartsuit}_{J}) \circ c^{\heartsuit}_{J,g_\bullet\circ f_\bullet} \circ ((?)^{\heartsuit}_{J}(c^{\heartsuit}_{f_\bullet,g_\bullet})^{-1}),$$

where $c^{\heartsuit}_{f_\bullet,g_\bullet} : (g_\bullet \circ f_\bullet)^{\heartsuit}_{*} \cong (g_\bullet)^{\heartsuit}_{*} \circ (f_\bullet)^{\heartsuit}_{*}$ is the canonical isomorphism, and similarly for $c^{\heartsuit}_{f_\bullet|_J,g_\bullet|_J}$.

5 The canonical map

$$m_J : \mathcal{M}_J \otimes_{\mathcal{O}_{X_\bullet|_J}} \mathcal{N}_J \to (\mathcal{M} \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{N})_J$$

is an isomorphism, as can be seen easily (the corresponding assertion for PM is obvious. Utilize Lemma 5.7 below to show the case of Mod). The canonical map

$$\Delta : L_J(\mathcal{M} \otimes_{\mathcal{O}_{X_\bullet|_J}} \mathcal{N}) \cong (L_J\mathcal{M}) \otimes_{\mathcal{O}_{X_\bullet}} (L_J\mathcal{N}).$$

is defined, which may not be an isomorphism.

5.7 Lemma. *Let I be a small category, S a scheme, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Let J be a subcategory of I. Then the natural map*

$$\theta = \bar{\theta} : a(?)^{\mathrm{PM}}_{J} \to (?)^{\mathrm{Mod}}_{J} a$$

is an isomorphism.

Proof. Obvious by Lemma 2.31. □

6 Operations on sheaves via the structure data

Let I be a small category, S a scheme, and $\mathcal{P} := \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. To study sheaves on objects of $\mathcal{P}$, it is convenient to utilize the structure data of them, and then utilize the usual sheaf theory on schemes.

(6.1) Let $X_\bullet \in \mathcal{P}$. Let $\heartsuit$ be any of $\mathrm{PA}, \mathrm{AB}, \mathrm{PM}, \mathrm{Mod}$, and $\mathcal{M}, \mathcal{N} \in \heartsuit(X_\bullet)$. An element (φ_i) in $\prod \mathrm{Hom}_{\heartsuit(X_i)}(\mathcal{M}_i, \mathcal{N}_i)$ is given by some $\varphi \in \mathrm{Hom}_{\heartsuit(X_\bullet)}(\mathcal{M}, \mathcal{N})$ (by the canonical faithful functor $\heartsuit(X_\bullet) \rightarrow \prod \heartsuit(X_i)$), if and only if

$$\varphi_j \circ \alpha_\phi(\mathcal{M}) = \alpha_\phi(\mathcal{N}) \circ (X_\phi)^*_\heartsuit(\varphi_i) \tag{6.2}$$

holds (or equivalently, $\beta_\phi(\mathcal{N}) \circ \varphi_i = (X_\phi)_* \varphi_j \circ \beta_\phi(\mathcal{M})$ holds) for any $(\phi : i \rightarrow j) \in \mathrm{Mor}(I)$.

We say that a family of morphisms $(\varphi_i)_{i \in I}$ between structure data

$$\varphi_i : \mathcal{M}_i \rightarrow \mathcal{N}_i$$

is a morphism of structure data if φ_i is a morphism in $\heartsuit(X_i)$ for each i, and (6.2) is satisfied for any ϕ. Thus the categories of structure data of sheaves, presheaves, modules, and premodules on $X_\bullet$, denoted by $\mathfrak{D}_\heartsuit(X_\bullet)$ are defined, and the equivalence $\mathrm{Dat}_\heartsuit : \heartsuit(X_\bullet) \cong \mathfrak{D}_\heartsuit(X_\bullet)$ are given. This is the precise meaning of Lemma 4.8.

(6.3) Let $X_\bullet \in \mathcal{P}$ and $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(X_\bullet)$. As in Example 5.6, **5**, we have an isomorphism

$$m_i : \mathcal{M}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{N}_i \cong (\mathcal{M} \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{N})_i.$$

This is trivial for presheaves, and utilize the fact the sheafification is compatible with $(?)_i$ for sheaves. At the section level, for $\mathcal{M}, \mathcal{N} \in \mathrm{PM}(X_\bullet)$, $i \in I$, and $U \in \mathrm{Zar}(X_i)$,

$$m_i^p : \Gamma(U, \mathcal{M}_i \otimes^p_{\mathcal{O}_{X_i}} \mathcal{N}_i) \rightarrow \Gamma(U, (\mathcal{M} \otimes^p_{\mathcal{O}_{X_\bullet}} \mathcal{N})_i)$$

is nothing but the identification

$$\begin{aligned}\Gamma(U, \mathcal{M}_i) \otimes_{\Gamma(U, \mathcal{O}_{X_i})} \Gamma(U, \mathcal{N}_i) = \Gamma((i,U), \mathcal{M}) \otimes_{\Gamma((i,U), \mathcal{O}_{X_\bullet})} \Gamma((i,U), \mathcal{N}) \\ = \Gamma((i,U), \mathcal{M} \otimes^p_{\mathcal{O}_{X_\bullet}} \mathcal{N}).\end{aligned}$$

For $\mathcal{M},\mathcal{N} \in \mathrm{Mod}(X_\bullet)$ and $i \in I$, m_i is given as the composite

$$\mathcal{M}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{N}_i = a(q\mathcal{M}_i \otimes^p_{\mathcal{O}_{X_i}} q\mathcal{N}_i) \xrightarrow{c} a((q\mathcal{M})_i \otimes^p_{\mathcal{O}_{X_i}} (q\mathcal{N})_i) \xrightarrow{m_i^p}$$
$$a(q\mathcal{M} \otimes^p_{\mathcal{O}_{X_\bullet}} q\mathcal{N})_i \xrightarrow{\theta} (a(q\mathcal{M} \otimes^p_{\mathcal{O}_{X_\bullet}} q\mathcal{N}))_i = (\mathcal{M} \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{N})_i,$$

see (2.52). Utilizing this identification, the structure map α_ϕ of $\mathcal{M} \otimes \mathcal{N}$ can be completely described via those of $\mathcal{M}$ and $\mathcal{N}$. Namely,

6.4 Lemma. *Let $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$, and $\mathcal{M},\mathcal{N} \in \heartsuit(X_\bullet)$, where $\heartsuit$ is* PM *or* Mod*. For $\phi \in I(i,j)$, $\alpha_\phi(\mathcal{M} \otimes \mathcal{N})$ agrees with the composite map*

$$X_\phi^*(\mathcal{M}\otimes\mathcal{N})_i \xrightarrow{m_i^{-1}} X_\phi^*(\mathcal{M}_i\otimes\mathcal{N}_i) \xrightarrow{\Delta} X_\phi^*\mathcal{M}_i\otimes X_\phi^*\mathcal{N}_i \xrightarrow{\alpha_\phi\otimes\alpha_\phi} \mathcal{M}_j\otimes\mathcal{N}_j \xrightarrow{m_j} (\mathcal{M}\otimes\mathcal{N})_j,$$

where $\otimes$ should be replaced by $\otimes^p$ when $\heartsuit =$ PM*.*

Proof (*sketch*). It is not so difficult to show that it suffices to show that $\beta_\phi(\mathcal{M} \otimes \mathcal{N})$ agrees with the composite

$$(\mathcal{M} \otimes \mathcal{N})_i \xrightarrow{m_i^{-1}} \mathcal{M}_i \otimes \mathcal{N}_i \xrightarrow{\beta\otimes\beta} (X_\phi)_*\mathcal{M}_j \otimes (X_\phi)_*\mathcal{N}_j \xrightarrow{m}$$
$$(X_\phi)_*(\mathcal{M}_j \otimes \mathcal{N}_j) \xrightarrow{m_j} (X_\phi)_*(\mathcal{M} \otimes \mathcal{N})_j. \quad (6.5)$$

First we prove this for the case that $\heartsuit =$ PM. For an open subset U of X_i, this composite map evaluated at U is

$$\Gamma((i,U),(\mathcal{M}\otimes\mathcal{N})) = \Gamma((i,U),\mathcal{M}) \otimes_{\Gamma((i,U),\mathcal{O}_{X_\bullet})} \Gamma((i,U),\mathcal{N}) \xrightarrow{\mathrm{res}\otimes\mathrm{res}}$$
$$\Gamma((j,X_\phi^{-1}(U)),\mathcal{M}) \otimes_{\Gamma((i,U),\mathcal{O}_{X_\bullet})} \Gamma((j,X_\phi^{-1}(U)),\mathcal{N}) \xrightarrow{p}$$
$$\Gamma((j,X_\phi^{-1}(U)),\mathcal{M}) \otimes_{\Gamma((j,X_\phi^{-1}(U)),\mathcal{O}_{X_\bullet})} \Gamma((j,X_\phi^{-1}(U)),\mathcal{N})$$
$$= \Gamma((j,X_\phi^{-1}(U)),\mathcal{M}\otimes\mathcal{N}),$$

where $p(m \otimes n) = m \otimes n$. This composite map is nothing but the restriction map of $\mathcal{M} \otimes \mathcal{N}$. So by definition, it agrees with

$$\beta_\phi : \Gamma(U,(\mathcal{M}\otimes\mathcal{N})_i) \to \Gamma(U,(X_\phi)_*(\mathcal{M}\otimes\mathcal{N})_j).$$

Next we consider the case $\heartsuit =$ Mod. First note that the diagram

$$\begin{array}{ccccc}
(a(q\mathcal{M} \otimes^p q\mathcal{N}))_i & & \xrightarrow{\qquad\qquad\beta\qquad\qquad} & & (X_\phi)_*(a(q\mathcal{M} \otimes^p q\mathcal{N}))_j \\
\uparrow \theta & & & & \uparrow \theta \\
a(q\mathcal{M} \otimes^p q\mathcal{N})_i & \xrightarrow{\beta} & a(X_\phi)_*(q\mathcal{M} \otimes^p q\mathcal{N})_j & \xrightarrow{\theta} & (X_\phi)_* a(q\mathcal{M} \otimes^p q\mathcal{N})_j
\end{array}$$

is commutative by Lemma 4.17. By the presheaf version of the lemma, which has been proved in the last paragraph, the diagram

$$\begin{tikzcd}
a(q\mathcal{M} \otimes^p q\mathcal{N})_i \arrow[dd, "\beta"] & a((q\mathcal{M})_i \otimes^p (q\mathcal{N})_i) \arrow[l, "m_i"'] \arrow[d, "\beta\otimes\beta"] \\
 & a((X_\phi)_*(q\mathcal{M})_j \otimes^p (X_\phi)_*(q\mathcal{N})_j) \arrow[d, "m"] \\
a(X_\phi)_*(q\mathcal{M} \otimes^p q\mathcal{N})_j & a(X_\phi)_*((q\mathcal{M})_j \otimes^p (q\mathcal{N})_j) \arrow[l, "m_j"']
\end{tikzcd}$$

is commutative. By the commutativity of the diagram (4.16), the diagram

$$\begin{tikzcd}
a((q\mathcal{M})_i \otimes^p (q\mathcal{N})_i) \arrow[r, "c\otimes c"] \arrow[dd, "\beta\otimes\beta"] & a(q\mathcal{M}_i \otimes^p q\mathcal{N}_i) \arrow[d, "\beta\otimes\beta"] \\
 & a(q(X_\phi)_*\mathcal{M}_j \otimes^p q(X_\phi)_*\mathcal{N}_j) \arrow[d, "c\otimes c"] \\
a((X_\phi)_*(q\mathcal{M})_j \otimes^p (X_\phi)_*(q\mathcal{N})_j) \arrow[r, "c\otimes c"] & a((X_\phi)_*q\mathcal{M}_j \otimes^p (X_\phi)_*q\mathcal{N}_j)
\end{tikzcd}$$

is commutative. Combining the commutativity of these three diagrams (and some other easy commutativity), it is not so difficult to show that the map

$$\beta : (\mathcal{M}\otimes\mathcal{N})_i = (a(q\mathcal{M}\otimes^p q\mathcal{N}))_i \to (X_\phi)_*(a(q\mathcal{M}\otimes^p q\mathcal{N}))_j = (X_\phi)_*(\mathcal{M}\otimes\mathcal{N})_j$$

agrees with the composite

$$\begin{aligned}
(\mathcal{M}\otimes\mathcal{N})_i = (a(q\mathcal{M}\otimes^p q\mathcal{N}))_i &\xrightarrow{\theta^{-1}} a(q\mathcal{M}\otimes^p q\mathcal{N})_i \xrightarrow{m_i^{-1}} a((q\mathcal{M})_i \otimes^p (q\mathcal{N})_i) \\
&\xrightarrow{c\otimes c} a(q\mathcal{M}_i \otimes^p q\mathcal{N}_i) \xrightarrow{\beta\otimes\beta} a(q(X_\phi)_*\mathcal{M}_j \otimes^p q(X_\phi)_*\mathcal{N}_j) \xrightarrow{c\otimes c} \\
a((X_\phi)_*q\mathcal{M}_j \otimes^p (X_\phi)_*q\mathcal{N}_j) &\xrightarrow{m} a(X_\phi)_*(q\mathcal{M}_j \otimes^p q\mathcal{N}_j) \xrightarrow{\theta} (X_\phi)_*a(q\mathcal{M}_j \otimes^p q\mathcal{N}_j) \\
&\xrightarrow{c\otimes c} (X_\phi)_*a((q\mathcal{M})_j \otimes^p (q\mathcal{N})_j) \xrightarrow{m_j} (X_\phi)_*a((q\mathcal{M}\otimes^p q\mathcal{N})_j) \xrightarrow{\theta} \\
&\qquad (X_\phi)_*(a(q\mathcal{M}\otimes^p q\mathcal{N}))_j = (X_\phi)_*(\mathcal{M}\otimes\mathcal{N})_j.
\end{aligned}$$

This composite map agrees with the composite map (6.5). This proves the lemma. $\square$

(6.6) Let $X_\bullet \in \mathcal{P}$, and J a subcategory of I. The left adjoint functor $L_J^\heartsuit = Q(X_\bullet, J)_\#^\heartsuit$ of $(?)_J^\heartsuit$ is given by the structure data as follows explicitly. For $\mathcal{M} \in \heartsuit(X_\bullet|_J)$ and $i \in I$, we have

6.7 Lemma. *There is an isomorphism*

$$\lambda_{J,i} : (L_J^\heartsuit(\mathcal{M}))_i^\heartsuit \cong \varinjlim (X_\phi)_\heartsuit^*(\mathcal{M}_j),$$

where the colimit is taken over the subcategory $(I_i^{(J^{\mathrm{op}} \to I^{\mathrm{op}})})^{\mathrm{op}}$ *of* I/i *whose objects are* $(\phi : j \to i) \in I/i$ *with* $j \in \mathrm{ob}(J)$ *and morphisms are morphisms* φ *of* I/i *such that* $\varphi \in \mathrm{Mor}(J)$*. The translation map of the direct system is given as follows. For morphisms* $\phi : j \to i$ *and* $\psi : j' \to j$*, the translation map* $X_{\phi\psi}^*\mathcal{M}_{j'} \to X_\phi^*\mathcal{M}_j$ *is the composite*

$$X_{\phi\psi}^*\mathcal{M}_{j'} \xrightarrow{d} X_\phi^* X_\psi^* \mathcal{M}_{j'} \xrightarrow{\alpha_\psi} X_\phi^*\mathcal{M}_j.$$

Proof. We prove the lemma for the case that $\heartsuit = \mathrm{PM}, \mathrm{Mod}$. The case that $\heartsuit = \mathrm{PA}, \mathrm{AB}$ is similar and easier.

Consider the case $\heartsuit = \mathrm{PM}$ first. For any object $(\phi, h) : (i, U) \to (j, V)$ of $I_{(i,U)}^{\mathrm{Zar}(X_\bullet|_J) \hookrightarrow \mathrm{Zar}(X_\bullet)}$, consider the obvious map

$$\begin{aligned} \Gamma((i,U), \mathcal{O}_{X_\bullet}) \otimes_{\Gamma((j,V), \mathcal{O}_{X_\bullet|_J})} \Gamma((j,V), \mathcal{M}) &= \Gamma(U, \mathcal{O}_{X_i}) \otimes_{\Gamma(V, \mathcal{O}_{X_j})} \Gamma(V, \mathcal{M}_j) \\ \to \varinjlim_{X_\phi^{-1}(V') \supset U} &\Gamma(U, \mathcal{O}_{X_i}) \otimes_{\Gamma(V', \mathcal{O}_{X_j})} \Gamma(V', \mathcal{M}_j) \\ &= \Gamma(U, X_\phi^*\mathcal{M}_j) \to \varinjlim \Gamma(U, X_{\phi'}^*\mathcal{M}_{j'}), \end{aligned}$$

where the last $\varinjlim$ is taken over $(\phi' : j' \to i) \in (I_i^{(J^{\mathrm{op}} \to I^{\mathrm{op}})})^{\mathrm{op}}$. This map induces a unique map

$$\begin{aligned} &\Gamma(U, (L_J\mathcal{M})_i) = \Gamma((i,U), L_J\mathcal{M}) = \\ &\quad \varinjlim \Gamma((i,U), \mathcal{O}_{X_\bullet}) \otimes_{\Gamma((j,V), \mathcal{O}_{X_\bullet|_J})} \Gamma((j,V), \mathcal{M}) \to \varinjlim \Gamma(U, X_{\phi'}^*\mathcal{M}_{j'}). \end{aligned}$$

It is easy to see that this defines $\lambda_{J,i}$.

We define the inverse of $\lambda_{J,i}$ explicitly. Let $(\phi : j \to i) \in (I_i^{(J^{\mathrm{op}} \to I^{\mathrm{op}})})^{\mathrm{op}}$. Let $U \in \mathrm{Zar}(X_i)$ and $V \in \mathrm{Zar}(X_j)$ such that $U \subset X_\phi^{-1}(V)$. We have an obvious map

$$\begin{aligned} \Gamma(U, \mathcal{O}_{X_i}) \otimes_{\Gamma(V, \mathcal{O}_{X_j})} \Gamma(V, \mathcal{M}_j) &= \Gamma((i,U), \mathcal{O}_{X_\bullet}) \otimes_{\Gamma((j,V), \mathcal{O}_{X_\bullet|_J})} \Gamma((j,V), \mathcal{M}) \\ \to \varinjlim \Gamma((i,U), &\mathcal{O}_{X_\bullet}) \otimes_{\Gamma((j,V), \mathcal{O}_{X_\bullet|_J})} \Gamma((j,V), \mathcal{M}) \\ &= \Gamma((i,U), L_J\mathcal{M}) = \Gamma(U, (L_J\mathcal{M})_i), \end{aligned}$$

which induces

$$\Gamma(U, X_\phi^*\mathcal{M}) = \varinjlim \Gamma(U, \mathcal{O}_{X_i}) \otimes_{\Gamma(V,\mathcal{O}_{X_j})} \Gamma(V, \mathcal{M}_j) \to \Gamma(U, (L_J\mathcal{M})_i).$$

This gives a morphism $X_\phi^*\mathcal{M} \to (L_J\mathcal{M})_i$. It is easy to see that this defines $\varinjlim X_\phi^*\mathcal{M} \to (L_J\mathcal{M})_i$, which is the inverse of $\lambda_{J,i}$. This completes the proof for the case that $\heartsuit = \mathrm{PM}$.

Now consider the case $\heartsuit = \mathrm{Mod}$. Define $\lambda_{J,i}^{\mathrm{Mod}}$ to be the composite

$$\begin{aligned}(L_J^{\mathrm{Mod}}\mathcal{M})_i = (?)_i a L_J^{\mathrm{PM}} q\mathcal{M} &\xrightarrow{\theta^{-1}} a(?)_i L_J^{\mathrm{PM}} q\mathcal{M} \xrightarrow{\lambda_{J,i}^{\mathrm{PM}}} a \varinjlim X_\phi^*(q\mathcal{M})_j \\ &\cong \varinjlim aX_\phi^*(q\mathcal{M})_j \xrightarrow{c} \varinjlim aX_\phi^* q\mathcal{M}_j = \varinjlim X_\phi^*\mathcal{M}_j.\end{aligned}$$

As the morphisms appearing in the composition are all isomorphisms, $\lambda_{J,i}^{\mathrm{Mod}}$ is an isomorphism. $\square$

In particular, we have an isomorphism

$$\lambda_{j,i} : (L_j^\heartsuit(\mathcal{M}))_i^\heartsuit \cong \bigoplus_{\phi \in I(j,i)} (X_\phi)_\heartsuit^*(\mathcal{M}). \tag{6.8}$$

(6.9) As announced in (2.61), we show that the monoidal adjoint pair $((?)_\#^{\mathrm{Mod}}, (?)^\#_{\mathrm{Mod}})$ in Lemma 2.55 is not Lipman.

We define a finite category $\mathcal{K}$ by $\mathrm{ob}(\mathcal{K}) = \{s, t\}$, and $\mathcal{K}(s,t) = \{u, v\}$, $\mathcal{K}(s,s) = \{\mathrm{id}_s\}$, and $\mathcal{K}(t,t) = \{\mathrm{id}_t\}$. Pictorially, $\mathcal{K}$ looks like $t \underset{v}{\overset{u}{\leftleftarrows}} s$. Let k be a field, and define $X_\bullet \in \mathcal{P}(\mathcal{K}, \underline{\mathrm{Sch}})$ by $X_s = X_t = \operatorname{Spec} k$, and $X_u = X_v = \mathrm{id}$. Then $\Gamma(X_t, (L_s\mathcal{O}_{X_s})_t)$ is two-dimensional by (6.8). So $L_s\mathcal{O}_{X_s}$ and $\mathcal{O}_{X_\bullet}$ are not isomorphic by the dimension reason. Similarly, $L_s(\mathcal{O}_{X_s} \otimes_{\mathcal{O}_{X_s}} \mathcal{O}_{X_s})$ cannot be isomorphic to $L_s\mathcal{O}_{X_s} \otimes_{\mathcal{O}_{X_\bullet}} L_s\mathcal{O}_{X_s}$.

Similarly, $((?)_\#^{\mathrm{PM}}, (?)^\#_{\mathrm{PM}})$ in Lemma 2.55 is not Lipman.

(6.10) Let $\psi : i \to i'$ be a morphism. The structure map

$$\alpha_\psi : (X_\psi)_\heartsuit^*((L_J^\heartsuit(\mathcal{M}))_i^\heartsuit) \to (L_J^\heartsuit(\mathcal{M}))_{i'}^\heartsuit$$

is induced by

$$(X_\psi)_\heartsuit^*((X_\phi)_\heartsuit^*(\mathcal{M}_j)) \cong (X_{\psi\phi})_\heartsuit^*(\mathcal{M}_j).$$

More precisely, for $\psi : i \to i'$, the diagram

$$\begin{array}{ccccc} X_\psi^*((L_J\mathcal{M})_i) & \xrightarrow{\lambda_{J,i}} & X_\psi^* \varinjlim X_\phi^*\mathcal{M}_j & \cong & \varinjlim X_\psi^* X_\phi^*\mathcal{M}_j \\ \downarrow \alpha_\psi & & & & \downarrow h \\ (L_J\mathcal{M})_{i'} & & \xrightarrow{\qquad\lambda_{J,i'}\qquad} & & \varinjlim X_{\phi'}^*\mathcal{M}_{j'} \end{array}$$

is commutative, where $\phi : i \to j$ runs through $(I_i^f)^{\mathrm{op}}$, and $\phi' : i' \to j'$ runs through $(I_{i'}^f)^{\mathrm{op}}$, where $f : J^{\mathrm{op}} \to I^{\mathrm{op}}$ is the inclusion. The map h is induced by $d : X_\psi^* X_\phi^* \to (X_\phi X_\psi)^* = X_{\psi\phi}^*$. This is checked at the section level directly when $\heartsuit = \mathrm{PM}$.

We consider the case that $\heartsuit = \mathrm{Mod}$. Then the composite

$$X_\psi^*(?)_i L_J \xrightarrow{\lambda_{J,i}} X_\psi^* \varinjlim X_\phi^*(?)_j \cong \varinjlim X_\psi^* X_\phi^*(?)_j \xrightarrow{h} \varinjlim X_{\phi'}^*(?)_{j'}$$

agrees with the composite

$$\begin{aligned} X_\psi^*(?)_i L_J = aX_\psi^* q(?)_i a L_J q \xrightarrow{\theta^{-1}} aX_\psi^* qa(?)_i L_J q \xrightarrow{\lambda_{J,i}^{\mathrm{PM}}} aX_\psi^* qa \varinjlim X_\phi^*(?)_j q \\ \xrightarrow{\cong} aX_\psi^* q \varinjlim aX_\phi^*(?)_j q \xrightarrow{c} aX_\psi^* q \varinjlim aX_\phi^* q(?)_j \xrightarrow{\cong} \varinjlim aX_\psi^* qaX_\phi^* q(?)_j \xrightarrow{u^{-1}} \\ \varinjlim aX_\psi^* X_\phi^* q(?)_j \xrightarrow{d} \varinjlim aX_{\psi\phi}^* q(?)_j \to \varinjlim aX_{\phi'}^* q(?)_{j'} = \varinjlim X_{\phi'}^*(?)_{j'}. \end{aligned}$$

Using Lemma 2.60, it is straightforward to show that this map agrees with

$$\begin{aligned} X_\psi^*(?)_i L_J = aX_\psi^* q(?)_i a L_J q \xrightarrow{c} aX_\psi^*(?)_i qa L_J q \xrightarrow{\alpha_\psi} a(?)_{i'} qa L_J q \xrightarrow{c} \\ aq(?)_{i'} a L_J q \xrightarrow{\varepsilon} (?)_{i'} a L_J q \xrightarrow{\theta^{-1}} a(?)_{i'} L_J q \xrightarrow{\lambda_{J,i'}} a \varinjlim X_{\phi'}^*(?)_{j'} q \xrightarrow{\cong} \\ \varinjlim aX_{\phi'}^*(?)_{j'} q \xrightarrow{c} \varinjlim aX_{\phi'}^* q(?)_{j'} = \varinjlim X_{\phi'}^*(?)_{j'}. \end{aligned}$$

This composite map agrees with

$$X_\psi^*(?)_i L_J \xrightarrow{\alpha_\psi} (?)_{i'} L_J \xrightarrow{\lambda_{J,i'}} \varinjlim X_{\phi'}^*(?)_{j'}$$

by (4.20) and the definition of $\lambda_{J,i'}$ for sheaves (see the proof of Lemma 6.7). This is what we wanted to prove.

The case that $\heartsuit = \mathrm{PA}, \mathrm{AB}$ is proved similarly.

(6.11) In the remainder of this chapter, we do not give detailed proofs, since the strategy is similar to the above (just check the commutativity at the section level for presheaves, and sheafify it).

(6.12) The counit map $\varepsilon : L_J(?)_J \to \mathrm{Id}$ is given as a morphism of structure data as follows.

$$\varepsilon_i : (?)_i L_J(?)_J \to (?)_i$$

agrees with

$$(?)_i L_J(?)_J \xrightarrow{\lambda_{J,i}} \varinjlim X_\phi^*(?)_j(?)_J \xrightarrow{c} \varinjlim X_\phi^*(?)_j \xrightarrow{\alpha} (?)_i,$$

where α is induced by $\alpha_\phi : X_\phi^*(?)_j \to (?)_i$.

(6.13) The unit map $u : \mathrm{Id} \to (?)_J L_J$ is also described, as follows.

$$u_j : (?)_j \to (?)_j (?)_J L_J$$

agrees with

$$(?)_j \xrightarrow{\mathfrak{f}^{-1}} X^*_{\mathrm{id}_j}(?)_j \to \varinjlim X^*_\phi(?)_k \xrightarrow{\lambda^{-1}_{J,j}} (?)_j L_J \cong (?)_j(?)_J L_J,$$

where the colimit is taken over $(\phi : k \to j) \in (I_j^{(J^{\mathrm{op}} \subset I^{\mathrm{op}})})^{\mathrm{op}}$.

(6.14) Let $X_\bullet \in \mathcal{P}$, and J a subcategory of I. The right adjoint functor $R_J^\heartsuit$ of $(?)_J^\heartsuit$ is given as follows explicitly. For $\mathcal{M} \in \heartsuit(X_\bullet|_J)$ and $i \in I$, we have

$$\rho^{J,i} : (R_J^\heartsuit(\mathcal{M}))_i^\heartsuit \cong \varprojlim (X_\phi)_*^\heartsuit(\mathcal{M}_j),$$

where the limit is taken over $I_i^{(J\to I)}$, see (2.6) for the notation. The descriptions of α, u, and ε for the right induction are left to the reader.

6.15 Lemma. *Let $X_\bullet \in \mathcal{P}$, and J a full subcategory of I. Then we have the following.*

1 *The counit of adjunction $\varepsilon : (?)_J^\heartsuit \circ R_J^\heartsuit \to \mathrm{Id}$ is an isomorphism. In particular, $R_J^\heartsuit$ is full and faithful.*

2 *The unit of adjunction $u : \mathrm{Id} \to (?)_J^\heartsuit \circ L_J^\heartsuit$ is an isomorphism. In particular, $L_J^\heartsuit$ is full and faithful.*

Proof. **1** For $i \in J$, the restriction

$$\varepsilon_i : (?)_i^\heartsuit (?)_J^\heartsuit R_J^\heartsuit \mathcal{M} = \varprojlim (X_\phi)_*^\heartsuit(\mathcal{M}_j) \to (X_{\mathrm{id}_i})_* \mathcal{M}_i = \mathcal{M}_i = (?)_i \mathcal{M}$$

is nothing but the canonical map from the projective limit, where the limit is taken over $(\phi : i \to j) \in I_i^{(J\to I)}$. As J is a full subcategory, we have $I_i^{(J\to I)}$ equals i/J, and hence id_i is its initial object. So the limit is equal to $\mathcal{M}_i$, and ε_i is the identity map. Since ε_i is an isomorphism for each $i \in J$, we have that ε is an isomorphism.

The proof of **2** is similar, and we omit it. □

Let $\mathcal{C}$ be a small category. A connected component of $\mathcal{C}$ is a full subcategory of $\mathcal{C}$ whose object set is one of the equivalence classes of $\mathrm{ob}(\mathcal{C})$ with respect to the transitive symmetric closure of the relation $\sim$ given by

$$c \sim c' \iff \mathcal{C}(c, c') \neq \emptyset.$$

6.16 Definition. We say that a subcategory J of I is *admissible* if

1 For $i \in I$, the category $(I_i^{(J^{\text{op}} \subset I^{\text{op}})})^{\text{op}}$ is pseudofiltered.

2 For $j \in J$, we have id_j is the initial object of one of the connected components of $I_j^{(J^{\text{op}} \subset I^{\text{op}})}$ (i.e., id_j is the terminal object of one of the connected components of $(I_j^{(J^{\text{op}} \subset I^{\text{op}})})^{\text{op}}$).

Note that for $j \in I$, the subcategory $j = (\{j\}, \{\text{id}_j\})$ of I is admissible.

In Lemma 6.7, the colimit in the right hand side is pseudo-filtered and hence it preserves exactness, if **1** is satisfied. In particular, if **1** is satisfied, then $Q(X_\bullet, J) : \text{Zar}(X_\bullet|_J) \to \text{Zar}(X_\bullet)$ is an admissible functor. As in the proof of Lemma 6.15, $(?)_j$ is a direct summand of $(?)_j \circ L_J$ for $j \in J$ so that L_J is faithful, if **2** is satisfied. We have the following.

6.17 Lemma. *Let $X_\bullet \in \mathcal{P}(I, \underline{\text{Sch}}/S)$, and $K \subset J \subset I$ be admissible subcategories of I. Then $L_{J,K}^{\text{PA}}$ is faithful and exact. The morphism of sites $Q(X_\bullet|_J, K)$ is admissible. If, moreover, X_ϕ is flat for any $\phi \in I(k, j)$ with $j \in J$ and $k \in K$, then $L_{J,K}^{\heartsuit}$ is faithful and exact for $\heartsuit = \text{Mod}$.*

Proof. Assume that $\mathcal{M} \in \heartsuit(X_\bullet|_K)$, $\mathcal{M} \neq 0$, and $L_{J,K}\mathcal{M} = 0$. There exists some $k \in K$ such that $M_k \neq 0$. Since $L_{J,K}\mathcal{M} = 0$, we have that $0 \cong (?)_k L_{I,J} L_{J,K}\mathcal{M} \cong (?)_k L_{I,K}\mathcal{M}$. This contradicts the fact that $\mathcal{M}_k$ is a direct summand of $(L_{I,K}\mathcal{M})_k$. Hence $L_{J,K}$ is faithful.

We prove that $L_{J,K}^{\heartsuit}$ is exact. It suffices to show that for any $j \in J$, $(?)_j L_{J,K}$ is exact. As J is admissible, $(?)_j$ is a direct summand of $(?)_j L_{I,J}$. Hence it suffices to show that $(?)_j L_{I,K} \cong (?)_j L_{I,J} L_{J,K}$ is exact. By Lemma 6.7, $(?)_j L_{I,K} \cong \varinjlim (X_\phi)^*_\heartsuit (?)_k$, where the colimit is taken over $(\phi\colon k \to j) \in (I_j^{K^{\text{op}} \subset I^{\text{op}}})^{\text{op}}$. By assumption, $(X_\phi)^*_\heartsuit$ is exact for any ϕ in the colimit. As $(I_j^{K^{\text{op}} \subset I^{\text{op}}})^{\text{op}}$ is pseudo-filtered by assumption, $(?)_j L_{I,K}$ is exact, as desired. □

(6.18) As in Example 5.6, **2**, we have an isomorphism

$$c_{i,f_\bullet} : (?)_i \circ (f_\bullet)_* \cong (f_i)_* \circ (?)_i. \tag{6.19}$$

The translation α_ϕ is described as follows.

6.20 Lemma. *Let $f_\bullet\colon X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}(I, \underline{\text{Sch}}/S)$. For $\phi \in I(i, j)$,*

$$\alpha_\phi(f_\bullet)_* : Y_\phi^*(?)_i(f_\bullet)_* \to (?)_j(f_\bullet)_*$$

agrees with

$$
\begin{aligned}
Y_\phi^*(?)_i(f_\bullet)_* \xrightarrow{c_{i,f_\bullet}} Y_\phi^*(f_i)_*(?)_i \xrightarrow{\text{via } \theta} (f_j)_*X_\phi^*(?)_i& \\
\xrightarrow{(f_j)_*\alpha_\phi} (f_j)_*(?)_j \xrightarrow{c_{j,f_\bullet}^{-1}} (?)_j(f_\bullet)_*,& \qquad (6.21)
\end{aligned}
$$

where θ is Lipman's theta [26, (3.7.2)].

One of the definitions of θ is the composite

$$\theta : Y_\phi^*(f_i)_* \xrightarrow{\text{via } u} Y_\phi^*(f_i)_*(X_\phi)_*X_\phi^* \xrightarrow{c} Y_\phi^*(Y_\phi)_*(f_j)_*X_\phi^* \xrightarrow{\text{via } \varepsilon} (f_j)_*X_\phi^*.$$

Proof. Note that the diagram

$$
\begin{tikzcd}
(?)_i(f_\bullet)_* \arrow[r, "\beta"] \arrow[d, "c"] & (Y_\phi)_*(?)_j(f_\bullet)_* \arrow[r, "c"] & (Y_\phi)_*(f_j)_*(?)_j \arrow[d, "c"] \\
(f_i)_*(?)_i \arrow[rr, "(f_i)_*\beta"] & & (f_i)_*(X_\phi)_*(?)_j
\end{tikzcd}
\qquad (6.22)
$$

is commutative. Indeed, when we apply the functor $\Gamma(U, ?)$ for an open subset U of Y_i, then we get an obvious commutative diagram

$$
\begin{tikzcd}
\Gamma((i, f_i^{-1}(U)), ?) \arrow[r, "\mathrm{res}"] \arrow[d, "\mathrm{id}"] & \Gamma((j, f_j^{-1}(Y_\phi^{-1}(U))), ?) \arrow[r, "\mathrm{id}"] & \Gamma((j, f_j^{-1}(Y_\phi^{-1}(U))), ?) \arrow[d, "\mathrm{id}"] \\
\Gamma((i, f_i^{-1}(U)), ?) \arrow[rr, "\mathrm{res}"] & & \Gamma((j, X_\phi^{-1}(f_i^{-1}(U))), ?).
\end{tikzcd}
$$

Now the assertion of the lemma follows from the commutativity of the diagram

$$
\begin{tikzcd}
Y_\phi^*(?)_i(f_\bullet)_* \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"] \arrow[dr, phantom, "\text{(a)}"] & (?)_j(f_\bullet)_* \arrow[r, "c"] \arrow[dr, phantom, "\text{(b)}"] & (f_j)_*(?)_j \arrow[r, "\mathrm{id}"] & (f_j)_*(?)_j \arrow[ddd, "\mathrm{id}"] \\
Y_\phi^*(?)_i(f_\bullet)_* \arrow[r, "\beta"] \arrow[d, "c"] & Y_\phi^*(Y_\phi)_*(?)_j(f_\bullet)_* \arrow[r, "c"] \arrow[u, "\varepsilon"] \arrow[d, phantom, "\text{(c)}"] & Y_\phi^*(Y_\phi)_*(f_j)_*(?)_j \arrow[u, "\varepsilon"] \arrow[d, "c"] & \\
Y_\phi^*(f_i)_*(?)_i \arrow[rr, "\beta"] \arrow[d, "\theta"] & \arrow[d, phantom, "\text{(d)}"] & Y_\phi^*(f_i)_*(X_\phi)_*(?)_j \arrow[d, "\theta"] \arrow[r, phantom, "\text{(f)}"] & \\
(f_j)_*X_\phi^*(?)_i \arrow[rr, "\beta"] \arrow[d, "\mathrm{id}"] & \arrow[d, phantom, "\text{(e)}"] & (f_j)_*X_\phi^*(X_\phi)_*(?)_j \arrow[d, "\varepsilon"] & \\
(f_j)_*X_\phi^*(?)_i \arrow[rr, "\alpha"] & & (f_j)_*(?)_j & (f_j)_*(?)_j. \arrow[l, "\mathrm{id}"']
\end{tikzcd}
$$

Indeed, the commutativity of (a) and (e) is the definition of α. The commutativity of (b) follows from the naturality of ε. The commutativity of (c) follows from the commutativity of (6.22). The commutativity of (d) is the naturality of θ. The commutativity of (f) follows from the definition of θ and the fact that the composite

$$(X_\phi)_* \xrightarrow{u} (X_\phi)_* X_\phi^*(X_\phi)_* \xrightarrow{\varepsilon} (X_\phi)_*$$

is the identity. $\square$

6.23 Proposition. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}$, J a subcategory of I, and $i \in I$. Then the composite map*

$$(?)_i L_J(f_\bullet|_J)_* \xrightarrow{\text{via } \theta} (?)_i(f_\bullet)_* L_J \xrightarrow{\text{via } c_{i,f_\bullet}} (f_i)_*(?)_i L_J$$

agrees with the composite map

$$(?)_i L_J(f_\bullet|_J)_* \xrightarrow{\text{via } \lambda_{J,i}} \varinjlim Y_\phi^*(?)_j(f_\bullet|_J)_* \xrightarrow{\text{via } c_{j,f_\bullet|_J}} \varinjlim Y_\phi^*(f_j)_*(?)_j$$
$$\xrightarrow{\text{via } \theta} \varinjlim (f_i)_* X_\phi^*(?)_j \to (f_i)_* \varinjlim X_\phi^*(?)_j \xrightarrow{\text{via } \lambda_{J,i}^{-1}} (f_i)_*(?)_i L_J.$$

Proof. Note that θ in the first composite map is the composite

$$\theta = \theta(J, f_\bullet) : L_J(f_\bullet|_J)_* \xrightarrow{\text{via } u} L_J(f_\bullet|_J)_*(?)_J L_J \xrightarrow{c} L_J(?)_J(f_\bullet)_* L_J \xrightarrow{\varepsilon} (f_\bullet)_* L_J.$$

The description of u and ε are already given, and the proof is reduced to the iterative use of (6.10), (6.12), (6.13), and Lemma 6.20. The detailed argument is left to a patient reader. The reason why the second map involves θ is Lemma 6.20. $\square$

Similarly, we have the following.

6.24 Proposition. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}$, J a subcategory of I, and $i \in I$. Then the composite map*

$$(f_i)^*(?)_i L_J \xrightarrow{\text{via } \theta(f_\bullet, i)} (?)_i(f_\bullet)^* L_J \xrightarrow{\text{via } d_{f_\bullet, J}} (?)_i L_J(f_\bullet|_J)^*$$

agrees with the composite map

$$(f_i)^*(?)_i L_J \xrightarrow{\text{via } \lambda_{J,i}} (f_i)^* \varinjlim Y_\phi^*(?)_j \cong \varinjlim (f_i)^* Y_\phi^*(?)_j$$
$$\xrightarrow{d} \varinjlim X_\phi^*(f_j)^*(?)_j \xrightarrow{\text{via } \theta(f_\bullet|_J, j)} \varinjlim X_\phi^*(?)_j(f_\bullet|_J)^* \xrightarrow{\text{via } \lambda_{J,i}^{-1}} (?)_i L_J(f_\bullet|_J)^*.$$

The proof is left to the reader. The proof of Proposition 6.23 and Proposition 6.24 are formal, and the propositions are valid for $\heartsuit = \mathrm{PM}, \mathrm{Mod}, \mathrm{PA}$, and AB.

Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}$, and $J \subset I$ a subcategory. The inverse image $(f_\bullet)^*_\heartsuit$ is compatible with the restriction $(?)_J$.

6.25 Lemma. *The natural map*

$$\theta_\heartsuit = \theta_\heartsuit(f_\bullet, J) : ((f_\bullet)|_J)^*_\heartsuit \circ (?)_J \to (?)_J \circ (f_\bullet)^*_\heartsuit$$

is an isomorphism for $\heartsuit = \mathrm{PA}, \mathrm{AB}, \mathrm{PM}, \mathrm{Mod}$*. In particular,* $f_\bullet^{-1} : \mathrm{Zar}(Y_\bullet) \to \mathrm{Zar}(X_\bullet)$ *is an admissible continuous functor.*

Proof. We consider the case where $\heartsuit = \mathrm{PM}$.

Let $\mathcal{M} \in \mathrm{PM}(Y_\bullet)$, and $(j, U) \in \mathrm{Zar}(X_\bullet|_J)$. We have

$$\Gamma((j,U),(f_\bullet|_J)^*\mathcal{M}_J) = \varinjlim \Gamma((j,U),\mathcal{O}_{X_\bullet}) \otimes_{\Gamma((j',V),\mathcal{O}_{Y_\bullet})} \Gamma((j',V),\mathcal{M}),$$

where the colimit is taken over $(j',V) \in (I_{(j,U)}^{(f_\bullet|_J)^{-1}})^{\mathrm{op}}$. On the other hand, we have

$$\Gamma((j,U),(?)_J f_\bullet^*\mathcal{M}) = \varinjlim \Gamma((j,U),\mathcal{O}_{X_\bullet}) \otimes_{\Gamma((i,V),\mathcal{O}_{Y_\bullet})} \Gamma((i,V),\mathcal{M}),$$

where the colimit is taken over $(i,V) \in (I_{(j,U)}^{f_\bullet^{-1}})^{\mathrm{op}}$. There is an obvious map from the first to the second. This obvious map is θ, see (2.57).

To verify that this is an isomorphism, it suffices to show that the category $(I_{(j,U)}^{(f_\bullet|_J)^{-1}})^{\mathrm{op}}$ is final in the category $(I_{(j,U)}^{f_\bullet^{-1}})^{\mathrm{op}}$. In fact, any $(\phi, h) : (j,U) \to (i, f_i^{-1}(V))$ with $(i,V) \in \mathrm{Zar}(Y_\bullet)$ factors through

$$(\mathrm{id}_j, h) : (j,U) \to (j, f_j^{-1} Y_\phi^{-1}(V)).$$

Hence, $\theta_\heartsuit$ is an isomorphism for $\heartsuit = \mathrm{PM}$. The construction for the case where $\heartsuit = \mathrm{PA}$ is similar.

As $(?)_J$ is compatible with the sheafification by Lemma 2.31, we have that θ is an isomorphism for $\heartsuit = \mathrm{Mod}, \mathrm{AB}$ by Lemma 2.59. $\square$

6.26 Corollary. *The conjugate*

$$\xi_\heartsuit = \xi_\heartsuit(f_\bullet, J) : (f_\bullet)_*^\heartsuit R_J \to R_J (f_\bullet|_J)_*^\heartsuit$$

of $\theta_\heartsuit(f_\bullet, J)$ *is an isomorphism for* $\heartsuit = \mathrm{PA}, \mathrm{AB}, \mathrm{PM}, \mathrm{Mod}$.

Proof. Obvious by Lemma 6.25. $\square$

(6.27) By Corollary 6.26, we may define the composite

$$\mu_{\heartsuit} = \mu_{\heartsuit}(f_\bullet, J) : f_\bullet^* R_J \xrightarrow{u} f_\bullet^* R_J (f_\bullet|_J)_*(f_\bullet|_J)^* \xrightarrow{\xi^{-1}} f_\bullet^*(f_\bullet)_* R_J (f_\bullet|_J)^* \xrightarrow{\varepsilon} R_J (f_\bullet|_J)^*.$$

Observe that the diagram

$$\begin{tikzcd}
(?)_i f_\bullet^* R_J \arrow[r, "\theta^{-1}"] \arrow[d, "\mu"] & f_i^*(?)_i R_J \arrow[r, "\rho"] & f_i^* \varprojlim (Y_\phi)_*(?)_j \arrow[r] & \varprojlim f_i^*(Y_\phi)_*(?)_j \arrow[dll, "\theta"] \\
(?)_i R_J f_\bullet|_J^* \arrow[r, "\rho"] & \varprojlim (X_\phi)_*(?)_j f_\bullet|_J^* \arrow[r, "\theta^{-1}"] & \varprojlim (X_\phi)_* f_j^*(?)_j &
\end{tikzcd}$$

is commutative.

6.28 Lemma. *Let the notation be as above, and $\mathcal{M}, \mathcal{N} \in \heartsuit(Y_\bullet)$. Then the diagram*

$$\begin{array}{ccccc}
(f_\bullet|_J)_\heartsuit^*(\mathcal{M}_J \otimes \mathcal{N}_J) & \xrightarrow{m} & (f_\bullet|_J)_\heartsuit^*((\mathcal{M}\otimes\mathcal{N})_J) & \xrightarrow{\theta} & ((f_\bullet)_\heartsuit^*(\mathcal{M}\otimes\mathcal{N}))_J \\
\downarrow \Delta & & & & \downarrow (?)_J\Delta \\
(f_\bullet|_J)_\heartsuit^*\mathcal{M}_J \otimes (f_\bullet|_J)_\heartsuit^*\mathcal{N}_J & \xrightarrow{\theta\otimes\theta} & ((f_\bullet)_\heartsuit^*\mathcal{M})_J \otimes ((f_\bullet)_\heartsuit^*\mathcal{N})_J & \xrightarrow{m} & ((f_\bullet)_\heartsuit^*\mathcal{M} \otimes (f_\bullet)_\heartsuit^*\mathcal{N})_J
\end{array} \tag{6.29}$$

is commutative.

Proof. This is an immediate consequence of Lemma 1.44. □

6.30 Corollary. *The adjoint pair $((?)^*_{\mathrm{Mod}}, (?)_*^{\mathrm{Mod}})$ over the category $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$ is Lipman.*

Proof. Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism of $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. It is easy to see that the diagram

$$\begin{tikzcd}
(?)_i \mathcal{O}_{Y_\bullet} \arrow[r, "="] \arrow[d, "(?)_i\eta"] & \mathcal{O}_{Y_i} \arrow[dr, "\eta"] & \\
(?)_i (f_\bullet)_* \mathcal{O}_{X_\bullet} \arrow[r, "c"] & (f_i)_*(?)_i \mathcal{O}_{X_\bullet} \arrow[r, "="] & (f_i)_* \mathcal{O}_{X_i}
\end{tikzcd}$$

is commutative. So utilizing Lemma 1.25, it is easy to see that

$$\begin{tikzcd}
(?)_i f_\bullet^* \mathcal{O}_{Y_\bullet} \arrow[r, "\theta^{-1}"] \arrow[d, "(?)_i C"] & f_i^*(?)_i \mathcal{O}_{Y_\bullet} \arrow[r, "="] & f_i^* \mathcal{O}_{Y_i} \arrow[dl, "C"] \\
(?)_i \mathcal{O}_{X_\bullet} \arrow[r, "="] & \mathcal{O}_{X_i} &
\end{tikzcd}$$

is also commutative. Since $C\colon f_i^*\mathcal{O}_{Y_i} \to \mathcal{O}_{X_i}$ is an isomorphism by Corollary 2.65, $(?)_i C$ is an isomorphism for any $i \in I$. Hence $C\colon f_\bullet^*\mathcal{O}_{Y_\bullet} \to \mathcal{O}_{X_\bullet}$ is also an isomorphism.

Let us consider $\mathcal{M}, \mathcal{N} \in \heartsuit(Y_\bullet)$. To verify that Δ is an isomorphism, it suffices to show that

$$(?)_i\Delta : (f_\bullet^*(\mathcal{M} \otimes \mathcal{N}))_i \to (f_\bullet^*\mathcal{M} \otimes f_\bullet^*\mathcal{N})_i$$

is an isomorphism for any $i \in \mathrm{ob}(I)$. Now consider the diagram (6.29) for $J = i$. Horizontal maps in the diagram are isomorphisms by (6.3) and Lemma 6.25. The left Δ is an isomorphism, since f_i is a morphism of single schemes. By Lemma 6.28, $(?)_i\Delta$ is also an isomorphism. $\square$

(6.31) The description of the translation map α_ϕ for $f_\bullet^*$ is as follows. For $\phi \in I(i, j)$,

$$\alpha_\phi : X_\phi^*(?)_i f_\bullet^* \to (?)_j f_\bullet^*$$

is the composite

$$X_\phi^*(?)_i f_\bullet^* \xrightarrow{X_\phi^*\theta^{-1}} X_\phi^* f_i^*(?)_i \xrightarrow{d} f_j^* Y_\phi^*(?)_i \xrightarrow{f_j^*\alpha_\phi} f_j^*(?)_j \xrightarrow{\theta} (?)_j f_\bullet^*.$$

(6.32) Let $X_\bullet \in \mathcal{P}$, and $\mathcal{M}, \mathcal{N} \in \heartsuit(X_\bullet)$. Although there is a canonical map

$$H_i : \underline{\mathrm{Hom}}_{\heartsuit(X_\bullet)}(\mathcal{M}, \mathcal{N})_i \to \underline{\mathrm{Hom}}_{\heartsuit(X_i)}(\mathcal{M}_i, \mathcal{N}_i)$$

arising from the closed structure for $i \in I$, this may not be an isomorphism. However, we have the following.

6.33 Lemma. *Let $i \in I$. If $\mathcal{M}$ is equivariant, then the canonical map*

$$H_i : \underline{\mathrm{Hom}}_{\heartsuit(X_\bullet)}(\mathcal{M}, \mathcal{N})_i \to \underline{\mathrm{Hom}}_{\heartsuit(X_i)}(\mathcal{M}_i, \mathcal{N}_i)$$

is an isomorphism of presheaves. In particular, it is an isomorphism in $\heartsuit(X_i)$.

Proof. It suffices to prove that

$$H_i : \mathrm{Hom}_{\heartsuit(\mathrm{Zar}(X_\bullet)/(i,U))}(\mathcal{M}|_{(i,U)}, \mathcal{N}|_{(i,U)}) \to \mathrm{Hom}_{\heartsuit(U)}(\mathcal{M}_i|_U, \mathcal{N}_i|_U)$$

is an isomorphism for any Zariski open set U in X_i.

To give an element of $\varphi \in \mathrm{Hom}_{\heartsuit(\mathrm{Zar}(X_\bullet)/(i,U))}(\mathcal{M}|_{(i,U)}, \mathcal{N}|_{(i,U)})$ is the same as to give a family $(\varphi_\phi)_{\phi : i \to j}$ with

$$\varphi_\phi \in \mathrm{Hom}_{\heartsuit(X_\phi^{-1}(U))}(\mathcal{M}_j|_{X_\phi^{-1}(U)}, \mathcal{N}_j|_{X_\phi^{-1}(U)})$$

such that for any $\phi \colon i \to j$ and $\psi \colon j \to j'$,

$$\varphi_{\psi\phi} \circ (\alpha_\psi(\mathcal{M}))|_{X_{\psi\phi}^{-1}(U)} = (\alpha_\psi(\mathcal{N}))|_{X_{\psi\phi}^{-1}(U)} \circ ((X_\psi)|_{X_{\psi\phi}^{-1}(U)})^*_\heartsuit(\varphi_\phi). \tag{6.34}$$

As $\alpha_\phi(\mathcal{M})$ is an isomorphism for any $\phi : i \to j$, we have that such a (φ_ϕ) is uniquely determined by φ_{id_i} by the formula

$$\varphi_\phi = (\alpha_\phi(\mathcal{N}))|_{X_\phi^{-1}(U)} \circ ((X_\phi)|_{X_\phi^{-1}(U)})^*_\heartsuit(\varphi_{\mathrm{id}_i}) \circ (\alpha_\phi(\mathcal{M}))|^{-1}_{X_\phi^{-1}(U)}. \tag{6.35}$$

Conversely, fix φ_{id_i}, and define φ_ϕ by (6.35). Consider the diagram

$$\begin{tikzcd}
X^*_{\psi\phi}\mathcal{M}_i \arrow[r, "d^{-1}"] \arrow[d, "\varphi_{\mathrm{id}_i}"'] \arrow[dr, phantom, "\text{(a)}"] & X^*_\psi X^*_\phi \mathcal{M}_i \arrow[r, "\alpha_\phi"] \arrow[d, "\varphi_{\mathrm{id}_i}"] \arrow[dr, phantom, "\text{(b)}"] & X^*_\psi \mathcal{M}_j \arrow[r, "\alpha_\psi"] \arrow[d, "\varphi_\phi"] \arrow[dr, phantom, "\text{(c)}"] & \mathcal{M}_{j'} \arrow[d, "\varphi_{\psi\varphi}"] \\
X^*_{\psi\varphi}\mathcal{N}_i \arrow[r, "d^{-1}"] & X^*_\psi X^*_\phi \mathcal{N}_i \arrow[r, "\alpha_\phi"] & X^*_\psi \mathcal{N}_j \arrow[r, "\alpha_\psi"] & \mathcal{N}_{j'}\ .
\end{tikzcd}$$

The diagram (a) is commutative by the naturality of d^{-1}. The diagram (b) and (a)+(b)+(c) are commutative, by the definition of φ_ϕ and $\varphi_{\psi\phi}$ (6.35), respectively. Since d^{-1} and $\alpha_\phi(\mathcal{M})$ are isomorphisms, the diagram (c) is commutative, and hence (6.34) holds. Hence H_i is bijective, as desired. □

6.36 Lemma. *Let J be a subcategory of I. If $\mathcal{M}$ is equivariant, then the canonical map*

$$H_J : \underline{\mathrm{Hom}}_{\heartsuit(X_\bullet)}(\mathcal{M}, \mathcal{N})_J \to \underline{\mathrm{Hom}}_{\heartsuit(X_\bullet|_J)}(\mathcal{M}_J, \mathcal{N}_J)$$

is an isomorphism of presheaves. In particular, it is an isomorphism in $\heartsuit(X_\bullet|_J)$.

Proof. It suffices to show that

$$(H_J)_i : (\underline{\mathrm{Hom}}_{\heartsuit(X_\bullet)}(\mathcal{M}, \mathcal{N})_J)_i \to \underline{\mathrm{Hom}}_{\heartsuit(X_J)}(\mathcal{M}_J, \mathcal{N}_J)_i$$

is an isomorphism for each $i \in J$. By Lemma 1.39, the composite map

$$\underline{\mathrm{Hom}}_{\heartsuit(X_\bullet)}(\mathcal{M}, \mathcal{N})_i \cong (\underline{\mathrm{Hom}}_{\heartsuit(X_\bullet)}(\mathcal{M}, \mathcal{N})_J)_i$$
$$\xrightarrow{(H_J)_i} \underline{\mathrm{Hom}}_{\heartsuit(X_J)}(\mathcal{M}_J, \mathcal{N}_J)_i \xrightarrow{H_i} \underline{\mathrm{Hom}}_{\heartsuit(X_i)}(\mathcal{M}_i, \mathcal{N}_i)$$

agrees with H_i. As $\mathcal{M}_J$ is also equivariant, we have that the two H_i are isomorphisms by Lemma 6.33, and hence $(H_J)_i$ is an isomorphism for any $i \in J$. □

(6.37) By the lemma, the sheaf $\underline{\mathrm{Hom}}_{\heartsuit(X_\bullet)}(\mathcal{M},\mathcal{N})$ is given by the collection

$$(\underline{\mathrm{Hom}}_{\heartsuit(X_i)}(\mathcal{M}_i,\mathcal{N}_i))_{i\in I}$$

provided $\mathcal{M}$ is equivariant. The structure map is the canonical composite map

$$\alpha_\phi : (X_\phi)^*_\heartsuit \underline{\mathrm{Hom}}_{\heartsuit(X_i)}(\mathcal{M}_i,\mathcal{N}_i) \xrightarrow{P} \underline{\mathrm{Hom}}_{\heartsuit(X_j)}((X_\phi)^*_\heartsuit\mathcal{M}_i,(X_\phi)^*_\heartsuit\mathcal{N}_i)$$
$$\xrightarrow{\underline{\mathrm{Hom}}_{\heartsuit(X_j)}(\alpha_\phi^{-1},\alpha_\phi)} \underline{\mathrm{Hom}}_{\heartsuit(X_j)}(\mathcal{M}_j,\mathcal{N}_j).$$

Similarly, the following is also easy to prove.

6.38 Lemma. *Let $i\in I$ be an initial object of I. Then the following hold:*

1 *If $\mathcal{M}\in\heartsuit(X_\bullet)$ is equivariant, then*

$$(?)_i : \mathrm{Hom}_{\heartsuit(X_\bullet)}(\mathcal{M},\mathcal{N}) \to \mathrm{Hom}_{\heartsuit(X_i)}(\mathcal{M}_i,\mathcal{N}_i)$$

is an isomorphism.

2 $(?)_i : \mathrm{EM}(X_\bullet)\to\mathrm{Mod}(X_i)$ *is an equivalence, whose quasi-inverse is L_i.*

The fact that $L_i(\mathcal{M})$ is equivariant for $\mathcal{M}\in\mathrm{Mod}(X_i)$ is checked directly from the definition.

7 Quasi-coherent sheaves over a diagram of schemes

Let I be a small category, S a scheme, and $X_\bullet\in\mathcal{P}(I,\underline{\mathrm{Sch}}/S)$.

(7.1) Let $\mathcal{M}\in\mathrm{Mod}(X_\bullet)$. We say that $\mathcal{M}$ is *locally quasi-coherent* (resp. *locally coherent*) if $\mathcal{M}_i$ is quasi-coherent (resp. coherent) for any $i\in I$. We say that $\mathcal{M}$ is *quasi-coherent* if for any $(i,U)\in\mathrm{Zar}(X_\bullet)$ with $U=\operatorname{Spec}A$ being affine, there exists an exact sequence in $\mathrm{Mod}(\mathrm{Zar}(X_\bullet)/(i,U))$ of the form

$$(O_{X_\bullet}|_{(i,U)})^{(T)} \to (O_{X_\bullet}|_{(i,U)})^{(\Sigma)} \to \mathcal{M}|_{(i,U)} \to 0, \tag{7.2}$$

where T and Σ are arbitrary small sets.

7.3 Lemma. *Let $\mathcal{M}\in\mathrm{Mod}(X_\bullet)$. Then the following are equivalent.*

1 $\mathcal{M}$ *is quasi-coherent.*

2 $\mathcal{M}$ *is locally quasi-coherent and equivariant.*

3 *For any morphism* $(\phi, h) : (j, V) \to (i, U)$ *in* $\mathrm{Zar}(X_\bullet)$ *such that* $V = \operatorname{Spec} B$ *and* $U = \operatorname{Spec} A$ *are affine, the canonical map* $B \otimes_A \Gamma((i,U),\mathcal{M}) \to \Gamma((j,V),\mathcal{M})$ *is an isomorphism.*

Proof. **1⇒2** Let $i \in I$ and U an affine open subset of X_i. Then there is an exact sequence of the form (7.2). Applying the restriction functor $\mathrm{Mod}(\mathrm{Zar}(X_\bullet)/(i,U)) \to \mathrm{Mod}(U)$, we get an exact sequence

$$\mathcal{O}_U^{(T)} \to \mathcal{O}_U^{(\Sigma)} \to (\mathcal{M}_i)|_U \to 0,$$

which shows that $\mathcal{M}_i$ is quasi-coherent for any $i \in I$. We prove that $\alpha_\phi(\mathcal{M})$ is an isomorphism for any $\phi : i \to j$, to show that $\mathcal{M}$ is equivariant. Take an affine open covering (U_λ) of X_i, and we prove that $\alpha_\phi(\mathcal{M})$ is an isomorphism over $X_\phi^{-1}(U_\lambda)$ for each λ. But this is obvious by the existence of an exact sequence of the form (7.2) and the five lemma.

2⇒3 Set $W := X_\phi^{-1}(U)$, and let $\iota : V \hookrightarrow W$ be the inclusion map. Obviously, we have $h = (X_\phi)|_W \circ \iota$. As $\mathcal{M}$ is equivariant, we have that the canonical map

$$\alpha_\phi|_W(\mathcal{M}) : (X_\phi)|_W{}^*_{\mathrm{Mod}}(\mathcal{M}_i)|_U \to (\mathcal{M}_j)|_W$$

is an isomorphism. Applying ι^*_{Mod} to the isomorphism, we have that $h^*_{\mathrm{Mod}}((\mathcal{M}_i)|_U) \cong (\mathcal{M}_j)|_V$. The assertion follows from the assumption that $\mathcal{M}_i$ is quasi-coherent.

3⇒1 Let $(i,U) \in \mathrm{Zar}(X_\bullet)$ with $U = \operatorname{Spec} A$ affine. There is a presentation of the form

$$A^{(T)} \to A^{(\Sigma)} \to \Gamma((i,U),\mathcal{M}) \to 0.$$

It suffices to prove that the induced sequence (7.2) is exact. To verify this, it suffices to prove that the sequence is exact after taking the section at $((\phi,h) : (j,V) \to (i,U)) \in \mathrm{Zar}(X_\bullet)/(i,U)$ with $V = \operatorname{Spec} B$ being affine. We have a commutative diagram

$$\begin{array}{ccccccc} B \otimes_A A^{(T)} & \to & B \otimes_A A^{(\Sigma)} & \to & B \otimes_A \Gamma((i,U),\mathcal{M}) & \to & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ B^{(T)} & \to & B^{(\Sigma)} & \to & \Gamma((j,V),\mathcal{M}) & \to & 0 \end{array}$$

whose first row is exact and vertical arrows are isomorphisms. Hence, the second row is also exact, and (7.2) is exact. □

7.4 Definition. We say that $\mathcal{M} \in \mathrm{Mod}(X_\bullet)$ is *coherent* if it is equivariant and locally coherent. We denote the full subcategory of $\mathrm{Mod}(X_\bullet)$ consisting of coherent objects by $\mathrm{Coh}(X_\bullet)$.

(7.5) Let $J \subset I$ be a subcategory. We say that J is *big* in I if for any $(\psi : j \to k) \in \mathrm{Mor}(I)$, there exists some $(\phi : i \to j) \in \mathrm{Mor}(J)$ such that $\psi \circ \phi \in \mathrm{Mor}(J)$. Note that $\mathrm{ob}(J) = \mathrm{ob}(I)$ if J is big in I. Let $\mathbb{Q}$ be a property of morphisms of schemes. We say that $X_\bullet$ has $\mathbb{Q}$ J-arrows if $(X_\bullet)|_J$ has $\mathbb{Q}$-arrows.

7.6 Lemma. *Let $J \subset I$ be a subcategory, and $\mathcal{M} \in \mathrm{Mod}(X_\bullet)$.*

1 *The full subcategory $\mathrm{Lqc}(X_\bullet)$ of $\mathrm{Mod}(X_\bullet)$ consisting of locally quasi-coherent objects is a plump subcategory.*

2 *If $\mathcal{M}$ is equivariant (resp. locally quasi-coherent, quasi-coherent), then so is $\mathcal{M}_J^{\mathrm{Mod}}$.*

3 *If J is big in I and $\mathcal{M}_J$ is equivariant (resp. locally quasi-coherent, quasi-coherent), then so is $\mathcal{M}$.*

4 *If J is big in I and $X_\bullet$ has flat J-arrows, then the full subcategory $\mathrm{EM}(X_\bullet)$ (resp. $\mathrm{Qch}(X_\bullet)$) of $\mathrm{Mod}(X_\bullet)$ consisting of equivariant (resp. quasi-coherent) objects is a plump subcategory.*

5 *If J is big in I, then $(?)_J$ is faithful and exact.*

Proof. **1** and **2** are trivial.

We prove **3**. The assertion for the local quasi-coherence is obvious, because we have $\mathrm{ob}(J) = \mathrm{ob}(I)$. By Lemma 7.3, it remains to show the assertion for the equivariance. Let us assume that $\mathcal{M}_J$ is equivariant and $\psi : j \to k$ is a morphism in I, and take $\phi : i \to j$ such that $\phi, \psi\phi \in \mathrm{Mor}(J)$. Then the composite map

$$(X_{\psi\phi})^*_{\mathrm{Mod}}(\mathcal{M}_i) \cong (X_\psi)^*_{\mathrm{Mod}}(X_\phi)^*_{\mathrm{Mod}}(\mathcal{M}_i) \xrightarrow{(X_\psi)^*_{\mathrm{Mod}}\alpha_\phi^{\mathrm{Mod}}} (X_\psi)^*_{\mathrm{Mod}}(\mathcal{M}_j) \xrightarrow{\alpha_\psi^{\mathrm{Mod}}} \mathcal{M}_k,$$

which agrees with $\alpha_{\psi\phi}^{\mathrm{Mod}}$, is an isomorphism by assumption. As we have $\alpha_\phi^{\mathrm{Mod}}$ is also an isomorphism, we have that $\alpha_\psi^{\mathrm{Mod}}$ is an isomorphism. Thus $\mathcal{M}$ is equivariant.

We prove **4**. By **1** and Lemma 7.3, it suffices to prove the assertion only for the equivariance. Let

$$\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to \mathcal{M}_4 \to \mathcal{M}_5$$

be an exact sequence in $\mathrm{Mod}(X_\bullet)$, and assume that $\mathcal{M}_i$ is equivariant for $i = 1, 2, 4, 5$. We prove that $\mathcal{M}_3$ is equivariant. The sequence remains exact after applying the functor $(?)_J^{\mathrm{Mod}}$. By **3**, replacing I by J and $X_\bullet$ by $(X_\bullet)|_J$, we may assume that $X_\bullet$ has flat arrows. Now the assertion follows easily from the five lemma.

The assertion **5** is obvious, because $\mathrm{ob}(J) = \mathrm{ob}(I)$. $\square$

7.7 Lemma. *Let $(\mathcal{M}_\lambda)$ be a diagram in $\mathrm{Mod}(X_\bullet)$. If each $\mathcal{M}_\lambda$ is locally quasi-coherent (resp. equivariant, quasi-coherent), then so is $\varinjlim \mathcal{M}_\lambda$.*

Proof. As $(?)_i$ preserves colimits, the assertion for local quasi-coherence is trivial. Assume that each $\mathcal{M}_\lambda$ is equivariant. For $(\phi : i \to j) \in \mathrm{Mor}(I)$, $\alpha_\phi(\mathcal{M}_\lambda)$ is an isomorphism. As $\alpha_\phi(\varinjlim \mathcal{M}_\lambda)$ is nothing but the composite

$$(X_\phi)^*_{\mathrm{Mod}}((\varinjlim \mathcal{M}_\lambda)_i) \cong \varinjlim (X_\phi)^*_{\mathrm{Mod}}(\mathcal{M}_\lambda)_i \xrightarrow{\varinjlim \alpha_\phi(\mathcal{M}_\lambda)} \varinjlim (\mathcal{M}_\lambda)_j \cong (\varinjlim \mathcal{M}_\lambda)_j,$$

it is an isomorphism. The rest of the assertions follow. $\square$

By Lemma 6.7, we have the following.

7.8 Lemma. *Let $J \subset I$ be a subcategory, and $\mathcal{M} \in \mathrm{Lqc}(X_\bullet|_J)$. Then we have $L_J^{\mathrm{Mod}}(\mathcal{M}) \in \mathrm{Lqc}(X_\bullet)$.*

Similarly, we have the next lemma. We say that a morphism $f : X \to Y$ of schemes is *quasi-separated* if the diagonal map $X \to X \times_Y X$ is quasi-compact. A quasi-compact quasi-separated morphism is said to be *concentrated*. If $f : X \to Y$ is concentrated, and $\mathcal{M} \in \mathrm{Qch}(X)$, then $f_*\mathcal{M} \in \mathrm{Qch}(Y)$ [14, (9.2.1)], where $\mathrm{Qch}(X)$ and $\mathrm{Qch}(Y)$ denote the category of quasi-coherent sheaves on X and Y, respectively.

7.9 Lemma. *Let $j \in I$. Assume that $X_\bullet$ has concentrated arrows, and that $I(i, j)$ is finite for any $i \in I$. If $\mathcal{M} \in \mathrm{Qch}(X_j)$, then we have $R_j\mathcal{M} \in \mathrm{Lqc}(X_\bullet)$.*

The following is also proved easily, using (6.3) and Lemma 6.4.

7.10 Lemma. *Let $\mathcal{M}$ and $\mathcal{N}$ be locally quasi-coherent (resp. equivariant, quasi-coherent) $\mathcal{O}_{X_\bullet}$-modules. Then $\mathcal{M} \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{N}$ is also locally quasi-coherent (resp. equivariant, quasi-coherent).*

The following is a consequence of the observation in (6.37).

7.11 Lemma. *Let $\mathcal{M}$ be a coherent $\mathcal{O}_{X_\bullet}$-module, and $\mathcal{N}$ a locally quasi-coherent $\mathcal{O}_{X_\bullet}$-module. Then $\underline{\mathrm{Hom}}_{\mathrm{Mod}(X_\bullet)}(\mathcal{M},\mathcal{N})$ is locally quasi-coherent. If, moreover, there is a big subcategory J of I such that $X_\bullet$ has flat J-arrows and $\mathcal{N}$ is quasi-coherent, then $\underline{\mathrm{Hom}}_{\mathrm{Mod}(X_\bullet)}(\mathcal{M},\mathcal{N})$ is quasi-coherent.*

7.12 Lemma. *Let $f : X \to Y$ be a concentrated morphism of schemes, and $g_Y : Y' \to Y$ a flat morphism of schemes. Set $X' := X \times_Y Y'$, $g_X : X' \to X$ the first projection, and $f' : X' \to Y'$ the second projection. Then for $\mathcal{M} \in \mathrm{Qch}(X)$, the canonical morphism*

$$\theta : g_Y^* f_* \mathcal{M} \to f'_* g_X^* \mathcal{M}$$

is an isomorphism.

Proof. First note that the assertion is true if g_Y is an open immersion. Indeed, it is easy to check that θ_{PM} and θ in the composition in Lemma 2.59 are isomorphisms in this case.

Using Lemma 1.23, we may assume that both Y and Y' are affine. Thus X is quasi-compact. Let (U_i) be a finite affine open covering of X, which exists. Set $\tilde{X} = \coprod_i U_i$, and let $p : \tilde{X} \to X$ be the obvious map. Since f is quasi-separated and Y is affine, $U_i \cap U_j$ is quasi-compact for any i, j. Thus p is quasi-compact. Note also that p is separated, since $\tilde{X}$ is affine. Let $p_i : \tilde{X} \times_X \tilde{X} \to \tilde{X}$ be the ith projection for $i = 1, 2$, and set $q = pp_1 = pp_2$. Note that p_1, p_2 and q are quasi-compact separated. Almost by the definition of a sheaf, there is an exact sequence of the form

$$0 \to \mathcal{M} \xrightarrow{u} p_* p^* \mathcal{M} \to q_* q^* \mathcal{M}.$$

Since $q_* q^* \mathcal{M} \cong p_*((p_1)_* q^* \mathcal{M})$, and $p^* \mathcal{M}$ and $(p_1)_* q^* \mathcal{M}$ are quasi-coherent, we may assume that $\mathcal{M} = p_* \mathcal{N}$ for some $\mathcal{N} \in \mathrm{Qch}(\tilde{X})$ by the five lemma. By Lemma 1.22, replacing f by p and fp, we may assume that f is quasi-compact separated. Then repeating the same argument as above, we may assume that p is affine now. Replacing f by p and fp again, we may assume that f is affine. That is, X is affine. But this case is trivial. $\square$

(7.13) Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P} = \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$.

As θ in (6.21) is not an isomorphism in general, $(f_\bullet)_*^\heartsuit(\mathcal{M})$ need not be equivariant even if $\mathcal{M}$ is equivariant. However, we have

7.14 Lemma. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}$, and J a big subcategory of I. Then we have the following:*

1 *$f_\bullet$ is cartesian if and only if $(f_\bullet)|_J$ is cartesian.*

2 *If $f_\bullet$ is concentrated and $\mathcal{M} \in \mathrm{Lqc}(X_\bullet)$, then $(f_\bullet)_*(\mathcal{M}) \in \mathrm{Lqc}(Y_\bullet)$.*

3 *If $f_\bullet$ is cartesian concentrated, $Y_\bullet$ has flat J-arrows, and $\mathcal{M} \in \mathrm{Qch}(X_\bullet)$, then we have $(f_\bullet)_*(\mathcal{M}) \in \mathrm{Qch}(Y_\bullet)$.*

Proof. **1** Assume that $f_\bullet|_J$ is cartesian, and let $\psi : j \to k$ be a morphism in I. Take $\phi : i \to j$ such that $\phi, \psi\phi \in \mathrm{Mor}(J)$. Consider the commutative diagram

$$\begin{array}{ccccc} X_k & \xrightarrow{X_\psi} & X_j & \xrightarrow{X_\phi} & X_i \\ \downarrow f_k & \text{(a)} & \downarrow f_j & \text{(b)} & \downarrow f_i \\ Y_k & \xrightarrow{Y_\psi} & Y_j & \xrightarrow{Y_\phi} & Y_i. \end{array}$$

By assumption, the square (b) and the whole rectangle ((a)+(b)) are fiber squares. Hence (a) is also a fiber square. This shows that $f_\bullet$ is cartesian. The converse is obvious.

The assertion **2** is obvious by the isomorphism $((f_\bullet)_*\mathcal{M})_i \cong (f_i)_*(\mathcal{M}_i)$ for $i \in I$.

We prove **3**. By Lemma 7.6, we may assume that $J = I$. Then $(f_\bullet)_*(\mathcal{M})$ is locally quasi-coherent by **2**. As $\mathcal{M}$ is equivariant and θ in (6.21) is an isomorphism by Lemma 7.12, we have that $(f_\bullet)_*(\mathcal{M})$ is equivariant. Hence by Lemma 7.3, $(f_\bullet)_*(\mathcal{M})$ is quasi-coherent. □

(7.15) Let the notation be as in Lemma 7.14. If $f_\bullet$ is concentrated, then $(f_\bullet)_*^{\mathrm{Lqc}} : \mathrm{Lqc}(X_\bullet) \to \mathrm{Lqc}(Y_\bullet)$ is defined as the restriction of $(f_\bullet)_*^{\mathrm{Mod}}$. If $f_\bullet$ is concentrated cartesian and $Y_\bullet$ has flat J-arrows, then $(f_\bullet)_*^{\mathrm{Qch}} : \mathrm{Qch}(X_\bullet) \to \mathrm{Qch}(Y_\bullet)$ is induced.

7.16 Lemma. *Let $f_\bullet : X_\bullet \to Y_\bullet$ and $g_\bullet : Y_\bullet \to Z_\bullet$ be morphisms in $\mathcal{P}$. Then the following hold.*

0 *An isomorphism is a cartesian morphism.*

1 *If $f_\bullet$ and $g_\bullet$ are cartesian, then so is $g_\bullet \circ f_\bullet$.*

2 *If $g_\bullet$ and $g_\bullet \circ f_\bullet$ are cartesian, then so is $f_\bullet$.*

3 *If $f_\bullet$ is faithfully flat cartesian and $g_\bullet \circ f_\bullet$ is cartesian, then $g_\bullet$ is cartesian.*

Proof. Trivial. □

7.17 Lemma. *Let $f_\bullet : X_\bullet \to Y_\bullet$ and $g_\bullet : Y'_\bullet \to Y_\bullet$ be morphisms in $\mathcal{P}$. Let $f'_\bullet : X'_\bullet \to Y'_\bullet$ be the base change of $f_\bullet$ by $g_\bullet$.*

1 *If $f_\bullet$ is cartesian, then so is $f'_\bullet$.*

2 *If $f'_\bullet$ is cartesian and $g_\bullet$ is faithfully flat, then $f_\bullet$ is cartesian.*

Proof. Obvious. □

(7.18) Let $f : X \to Y$ be a morphism of schemes. If f is concentrated, then f_* is compatible with pseudo-filtered inductive limits.

7.19 Lemma ([23, p.641, Proposition 6]). *Let $f : X \to Y$ be a concentrated morphism of schemes, and $(\mathcal{M}_i)$ a pseudo-filtered inductive system of $\mathcal{O}_X$-modules. Then the canonical map*

$$\varinjlim f_* \mathcal{M}_i \to f_* \varinjlim \mathcal{M}_i$$

is an isomorphism.

By the lemma, the following follows immediately.

7.20 Lemma. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. If $f_\bullet$ is concentrated, then $(f_\bullet)_*^{\mathrm{Mod}}$ and $(f_\bullet)_*^{\mathrm{Lqc}}$ preserve pseudo-filtered inductive limits. If, moreover, $f_\bullet$ is cartesian and $Y_\bullet$ has flat arrows, then $(f_\bullet)_*^{\mathrm{Qch}}$ preserves pseudo-filtered inductive limits.*

7.21 Lemma. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}$. Let J be an admissible subcategory of I. If $Y_\bullet$ has flat arrows and $f_\bullet$ is cartesian and concentrated, then the canonical map*

$$\theta(J, f_\bullet) : L_J \circ (f_\bullet|_J)_* \to (f_\bullet)_* \circ L_J$$

is an isomorphism of functors from $\mathrm{Lqc}(X_\bullet|_J)$ to $\mathrm{Lqc}(Y_\bullet)$.

Proof. This is obvious by Proposition 6.23, Lemma 7.19, and Lemma 7.12. □

The following is obvious by Lemma 6.25 and (6.31).

7.22 Lemma. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}$. If $\mathcal{M} \in \mathrm{Mod}(Y_\bullet)$ is equivariant (resp. locally quasi-coherent, quasi-coherent), then so is $(f_\bullet)^*_{\mathrm{Mod}}(\mathcal{M})$. If $\mathcal{M} \in \mathrm{Mod}(Y_\bullet)$, $f_\bullet$ is faithfully flat, and $(f_\bullet)^*_{\mathrm{Mod}}(\mathcal{M})$ is equivariant, then we have $\mathcal{M}$ is equivariant.*

The restriction $(f_\bullet)^* : \mathrm{Qch}(Y_\bullet) \to \mathrm{Qch}(X_\bullet)$ is sometimes denoted by $(f_\bullet)^*_{\mathrm{Qch}}$.

8 Derived functors of functors on sheaves of modules over diagrams of schemes

(8.1) Let I be a small category, and S a scheme. Set $\mathcal{P} := \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$, and let $X_\bullet \in \mathcal{P}$. In these notes, we use some abbreviated notation for derived categories of modules over diagrams of schemes. In the sequel, $D(\mathrm{Mod}(X_\bullet))$ may be denoted by $D(X_\bullet)$. $D^+_{\mathrm{EM}(X_\bullet)}(\mathrm{Mod}(X_\bullet))$ may be denoted by $D^+_{\mathrm{EM}}(X_\bullet)$. $D^b_{\mathrm{Coh}(X_\bullet)}(\mathrm{Qch}(X_\bullet))$ may be denoted by $D^b_{\mathrm{Coh}}(\mathrm{Qch}(X_\bullet))$, and so on. This notation will be also used for a single scheme. For a scheme X, $D^+_{\mathrm{Qch}(X)}(\mathrm{Mod}(X))$ will be denoted by $D^+_{\mathrm{Qch}}(X)$, where $\mathrm{Mod}(X)$ is the category of $\mathcal{O}_X$-modules.

8.2 Proposition. *Let $X_\bullet \in \mathcal{P}$, and $\mathbb{I} \in K(\mathrm{Mod}(X_\bullet))$. We have $\mathbb{I}$ is K-limp if and only if so is $\mathbb{I}_i$ for $i \in I$.*

Proof. The only if part follows from Lemma 3.31 and Lemma 3.25, **4**.

We prove the if part. Let $\mathbb{I} \to \mathbb{J}$ be a K-injective resolution, and let $\mathbb{C}$ be the mapping cone. Note that $\mathbb{C}_i$ is exact for each i.

Let $(U, i) \in \mathrm{Zar}(X_\bullet)$. We have an isomorphism

$$\Gamma((U, i), \mathbb{C}) \cong \Gamma(U, \mathbb{C}_i).$$

As $\mathbb{C}_i$ is K-limp by the only if part, these are exact for each (U, i). It follows that $\mathbb{I}$ is K-limp. □

8.3 Corollary. *Let J be a subcategory of I, and $f_\bullet : X_\bullet \to Y_\bullet$ a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Then there is a canonical isomorphism*

$$c(J, f_\bullet) : (?)_J R(f_\bullet)_* \cong R(f_\bullet|_J)_*(?)_J.$$

8.4 Lemma. *Let J be an admissible subcategory of I. Assume that $X_\bullet$ has flat arrows. If $\mathbb{I}$ is a K-injective complex in $\mathrm{Mod}(X_\bullet)$, then $\mathbb{I}_J$ is K-injective.*

Proof. This is simply because $(?)_J$ has an exact left adjoint L_J. □

8.5 Lemma. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a concentrated morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Then $R(f_\bullet)_*$ takes $D_{\mathrm{Lqc}}(X_\bullet)$ to $D_{\mathrm{Lqc}}(Y_\bullet)$. $R(f_\bullet)_* : D_{\mathrm{Lqc}}(X_\bullet) \to D_{\mathrm{Lqc}}(Y_\bullet)$ is way-out in both directions if $Y_\bullet$ is quasi-compact and I is finite.*

Proof. Follows from [26, (3.9.2)] and Corollary 8.3 easily. □

8.6 Lemma. *Let $X_\bullet \in \mathcal{P}$. Assume that $X_\bullet$ has flat arrows. For a complex $\mathbb{F}$ in $\mathrm{Mod}(X_\bullet)$, $\mathbb{F}$ has equivariant cohomology groups if and only if $\alpha_\phi \colon X_\phi^* \mathbb{F}_i \to \mathbb{F}_j$ is a quasi-isomorphism for any morphism $\phi \colon i \to j$ in I.*

Proof. This is easy, since X_ϕ^* is an exact functor. □

8.7 Lemma. *Let $f_\bullet \colon X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}$. Assume that $f_\bullet$ is concentrated and cartesian, and $Y_\bullet$ has flat arrows. If $\mathbb{F} \in D_{\mathrm{Qch}}(X_\bullet)$, then $R(f_\bullet)_* \mathbb{F} \in D_{\mathrm{Qch}}(Y_\bullet)$.*

Proof. By the derived version of Lemma 6.20,

$$\alpha_\phi \colon Y_\phi^* (?)_i R(f_\bullet)_* \mathbb{F} \to (?)_j R(f_\bullet)_* \mathbb{F} \tag{8.8}$$

agrees with the composite

$$Y_\phi^*(?)_i R(f_\bullet)_* \mathbb{F} \xrightarrow{c} Y_\phi^* R(f_i)_* \mathbb{F}_i \xrightarrow{\theta} R(f_j)_* X_\phi^* \mathbb{F}_i \xrightarrow{\alpha_\phi} R(f_j)_* \mathbb{F}_j \xrightarrow{c} (?)_j R(f_\bullet)_* \mathbb{F}.$$

The first and the fourth map c's are isomorphisms. The second map θ is an isomorphism by [26, (3.9.5)]. The third map α_ϕ is an isomorphism by assumption and Lemma 8.6. Thus (8.8) is an isomorphism. Again by Lemma 8.6, we have the desired assertion. □

(8.9) Let X be a scheme, $x \in X$, and M an $\mathcal{O}_{X,x}$-module. We define $\xi_x(M) \in \mathrm{Mod}(X)$ by $\Gamma(U, \xi_x(M)) = M$ if $x \in U$, and zero otherwise. The restriction maps are defined in an obvious way. For an exact complex $\mathbb{H}$ of $\mathcal{O}_{X,x}$-modules, $\xi_x(\mathbb{H})$ is exact not only as a complex of sheaves, but also as a complex of presheaves. For a morphism of schemes $f : X \to Y$, we have that $f_* \xi_x(M) \cong \xi_{f(x)}(M)$.

8.10 Lemma. *Let $\mathbb{F} \in C(\mathrm{Mod}(X_\bullet))$. The following are equivalent.*

1 $\mathbb{F}$ *is K-flat.*

2 $\mathbb{F}_i$ *is K-flat for $i \in \mathrm{ob}(I)$.*

3 $\mathbb{F}_{i,x}$ *is a K-flat complex of $\mathcal{O}_{X_i,x}$-modules for any $i \in \mathrm{ob}(I)$ and $x \in X_i$.*

Proof. **3⇒1** Let $\mathbb{G} \in C(\mathrm{Mod}(X_\bullet))$ be exact. We are to prove that $\mathbb{F} \otimes_{\mathcal{O}_{X_\bullet}} \mathbb{G}$ is exact. For $i \in \mathrm{ob}(I)$ and $x \in X_i$, we have

$$(\mathbb{F} \otimes_{\mathcal{O}_{X_\bullet}} \mathbb{G})_{i,x} \cong (\mathbb{F}_i \otimes_{\mathcal{O}_{X_i}} \mathbb{G}_i)_x \cong \mathbb{F}_{i,x} \otimes_{\mathcal{O}_{X_{i,x}}} \mathbb{G}_{i,x}.$$

Since $\mathbb{G}_{i,x}$ is exact, $(\mathbb{F} \otimes_{\mathcal{O}_{X_{i,x}}} \mathbb{G})_{i,x}$ is exact. So $\mathbb{F} \otimes_{\mathcal{O}_{X_\bullet}} \mathbb{G}$ is exact.

1⇒3 Let $\mathbb{H} \in C(\mathrm{Mod}(\mathcal{O}_{X_{i,x}}))$ be an exact complex, and we are to prove that $\mathbb{F}_{i,x} \otimes_{\mathcal{O}_{X_{i,x}}} \mathbb{H}$ is exact. For each $j \in \mathrm{ob}(I)$,

$$(?)_j R_i \xi_x(\mathbb{H}) \cong \prod_{\phi \in I(j,i)} (X_\phi)_* \xi_x(\mathbb{H}) \cong \prod_{\phi \in I(j,i)} \xi_{X_\phi(x)}(\mathbb{H})$$

is exact, since a direct product of exact complexes of *presheaves* is exact. So $R_i \xi_x(\mathbb{H})$ is exact. It follows that $\mathbb{F} \otimes_{\mathcal{O}_{X_\bullet}} R_i \xi_x(\mathbb{H})$ is exact. Hence

$$(\mathbb{F} \otimes_{\mathcal{O}_{X_\bullet}} R_i \xi_x(\mathbb{H}))_i \cong \mathbb{F}_i \otimes_{\mathcal{O}_{X_i}} \prod_{\phi \in I(i,i)} \xi_{X_\phi(x)} \mathbb{H}$$

is also exact. So $\mathbb{F}_i \otimes_{\mathcal{O}_{X_i}} \xi_{\mathrm{id}_{X_i}(x)}(\mathbb{H}) = \mathbb{F}_i \otimes_{\mathcal{O}_{X_i}} \xi_x \mathbb{H}$ is exact. So

$$(\mathbb{F}_i \otimes_{\mathcal{O}_{X_i}} \xi_x \mathbb{H})_x \cong \mathbb{F}_{i,x} \otimes_{\mathcal{O}_{X_{i,x}}} (\xi_x \mathbb{H})_x \cong \mathbb{F}_{i,x} \otimes_{\mathcal{O}_{X_{i,x}}} \mathbb{H}$$

is also exact.

Applying **1⇔3**, which has already been proved, to the complex $\mathbb{F}_i$ over the single scheme X_i, we get **2⇔3**. □

Hence by [39], we have the following.

8.11 Lemma. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}$. Then we have the following.*

1 *If $\mathbb{F} \in C(\mathrm{Mod}(Y_\bullet))$ is K-flat, then so is $f_\bullet^* \mathbb{F}$.*

2 *If $\mathbb{F} \in C(\mathrm{Mod}(Y_\bullet))$ is K-flat exact, then so is $f_\bullet^* \mathbb{F}$.*

3 *If $\mathbb{I} \in C(\mathrm{Mod}(X_\bullet))$ is weakly K-injective, then so is $(f_\bullet)_* \mathbb{I}$.*

(8.12) By the lemma, the left derived functor $Lf_\bullet^*$, which we already know its existence by Lemma 6.25, can also be calculated by K-flat resolutions.

8.13 Lemma. *Let J be a subcategory of I, and $f_\bullet : X_\bullet \to Y_\bullet$ a morphism in $\mathcal{P}$. Then we have the following.*

1 *The canonical map*

$$\theta(f_\bullet, J) : L(f_\bullet|_J)^*(?)_J \to (?)_J Lf_\bullet^*$$

is an isomorphism.

2 *The diagram*

$$\begin{tikzcd}
(?)_J \arrow[rr, "\mathrm{id}"] \arrow[d, "u"] & & (?)_J \arrow[d, "u"] \\
R(f_\bullet|_J)_* L(f_\bullet|_J)^*(?)_J \arrow[r, "\theta"] & R(f_\bullet|_J)_*(?)_J Lf_\bullet^* \arrow[r, "c^{-1}"] & (?)_J R(f_\bullet)_* Lf_\bullet^*
\end{tikzcd}$$

is commutative.

3 *The diagram*

$$\begin{tikzcd}
(?)_J \arrow[rr, "\mathrm{id}"] & & (?)_J \\
L(f_\bullet|_J)^* R(f_\bullet|_J)_*(?)_J \arrow[r, "c^{-1}"] \arrow[u, "\varepsilon"] & L(f_\bullet|_J)^*(?)_J R(f_\bullet)_* \arrow[r, "\theta"] & (?)_J L(f_\bullet)^* R(f_\bullet)_* \arrow[u, "\varepsilon"]
\end{tikzcd}$$

is commutative.

Proof. Since $(?)_J$ preserves K-flat complexes by Lemma 8.10, we have

$$L(f_\bullet|_J)^*(?)_J \cong L((f_\bullet|_J) \circ (?)_J).$$

On the other hand, it is obvious that we have $(?)_J Lf_\bullet^* \cong L((?)_J f_\bullet^*)$. By Lemma 6.25, we have a composite isomorphism

$$\theta : L(f_\bullet|_J)^*(?)_J \cong L((f_\bullet|_J)^* \circ (?)_J) \xrightarrow{L\theta} L((?)_J \circ f_\bullet^*) \cong (?)_J L(f_\bullet)^*,$$

and **1** is proved.

2 and **3** follow from the proofs of Lemma 1.24 and Lemma 1.25, respectively. □

8.14 Lemma. *Let $X_\bullet \in \mathcal{P}$, and $\mathbb{F}, \mathbb{G} \in D(X_\bullet)$. Then we have the following.*

1 $\mathbb{F}_J \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet|_J}} \mathbb{G}_J \cong (\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} \mathbb{G})_J$ *for any subcategory $J \subset I$.*

2 *If* $\mathbb{F}$ *and* $\mathbb{G}$ *have locally quasi-coherent cohomology groups, then* $\underline{\mathrm{Tor}}_i^{\mathcal{O}_{X_\bullet}}(\mathbb{F},\mathbb{G})$ *is also locally quasi-coherent for any* $i \in \mathbb{Z}$.

3 *Assume that there exists some big subcategory* J *of* I *such that* $X_\bullet$ *has flat* J*-arrows. If both* $\mathbb{F}$ *and* $\mathbb{G}$ *have equivariant* (*resp. quasi-coherent*) *cohomology groups, then* $\underline{\mathrm{Tor}}_i^{\mathcal{O}_{X_\bullet}}(\mathbb{F},\mathbb{G})$ *is also equivariant* (*resp. quasi-coherent*).

Proof. The assertion **1** is an immediate consequence of Lemma 8.10 and Example 5.6, **5**.

2 In view of **1**, we may assume that $X = X_\bullet$ is a single scheme. As the question is local, we may assume that X is even affine.

We may assume that $\mathbb{F} = \varinjlim \mathbb{F}_n$, where $(\mathbb{F}_n)$ is the $\mathfrak{P}(X_\bullet)$-special direct system such that each $\mathbb{F}_n$ is bounded above and has locally quasi-coherent cohomology groups as in Lemma 3.25, **3**. Similarly, we may assume that $\mathbb{G} = \varinjlim \mathbb{G}_n$. As filtered inductive limits are exact and compatible with tensor products, and the colimit of locally quasi-coherent sheaves is locally quasi-coherent, we may assume that both $\mathbb{F}$ and $\mathbb{G}$ are bounded above, flat, and has locally quasi-coherent cohomology groups. By [17, Proposition I.7.3], we may assume that both $\mathbb{F}$ and $\mathbb{G}$ are single quasi-coherent sheaves. This case is trivial.

3 In view of **1**, we may assume that $J = I$ and $X_\bullet$ has flat arrows. By **2**, it suffices to show the assertion for equivariance. Assuming that $\mathbb{F}$ and $\mathbb{G}$ are K-flat with equivariant cohomology groups, we prove that $\mathbb{F} \otimes \mathbb{G}$ has equivariant cohomology groups. This is enough.

Let $\phi : i \to j$ be a morphism of I. As X_ϕ is flat and $\mathbb{F}$ and $\mathbb{G}$ have equivariant cohomology groups, $\alpha_\phi : X_\phi^*\mathbb{F}_i \to \mathbb{F}_j$ and $\alpha_\phi : X_\phi^*\mathbb{G}_i \to \mathbb{G}_j$ are quasi-isomorphisms. The composite

$$X_\phi^*(\mathbb{F} \otimes^\bullet_{\mathcal{O}_{X_\bullet}} \mathbb{G})_i \cong X_\phi^*\mathbb{F}_i \otimes^\bullet_{\mathcal{O}_{X_j}} X_\phi^*\mathbb{G}_i \xrightarrow{\alpha_\phi \otimes \alpha_\phi} \mathbb{F}_j \otimes^\bullet_{\mathcal{O}_{X_j}} \mathbb{G}_j \cong (\mathbb{F} \otimes^\bullet_{\mathcal{O}_{X_\bullet}} \mathbb{G})_j$$

is a quasi-isomorphism, since $X_\phi^*\mathbb{G}_i$ and $\mathbb{F}_j$ are K-flat. By (6.3), $\alpha_\phi(\mathbb{F} \otimes^\bullet_{\mathcal{O}_{X_\bullet}} \mathbb{G})$ is a quasi-isomorphism.

As $X_\bullet$ has flat arrows, this shows that $\mathbb{F} \otimes^\bullet_{\mathcal{O}_{X_\bullet}} \mathbb{G}$ has equivariant cohomology groups. $\square$

(8.15) Let $X_\bullet \in \mathcal{P}$, and J an admissible subcategory of I. By Lemma 6.17, the left derived functor

$$LL_J^{\mathrm{Mod}} : D(X_\bullet|_J) \to D(X_\bullet)$$

of L_J^{Mod} is defined, since $Q(X_\bullet, J)$ is admissible. This is also calculated using K-flat resolutions. Namely,

8.16 Lemma. *Let $X_\bullet$ and J be as above. If $\mathbb{F} \in K(\mathrm{Mod}(X_\bullet|_J))$ is K-flat, then so is $L_J\mathbb{F}$. If $\mathbb{F}$ is K-flat exact, then so is $L_J\mathbb{F}$.*

Proof. This is trivial by Lemma 6.7. □

8.17 Corollary. *Let $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$, J an admissible subcategory of I, and $\mathbb{I} \in K(\mathrm{Mod}(X_\bullet))$. If $\mathbb{I}$ is weakly K-injective, then $\mathbb{I}_J$ is weakly K-injective.*

Proof. Let $\mathbb{F}$ be a K-flat exact complex in $K(\mathrm{Mod}(X_\bullet|_J))$. Then,

$$\mathrm{Hom}^\bullet_{\mathrm{Mod}(X_\bullet|_J)}(\mathbb{F}, \mathbb{I}_J) \cong \mathrm{Hom}^\bullet_{\mathrm{Mod}(X_\bullet)}(L_J\mathbb{F}, \mathbb{I})$$

is exact by the lemma. By Lemma 3.25, **5**, we are done. □

8.18 Lemma. *Let $f\colon X \to Y$ be a morphism of schemes, and $\mathcal{M} \in \mathrm{Qch}(Y)$. Then for any $i \geq 0$, $L_i f^* \mathcal{M} \in \mathrm{Qch}(X)$.*

Proof. Note that Lf^* is computed using a flat resolution, and a flat object is preserved by f^*. If g is a flat morphism of schemes, then g^* is exact. Thus using a spectral sequence argument, it is easy to see that the question is local both on Y and X. So we may assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine. If $\Gamma(Y, \mathcal{M}) = M$ and $\mathbb{F} \to M$ is an A-projective resolution, then

$$L_i f^* \mathcal{M} = H_i(f^* \tilde{\mathbb{F}}) = H_i((B \otimes_A \mathbb{F})^\sim) = \mathrm{Tor}_i^A(B, M)^\sim.$$

Thus $L_i f^* \mathcal{M}$ is quasi-coherent for any $i \geq 0$, as desired. □

8.19 Lemma. *Let $X_\bullet \in \mathcal{P}$ and J an admissible subcategory of I. Let $\mathbb{F} \in D_{\mathrm{Lqc}}(X_\bullet|_J)$. Then, $LL_J\mathbb{F} \in D_{\mathrm{Lqc}}(X_\bullet)$.*

Proof. First we consider the case that $\mathbb{F} = \mathcal{M}$ is a single locally quasi-coherent sheaf. Then by the uniqueness of the derived functor,

$$(?)_i H^{-n}(LL_J\mathcal{M}) = L_n((?)_i L_J)\mathcal{M} = \varinjlim L_n X_\phi^* \mathcal{M}_j$$

for $i \in I$. Thus $LL_J\mathcal{M} \in D_{\mathrm{Lqc}}(X_\bullet)$ by Lemma 8.18.

Now using the standard spectral sequence argument (or the way-out lemma [17, (I.7.3)]), the case that $\mathbb{F}$ is bounded above follows. The general case follows immediately by Lemma 3.25, **3**. □

8.20 Lemma. *Let $f_\bullet \colon X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}$. If $\mathbb{F} \in D_{\mathrm{Lqc}}(Y_\bullet)$, then $Lf_\bullet^*\mathbb{F} \in D_{\mathrm{Lqc}}(X_\bullet)$. If $Y_\bullet$ and $X_\bullet$ have flat arrows and $\mathbb{F} \in D_{\mathrm{EM}}(Y_\bullet)$, then $Lf_\bullet^*\mathbb{F} \in D_{\mathrm{EM}}(X_\bullet)$.*

Proof. For the first assertion, we may assume that $f : X \to Y$ is a morphism of single schemes by Lemma 8.13. If $\mathbb{F}$ is a single quasi-coherent sheaf, this is obvious by Lemma 8.18. So the case that $\mathbb{F}$ is bounded above follows from the way-out lemma. The general case follows from Lemma 3.25, **3**.

We prove the second assertion. If $\mathbb{F}$ is a K-flat complex in $\mathrm{Mod}(Y_\bullet)$ with equivariant cohomology groups, then $\alpha_\phi : Y_\phi^*\mathbb{F}_i \to \mathbb{F}_j$ is a quasi-isomorphism for any morphism $\phi : i \to j$ of I by Lemma 8.6. As the mapping cone $\mathrm{Cone}(\alpha_\phi)$ is K-flat exact by Lemma 8.10, $f_j^* \mathrm{Cone}(\alpha_\phi)$ is also exact. Thus $f_j^*\alpha_\phi : f_j^*Y_\phi^*\mathbb{F}_i \to f_j^*\mathbb{F}_j$ is a quasi-isomorphism. This shows that $\alpha_\phi : X_\phi^*(f_\bullet^*\mathbb{F})_i \to (f_\bullet^*\mathbb{F})_j$ is a quasi-isomorphism for any ϕ. So $f_\bullet^*\mathbb{F}$ has equivariant cohomology groups by Lemma 8.6. This is what we wanted to prove. $\square$

8.21 Lemma. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a flat morphism in $\mathcal{P}$. If $\mathbb{F} \in D_{\mathrm{EM}}(Y_\bullet)$, then $Lf_\bullet^*\mathbb{F} \in D_{\mathrm{EM}}(X_\bullet)$.*

Proof. Let $\mathbb{F}$ be a K-flat complex in $\mathrm{Mod}(Y_\bullet)$ with equivariant cohomology groups. Then $H^n(f_\bullet^*\mathbb{F}) \cong f_\bullet^*(H^n\mathbb{F})$ is equivariant by Lemma 7.22. This is what we wanted to prove. $\square$

(8.22) Let I be a small category, and S a scheme. Set $\mathcal{P} := \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. As we have seen, for a morphism $f_\bullet : X_\bullet \to Y_\bullet$, $f_\bullet^{-1} : \mathrm{Zar}(Y_\bullet) \to \mathrm{Zar}(X_\bullet)$ is an admissible ringed continuous functor by Lemma 6.25. Moreover, if J and K are admissible subcategories of I such that $J \subset K$, then $Q(X_\bullet|_J, K) : \mathrm{Zar}(X_\bullet|_K) \to \mathrm{Zar}(X_\bullet|_J)$ is also admissible. Utilizing Lemma 3.33 and Lemma 5.4, we have the following.

8.23 Example. Let I be a small category, S a scheme, and $f_\bullet : X_\bullet \to Y_\bullet$ and $g_\bullet : Y_\bullet \to Z_\bullet$ are morphisms in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Let $K \subset J \subset I$ be admissible subcategories. Then we have the following.

1 There is a natural isomorphism

$$c_{I,J,K} : (?)_{K,I} \cong (?)_{K,J} \circ (?)_{J,I}.$$

Taking the conjugate,

$$d_{I,J,K} : LL_{I,J} \circ LL_{J,K} \cong LL_{I,K}$$

is induced.

2 There are natural isomorphism

$$c_{J,f_\bullet} : (?)_J \circ R(f_\bullet)_* \cong R(f_\bullet|_J)_* \circ (?)_J$$

and its conjugate

$$d_{J,f_\bullet} : LL_J \circ L(f_\bullet|_J)^* \cong L(f_\bullet)^* \circ LL_J.$$

3 We have

$$(c_{K,f_\bullet|_J}(?)_J) \circ ((?)_{K,J} c_{J,f_\bullet}) = (R(f_\bullet|_K)_* c_{I,J,K}) \circ c_{K,f_\bullet} \circ (c^{-1}_{I,J,K} R(f_\bullet)_*).$$

4 We have

$$(R(g_\bullet|_J)_* c_{J,f_\bullet}) \circ (c_{J,g_\bullet} R(f_\bullet)_*) = (c_{f_\bullet|_J,g_\bullet|_J}(?)_J) \circ c_{J,g_\bullet \circ f_\bullet} \circ ((?)_J c^{-1}_{f_\bullet,g_\bullet}),$$

where $c_{f_\bullet,g_\bullet} : R(g_\bullet \circ f_\bullet)_* \cong R(g_\bullet)_* \circ R(f_\bullet)_*$ is the canonical isomorphism, and similarly for $c_{f_\bullet|_J,g_\bullet|_J}$.

5 The adjoint pair $(L(?)^*_{\mathrm{Mod}}, R(?)^{\mathrm{Mod}}_*)$ over the category $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$ is Lipman.

9 Simplicial objects

(9.1) For $n \in \mathbb{Z}$ with $n \geq -1$, we define $[n]$ to be the totally ordered finite set $\{0 < 1 < \ldots < n\}$. Thus, $[-1] = \emptyset$, $[0] = \{0\}$, $[1] = \{0 < 1\}$, and so on. We define (Δ^+) to be the small category given by $\mathrm{ob}(\Delta^+) := \{[n] \mid n \in \mathbb{Z},\ n \geq -1\}$ and

$$\mathrm{Mor}(\Delta^+) := \{\text{monotone maps}\}.$$

For a subset S of $\{-1, 0, 1, \ldots\}$, we define $(\Delta^+)_S$ to be the full subcategory of (Δ^+) such that $\mathrm{ob}((\Delta^+)_S) = \{[n] \mid n \in S\}$. We define $(\Delta) := (\Delta^+)_{[0,\infty)}$. If $-1 \notin S$, then $(\Delta^+)_S$ is also denoted by $(\Delta)_S$.

We define $(\Delta^+)^{\mathrm{mon}}$ to be the subcategory of (Δ^+) by $\mathrm{ob}((\Delta^+)^{\mathrm{mon}}) := \mathrm{ob}(\Delta^+)$ and

$$\mathrm{Mor}((\Delta^+)^{\mathrm{mon}}) := \{\text{injective monotone maps}\}.$$

For $S \subset \{-1, 0, 1, \ldots\}$, the full subcategories $(\Delta^+)^{\mathrm{mon}}_S$ and $(\Delta)^{\mathrm{mon}}_S$ of $(\Delta^+)^{\mathrm{mon}}$ are defined similarly.

We denote $(\Delta)^{\mathrm{mon}}_{\{0,1,2\}}$ and $(\Delta^+)^{\mathrm{mon}}_{\{-1,0,1,2\}}$ by Δ_M and Δ^+_M, respectively.

Let $\mathcal{C}$ be a category. We call an object of $\mathcal{P}((\Delta^+), \mathcal{C})$ (resp. $\mathcal{P}((\Delta), \mathcal{C})$, an *augmented simplicial object* (resp. *simplicial object*) of $\mathcal{C}$.

For a subcategory $\mathcal{D}$ of (Δ^+) and an object $X_\bullet \in \mathcal{P}(\mathcal{D}, \mathcal{C})$, we denote $X_{[n]}$ by X_n.

As $[-1]$ is the initial object of (Δ^+), an augmented simplicial object $X_\bullet$ of $\mathcal{C}$ with $X_{-1} = c$ is identified with a simplicial object of $\mathcal{C}/c$.

We define some particular morphisms in (Δ^+). The unique map $[-1] \to [n]$ is denoted by $\varepsilon(n)$. The unique injective monotone map $[n-1] \to [n]$ such that i is not in the image is denoted by $\delta_i(n)$ for $i \in [n]$. The unique surjective monotone map $[n+1] \to [n]$ such that i has two inverse images is denoted by $\sigma_i(n)$ for $i \in [n]$. The unique map $[0] \to [n]$ such that i is in the image is denoted by $\rho_i(n)$. The unique map $[n] \to [0]$ is denoted by λ_n.

Let $\mathcal{D}$ be a subcategory of (Δ^+). For $X_\bullet \in \mathcal{P}(\mathcal{D}, \mathcal{C})$, we denote $X_\bullet(\varepsilon(n))$ (resp. $X_\bullet(\delta_i(n))$, $X_\bullet(\sigma_i(n))$, $X_\bullet(\rho_i(n))$, and $X_\bullet(\lambda_n)$) by $e(n, X_\bullet)$ (resp. $d_i(n, X_\bullet)$, $s_i(n, X_\bullet)$, $r_i(n, X_\bullet)$, and $l_n(X_\bullet)$), or simply by $e(n)$ (resp. $d_i(n)$, $s_i(n)$, $r_i(n)$, l_n), if there is no danger of confusion.

Note that (Δ) is generated by $\delta_i(n)$, $\sigma_i(n)$ for various i and n.

(9.2) Note that $(\Delta^+)([m],[n])$ is a finite set for any m, n. Assume that $\mathcal{C}$ has finite limits and let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then the *Čech nerve* is defined to be $\mathrm{Nerve}(f) := \mathrm{cosk}^{(\Delta^+)}_{(\Delta^+)_{\{-1,0\}}}(f)$, where $\mathrm{cosk}^{(\Delta^+)}_{(\Delta^+)_{\{-1,0\}}}$ is the right adjoint of the restriction. It is described as follows. $\mathrm{Nerve}(f)_n = X \times_Y \times \cdots \times_Y X$ ($(n+1)$-fold fiber product) for $n \geq 0$, and $\mathrm{Nerve}(f)_{-1} = Y$. Note that $d_i(n)$ is given by

$$d_i(n)(x_n, \ldots, x_1, x_0) = (x_n, \cdots \overset{i}{\check{}} \cdots, x_1, x_0),$$

and $s_i(n)$ is given by

$$s_i(n)(x_n, \ldots, x_1, x_0) = (x_n, \ldots, x_{i+1}, x_i, x_i, x_{i-1}, \ldots, x_1, x_0)$$

if $\mathcal{C} = \underline{\mathrm{Set}}$.

(9.3) Let S be a scheme. A simplicial object (resp. augmented simplicial object) in $\underline{\mathrm{Sch}}/S$, in other words, an object of $\mathcal{P}((\Delta), \underline{\mathrm{Sch}}/S)$ (resp. $\mathcal{P}((\Delta^+), \underline{\mathrm{Sch}}/S))$, is called a simplicial (resp. augmented simplicial) S-scheme.

If I is a subcategory of (Δ^+), $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/\mathbb{Z})$, $\heartsuit = \mathrm{Mod}, \mathrm{PM}, \mathrm{AB}, \mathrm{PA}$, $\mathcal{M} \in \heartsuit(X_\bullet)$ and $[n] \in I$, then we sometimes denote $\mathcal{M}_{[n]}$ by $\mathcal{M}_n$.

The following is well-known.

9.4 Lemma. *Let $X_\bullet \in \mathcal{P}((\Delta), \underline{\mathrm{Sch}}/S)$. Then the restriction $(?)_{\Delta_M} : \mathrm{EM}(X_\bullet) \to \mathrm{EM}(X_\bullet|_{\Delta_M})$ is an equivalence. With the equivalence, quasi-coherent sheaves correspond to quasi-coherent sheaves.*

Proof. We define a third category $\mathcal{A}$ as follows. An object of $\mathcal{A}$ is a pair $(\mathcal{M}_0, \varphi)$ such that, $\mathcal{M}_0 \in \mathrm{Mod}(X_0)$, $\varphi \in \mathrm{Hom}_{\mathrm{Mod}(X_1)}(d_0^*(\mathcal{M}_0), d_1^*(\mathcal{M}_0))$, φ an isomorphism, and that $d_1^*(\varphi) = d_2^*(\varphi) \circ d_0^*(\varphi)$ (more precisely, the composite map

$$r_2^*\mathcal{M}_0 \xrightarrow{d^{-1}} d_1^*d_0^*\mathcal{M}_0 \xrightarrow{d_1^*\varphi} d_1^*d_1^*\mathcal{M}_0 \xrightarrow{d} r_0^*\mathcal{M}_0$$

agrees with the composite map

$$r_2^*\mathcal{M}_0 \xrightarrow{d^{-1}} d_0^*d_0^*\mathcal{M}_0 \xrightarrow{d_0^*\varphi} d_0^*d_1^*\mathcal{M}_0 \xrightarrow{d} d_2^*d_0^*\mathcal{M}_0 \xrightarrow{d_2^*\varphi} d_2^*d_1^*\mathcal{M}_0 \xrightarrow{d} r_0^*\mathcal{M}_0.$$

We use such a simplified notation throughout the proof of this lemma). Note that applying l_2^* to the last equality, we get $l_1^*(\varphi) = l_1^*(\varphi) \circ l_1^*(\varphi)$. As φ is an isomorphism, we get $l_1^*(\varphi) = \mathrm{id}$.

A morphism $\gamma_0 : (\mathcal{M}_0, \varphi) \to (\mathcal{N}_0, \psi)$ is an element

$$\gamma_0 \in \mathrm{Hom}_{\mathrm{Mod}(X_0)}(\mathcal{M}_0, \mathcal{N}_0)$$

such that

$$\psi \circ d_0^*(\gamma_0) = d_1^*(\gamma_0) \circ \varphi.$$

We define a functor $\Phi : \mathrm{EM}(X_\bullet|_{\Delta_M}) \to \mathcal{A}$ by

$$\Phi(\mathcal{M}) := (\mathcal{M}_0, \alpha_{d_1(1)}^{-1} \circ \alpha_{d_0(1)}).$$

It is easy to verify that this gives a well-defined functor.

Now we define a functor $\Psi : \mathcal{A} \to \mathrm{EM}(X_\bullet)$. Note that an object $\mathcal{M}$ of $\mathrm{EM}(X_\bullet)$ is identified with a family $(\mathcal{M}_n, \alpha_w)_{[n]\in(\Delta),\, w\in\mathrm{Mor}((\Delta))}$ such that $\mathcal{M}_n \in \mathrm{Mod}(X_n)$,

$$\alpha_w \in \mathrm{Hom}_{\mathrm{Mod}(X_n)}((X_w)^*_{\mathrm{Mod}}(\mathcal{M}_m), \mathcal{M}_n)$$

for $w \in \Delta(m, n)$, α_w is an isomorphism, and

$$\alpha_{ww'} = \alpha_w \circ X_w^*\alpha_{w'} \circ d^{-1} \tag{9.5}$$

whenever ww' is defined, see (4.6).

For $(\mathcal{M}'_0, \varphi) \in \mathcal{A}$, we define $\mathcal{M}_{n,i} := (r_i(n))^*(\mathcal{M}'_0)$, and $\mathcal{M}_n := \mathcal{M}_{n,0}$ for $n \geq 0$ and $0 \leq i \leq n$. We define $\psi_i(n) : \mathcal{M}_{n,i+1} \to \mathcal{M}_{n,i}$ to be $(X_{q(i,n)})^*(\varphi)$ for $n \geq 1$ and $0 \leq i < n$, where $q(i,n) : [1] \to [n]$ is the unique injective monotone map with $\{i, i+1\} = \operatorname{Im} q(i,n)$. We define $\varphi_i(n) : \mathcal{M}_{n,i} \cong \mathcal{M}_n$ to be the composite map

$$\varphi_i(n) := \psi_0(n) \circ \psi_1(n) \circ \cdots \circ \psi_{i-1}(n)$$

for $n \geq 0$ and $0 \leq i \leq n$.

Now we define

$$\alpha_w \in \operatorname{Hom}_{\operatorname{Mod}(X_n)}(X_w^*(\mathcal{M}_m), \mathcal{M}_n)$$

to be the map

$$X_w^*\mathcal{M}_m = X_w^* r_0(m)^* \mathcal{M}'_0 \xrightarrow{d} r_{w(0)}(n)^* \mathcal{M}'_0 = \mathcal{M}_{n,w(0)} \xrightarrow{\varphi_{w(0)}(n)} \mathcal{M}_n$$

for $w \in \Delta([m],[n])$.

Thus $(\mathcal{M}'_0, \varphi)$ yields a family $(\mathcal{M}_n, \alpha_w)$, and this gives the definition of $\Psi : \mathcal{A} \to \operatorname{EM}(X_\bullet)$. The details of the proof of the well-definedness is left to the reader.

It is also straightforward to check that $(?)_{\Delta_M}$, Φ, and Ψ give the equivalence of these three categories. The proof is also left to the reader.

The last assertion is obvious from the construction. $\square$

10 Descent theory

Let S be a scheme.

(10.1) Consider the functor shift : $(\Delta^+) \to (\Delta)$ given by $\operatorname{shift}[n] := [n+1]$, $\operatorname{shift}(\delta_i(n)) := \delta_{i+1}(n+1)$, $\operatorname{shift}(\sigma_i(n)) := \sigma_{i+1}(n+1)$, and $\operatorname{shift}(\varepsilon(0)) := \delta_1(1)$. We have a natural transformation $(\delta_0^+) : \operatorname{Id}_{(\Delta^+)} \to \iota \circ \operatorname{shift}$ given by $(\delta_0^+)_n := \delta_0(n+1)$ for $n \geq 0$ and $(\delta_0^+)_{-1} := \varepsilon(0)$, where $\iota : (\Delta) \hookrightarrow (\Delta^+)$ is the inclusion. We denote $(\delta_0^+)\iota$ by (δ_0). Note that (δ_0) can be viewed as a natural map $(\delta_0) : \operatorname{Id}_{(\Delta)} \to \operatorname{shift} \iota$.

Let $X_\bullet \in \mathcal{P}((\Delta), \underline{\mathrm{Sch}}/S)$. We define $X'_\bullet$ to be the augmented simplicial scheme $\operatorname{shift}^\#(X_\bullet) = X_\bullet \operatorname{shift}$. The natural map

$$X_\bullet(\delta_0) : X'_\bullet|_{(\Delta)} = X_\bullet \operatorname{shift} \iota \to X_\bullet$$

is denoted by $(d_0)(X_\bullet)$ or (d_0). Similarly, if $Y_\bullet \in \mathcal{P}((\Delta^+), \underline{\mathrm{Sch}}/S)$, then

$$(d_0^+)(Y_\bullet) : (Y_\bullet|_{(\Delta)})' = Y_\bullet \iota\,\mathrm{shift} \xrightarrow{Y_\bullet(\delta_0^+)} Y_\bullet$$

is defined as well.

(10.2) We say that $X_\bullet \in \mathcal{P}((\Delta), \underline{\mathrm{Sch}}/S)$ is a *simplicial groupoid* of S-schemes if there is a faithfully flat morphism of S-schemes $g : Z \to Y$ such that there is a faithfully flat cartesian morphism $f_\bullet : Z_\bullet \to X_\bullet$ of $\mathcal{P}((\Delta), \underline{\mathrm{Sch}}/S)$, where $Z_\bullet = \mathrm{Nerve}(g)|_{(\Delta)}$.

10.3 Lemma. *Let $X_\bullet \in \mathcal{P}((\Delta), \underline{\mathrm{Sch}}/S)$.*

1 *If $X_\bullet \cong \mathrm{Nerve}(g)|_{(\Delta)}$ for some faithfully flat morphism g of S-schemes, then $X_\bullet$ is a simplicial groupoid.*

2 *If $f_\bullet : Z_\bullet \to X_\bullet$ is a faithfully flat cartesian morphism of simplicial S-schemes and $Z_\bullet$ is a simplicial groupoid, then we have $X_\bullet$ is also a simplicial groupoid.*

3 *$X_\bullet$ is a simplicial groupoid if and only if $(d_0) : X'_\bullet|_{(\Delta)} \to X_\bullet$ is cartesian, the canonical unit map*

$$X'_\bullet \to \mathrm{Nerve}(d_1(1)) = \mathrm{cosk}^{(\Delta^+)}_{(\Delta^+)^{\mathrm{mon}}_{\{-1,0\}}}(X'_\bullet|_{(\Delta^+)^{\mathrm{mon}}_{\{-1,0\}}})$$

is an isomorphism, and $d_0(1)$ and $d_1(1)$ are flat.

4 *If $f_\bullet : Z_\bullet \to X_\bullet$ is a cartesian morphism of simplicial S-schemes and $X_\bullet$ is a simplicial groupoid, then $Z_\bullet$ is a simplicial groupoid.*

5 *A simplicial groupoid has faithfully flat $(\Delta)^{\mathrm{mon}}$-arrows.*

6 *If $X_\bullet$ is a simplicial groupoid of S-schemes such that $d_0(1)$ and $d_1(1)$ are separated (resp. quasi-compact, quasi-separated, of finite type, smooth, étale), then $X_\bullet$ has separated (resp. quasi-compact, quasi-separated, of finite type, smooth,étale) $(\Delta)^{\mathrm{mon}}$-arrows, and $(d_0) : X'_\bullet|_{(\Delta)} \to X_\bullet$ is separated (resp. quasi-compact, quasi-separated, of finite type, smooth, étale).*

Proof. **1** and **2** are obvious by definition.

We prove **3**. We prove the 'if' part. As $d_0(1)s_0(0) = \mathrm{id} = d_1(1)s_0(0)$, we have that $d_0(1)$ and $d_1(1)$ are faithfully flat by assumption. As $d_0(1) = (d_0)_0$ is faithfully flat and (d_0) is cartesian, it is easy to see that (d_0) is also faithfully flat. So this direction is obvious.

We prove the 'only if' part. As $X_\bullet$ is a simplicial groupoid, there is a faithfully flat S-morphism $g : Z \to Y$ and a faithfully flat cartesian morphism $f_\bullet : Z_\bullet \to X_\bullet$ of simplicial S-schemes, where $Z_\bullet = \mathrm{Nerve}(g)|_{(\Delta)}$. It is easy to see that $(d_0) : Z'_\bullet|_{(\Delta)} \to Z_\bullet$ is nothing but the base change by g, and it is faithfully flat cartesian. It is also obvious that $Z'_\bullet \cong \mathrm{Nerve}(d_1(1)(Z_\bullet))$ and $d_0(1)(Z_\bullet)$ and $d_1(1)(Z_\bullet)$ are flat. It is obvious that $f'_\bullet : Z'_\bullet \to X'_\bullet$ is faithfully flat cartesian. Now by Lemma 7.16, $(d_0)(X_\bullet)$ is cartesian. As $f_\bullet$ is faithfully flat cartesian and $d_0(1)(Z_\bullet)$ and $d_1(1)(Z_\bullet)$ are flat, we have that $d_0(1)(X_\bullet)$ and $d_1(1)(X_\bullet)$ are flat. When we base change $X'_\bullet \to \mathrm{Nerve}(d_1(1)(X_\bullet))$ by $f_0 : Z_0 \to X_0$, then we have the isomorphism $Z'_\bullet \cong \mathrm{Nerve}(d_1(1)(Z_\bullet))$. As f_0 is faithfully flat, we have that $X'_\bullet \to \mathrm{Nerve}(d_1(1))$ is also an isomorphism.

The assertions **4**, **5** and **6** are proved easily. □

(10.4) Let $X_\bullet \in \mathcal{P}((\Delta), \underline{\mathrm{Sch}}/S)$. Then we define $F : \mathrm{Zar}(X'_\bullet) \to \mathrm{Zar}(X_\bullet)$ by $F(([n], U)) = (\mathrm{shift}[n], U)$ and $F((w, h)) = (\mathrm{shift}\, w, h)$. The corresponding pull-back $F^{\#}_{\mathrm{Mod}}$ is denoted by $(?)'$. It is easy to see that $(?)'$ has a left and a right adjoint. It also preserves equivariant and locally quasi-coherent sheaves.

Let $\mathcal{M} \in \mathrm{Mod}(X_\bullet)$. Then we define $(\alpha) : (d_0)^*\mathcal{M} \to \mathcal{M}'_{(\Delta)}$ by

$$(\alpha)_n : ((d_0)^*\mathcal{M})_n \xrightarrow{\theta^{-1}} d_0(n)^*\mathcal{M}_n \xrightarrow{\alpha_{\delta_0(n)}} \mathcal{M}_{n+1} = \mathcal{M}'_n.$$

It is easy to see that $(\alpha) : (d_0)^* \to (?)_{(\Delta)} \circ (?)'$ is a natural map. Similarly, for $Y_\bullet \in \mathcal{P}((\Delta^+), \underline{\mathrm{Sch}}/S)$,

$$(\alpha^+) : (d_0^+)^* \to (?)' \circ (?)_{(\Delta)}$$

is defined.

(10.5) Let $X_\bullet \in \mathcal{P}((\Delta^+), \underline{\mathrm{Sch}}/S)$, and $\mathcal{M} \in \mathrm{Mod}(X_\bullet|_{(\Delta)})$. Then, we have a cosimplicial object $\mathrm{Cos}(\mathcal{M})$ of $\mathrm{Mod}(X_{-1})$ (i.e., a simplicial object of $\mathrm{Mod}(X_{-1})^{\mathrm{op}}$). We have $\mathrm{Cos}(\mathcal{M})_n := e(n)_*(\mathcal{M}_n)$, and

$$\mathrm{Cos}(\mathcal{M})_w : e(m)_*(\mathcal{M}_m) \xrightarrow{\beta_w} e(m)_*(w_*(\mathcal{M}_n)) \xrightarrow{c^{-1}} e(n)_*(\mathcal{M}_n)$$

for a morphism $w : [m] \to [n]$ in Δ. Similarly, the augmented cosimplicial object $\mathrm{Cos}^+(\mathcal{N})$ of $\mathrm{Mod}(X_{-1})$ is defined for $\mathcal{N} \in \mathrm{Mod}(X_\bullet)$.

By (6.14), it is easy to see that $\mathcal{M}^+ := R^{\mathrm{Mod}}_{(\Delta)}(\mathcal{M})$ is $\mathcal{M}$ on $X_\bullet|_{(\Delta)}$, $\mathcal{M}^+_{-1}$ is $\varprojlim \mathrm{Cos}(\mathcal{M})$, and $\beta_{\varepsilon(n)}(\mathcal{M}^+)$ is nothing but the canonical map

$$\varprojlim \mathrm{Cos}(\mathcal{M}) \to \mathrm{Cos}(\mathcal{M})_n = e(n)_*(\mathcal{M}_n) = e(n)_*(\mathcal{M}^+_n).$$

Note that $\mathrm{Cos}(\mathcal{M})$ can be viewed as a (co)chain complex such that $\mathrm{Cos}(\mathcal{M})^n = e(n)_*(\mathcal{M}_n)$ for $n \geq 0$, and the boundary map $\partial^n : \mathrm{Cos}(\mathcal{M})^n \to \mathrm{Cos}(\mathcal{M})^{n+1}$ is given by $\partial^n = d_0 - d_1 + \cdots + (-1)^{n+1} d_{n+1}$, where $d_i = d_i(n+1) = \mathrm{Cos}(\mathcal{M})_{\delta_i(n+1)}$. Similarly, for $\mathcal{N} \in \mathrm{Mod}(X_\bullet)$, $\mathrm{Cos}^+(\mathcal{N})$ can be viewed as an augmented cochain complex.

Note also that for $\mathcal{M} \in \mathrm{Mod}(X_\bullet|_{(\Delta)})$, we have

$$\varprojlim \mathrm{Cos}(\mathcal{M}) = \mathrm{Ker}(d_0(1) - d_1(1)) = H^0(\mathrm{Cos}(\mathcal{M})), \tag{10.6}$$

which is determined only by $\mathcal{M}_{(\Delta)_{\{0,1\}}}$.

10.7 Lemma. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism of $\mathcal{P}((\Delta^+), \underline{\mathrm{Sch}}/S)$. If $f_\bullet|_{(\Delta^+)_{\{-1,0,1\}}}$ is flat cartesian, and $Y_\bullet$ has concentrated $(\Delta^+)^{\mathrm{mon}}_{\{-1,0,1\}}$-arrows, then the canonical map*

$$\mu : f_\bullet^* \circ R_{(\Delta)} \to R_{(\Delta)} \circ (f_\bullet|_{(\Delta)})^*$$

(*see* (6.27)) *is an isomorphism of functors from* $\mathrm{Lqc}(Y_\bullet|_{(\Delta)})$ *to* $\mathrm{Lqc}(X_\bullet)$.

Proof. To prove that the map in question is an isomorphism, it suffices to show that the map is an isomorphism after applying the functor $(?)_n$ for $n \geq -1$. This is trivial if $n \geq 0$. On the other hand, if $n = -1$, the map restricted at -1 and evaluated at $\mathcal{M} \in \mathrm{Lqc}(Y_\bullet|_{(\Delta)})$ is nothing but

$$f_{-1}^*(H^0(\mathrm{Cos}(\mathcal{M}))) \cong H^0(f_{-1}^*(\mathrm{Cos}(\mathcal{M}))) \to H^0(\mathrm{Cos}((f_\bullet|_{(\Delta)})^*(\mathcal{M}))).$$

The first map is an isomorphism as f_{-1} is flat. Although the map

$$f_{-1}^*(\mathrm{Cos}(\mathcal{M})) \to \mathrm{Cos}((f_\bullet|_{(\Delta)})^*(\mathcal{M}))$$

may not be a chain isomorphism, it is an isomorphism at the degrees $-1, 0, 1$, and it induces the isomorphism of H^0. $\square$

10.8 Lemma. *Let* $X_\bullet \in \mathcal{P}((\Delta), \underline{\mathrm{Sch}}/S)$*, and* $\mathcal{M} \in \mathrm{Mod}(X_\bullet)$*. Then the* (*associated chain complex of the*) *augmented cosimplicial object* $\mathrm{Cos}^+(\mathcal{M}')$ *of* $\mathcal{M}' \in \mathrm{Mod}(X'_\bullet)$ *is split exact. In particular, the unit map* $u : \mathcal{M}' \to R_{(\Delta)}\mathcal{M}'_{(\Delta)}$ *is an isomorphism.*

Proof. Define $s_n : \mathrm{Cos}^+(\mathcal{M}')_n \to \mathrm{Cos}^+(\mathcal{M}')_{n-1}$ to be

$$\mathrm{Cos}^+(\mathcal{M}')_n = (r_0)(n+1)_*(\mathcal{M}_{n+1}) \xrightarrow{(r_0)(n+1)_*\beta_{\sigma_0(n)}} (r_0)(n+1)_* s_0(n)_*(\mathcal{M}_n) \xrightarrow{c^{-1}} (r_0)(n)_*(\mathcal{M}_n) = \mathrm{Cos}^+(\mathcal{M}')_{n-1}$$

for $n \geq 0$, and $s_{-1} : \mathrm{Cos}^+(\mathcal{M}')_{-1} \to 0$ to be 0. It is easy to verify that s is a chain deformation of $\mathrm{Cos}^+(\mathcal{M}')$. □

10.9 Corollary. *Let the notation be as in the lemma. Then there is a functorial isomorphism*

$$R_{(\Delta)}(d_0)^*(\mathcal{M}) \to \mathcal{M}' \qquad (10.10)$$

for $\mathcal{M} \in \mathrm{EM}(X_\bullet)$*. In particular, there is a functorial isomorphism*

$$(R_{(\Delta)}(d_0)^*(\mathcal{M}))_{-1} \to \mathcal{M}_0. \qquad (10.11)$$

Proof. The first map (10.10) is defined to be the composite

$$R_{(\Delta)}(d_0)^* \xrightarrow{R_{(\Delta)}(\alpha)} R_{(\Delta)}(?)_{(\Delta)}(?)' \xrightarrow{u^{-1}} (?)'.$$

As $(\alpha)(\mathcal{M})$ is an isomorphism if $\mathcal{M}$ is equivariant, this is an isomorphism. The second map (10.11) is obtained from (10.10), applying $(?)_{-1}$. □

The following well-known theorem in descent theory contained in [33] is now easy to prove.

10.12 Proposition. *Let* $f : X \to Y$ *be a morphism of* S*-schemes, and set* $X^+_\bullet := \mathrm{Nerve}(f)$*, and* $X_\bullet := X^+_\bullet|_{(\Delta)}$*. Let* $\mathcal{M} \in \mathrm{Mod}(X_\bullet)$*. Then we have the following.*

0 *The counit of adjunction*

$$\varepsilon : (R_{(\Delta)}\mathcal{M})_{(\Delta)} \to \mathcal{M}$$

is an isomorphism.

1 *If* f *is concentrated and* $\mathcal{M} \in \mathrm{Lqc}(X_\bullet)$*, then* $R_{(\Delta)}\mathcal{M} \in \mathrm{Lqc}(X^+_\bullet)$*.*

2 *If f is faithfully flat concentrated and $\mathcal{M} \in \mathrm{Qch}(X_\bullet)$, then we have $R_{(\Delta)}\mathcal{M} \in \mathrm{Qch}(X_\bullet^+)$.*

3 *If f is faithfully flat concentrated, $\mathcal{N} \in \mathrm{EM}(X_\bullet^+)$, and $\mathcal{N}_{(\Delta)} \in \mathrm{Qch}(X_\bullet)$, then the unit of adjunction*

$$u : \mathcal{N} \to R_{(\Delta)}(\mathcal{N}_{(\Delta)})$$

is an isomorphism. In particular, $\mathcal{N}$ is quasi-coherent.

4 *If f is faithfully flat concentrated, then the restriction functor*

$$(?)_{(\Delta)} : \mathrm{Qch}(X_\bullet^+) \to \mathrm{Qch}(X_\bullet)$$

is an equivalence, with $R_{(\Delta)}$ its quasi-inverse.

Proof. The assertion **0** follows from Lemma 6.15.

We prove **1**. By **0**, it suffices to prove that

$$(R_{(\Delta)}\mathcal{M})_{-1} = \mathrm{Ker}(e(0)_*\beta_{\delta_0(1)} - e(0)_*\beta_{\delta_1(1)})$$

is quasi-coherent. This is obvious by [14, (9.2.2)].

Now we assume that f is faithfully flat concentrated, to prove the assertions **2**, **3**, and **4**.

We prove **2**. As we already know that $R_{(\Delta)}\mathcal{M}$ is locally quasi-coherent, it suffices to show that it is equivariant. As $(d_0^+) : (X_\bullet|_{(\Delta)})' \to X_\bullet$ is faithfully flat, it suffices to show that $(d_0^+)^* R_{(\Delta)}\mathcal{M}$ is equivariant, by Lemma 7.22. Now the assertion is obvious by Lemma 10.7 and Corollary 10.9, as $\mathcal{M}'$ is quasi-coherent.

We prove **3**. Note that the composite map

$$(d_0^+)^*\mathcal{N} \xrightarrow{(d_0^+)^*u} (d_0^+)^* R_{(\Delta)}(\mathcal{N}_{(\Delta)}) \xrightarrow{\mu} R_{(\Delta)}(d_0)^*\mathcal{N}_{(\Delta)} \cong R_{(\Delta)}((d_0^+)^*\mathcal{N})_{(\Delta)} \tag{10.13}$$

is nothing but the unit of adjunction $u((d_0^+)^*\mathcal{N})$. As $(\alpha^+) : (d_0^+)^*\mathcal{N} \to (\mathcal{N}_{(\Delta)})'$ is an isomorphism since $\mathcal{N}$ is equivariant, we have that $u((d_0^+)^*\mathcal{N})$ is an isomorphism by Lemma 10.8. As μ in (10.13) is an isomorphism by Lemma 10.7, we have that $(d_0^+)^*u$ is an isomorphism. As (d_0^+) is faithfully flat, we have that $u : \mathcal{N} \to R_{(\Delta)}(\mathcal{N}_{(\Delta)})$ is an isomorphism, as desired. The last assertion is obvious by **2**, and **3** is proved.

The assertion **4** is a consequence of **0**, **2** and **3**. □

10.14 Corollary. *Let $f : X \to Y$ be a faithfully flat quasi-compact morphism of schemes, and $\mathcal{M} \in \mathrm{Mod}(Y)$. Then $\mathcal{M}$ is quasi-coherent if and only if $f^*\mathcal{M}$ is.*

Proof. The 'only if' part is trivial.

We prove the 'if' part. We may assume that Y is affine. So X is quasi-compact, and has a finite affine open covering (U_i). Replacing X by $\coprod_i U_i$, we may assume that X is also affine. Thus f is faithfully flat concentrated. If $f^*\mathcal{M}$ is quasi-coherent, then $\mathcal{N} := L_{-1}\mathcal{M}$ satisfies the assumption of **3** of the proposition, as can be seen easily. So $\mathcal{M} \cong (\mathcal{N})_{-1}$ is quasi-coherent. $\square$

10.15 Corollary. *Let the notation be as in the proposition, and assume that f is faithfully flat concentrated. The composite functor*

$$\mathbb{A} := (?)_{(\Delta)} \circ L_{-1} : \mathrm{Qch}(Y) \to \mathrm{Qch}(X_\bullet)$$

is an equivalence with

$$\mathbb{D} := (?)_{-1} \circ R_{(\Delta)}$$

its quasi-inverse.

Proof. Follows immediately by the proposition and Lemma 6.38, **2**, since $[-1]$ is the initial object of (Δ^+). $\square$

We call $\mathbb{A}$ in the corollary the *ascent functor*, and $\mathbb{D}$ the *descent functor*.

10.16 Corollary. *Let the notation be as in the proposition. Then the composite functor*

$$\mathbb{A} \circ \mathbb{D} : \mathrm{Lqc}(X_\bullet) \to \mathrm{Qch}(X_\bullet)$$

is the right adjoint functor of the inclusion $\mathrm{Qch}(X_\bullet) \hookrightarrow \mathrm{Lqc}(X_\bullet)$.

Proof. Note that $\mathbb{D} : \mathrm{Lqc}(X_\bullet) \to \mathrm{Qch}(Y)$ is a well-defined functor, and hence $\mathbb{A} \circ \mathbb{D}$ is a functor from $\mathrm{Lqc}(X_\bullet)$ to $\mathrm{Qch}(X_\bullet)$.

For $\mathcal{M} \in \mathrm{Qch}(X_\bullet)$ and $\mathcal{N} \in \mathrm{Lqc}(X_\bullet)$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Qch}(X_\bullet)}(\mathcal{M}, \mathbb{A}\mathbb{D}\mathcal{N}) &\cong \mathrm{Hom}_{\mathrm{Qch}(Y)}(\mathbb{D}\mathcal{M}, \mathbb{D}\mathcal{N}) \\ &\cong \mathrm{Hom}_{\mathrm{Lqc}(X_\bullet^+)}(R_{(\Delta)}\mathcal{M}, R_{(\Delta)}\mathcal{N}) \\ &\cong \mathrm{Hom}_{\mathrm{Lqc}(X_\bullet)}((R_{(\Delta)}\mathcal{M})_{(\Delta)}, \mathcal{N}) \cong \mathrm{Hom}_{\mathrm{Lqc}(X_\bullet)}(\mathcal{M}, \mathcal{N})\end{aligned}$$

by the proposition, Corollary 10.15, and Lemma 6.38, **1**. $\square$

10.17 Corollary. *Let $X_\bullet$ be a simplicial groupoid of S-schemes, and assume that $d_0(1)$ and $d_1(1)$ are concentrated. Then*

$$(d_0)_*^{\mathrm{Qch}} \circ \mathbb{A} : \mathrm{Qch}(X_0) \to \mathrm{Qch}(X_\bullet)$$

is a right adjoint of $(?)_0 : \mathrm{Qch}(X_\bullet) \to \mathrm{Qch}(X_0)$, where $\mathbb{A} : \mathrm{Qch}(X_0) \to \mathrm{Qch}(X'_\bullet|_{(\Delta)})$ is the ascent functor defined in Corollary 10.15.

Proof. Note that $(d_0)_*^{\mathrm{Qch}}$ is well-defined, because (d_0) is concentrated cartesian, and the simplicial groupoid $X_\bullet$ has flat $(\Delta)^{\mathrm{mon}}$-arrows, see (7.15) and Lemma 10.3. It is obvious that $\mathbb{D} \circ (d_0)^*_{\mathrm{Qch}}$ is the left adjoint of $(d_0)_*^{\mathrm{Qch}} \circ \mathbb{A}$ by Corollary 10.15. On the other hand, we have $(?)_0 \cong \mathbb{D} \circ (d_0)^*_{\mathrm{Qch}}$ by Corollary 10.9. Hence, $(d_0)_*^{\mathrm{Qch}} \circ \mathbb{A}$ is a right adjoint of $(?)_0$, as desired. $\square$

(10.18) Let $f : X \to Y$ be a morphism of S-schemes, and set $X_\bullet^+ := \mathrm{Nerve}(f)$, and $X_\bullet = X_\bullet^+|_{(\Delta)}$. It seems that even if f is concentrated and faithfully flat, the canonical descent functor $\mathrm{EM}(X_\bullet) \to \mathrm{Mod}(Y)$ may not be an isomorphism. However, we have this kind of isomorphism for special morphisms.

Let $f : X \to Y$ be a morphism of schemes. We say that f is a *locally an open immersion* if there exists some open covering (U_i) of X such that $f|_{U_i}$ is an open immersion for any i. Assume that f is locally an open immersion.

10.19 Lemma. *Let $f : X \to Y$ be locally an open immersion. Let $g : Y' \to Y$ be any morphism, $X' := Y' \times_Y X$, $g' : X' \to X$ the second projection, and $f' : X' \to Y'$ the first projection. Then the canonical map $\theta : f^* g_* \to g'_*(f')^*$ between the functors from* $\mathrm{Mod}(Y')$ *to* $\mathrm{Mod}(X)$ *is an isomorphism.*

Proof. Use Lemma 2.59. $\square$

10.20 Lemma. *Let $f : X \to Y$, $X_\bullet^+$, and $X_\bullet$ be as in* (10.18). *Assume that f is faithfully flat and locally an open immersion. Then the descent functor $\mathbb{D} = (?)_{-1} R_{(\Delta)} : \mathrm{EM}(X_\bullet) \to \mathrm{Mod}(Y)$ is an equivalence with $\mathbb{A} = (?)_{(\Delta)} L_{-1}$ its quasi-inverse.*

Proof. Similar to the proof of Proposition 10.12. $\square$

10.21 Lemma. *Let $f : X \to Y$, $X_\bullet^+$, and $X_\bullet$ be as in* (10.18). *Assume that f is faithfully flat and locally an open immersion. Then for $\mathcal{M} \in \mathrm{EM}(X'_\bullet)$, the direct image $(d_0)_* \mathcal{M}$ is equivariant. The restriction $\mathrm{EM}(X_\bullet) \to \mathrm{Mod}(X_0) = \mathrm{Mod}(X)$ has the right adjoint $(d_0)_* \mathbb{A}$.*

Proof. Easy. $\square$

11 Local noetherian property

An abelian category $\mathcal{A}$ is called *locally noetherian* if it is a $\mathcal{U}$-category, satisfies the (AB5) condition, and has a small set of noetherian generators [11]. For a locally noetherian category $\mathcal{A}$, we denote the full subcategory of $\mathcal{A}$ consisting of its noetherian objects by $\mathcal{A}_f$.

11.1 Lemma. *Let $\mathcal{A}$ be an abelian $\mathcal{U}$-category which satisfies the* (AB3) *condition, and $\mathcal{B}$ a locally noetherian category. Let $F : \mathcal{A} \to \mathcal{B}$ be a faithful exact functor, and G its right adjoint. If G preserves filtered inductive limits, then the following hold.*

1 *$\mathcal{A}$ is locally noetherian.*

2 *$a \in \mathcal{A}$ is a noetherian object if and only if Fa is.*

Proof. The 'if' part of **2** is obvious, as F is faithful and exact. Note that $\mathcal{A}$ satisfies the (AB5) condition, as F is faithful exact and colimit preserving, and $\mathcal{B}$ satisfies the (AB5) condition.

Note also that, for $a \in \mathcal{A}$, the set of subobjects of a is small, because the set of subobjects of Fa is small [13] and F is faithful exact.

Let S be a small set of noetherian generators of $\mathcal{B}$. As any noetherian object is a quotient of a finite sum of objects in S, we may assume that any noetherian object in $\mathcal{B}$ is isomorphic to an element of S, replacing S by some larger small set, if necessary. For each $s \in S$, the set of subobjects of Gs is small by the last paragraph. Hence, there is a small subset T of $\mathrm{ob}(\mathcal{A})$ such that, any element $t \in T$ admits a monomorphism $t \to Gs$ for some $s \in S$, Ft is noetherian, and that if $a \in \mathcal{A}$ admits a monomorphism $a \to Gs$ for some $s \in S$ and Fa noetherian then $a \cong t$ for some $t \in T$.

We claim that any $a \in \mathcal{A}$ is a filtered inductive limit $\varinjlim a_\lambda$ of subobjects a_λ of a, with each a_λ is isomorphic to some element in T.

If the claim is true, then **1** is obvious, as T is a small set of noetherian generators of $\mathcal{A}$, and $\mathcal{A}$ satisfies the (AB5) condition, as we have already seen.

The 'only if' part of **2** is also true if the claim is true, since if $a \in \mathcal{A}$ is noetherian, then it is a quotient of a finite sum of elements of T, and hence Fa is noetherian.

It suffices to prove the claim. As $\mathcal{B}$ is locally noetherian, we have $Fa = \varinjlim b_\lambda$, where (b_λ) is the filtered inductive system of noetherian subobjects of Fa.

Let $u : \mathrm{Id} \to GF$ be the unit of adjunction, and $\varepsilon : FG \to \mathrm{Id}$ be the counit of adjunction. It is well-known that we have $(\varepsilon F) \circ (Fu) = \mathrm{id}_F$. As Fu is a split monomorphism, u is also a monomorphism. We define $a_\lambda := u(a)^{-1}(Gb_\lambda)$. As G preserves filtered inductive limits and $\mathcal{A}$ satisfies the (AB5) condition, we have

$$\varinjlim a_\lambda = u(a)^{-1}(G \varinjlim b_\lambda) = u(a)^{-1}(GFa) = a.$$

Note that $a_\lambda \to Gb_\lambda$ is a monomorphism, with b_λ being noetherian.

It remains to show that Fa_λ is noetherian. Let $i_\lambda : a_\lambda \hookrightarrow a$ be the inclusion map, and $j_\lambda : b_\lambda \to Fa$ the inclusion. Then the diagram

$$\begin{array}{ccccc} Fa & \xrightarrow{Fu(a)} & FGFa & \xrightarrow{\varepsilon F(a)} & Fa \\ Fi_\lambda \uparrow & & \uparrow FGj_\lambda & & \uparrow j_\lambda \\ Fa_\lambda & \longrightarrow & FGb_\lambda & \xrightarrow{\varepsilon(b_\lambda)} & b_\lambda \end{array}$$

is commutative. As the composite of the first row is the identity map and Fi_λ is a monomorphism, we have that the composite of the second row $Fa_\lambda \to b_\lambda$ is a monomorphism. As b_λ is noetherian, we have that Fa_λ is also noetherian, as desired. □

11.2 Lemma. *Let $\mathcal{A}$ be an abelian $\mathcal{U}$-category which satisfies the* (AB3) *condition, and $\mathcal{B}$ a Grothendieck category. Let $\mathcal{A} \to \mathcal{B}$ be a faithful exact functor, and G its right adjoint. If G preserves filtered inductive limits, then $\mathcal{A}$ is Grothendieck.*

Proof. Similar. □

(11.3) Let S be a scheme, and $X_\bullet \in \mathcal{P}((\Delta), \underline{\mathrm{Sch}}/S)$.

11.4 Lemma. *The restriction functor* $(?)_0 : \mathrm{EM}(X_\bullet) \to \mathrm{Mod}(X_0)$ *is faithful exact.*

Proof. This is obvious, because for any $[n] \in (\Delta)$, there is a morphism $[0] \to [n]$.

11.5 Lemma. *Let $X_\bullet$ be a simplicial groupoid of S-schemes, and assume that $d_0(1)$ and $d_1(1)$ are concentrated. If* $\mathrm{Qch}(X_0)$ *is Grothendieck, then* $\mathrm{Qch}(X_\bullet)$ *is Grothendieck. Assume moreover that* $\mathrm{Qch}(X_0)$ *is locally noetherian. Then we have*

1 $\mathrm{Qch}(X_\bullet)$ *is locally noetherian.*

2 $\mathcal{M} \in \mathrm{Qch}(X_\bullet)$ *is a noetherian object if and only if* $\mathcal{M}_0$ *is a noetherian object.*

Proof. Let $F := (?)_0 : \mathrm{Qch}(X_\bullet) \to \mathrm{Qch}(X_0)$ be the restriction. By Lemma 11.4, F is faithful exact. Let $G := (d_0)_*^{\mathrm{Qch}} \circ \mathbb{A}$ be the right adjoint of F, see Corollary 10.17. As $\mathbb{A}$ is an equivalence and $(d_0)_*^{\mathrm{Qch}}$ preserves filtered inductive limits by Lemma 7.20, G preserves filtered inductive limits. As $\mathrm{Qch}(X_\bullet)$ satisfies (AB3) by Lemma 7.7, the assertion is obvious by Lemma 11.1 and Lemma 11.2. $\square$

The following is well-known, see [18, pp. 126–127].

11.6 Corollary. *Let Y be a noetherian scheme. Then $\mathrm{Qch}(Y)$ is locally noetherian, and $\mathcal{M} \in \mathrm{Qch}(Y)$ is a noetherian object if and only if it is coherent.*

Proof. This is obvious if $Y = \operatorname{Spec} A$ is affine. Now consider the general case. Let $(U_i)_{1 \leq i \leq r}$ be an affine open covering of Y, and set $X := \coprod_i U_i$. Let $p : X \to Y$ be the canonical map, and set $X_\bullet := \mathrm{Nerve}(f)|_{(\Delta)}$. Note that p is faithfully flat quasi-compact separated. By assumption and the lemma, we have that $\mathrm{Qch}(X_\bullet)$ is locally noetherian, and $\mathcal{M} \in \mathrm{Qch}(X_\bullet)$ is a noetherian object if and only if $\mathcal{M}_0$ is noetherian, i.e., coherent. As $\mathbb{A} : \mathrm{Qch}(Y) \to \mathrm{Qch}(X_\bullet)$ is an equivalence, we have that $\mathrm{Qch}(Y)$ is locally noetherian, and $\mathcal{M} \in \mathrm{Qch}(Y)$ is a noetherian object if and only if $(\mathbb{A}\mathcal{M})_0 = p^*\mathcal{M}$ is coherent if and only if $\mathcal{M}$ is coherent. $\square$

A scheme X is said to be *concentrated* if the structure map $X \to \operatorname{Spec} \mathbb{Z}$ is concentrated.

11.7 Corollary. *Let Y be a concentrated scheme. Then $\mathrm{Qch}(Y)$ is Grothendieck.*

Proof. Similar. $\square$

Now the following is obvious.

11.8 Corollary. *Let $X_\bullet$ be a simplicial groupoid of S-schemes, with $d_0(1)$ and $d_1(1)$ concentrated. If X_0 is concentrated, then $\mathrm{Qch}(X_\bullet)$ is Grothendieck. If, moreover, X_0 is noetherian, then $\mathrm{Qch}(X_\bullet)$ is locally noetherian, and $\mathcal{M} \in \mathrm{Qch}(X_\bullet)$ is a noetherian object if and only if $\mathcal{M}_0$ is coherent.*

11.9 Lemma. *Let I be a finite category, S a scheme, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. If $X_\bullet$ is noetherian, then $\mathrm{Mod}(X_\bullet)$ and $\mathrm{Lqc}(X_\bullet)$ are locally noetherian. $\mathcal{M} \in \mathrm{Lqc}(X_\bullet)$ is a noetherian object if and only if $\mathcal{M}$ is locally coherent.*

Proof. Let J be the discrete subcategory of I such that $\mathrm{ob}(J) = \mathrm{ob}(I)$. Obviously, the restriction $(?)_J$ is faithful and exact. For $i \in \mathrm{ob}(I)$, there is an isomorphism of functors $(?)_i R_J \cong \prod_{j\in\mathrm{ob}(J)} \prod_{\phi\in I(i,j)} (X_\phi)_*(?)_j$. The product is a finite product, as I is finite. As each X_ϕ is concentrated, $(X_\phi)_*(?)_j$ preserves filtered inductive limits by Lemma 7.19. Hence R_J preserves filtered inductive limits. Note also that R_J preserves local quasi-coherence.

Hence we may assume that I is a discrete finite category, which case is trivial by [17, Theorem II.7.8] and Corollary 11.6. □

11.10 Lemma. *Let I be a finite category, S a scheme, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. If $X_\bullet$ is concentrated, then* $\mathrm{Lqc}(X_\bullet)$ *is Grothendieck.*

Proof. Similar.

12 Groupoid of schemes

(12.1) Let $\mathcal{C}$ be a category with finite limits. A *$\mathcal{C}$-groupoid* X_* is a functor from $\mathcal{C}^{\mathrm{op}}$ to the category of groupoids $\in \mathcal{U}$ (i.e., category $\in \mathcal{U}$ all of whose morphisms are isomorphisms) such that the set valued functors $X_0 := \mathrm{ob} \circ X_*$ and $X_1 := \mathrm{Mor} \circ X_*$ are representable.

Let X_* be a $\mathcal{C}$-groupoid. Let us denote the source (resp. target) $X_1 \to X_0$ by d_1 (resp. d_0). Then $X_2 := X_1 \, {}_{d_1}\!\times_{d_0} X_1$ represents the functor of pairs (f, g) of morphisms of X_* such that $f \circ g$ is defined. Let $d_0' : X_2 \to X_1$ (resp. $d_2' : X_2 \to X_1$) be the first (resp. second) projection , and $d_1' : X_2 \to X_1$ the composition.

By Yoneda's lemma, d_0, d_1, d_0', d_1', and d_2' are morphisms of $\mathcal{C}$. $d_1 : X_1(T) \to X_0(T)$ is surjective for any $T \in \mathrm{ob}(\mathcal{C})$. Note that the squares

$$\begin{array}{ccc} X_2 & \xrightarrow{d_0'} & X_1 \\ \downarrow d_2' & & \downarrow d_1 \\ X_1 & \xrightarrow{d_0} & X_0 \end{array} \qquad \begin{array}{ccc} X_2 & \xrightarrow{d_1'} & X_1 \\ \downarrow d_2' & & \downarrow d_1 \\ X_1 & \xrightarrow{d_1} & X_0 \end{array} \qquad \begin{array}{ccc} X_2 & \xrightarrow{d_1'} & X_1 \\ \downarrow d_0' & & \downarrow d_0 \\ X_1 & \xrightarrow{d_0} & X_0 \end{array} \tag{12.2}$$

are fiber squares. In particular,

$$X_* := X_2 \; \begin{array}{c} \xrightarrow{d_0'} \\ \xrightarrow{d_1'} \\ \xrightarrow{d_2'} \end{array} \; X_1 \; \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} \; X_0 \tag{12.3}$$

forms an object of $\mathcal{P}(\Delta_M, \mathcal{C})$. Finally, by the associativity,

$$\circ(\circ \times 1) = \circ(1 \times \circ), \tag{12.4}$$

where $\circ : X_1 \,{}_{d_1}\!\times_{d_0} X_1 \to X_1$ denotes the composition, or $\circ$ is the composite

$$X_1 \,{}_{d_1}\!\times_{d_0} X_1 \cong X_2 \xrightarrow{d_1'} X_1.$$

Conversely, a diagram $X_* \in \mathcal{P}(\Delta_M, \mathcal{C})$ as in (12.3) such that the squares in (12.2) are fiber squares, $d_1(T) : X_1(T) \to X_0(T)$ are surjective for all $T \in \mathrm{ob}(\mathcal{C})$, and the associativity (12.4) holds gives a $\mathcal{C}$-groupoid [12]. In the sequel, we mainly consider that a $\mathcal{C}$-groupoid is an object of $\mathcal{P}(\Delta_M, \mathcal{C})$.

Let S be a scheme. We say that $X_\bullet \in \mathcal{P}(\Delta_M, \underline{\mathrm{Sch}}/S)$ is an *S-groupoid*, if $X_\bullet$ is a $(\underline{\mathrm{Sch}}/S)$-groupoid with flat arrows.

(12.5) Let $X_\bullet$ be a $(\underline{\mathrm{Sch}}/S)$-groupoid, and set $X_n := X_1 \,{}_{d_1}\!\times_{d_0} X_1 \,{}_{d_1}\!\times_{d_0} \cdots {}_{d_1}\!\times_{d_0} X_1$ (X_1 appears n times) for $n \geq 2$. For $n \geq 2$, $d_i : X_n \to X_{n-1}$ is defined by $d_0(x_{n-1}, \ldots, x_1, x_0) = (x_{n-1}, \ldots, x_1)$, $d_n(x_{n-1}, \ldots, x_1, x_0) = (x_{n-2}, \ldots, x_0)$, and $d_i(x_{n-1}, \ldots, x_1, x_0) = (x_{n-1}, \ldots, x_i \circ x_{i-1}, \ldots, x_0)$ for $0 < i < n$. $s_i : X_n \to X_{n+1}$ is defined by

$$s_i(x_{n-1}, \ldots, x_1, x_0) = (x_{n-1}, \ldots, x_i, \mathrm{id}, x_{i-1}, \ldots, x_0).$$

It is easy to see that this gives a simplicial S-scheme $\Sigma(X_\bullet)$ such that $\Sigma(X_\bullet)|_{\Delta_M} = X_\bullet$.

For any simplicial S-scheme $Z_\bullet$ and $\psi_\bullet : Z_\bullet|_{\Delta_M} \to X_\bullet$, there exists some unique $\varphi_\bullet : Z_\bullet \to \Sigma(X_\bullet)$ such that $\varphi|_{\Delta_M} : Z_\bullet|_{\Delta_M} \to \Sigma(X_\bullet)|_{\Delta_M} = X_\bullet$ equals ψ. Indeed, φ is given by

$$\varphi_n(z) = (\psi_1(Q_{n-1}(z)), \ldots, \psi_1(Q_0(z))),$$

where $q_i : [1] \to [n]$ is the injective monotone map such that $\operatorname{Im} q_i = \{i, i+1\}$ for $0 \leq i < n$, and $Q_i : Z_n \to Z_1$ is the associated morphism. This shows that $\Sigma(X_\bullet) \cong \operatorname{cosk}^{(\Delta)}_{\Delta_M} X_\bullet$, and the counit map $(\operatorname{cosk}^{(\Delta)}_{\Delta_M} X_\bullet)|_{\Delta_M} \to X_\bullet$ is an isomorphism.

Note that under the identification $\Sigma(X_\bullet)_{n+1} \cong X_n \,{}_{r_0}\!\times_{d_0} X_1$, the morphism $d_0 : \Sigma(X_\bullet)_{n+1} \to \Sigma(X_\bullet)_n$ is nothing but the first projection. So $(d_0) : \Sigma(X_\bullet)' \to \Sigma(X_\bullet)$ is cartesian. If, moreover, $d_0(1)$ is flat, then (d_0) is faithfully flat.

We construct an isomorphism $h_\bullet : \Sigma(X_\bullet)' \to \mathrm{Nerve}(d_1(1))$. Define $h_{-1} = \mathrm{id}$ and $h_0 = \mathrm{id}$. Define h_1 to be the composite

$$X_1 \,{}_{d_1}\!\times_{d_0} X_1 \xrightarrow{(d_0 \boxtimes d_2)^{-1}} X_2 \xrightarrow{d_1 \boxtimes d_2} X_1 \,{}_{d_1}\!\times_{d_1} X_1.$$

Now define h_n to be the composite

$$\begin{aligned} X_1 \,{}_{d_1}\!\times_{d_0} X_1 \,{}_{d_1}\!\times_{d_0} \cdots {}_{d_1}\!\times_{d_0} X_1 \,{}_{d_1}\!\times_{d_0} X_1 &\xrightarrow{1 \times h_1} X_1 \,{}_{d_1}\!\times_{d_0} X_1 \,{}_{d_1}\!\times_{d_0} \cdots {}_{d_1}\!\times_{d_0} X_1 \,{}_{d_1}\!\times_{d_1} X_1 \\ &\xrightarrow{\text{via } h_1} \cdots \xrightarrow{h_1 \times 1} X_1 \,{}_{d_1}\!\times_{d_1} X_1 \,{}_{d_1}\!\times_{d_1} \cdots {}_{d_1}\!\times_{d_1} X_1 \,{}_{d_1}\!\times_{d_1} X_1. \end{aligned}$$

It is straightforward to check that this gives a well-defined isomorphism $h_\bullet : \Sigma(X_\bullet)' \to \mathrm{Nerve}(d_1(1))$. In conclusion, we have

12.6 Lemma. *If $X_\bullet$ is an S-groupoid, then $\mathrm{cosk}^{(\Delta)}_{\Delta_M} X_\bullet$ is a simplicial S-groupoid, and the counit $\varepsilon : (\mathrm{cosk}^{(\Delta)}_{\Delta_M} X_\bullet)|_{\Delta_M} \to X_\bullet$ is an isomorphism.*

Conversely, the following holds.

12.7 Lemma. *If $Y_\bullet$ is a simplicial S-groupoid, then $Y_\bullet|_{\Delta_M}$ is an S-groupoid, and the unit map $u : Y_\bullet \to \mathrm{cosk}^{(\Delta)}_{\Delta_M}(Y_\bullet|_{\Delta_M})$ is an isomorphism.*

Proof. It is obvious that $Y_\bullet|_{\Delta_M}$ has flat arrows. So it suffices to show that $Y_\bullet|_{\Delta_M}$ is a $(\underline{\mathrm{Sch}}/S)$-groupoid. Since $Y'_\bullet$ is isomorphic to $\mathrm{Nerve}\, d_1$, the square

$$\begin{array}{ccc} Y_2 & \xrightarrow{d_2} & Y_1 \\ \downarrow d_1 & & \downarrow d_1 \\ Y_1 & \xrightarrow{d_1} & Y_0 \end{array}$$

is a fiber square. Since $(d_0) : Y'_\bullet|_{(\Delta)} \to Y_\bullet$ is a cartesian morphism, the squares

$$\begin{array}{ccc} Y_2 & \xrightarrow{d_1} & Y_1 \\ \downarrow d_0 & & \downarrow d_0 \\ Y_1 & \xrightarrow{d_0} & Y_0 \end{array} \qquad\qquad \begin{array}{ccc} Y_2 & \xrightarrow{d_2} & Y_1 \\ \downarrow d_0 & & \downarrow d_0 \\ Y_1 & \xrightarrow{d_1} & Y_0 \end{array}$$

are fiber squares.

As $d_1 s_0 = \mathrm{id}$, $d_1(T) : Y_1(T) \to Y_0(T)$ is surjective for any S-scheme T.

Let us denote the composite

$$Y_1 \,{}_{d_1}\!\times_{d_0} Y_1 \cong Y_2 \xrightarrow{d_1} Y_1$$

by $\circ$. It remains to show the associativity.

As the three squares in the diagram

$$\begin{array}{ccccc} Y_3 & \xrightarrow{d_3} & Y_2 & \xrightarrow{d_2} & Y_1 \\ \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 \\ Y_2 & \xrightarrow{d_2} & Y_1 & \xrightarrow{d_1} & Y_0 \\ \downarrow d_0 & & \downarrow d_0 & & \\ Y_1 & \xrightarrow{d_1} & Y_0 & & \end{array}$$

are all fiber squares, the canonical map $Q := Q_2 \boxtimes Q_1 \boxtimes Q_0 : Y_3 \to Y_1 {}_{d_1}\!\times_{d_0} Y_1 {}_{d_1}\!\times_{d_0} Y_1$ is an isomorphism. So it suffices to show that the maps

$$Y_3 \xrightarrow{Q} Y_1 {}_{d_1}\!\times_{d_0} Y_1 {}_{d_1}\!\times_{d_0} Y_1 \xrightarrow{\circ\times 1} Y_1 {}_{d_1}\!\times_{d_0} Y_1 \xrightarrow{\circ} Y_1$$

and

$$Y_3 \xrightarrow{Q} Y_1 {}_{d_1}\!\times_{d_0} Y_1 {}_{d_1}\!\times_{d_0} Y_1 \xrightarrow{1\times\circ} Y_1 {}_{d_1}\!\times_{d_0} Y_1 \xrightarrow{\circ} Y_1$$

agree. But it is not so difficult to show that the first map is $d_1 d_2$, while the second one is $d_1 d_1$. So $Y_\bullet|_{\Delta_M}$ is an S-groupoid.

Set $Z_\bullet := \operatorname{cosk}^{(\Delta)}_{\Delta_M}(Y_\bullet|_{\Delta_M})$, and we are to show that the unit $u_\bullet : Y_\bullet \to Z_\bullet$ is an isomorphism. Since $Y_\bullet|_{\Delta_M}$ is an S-groupoid, $\varepsilon_\bullet : Z_\bullet|_{\Delta_M} \to Y_\bullet|_{\Delta_M}$ is an isomorphism. It follows that $u_\bullet|_{\Delta_M} : Y_\bullet|_{\Delta_M} \to Z_\bullet|_{\Delta_M}$ is also an isomorphism. Hence $\operatorname{Nerve}(d_1(1)(u_\bullet)) : \operatorname{Nerve}(d_1(1)(Y_\bullet)) \to \operatorname{Nerve}(d_1(1)(Z_\bullet))$ is also an isomorphism. As both $Y_\bullet$ and $Z_\bullet$ are simplicial S-groupoids by Lemma 12.6, $u'_\bullet : Y'_\bullet \to Z'_\bullet$ is an isomorphism. So $u_n : Y_n \to Z_n$ are all isomorphisms, and we are done. $\square$

12.8 Lemma. *Let S be a scheme, and $X_\bullet$ an S-groupoid, with $d_0(1)$ and $d_1(1)$ concentrated. If X_0 is concentrated, then $\operatorname{Qch}(X_\bullet)$ is Grothendieck. If, moreover, X_0 is noetherian, then $\operatorname{Qch}(X_\bullet)$ is locally noetherian, and $\mathcal{M} \in \operatorname{Qch}(X_\bullet)$ is a noetherian object if and only if $\mathcal{M}_0$ is coherent.*

Proof. This is immediate by Corollary 11.8 and Lemma 9.4. $\square$

(12.9) Let $f : X \to Y$ be a faithfully flat concentrated S-morphism. Set $X^+_\bullet := (\operatorname{Nerve}(f))|_{\Delta^+_M}$ and $X_\bullet := (X^+_\bullet)|_{\Delta_M}$. We define the descent functor

$$\mathbb{D} : \operatorname{Lqc}(X_\bullet) \to \operatorname{Qch}(Y)$$

to be the composite $(?)_{[-1]} R_{\Delta_M}$. The left adjoint $(?)_{\Delta_M} L_{[-1]}$ is denoted by $\mathbb{A}$, and called the ascent functor.

12.10 Lemma. *Let the notation be as above. Then* $\mathbb{D} : \mathrm{Qch}(X_\bullet) \to \mathrm{Qch}(Y)$ *is an equivalence, with* $\mathbb{A}$ *its quasi-inverse. The composite*

$$\mathbb{A} \circ \mathbb{D} : \mathrm{Lqc}(X_\bullet) \to \mathrm{Qch}(X_\bullet)$$

is the right adjoint of the inclusion $\mathrm{Qch}(X_\bullet) \hookrightarrow \mathrm{Lqc}(X_\bullet)$.

Proof. Follows easily from Lemma 9.4, Corollary 10.15, and Corollary 10.16. □

12.11 Lemma. *Let* $X_\bullet$ *be an* S*-groupoid, and assume that* $d_0(1)$ *and* $d_1(1)$ *are concentrated. Set* $Y_\bullet^+ := ((\mathrm{cosk}^{(\Delta)}_{\Delta_M} X_\bullet)')|_{\Delta_M^+}$, *and* $Y_\bullet := (Y_\bullet^+)|_{\Delta_M}$. *Let* $(d_0) : Y_\bullet \to X_\bullet$ *be the canonical map*

$$Y_\bullet = ((\mathrm{cosk}^{(\Delta)}_{\Delta_M} X_\bullet)')|_{\Delta_M} \xrightarrow{(d_0)|_{\Delta_M}} (\mathrm{cosk}^{(\Delta)}_{\Delta_M} X_\bullet)|_{\Delta_M} \cong X_\bullet.$$

Then (d_0) *is concentrated faithfully flat cartesian, and*

$$(d_0)_*^{\mathrm{Qch}} \circ \mathbb{A} : \mathrm{Qch}(X_0) \to \mathrm{Qch}(X_\bullet)$$

is a right adjoint of $(?)_0 : \mathrm{Qch}(X_\bullet) \to \mathrm{Qch}(X_0)$, *where* $\mathbb{A} : \mathrm{Qch}(X_0) \to \mathrm{Qch}(Y_\bullet)$ *is the ascent functor.*

Proof. Follows easily from Corollary 10.17. □

Utilizing Lemma 9.4, Lemma 10.20 and Lemma 10.21, we have the following easily.

12.12 Lemma. *Let* $f : X \to Y$ *be a faithfully flat locally an open immersion of schemes. Set* $X_\bullet = \mathrm{Nerve}(f)|_{\Delta_M}$. *Then the descent functor* $\mathbb{D} = (?)_{-1}R_{(\Delta_M)} : \mathrm{EM}(X_\bullet) \to \mathrm{Mod}(Y)$ *is an equivalence with the ascent functor* $\mathbb{A} = (?)_{\Delta_M}L_{-1}$ *its quasi-inverse. The restriction* $(?)_0 : \mathrm{EM}(X_\bullet) \to \mathrm{Mod}(X_0) = \mathrm{Mod}(X)$ *is faithfully exact with* $(d_0)_*\mathbb{A}$ *its right adjoint.*

Proof. Easy. □

(12.13) We say that $X_\bullet \in \mathcal{P}(\Delta_M, \underline{\mathrm{Sch}}/S)$ is an *almost-*S*-groupoid* if the three squares in (12.2) are cartesian, (12.4) holds, and $d_0(1)$ and $d_1(1)$ are faithfully flat. By definition, an S-groupoid is an almost-S-groupoid. If $X_\bullet$ is an almost-S-groupoid, then there is a faithfully flat cartesian morphism $p_\bullet : \mathrm{Nerve}(d_1(1))|_{\Delta_M} \to X_\bullet$ such that $p_0 : X_1 \to X_0$ is $d_0(1)$ (prove it). So Lemma 12.8 is true when we replace S-groupoid by almost-S-groupoid.

13 Bökstedt–Neeman resolutions and hyper-Ext sheaves

(13.1) Let $\mathcal{T}$ be a triangulated category with small direct products. Note that a direct product of distinguished triangles is again a distinguished triangle (Lemma 3.1).

Let

$$\cdots \to t_3 \xrightarrow{s_3} t_2 \xrightarrow{s_2} t_1 \tag{13.2}$$

be a sequence of morphisms in $\mathcal{T}$. We define $d : \prod_{i\geq 1} t_i \to \prod_{i\geq 1} t_i$ by $p_i \circ d = p_i - s_{i+1} \circ p_{i+1}$, where $p_i : \prod_i t_i \to t_i$ is the projection. Consider a distinguished triangle of the form

$$M \xrightarrow{m} \prod_{i\geq 1} t_i \xrightarrow{d} \prod_{i\geq 1} t_i \xrightarrow{q} \Sigma M,$$

where Σ denotes the suspension.

We call M, which is determined uniquely up to isomorphisms, the *homotopy limit* of (13.2) and denote it by $\operatorname{holim} t_i$.

(13.3) Dually, *homotopy colimit* is defined and denoted by hocolim, if $\mathcal{T}$ has small coproducts.

(13.4) Let $\mathcal{A}$ be an abelian category which satisfies (AB3*). Let $(\mathbb{F}_\lambda)_{\lambda\in\Lambda}$ be a small family of objects in $K(\mathcal{A})$. Then for any $\mathbb{G} \in K(\mathcal{A})$, we have that

$$\begin{aligned}\operatorname{Hom}_{K(\mathcal{A})}(\mathbb{G}, \prod_\lambda \mathbb{F}_\lambda) = H^0(\operatorname{Hom}^\bullet_{\mathcal{A}}(\mathbb{G}, \prod_\lambda \mathbb{F}_\lambda)) &\cong H^0(\prod_\lambda \operatorname{Hom}^\bullet_{\mathcal{A}}(\mathbb{G}, \mathbb{F}_\lambda)) \\ &\cong \prod_\lambda H^0(\operatorname{Hom}^\bullet_{\mathcal{A}}(\mathbb{G}, \mathbb{F}_\lambda)) = \prod_\lambda \operatorname{Hom}_{K(\mathcal{A})}(\mathbb{G}, \mathbb{F}_\lambda).\end{aligned}$$

That is, the direct product $\prod_\lambda \mathbb{F}_\lambda$ in $C(\mathcal{A})$ is also a direct product in $K(\mathcal{A})$.

(13.5) Let $\mathcal{A}$ be a Grothendieck abelian category, and (t_λ) a small family of objects of $D(\mathcal{A})$. Let $(\mathbb{F}_\lambda)$ be a family of K-injective objects of $K(\mathcal{A})$ such that $\mathbb{F}_\lambda$ represents t_λ for each λ. Then $Q(\prod_\lambda \mathbb{F}_\lambda)$ is a direct product of t_λ in $D(\mathcal{A})$ (note that the direct product $\prod_\lambda \mathbb{F}_\lambda$ exists, see [37, Corollary 7.10]). Hence $D(\mathcal{A})$ has small products.

13.6 Lemma. *Let I be a small category, S be a scheme, and let $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Let $\mathbb{F}$ be an object of $C(\mathrm{Mod}(X_\bullet))$. Assume that $\mathbb{F}$ has locally quasi-coherent cohomology groups. Then the following hold.*

1 *Let $\mathfrak{I}$ denote the full subcategory of $C(\mathrm{Mod}(X_\bullet))$ consisting of bounded below complexes of injective objects of $\mathrm{Mod}(X_\bullet)$ with locally quasi-coherent cohomology groups. There is an $\mathfrak{I}$-special inverse system $(I_n)_{n\in\mathbb{N}}$ with the index set $\mathbb{N}$ and an inverse system of chain maps $(f_n : \tau_{\geq -n}\mathbb{F} \to I_n)$ such that*

i *f_n is a quasi-isomorphism for any $n \in \mathbb{N}$.*

ii *$I_n^i = 0$ for $i < -n$.*

2 *If (I_n) and (f_n) are as in* **1***, then the following hold.*

i *For each $i \in \mathbb{Z}$, the canonical map $H^i(\varprojlim I_n) \to H^i(I_n)$ is an isomorphism for $n \geq \max(1, -i)$, where the projective limit is taken in the category $C(\mathrm{Mod}(X_\bullet))$, and $H^i(?)$ denotes the ith cohomology sheaf of a complex of sheaves.*

ii *$\varprojlim f_n : \mathbb{F} \to \varprojlim I_n$ is a quasi-isomorphism.*

iii *The projective limit $\varprojlim I_n$, viewed as an object of $K(\mathrm{Mod}(X))$, is the homotopy limit of (I_n).*

iv *$\varprojlim I_n$ is K-injective.*

Proof. The assertion **1** is [39, (3.7)].

We prove **2**, **i**. Let $j \in \mathrm{ob}(I)$ and U an affine open subset of X_j. Then for any $n \geq 1$, I_n^i and $H^i(I_n)$ are $\Gamma((j, U), ?)$-acyclic for each $i \in \mathbb{Z}$. As I_n is bounded below, each $Z^i(I_n)$ and $B^i(I_n)$ are also $\Gamma((j, U), ?)$-acyclic, and the sequence

$$0 \to \Gamma((j, U), Z^i(I_n)) \to \Gamma((j, U), I_n^i) \to \Gamma((j, U), B^{i+1}(I_n)) \to 0 \qquad (13.7)$$

and

$$0 \to \Gamma((j, U), B^i(I_n)) \to \Gamma((j, U), Z^i(I_n)) \to \Gamma((j, U), H^i(I_n)) \to 0 \qquad (13.8)$$

are exact for each i, as can be seen easily, where B^i and Z^i respectively denote the ith coboundary and the cocycle sheaves.

In particular, the inverse system $(\Gamma((j, U), B^i(I_n)))$ is a Mittag-Leffler inverse system of abelian groups by (13.7), since $(\Gamma((j, U), I_n^i))$ is. On the other hand, as we have $H^i(I_n) \cong H^i(\mathbb{F})$ for $n \geq \max(1, -i)$, the inverse system $(\Gamma((j, U), H^i(I_n)))$ stabilizes, and hence we have $(\Gamma((j, U), Z^i(I_n)))$ is also Mittag-Leffler.

Passing through the projective limit,

$$0 \to \Gamma((j,U), Z^i(\varprojlim I_n)) \to \Gamma((j,U), \varprojlim I_n) \to \Gamma((j,U), \varprojlim B^{i+1}(I_n)) \to 0$$

is exact. Hence, the canonical map $B^i(\varprojlim I_n) \to \varprojlim B^i(I_n)$ is an isomorphism, since (j,U) with U an affine open subset of X_j generates the topology of $\mathrm{Zar}(X_\bullet)$.

Taking the projective limit of (13.8), we have

$$0 \to \Gamma((j,U), B^i(\varprojlim I_n)) \to \Gamma((j,U), Z^i(\varprojlim I_n)) \to \Gamma((j,U), \varprojlim H^i(I_n)) \to 0$$

is an exact sequence for any j and any affine open subset U of X_j.

Hence, the canonical maps

$$\Gamma((j,U), H^i(I_n)) \cong \Gamma((j,U), \varprojlim H^i(I_n)) \leftarrow \Gamma((j,U), H^i(\varprojlim I_n))$$

are all isomorphisms for $n \geq \max(1,-i)$, and we have $H^i(I_n) \cong H^i(\varprojlim I_n)$ for $n \geq \max(1,-i)$.

The assertion **ii** is now trivial.

The assertion **iii** is now a consequence of [7, Remark 2.3] (one can work at the presheaf level where we have the (AB4*) property). The assertion **iv** is now obvious. $\square$

Let I be a small category, S a scheme, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$.

13.9 Lemma. *Assume that $X_\bullet$ has flat arrows. Let J be a subcategory of I, and let $\mathbb{F} \in D_{\mathrm{EM}}(X_\bullet)$ and $\mathbb{G} \in D(X_\bullet)$. Assume one of the following.*

a $\mathbb{G} \in D^+(X_\bullet)$.

b $\mathbb{F} \in D^+_{\mathrm{EM}}(X_\bullet)$.

c $\mathbb{G} \in D_{\mathrm{Lqc}}(X_\bullet)$.

Then the canonical map

$$H_J : (?)_J R\,\underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(X_\bullet)}(\mathbb{F}, \mathbb{G}) \to R\,\underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(X_\bullet|_J)}(\mathbb{F}_J, \mathbb{G}_J)$$

is an isomorphism of functors to $D(\mathrm{PM}(X_\bullet|_J))$ (here $\underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(X_\bullet)}(?,*)$ is viewed as a functor to $\mathrm{PM}(X_\bullet)$, and similarly for $\underline{\mathrm{Hom}}_{\mathrm{Mod}(X_\bullet|_J)}(?,*)$). In particular, it is an isomorphism of functors to $D(X_\bullet|_J)$.

Proof. By Lemma 1.39, we may assume that $J = i$ for an object i of I.

So what we want to prove is for any complex in $\mathrm{Mod}(X_\bullet)$ with equivariant cohomology groups $\mathbb{F}$ and any K-injective complex $\mathbb{G}$ in $\mathrm{Mod}(X_\bullet)$,

$$H_i : \underline{\mathrm{Hom}}_{\mathrm{Mod}(X_\bullet)}(\mathbb{F}, \mathbb{G})_i \to \underline{\mathrm{Hom}}_{\mathrm{Mod}(X_i)}(\mathbb{F}_i, \mathbb{G}_i)$$

is a quasi-isomorphism of complexes in $\mathrm{PM}(X_i)$ (in particular, it is a quasi-isomorphism of complexes in $\mathrm{Mod}(X_i)$), under the additional assumptions corresponding to **a**, **b**, or **c**. Indeed, if so, $\mathbb{G}_i$ is K-injective by Lemma 8.4.

First consider the case that $\mathbb{F}$ is a single equivariant object. Then the assertion is true by Lemma 6.36. By the way-out lemma [17, Proposition I.7.1], the case that $\mathbb{F}$ is bounded holds. Under the assumption of **a**, the case that $\mathbb{F}$ is bounded above holds.

Now consider the general case for **a**. As the functors in question on $\mathbb{F}$ changes coproducts to products, the map in question is a quasi-isomorphism if $\mathbb{F}$ is a direct sum of complexes bounded above with equivariant cohomology groups. Indeed, a direct product of quasi-isomorphisms of complexes of $\mathrm{PM}(X_i)$ is again quasi-isomorphic. In particular, the lemma holds if $\mathbb{F}$ is a homotopy colimit of objects of $D^-_{\mathrm{EM}}(X_\bullet)$. As any object $\mathbb{F}$ of $D_{\mathrm{EM}}(X_\bullet)$ is the homotopy colimit of $(\tau_{\leq n}\mathbb{F})$, we are done.

The proof for the case **b** is similar. As $\mathbb{F}$ has bounded below cohomology groups, $\tau_{\leq n}\mathbb{F}$ has bounded cohomology groups for each n.

We prove the case **c**. By Lemma 13.6, we may assume that $\mathbb{G}$ is a homotopy limit of K-injective complexes with locally quasi-coherent bounded below cohomology groups. As the functors on $\mathbb{G}$ in consideration commute with homotopy limits, the problem is reduced to the case **a**. □

13.10 Lemma. *Let I be a small category, S a scheme, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Assume that $X_\bullet$ has flat arrows and is locally noetherian. Let $\mathbb{F} \in D^-_{\mathrm{Coh}}(X_\bullet)$ and $\mathbb{G} \in D^+_{\mathrm{Lqc}}(X_\bullet)$ (resp. $D^+_{\mathrm{Lch}}(X_\bullet)$), where* Lch *denotes the plump subcategory of* Mod *consisting of locally coherent sheaves. Then $\underline{\mathrm{Ext}}^i_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathbb{G})$ is locally quasi-coherent (resp. locally coherent) for $i \in \mathbb{Z}$. If, moreover, $\mathbb{G}$ has quasi-coherent (resp. coherent) cohomology groups, then $\underline{\mathrm{Ext}}^i_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathbb{G})$ is quasi-coherent (resp. coherent) for $i \in \mathbb{Z}$.*

Proof. We prove the assertion for the local quasi-coherence and the local coherence. By Lemma 13.9, we may assume that $X_\bullet$ is a single scheme. This case is [17, Proposition II.3.3].

We prove the assertion for the quasi-coherence (resp. coherence), assuming that $\mathbb{G}$ has quasi-coherent (resp. coherent) cohomology groups. By [17, Proposition I.7.3], we may assume that $\mathbb{F}$ is a single coherent sheaf, and $\mathbb{G}$ is an injective resolution of a single quasi-coherent (resp. coherent) sheaf.

As $X_\bullet$ has flat arrows and the restrictions are exact, it suffices to show that

$$\alpha_\phi : X_\phi^*(?)_i \, \underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(X_\bullet)}(\mathbb{F}, \mathbb{G}) \to (?)_j \, \underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(X_\bullet)}(\mathbb{F}, \mathbb{G})$$

is a quasi-isomorphism for any morphism $\phi : i \to j$ in I.

As X_ϕ is flat, $\alpha_\phi : X_\phi^* \mathbb{F}_i \to \mathbb{F}_j$ and $\alpha_\phi : X_\phi^* \mathbb{G}_i \to \mathbb{G}_j$ are quasi-isomorphisms. In particular, the latter is a K-injective resolution.

By the derived version of (6.37), it suffices to show that

$$P : X_\phi^* R \, \underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_i}}(\mathbb{F}_i, \mathbb{G}_i) \to R \, \underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_j}}(X_\phi^* \mathbb{F}_i, X_\phi^* \mathbb{G}_i)$$

is an isomorphism. This is [17, Proposition II.5.8]. $\square$

14 The right adjoint of the derived direct image functor

(14.1) Let X be a scheme. A right adjoint of the inclusion $F_X : \mathrm{Qch}(X) \hookrightarrow \mathrm{Mod}(X)$ is called the *quasi-coherator* of X, and is denoted by $\mathrm{qch} = \mathrm{qch}(X)$.

If $f : Y \to X$ is a concentrated morphism of schemes and $\mathrm{qch}(X)$ and $\mathrm{qch}(Y)$ exist, then there is a canonical isomorphism $f_* \, \mathrm{qch}(Y) \cong \mathrm{qch}(X) f_*$, which is the conjugate to $f^* F_X \cong F_Y f^*$. Note also that if $\mathrm{qch}(X)$ exists, then the unit $u : \mathrm{Id} \to \mathrm{qch}(X) F_X$ is an isomorphism, see [19, (I.1.2.6)].

(14.2) Let S be a scheme, I a small category, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Assume that for each $i \in I$, there exists some $\mathrm{qch}(X_i)$ and that $X_\bullet$ has concentrated arrows. Then we define $\mathrm{lqc}(X_\bullet) : \mathrm{Mod}(X_\bullet) \to \mathrm{Lqc}(X_\bullet)$ as follows. Let $\mathcal{M} \in \mathrm{Mod}(X_\bullet)$. Then $\mathcal{M}$ is expressed in terms of the data $((\mathcal{M}_i)_{i \in I}, (\beta_\phi)_{\phi \in \mathrm{Mor}(I)})$. $\mathrm{lqc}(\mathcal{M})$ is then defined in terms of the data as follows. $(\mathrm{lqc}(\mathcal{M}))_i = \mathrm{qch} \, \mathcal{M}_i$ for $i \in I$, and β_ϕ is the composite

$$\mathrm{qch} \, \mathcal{M}_i \xrightarrow{\mathrm{qch} \, \beta_\phi} \mathrm{qch}(X_\phi)_* \mathcal{M}_j \cong (X_\phi)_* \, \mathrm{qch} \, \mathcal{M}_j$$

for $\phi : i \to j$. It is easy to see that $\mathrm{lqc}(X_\bullet)$ is the right adjoint of the inclusion $F_X : \mathrm{Lqc}(X_\bullet) \to \mathrm{Mod}(X_\bullet)$. We call lqc the *local quasi-coherator*.

14.3 Lemma. *Let X be a concentrated scheme.*

1 *There is a right adjoint* $\mathrm{qch}(X) : \mathrm{Mod}(X) \to \mathrm{Qch}(X)$ *of the canonical inclusion* $F_X : \mathrm{Qch}(X) \hookrightarrow \mathrm{Mod}(X)$*.* $\mathrm{qch}(X)$ *preserves filtered inductive limits.*

1' $\mathrm{qch}(X)$ *preserves K-injective complexes.* $R\,\mathrm{qch}(X) : D(\mathrm{Mod}(X)) \to D(\mathrm{Qch}(X))$ *is right adjoint to* $F_X : D(\mathrm{Qch}(X)) \to D(\mathrm{Mod}(X))$*.*

2 *Assume that X is separated or noetherian. Then the functor* $F_X : D(\mathrm{Qch}(X)) \to D(\mathrm{Mod}(X))$ *is full and faithful, and induces an equivalence* $D(\mathrm{Qch}(X)) \to D_{\mathrm{Qch}(X)}(\mathrm{Mod}(X))$*.*

3 *Assume that X is separated or noetherian. Then the unit of adjunction* $u : \mathrm{Id} \to R\,\mathrm{qch}(X)F_X$ *is an isomorphism, and* $\varepsilon : F_X R\,\mathrm{qch}(X) \to \mathrm{Id}$ *is an isomorphism on* $D_{\mathrm{Qch}(X)}(\mathrm{Mod}(X))$*.*

Proof. The existence assertion of **1** is proved in [20, Lemme 3.2]. If $\mathrm{Spec}\,A = X$ is affine, then it is easy to see that the functor $\mathcal{M} \mapsto \Gamma(X,\mathcal{M})^{\sim}$ is a desired $\mathrm{qch}(X)$. In fact, for a quasi-coherent $\mathcal{N}$, a morphism $\mathcal{N} \to \mathcal{M}$ is uniquely determined by the A-linear map $\Gamma(X,\mathcal{N}) \to \Gamma(X,\mathcal{M})$. So $\mathrm{qch}(X)$ preserves filtered inductive limits by [23, Proposition 6].

Next consider the general case. As X is quasi-compact, there is a finite affine open covering (U_i) of X. Set $Y = \coprod_i U_i$, and let $p\colon Y \to X$ be the canonical map. Note that p is locally an open immersion and faithfully flat. Let $X_\bullet = \mathrm{Nerve}(p)|_{\Delta_M}$.

Assume that each X_i admits $\mathrm{qch}(X_i)$. This is the case if X is separated (and hence X_i is affine for each i). Note that the inclusion $\mathrm{Qch}(X) \hookrightarrow \mathrm{Mod}(X)$ is equivalent to the composite

$$\mathrm{Qch}(X) \xrightarrow{\mathbb{A}} \mathrm{Qch}(X_\bullet) \hookrightarrow \mathrm{Lqc}(X_\bullet) \hookrightarrow \mathrm{Mod}(X_\bullet) \xrightarrow{\mathbb{D}} \mathrm{Mod}(X).$$

By Lemma 12.10, $\mathbb{A}$ is an equivalence. So it suffices to show that $\mathbb{D} : \mathrm{Qch}(X_\bullet) \to \mathrm{Mod}(X)$ has a right adjoint. By Lemma 12.12 and Lemma 12.10, for $\mathcal{M} \in \mathrm{Qch}(X_\bullet)$ and $\mathcal{N} \in \mathrm{Mod}(X)$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Mod}(X)}(\mathbb{D}\mathcal{M},\mathcal{N}) &\cong \mathrm{Hom}_{\mathrm{EM}(X_\bullet)}(\mathcal{M},\mathbb{A}\mathcal{N}) \cong \mathrm{Hom}_{\mathrm{Mod}(X_\bullet)}(\mathcal{M},\mathbb{A}\mathcal{N})\\ &\cong \mathrm{Hom}_{\mathrm{Lqc}(X_\bullet)}(\mathcal{M},\mathrm{lqc}\,\mathbb{A}\mathcal{N}) \cong \mathrm{Hom}_{\mathrm{Qch}(X_\bullet)}(\mathcal{M},\mathbb{A}\mathbb{D}\,\mathrm{lqc}\,\mathbb{A}\mathcal{N}).\end{aligned}$$

Thus $\mathbb{D} : \mathrm{Qch}(X_\bullet) \to \mathrm{Mod}(X)$ has a right adjoint $\mathbb{A}\mathbb{D}\,\mathrm{lqc}\,\mathbb{A}$, as desired (so $\mathrm{qch}(X) = \mathbb{D}\,\mathrm{lqc}\,\mathbb{A}$, as can be seen easily).

So the case that X is quasi-compact separated is done. Now repeating the same argument for the general X (then X_i is quasi-compact separated for each i), the construction of qch is done.

We prove that qch is compatible with filtered inductive limits. Assume first that X is separated. Then $\mathrm{lqc} : \mathrm{Mod}(X_\bullet) \to \mathrm{Lqc}(X_\bullet)$ preserves filtered inductive limits by the affine case. As $\mathbb{A}$, lqc, and $\mathbb{D}$ preserves filtered inductive limits, so is $\mathrm{qch}(X) = \mathbb{D}\,\mathrm{lqc}\,\mathbb{A}$. Now repeating the same argument, the general case follows.

The assertion **1'** follows from **1** and Lemma 3.12.

Clearly, **2** and **3** are equivalent. **2** for the case that X is separated is proved in [7]. We remark that Verdier's example [20, Appendice I] shows that the assertions are *not* true for a general concentrated scheme X which is not separated or noetherian.

We give a proof for **3** for the case that X is noetherian, using the result for the case that X is separated. It suffices to show that if $\mathbb{I}$ is a K-injective complex in $K(\mathrm{Mod}(X))$ with quasi-coherent cohomology groups, then $\mathrm{qch}(\mathbb{I}) \to \mathbb{I}$ is a quasi-isomorphism. Since $\mathrm{Mod}(X)$ is Grothendieck, there is a strictly injective resolution $\mathbb{I} \to \mathbb{J}$ by Lemma 3.9. As $\mathrm{Cone}(\mathbb{I} \to \mathbb{J})$ is null-homotopic, replacing $\mathbb{I}$ by $\mathbb{J}$, we may assume that $\mathbb{I}$ is strictly injective.

Let $\mathfrak{U} = (U_i)_{1\leq i\leq m}$ be a finite affine open covering of X. For a finite subset I of $\{1, \ldots, m\}$, we denote $\bigcap_{i\in I} U_i$ by U_I. Note that each U_I is noetherian and separated. Let $g_I : U_I \hookrightarrow X$ be the inclusion. For $M \in \mathrm{Mod}(X)$, the *Čech complex* $\check{\mathrm{C}}\mathrm{ech}(M) = \check{\mathrm{C}}\mathrm{ech}_{\mathfrak{U}}(M)$ of M is defined to be

$$0 \to \bigoplus_{\#I=1} (g_I)_* g_I^* M \to \bigoplus_{\#I=2} (g_I)_* g_I^* M \to \cdots \to \bigoplus_{\#I=m} (g_I)_* g_I^* M \to 0,$$

where $(g_I)_* g_I^* M \to (g_J)_* g_J^* M$ is the $\pm$ of the unit of adjunction if $J \supset I$, and zero if $J \not\supset I$. The augmented Cech complex $0 \to M \to \check{\mathrm{C}}\mathrm{ech}(M)$ is denoted by $\check{\mathrm{C}}\mathrm{ech}^+(M) = \check{\mathrm{C}}\mathrm{ech}^+_{\mathfrak{U}}(M)$. The lth term $\bigoplus_{\#I=l+1} (g_I)_* g_I^* M$ is denoted by $\check{\mathrm{C}}\mathrm{ech}^l(M)$.

Note that if $U_i = X$ for some i, then $\check{\mathrm{C}}\mathrm{ech}^+_{\mathfrak{U}}(M)$ is split exact. In particular, $g_i^*(\check{\mathrm{C}}\mathrm{ech}^+_{\mathfrak{U}}(M)) \cong \check{\mathrm{C}}\mathrm{ech}^+_{U_i\cap\mathfrak{U}}(g_i^* M)$ is split exact, where $U_i \cap \mathfrak{U} = (U_i \cap U_j)_{1\leq j\leq m}$ is the open covering of U_i. Let $g : Y = \coprod_i U_i \to X$ be the canonical map. Since g is faithfully flat and $g^*(\check{\mathrm{C}}\mathrm{ech}^+_{\mathfrak{U}}(M))$ is split exact, $\check{\mathrm{C}}\mathrm{ech}^+_{\mathfrak{U}}(M)$ is exact. Note also that if $M = (g_i)_* N$ for some $N \in \mathrm{Mod}(U_i)$, then $\check{\mathrm{C}}\mathrm{ech}^+_{\mathfrak{U}}(M) \cong (g_i)_*(\check{\mathrm{C}}\mathrm{ech}^+_{U_i\cap\mathfrak{U}}(N))$ is split exact. In particular, if M is a direct summand of g_*N for some $N \in \mathrm{Mod}(Y)$, then $\check{\mathrm{C}}\mathrm{ech}^+(M)$ is split exact. This is the case if M is injective, since $M \hookrightarrow g_*g^*M$ splits.

Now we want to prove that $\mathrm{qch}(\mathbb{I}) \to \mathbb{I}$ is a quasi-isomorphism. Since $\mathbb{I}$ is strictly injective, $\check{\mathrm{C}}\mathrm{ech}^+(\mathbb{I})$ is split exact. So $\mathbb{I} \to \check{\mathrm{C}}\mathrm{ech}(\mathbb{I})$ is a quasi-isomorphism. As $\mathrm{qch}(\check{\mathrm{C}}\mathrm{ech}^+(\mathbb{I}))$ is split exact, $\mathrm{qch}(\mathbb{I}) \to \mathrm{qch}(\check{\mathrm{C}}\mathrm{ech}(\mathbb{I}))$ is a quasi-isomorphism. So it suffices to show that $\mathrm{qch}(\check{\mathrm{C}}\mathrm{ech}(\mathbb{I})) \to \check{\mathrm{C}}\mathrm{ech}(\mathbb{I})$ is a quasi-isomorphism. To verify this, it suffices to show that $\mathrm{qch}(\check{\mathrm{C}}\mathrm{ech}^l(\mathbb{I})) \to \check{\mathrm{C}}\mathrm{ech}^l(\mathbb{I})$ is a quasi-isomorphism for $l = 0, \ldots, m-1$. To verify this, it suffices to show that for each non-empty subset I of $\{1, \ldots, m\}$, $\varepsilon : F_X\,\mathrm{qch}((g_I)_*g_I^*\mathbb{I}) \to (g_I)_*g_I^*\mathbb{I}$ is a quasi-isomorphism. This map can be identified with $(g_I)_*\varepsilon : (g_I)_*F_{U_I}\,\mathrm{qch}\,g_I^*\mathbb{I} \to (g_I)_*g_I^*\mathbb{I}$. By the case that X is separated, $\varepsilon : F_{U_I}\,\mathrm{qch}\,g_I^*\mathbb{I} \to g_I^*\mathbb{I}$ is a quasi-isomorphism, since $g_I^*\mathbb{I}$ is a K-injective complex and U_I is noetherian separated. Note that $g_I^*\mathbb{I}$ is $(g_I)_*$-acyclic simply because it is K-injective. On the other hand, since each term of $\mathrm{qch}\,g_I^*\mathbb{I}$ is an injective object of $\mathrm{Qch}(U_I)$, it is also an injective object of $\mathrm{Mod}(U_I)$, see [17, (II.7)]. In particular, each term of $\mathrm{qch}\,g_I^*\mathbb{I}$ is quasi-coherent and $(g_I)_*$-acyclic. By [26, (3.9.3.5)], $\mathrm{qch}\,g_I^*\mathbb{I}$ is $(g_I)_*$-acyclic. Hence $\varepsilon : F_X\,\mathrm{qch}((g_I)_*g_I^*\mathbb{I}) \to (g_I)_*g_I^*\mathbb{I}$ is a quasi-isomorphism, as desired. $\square$

By the lemma, the following follows immediately.

14.4 Corollary. *Let X be a concentrated scheme. Then* $\mathrm{Qch}(X)$ *has arbitrary small direct products.*

This also follows from Corollary 11.7 and [37, Corollary 7.10].

(14.5) Let $f : X \to Y$ be a concentrated morphism of schemes. Then $f_*^{\mathrm{Qch}} : \mathrm{Qch}(X) \to \mathrm{Qch}(Y)$ is defined, and we have $F_Y \circ f_*^{\mathrm{Qch}} \cong f_*^{\mathrm{Mod}} \circ F_X$, where F_Y and F_X are the forgetful functors. Note that $Rf_*(D_{\mathrm{Qch}}(X)) \subset D_{\mathrm{Qch}}(Y)$, see [26, (3.9.2)]. If, moreover, X is concentrated, then there is a right derived functor Rf_*^{Qch} of f_*^{Qch} by Corollary 11.7.

14.6 Lemma. *Let $f : X \to Y$ be a morphism of schemes. Then, we have the following.*

1 *If X is noetherian or both Y and f are quasi-compact separated, then the canonical maps*

$$F_Y \circ Rf_*^{\mathrm{Qch}} \cong R(F_Y \circ f_*^{\mathrm{Qch}}) \cong R(f_*^{\mathrm{Mod}} \circ F_X) \to Rf_*^{\mathrm{Mod}} \circ F_X$$

are all isomorphisms.

2 *Assume that both X and Y are either noetherian or quasi-compact separated. Then there are a left adjoint F of Rf_*^{Qch} and an isomorphism $F_X F \cong Lf^*_{\mathrm{Mod}} F_Y$.*

3 *Let X and Y be as in* **2**. *Then there is an isomorphism*

$$Rf_*^{\mathrm{Qch}} R\,\mathrm{qch} \cong R\,\mathrm{qch}\, Rf_*^{\mathrm{Mod}}.$$

Proof. We prove **1**. It suffices to show that, if $\mathbb{I} \in K(\mathrm{Qch}(X))$ is K-injective, then $F_X\mathbb{I}$ is f_*^{Mod}-acyclic. By Corollary 11.7 and Lemma 3.9, we may assume that each term of $\mathbb{I}$ is injective. By [26, (3.9.3.5)] (applied to the plump subcategory $\mathcal{A}_\# = \mathrm{Qch}(X)$ of $\mathrm{Mod}(X)$), it suffices to show that an injective object $\mathcal{I}$ of $\mathrm{Qch}(X)$ is f_*^{Mod}-acyclic. This is trivial if X is noetherian, since then $\mathcal{I}$ is injective in $\mathrm{Mod}(X)$, see [17, (II.7)].

Now assume that both Y and f are quasi-compact separated. Let $g : Z \to X$ be a faithfully flat morphism such that Z is affine. Such a morphism exists, as X is quasi-compact. Then it is easy to see that $\mathcal{I}$ is a direct summand of $g_*\mathcal{J}$ for some injective object $\mathcal{J}$ in $\mathrm{Qch}(Z)$. So it suffices to show that $F_X g_*\mathcal{J}$ is f_*^{Mod}-acyclic. Let $F_Z\mathcal{J} \to \mathbb{J}$ be an injective resolution. As g is affine (since X is separated and Z is affine) and $\mathcal{J}$ is quasi-coherent, we have $R^i g_*^{\mathrm{Mod}} F_Z\mathcal{J} = 0$ for $i > 0$. Hence $g_*F_Z\mathcal{J} \to g_*\mathbb{J}$ is a quasi-isomorphism, and hence is a K-limp resolution of $F_X g_*\mathcal{J} \cong g_*F_Z\mathcal{J}$. As $f \circ g$ is also affine, $f_*g_*F_Z\mathcal{J} \to f_*g_*\mathbb{J}$ is still a quasi-isomorphism, and this shows that $F_X g_*\mathcal{J}$ is f_*-acyclic.

2 Define $F : D(\mathrm{Qch}(Y)) \to D(\mathrm{Qch}(X))$ by $F := R\,\mathrm{qch}\, Lf^*_{\mathrm{Mod}} F_Y$. Note that $Lf^*_{\mathrm{Mod}} F_Y(D(\mathrm{Qch}(Y))) \subset D_{\mathrm{Qch}}(X)$, see [26, (3.9.1)]. So we have

$$F_X F = F_X R\,\mathrm{qch}\, Lf^*_{\mathrm{Mod}} F_Y \xrightarrow{\text{via } \varepsilon} Lf^*_{\mathrm{Mod}} F_Y$$

is an isomorphism by Lemma 14.3. Hence

$$\begin{aligned}&\mathrm{Hom}_{D(\mathrm{Qch}(X))}(F\mathbb{F}, \mathbb{G}) \cong \mathrm{Hom}_{D(X)}(F_X F\mathbb{F}, F_X\mathbb{G}) \cong \\ &\mathrm{Hom}_{D(X)}(Lf^*_{\mathrm{Mod}} F_Y\mathbb{F}, F_X\mathbb{G}) \cong \mathrm{Hom}_{D(Y)}(F_Y\mathbb{F}, Rf_*^{\mathrm{Mod}} F_X\mathbb{G}) \cong \\ &\mathrm{Hom}_{D(Y)}(F_Y\mathbb{F}, F_Y Rf_*^{\mathrm{Qch}}\mathbb{G}) \cong \mathrm{Hom}_{D(\mathrm{Qch}(Y))}(\mathbb{F}, Rf_*^{\mathrm{Qch}}\mathbb{G}).\end{aligned}$$

This shows that F is left adjoint to Rf_*^{Qch}.

3 Take the conjugate of **2**. $\square$

14.7 Lemma. *Let I be a small category, and $f_\bullet : X_\bullet \to Y_\bullet$ an affine morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}})$. Let $\mathbb{F} \in D_{\mathrm{Lqc}}(X_\bullet)$. Then $R^0(f_\bullet)_*\mathbb{F} \cong (f_\bullet)_*(H^0(\mathbb{F}))$.*

Proof. Clearly, $(f_\bullet)_*(H^0(\mathbb{F})) \cong R^0(f_\bullet)_*(H^0(\mathbb{F}))$. Since $R^i(f_\bullet)_*(\tau_{>0}\mathbb{F}) = 0$ for $i \leq 0$ is obvious, $R^i(f_\bullet)_*(\tau_{\leq 0}\mathbb{F}) \cong R^i(f_\bullet)_*\mathbb{F}$ for $i \leq 0$. So it suffices to show that $R^i(f_\bullet)_*(\tau_{<0}\mathbb{F}) = 0$ for $i \geq 0$.

To verify this, we may assume that $f_\bullet = f : X \to Y$ is a map of single schemes, and Y is affine. By Lemma 14.3, we may assume that $\mathbb{F} = F_X\mathbb{G}$ for some $D(\mathrm{Qch}(X))$. By Lemma 14.6, it suffices to show that $R^i f_*^{\mathrm{Qch}}(\tau_{<0}\mathbb{G}) = 0$ for $i \geq 0$. But this is trivial, since f_*^{Qch} is an exact functor. □

14.8 Corollary. *Let I and $f_\bullet$ be as in the lemma. If $\mathbb{F} \in D_{\mathrm{Lqc}}(X_\bullet)$, then $R^n(f_\bullet)_*\mathbb{F} \cong (f_\bullet)_*(H^n(\mathbb{F}))$.*

(14.9) Let $\mathcal{C}$ be an additive category, and $c \in \mathcal{C}$. We say that c is a *compact object*, if for any small family of objects $(t_\lambda)_{\lambda\in\Lambda}$ of $\mathcal{C}$ such that the coproduct (direct sum) $\bigoplus_{\lambda\in\Lambda} t_\lambda$ exists, the canonical map

$$\bigoplus_\lambda \mathrm{Hom}_{\mathcal{C}}(c, t_\lambda) \to \mathrm{Hom}_{\mathcal{C}}(c, \bigoplus_\lambda t_\lambda)$$

is an isomorphism.

A triangulated category $\mathcal{T}$ is said to be *compactly generated*, if $\mathcal{T}$ has small coproducts, and there is a small set C of compact objects of $\mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(c, t) = 0$ for all $c \in C$ implies $t = 0$. The following was proved by A. Neeman [35].

14.10 Theorem. *Let $\mathcal{S}$ be a compactly generated triangulated category, $\mathcal{T}$ any triangulated category, and $F : \mathcal{S} \to \mathcal{T}$ a triangulated functor. Suppose that F preserves coproducts, that is to say, for any small family of objects (s_λ) of $\mathcal{S}$, the canonical maps $F(s_\lambda) \to F(\bigoplus_\lambda s_\lambda)$ make $F(\bigoplus_\lambda s_\lambda)$ the coproduct of $F(s_\lambda)$. Then F has a right adjoint $G : \mathcal{T} \to \mathcal{S}$.*

For the definition of triangulated category and triangulated functor, see [36]. Related to the theorem, we remark the following.

14.11 Lemma (Keller and Vossieck [22]). *Let $\mathcal{S}$ and $\mathcal{T}$ be triangulated categories, and $F\colon \mathcal{S} \to \mathcal{T}$ a triangulated functor. If G is a right adjoint of F, then G is also a triangulated functor.*

Proof. By the opposite assertion of [29, (IV.1), Theorem 3], G is additive.

Let $\phi_F\colon F\Sigma \to \Sigma F$ be the canonical isomorphism. Then its inverse induces an isomorphism

$$\psi : F\Sigma^{-1} = \Sigma^{-1}\Sigma F\Sigma^{-1} \xrightarrow{\phi_F^{-1}} \Sigma^{-1}F\Sigma\Sigma^{-1} = \Sigma^{-1}F.$$

Taking the conjugate of ψ, we get an isomorphism $\phi_G\colon G\Sigma \cong \Sigma G$.

Let $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} \Sigma A$ be a distinguished triangle in $\mathcal{T}$. Then there exists some distinguished triangle of the form

$$GA \xrightarrow{Ga} GB \xrightarrow{\alpha} X \xrightarrow{\beta} \Sigma(GA).$$

Then there exists some $d\colon FX \to C$ such that

$$\begin{tikzcd} FGA \arrow[r, "FGa"] \arrow[d, "\varepsilon"] & FGB \arrow[r, "F\alpha"] \arrow[d, "\varepsilon"] & FX \arrow[r, "\phi\circ F\beta"] \arrow[d, "d"] & \Sigma(FGA) \arrow[d, "\Sigma\varepsilon"] \\ A \arrow[r, "a"] & B \arrow[r, "b"] & C \arrow[r, "c"] & \Sigma A \end{tikzcd}$$

is a map of triangles. Then taking the adjoint, we get a commutative diagram

$$\begin{tikzcd} GA \arrow[r, "Ga"] \arrow[d, "\mathrm{id}"] & GB \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"] & X \arrow[r, "\beta"] \arrow[d, "\delta"] & \Sigma GA \arrow[d] \\ GA \arrow[r, "Ga"] & GB \arrow[r, "Gb"] & GC \arrow[r, "\phi_G\circ Gc"] & \Sigma GA, \end{tikzcd} \tag{14.12}$$

where δ is the adjoint of d, and the right most vertical arrow is the composite

$$\Sigma GA \xrightarrow{u} GF\Sigma GA \xrightarrow{\phi_F} G\Sigma FGA \xrightarrow{\varepsilon} G\Sigma A \xrightarrow{\phi_G} \Sigma GA,$$

which agrees with id, as can be seen easily. This induces a commutative diagram

$$\begin{tikzcd} \mathcal{S}(?, GA) \arrow[r, "(Ga)_*"] \arrow[d, "\mathrm{id}"] & \mathcal{S}(?, GB) \arrow[r, "\alpha_*"] \arrow[d, "\mathrm{id}"] & \mathcal{S}(?, X) \arrow[r, "\beta_*"] \arrow[d, "\delta_*"] & \mathcal{S}(?, \Sigma GA) \arrow[r, "(\Sigma Ga)_*"] \arrow[d, "\mathrm{id}"] & \mathcal{S}(?, \Sigma GB) \arrow[d, "\mathrm{id}"] \\ \mathcal{S}(?, GA) \arrow[r, "(Ga)_*"] & \mathcal{S}(?, GB) \arrow[r, "(Gb)_*"] & \mathcal{S}(?, GC) \arrow[r, "(\phi_G\circ Gc)_*"] & \mathcal{S}(?, \Sigma GA) \arrow[r, "(\Sigma Ga)_*"] & \mathcal{S}(?, \Sigma GB). \end{tikzcd}$$

The first row is exact, since it comes from a distinguished triangle, see [17, Proposition I.1.1]. As the second row is isomorphic to the sequence

$$\mathcal{T}(F?, A) \xrightarrow{a_*} \mathcal{T}(F?, B) \xrightarrow{b_*} \mathcal{T}(F?, C) \xrightarrow{c_*} \mathcal{T}(F?, \Sigma A) \xrightarrow{(\Sigma a)_*} \mathcal{T}(F?, \Sigma B)$$

and $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} \Sigma A$ is a distinguished triangle, it is also exact. By the five lemma, δ_* is an isomorphism. By Yoneda's lemma, δ is an isomorphism. This shows that (14.12) is an isomorphism, and hence the second row of (14.12) is distinguished. This is what we wanted to show. $\square$

14.13 Lemma. *Let $\mathcal{S}$ and $\mathcal{T}$ be triangulated categories, and $F\colon \mathcal{S} \to \mathcal{T}$ a triangulated functor with a right adjoint G. If both $\mathcal{S}$ and $\mathcal{T}$ have t-structures and F is way-out left (i.e., $F(\tau_{\leq 0}\mathcal{S}) \subset \tau_{\leq d}\mathcal{T}$ for some d), then G is way-out right.*

Proof. For the definition of t-structures on triangulated categories, see [5]. Assuming that $F(\tau_{\leq 0}\mathcal{S}) \subset \tau_{\leq d}\mathcal{T}$, we show $G(\tau_{\geq 0}\mathcal{T}) \subset \tau_{\geq -d}\mathcal{S}$.

Let $t \in \tau_{\geq 0}\mathcal{T}$ and $s \in \tau_{\leq -d-1}\mathcal{S}$. Then since $Fs \in \tau_{\leq -1}\mathcal{T}$, we have $\mathcal{S}(s, Gt) \cong \mathcal{T}(Fs, t) = 0$. By [5, (1.3.4)], $Gt \in \tau_{\geq -d}\mathcal{S}$. □

The following was proved by A. Neeman [35] for the quasi-compact separated case, and was proved generally by A. Bondal and M. van den Bergh [8].

14.14 Theorem. *Let X be a concentrated scheme. Then $c \in D_{\mathrm{Qch}}(X)$ is a compact object if and only if c is isomorphic to a* perfect complex, *where we say that $\mathbb{C} \in C(\mathrm{Qch}(X))$ is perfect if $\mathbb{C}$ is bounded, and each term of $\mathbb{C}$ is locally free of finite rank. Moreover, $D_{\mathrm{Qch}}(X)$ is compactly generated.*

14.15 Lemma. *Let $f : X \to Y$ be a concentrated morphism of schemes. Then $Rf_*^{\mathrm{Mod}} : D_{\mathrm{Qch}}(X) \to D(Y)$ preserves coproducts.*

Proof. See [35] or [26, (3.9.3.2), Remark (b)]. □

(14.16) Let $f : X \to Y$ be a concentrated morphism of schemes such that X is concentrated. Then by Theorem 14.10, Theorem 14.14 and Lemma 14.15, there is a right adjoint

$$f^{\times} : D(Y) \to D_{\mathrm{Qch}}(X)$$

of Rf_*^{Mod}.

By restriction, $f^{\times} : D_{\mathrm{Qch}}(Y) \to D_{\mathrm{Qch}}(X)$ is a right adjoint of $Rf_*^{\mathrm{Mod}} : D_{\mathrm{Qch}}(X) \to D_{\mathrm{Qch}}(Y)$. Note that $(R(?)_*, (?)^{\times})$ is an adjoint pair of Δ-pseudofunctors on the opposite of the category of concentrated schemes. In other words, $(R(?)_*, (?)^{\times})$ is an opposite adjoint pair (of Δ-pseudofunctors) on the category of concentrated schemes, see (1.18).

15 Comparison of local Ext sheaves

(15.1) Let S be a scheme, and $X_\bullet$ an almost-S-groupoid. Assume that $d_0(1)$ and $d_1(1)$ are affine, and X_0 is locally noetherian.

15.2 Lemma. *Let* $\mathbb{F} \in K^-_{\mathrm{Coh}}(\mathrm{Qch}(X_\bullet))$ *and* $\mathbb{G} \in K^+(\mathrm{Qch}(X_\bullet))$*. If* $\mathbb{G}$ *is a bounded below complex consisting of injective objects of* $\mathrm{Qch}(X_\bullet)$*, then* $\mathbb{G}$ *is* $\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, ?)$*-acyclic as a complex of* $\mathrm{Mod}(X_\bullet)$*.*

Proof. It is easy to see that we may assume that $\mathbb{F}$ is a single coherent sheaf, and $\mathbb{G}$ is a single injective object of $\mathrm{Qch}(X_\bullet)$. To prove this case, it suffices to show that

$$\underline{\mathrm{Ext}}^i_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathbb{G}) = 0$$

for $i > 0$.

Set $X'_\bullet := \mathrm{Nerve}(d_1(1))|_{\Delta_M}$, and let $p_\bullet : X'_\bullet \to X_\bullet$ be a cartesian morphism such that $p_0 : X_1 \to X_0$ is $d_0(1)$, see (12.13). In particular, $p_\bullet$ is affine and faithfully flat.

Let $\mathbb{A} : \mathrm{Mod}(X_0) \to \mathrm{Mod}(X'_\bullet)$ be the ascent functor, and $\mathbb{D} : \mathrm{Mod}(X'_\bullet) \to \mathrm{Mod}(X_0)$ be the descent functor.

As $(p_\bullet)_* : \mathrm{Qch}(X'_\bullet) \to \mathrm{Qch}(X_\bullet)$ has a faithful exact left adjoint $p_\bullet^*$, there exists some injective object $\mathbb{I}$ of $\mathrm{Qch}(X'_\bullet)$ such that $\mathbb{G}$ is a direct summand of $(p_\bullet)_*\mathbb{I}$. We may assume that $\mathbb{G} = (p_\bullet)_*\mathbb{I}$. As

$$R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, R(p_\bullet)_*\mathbb{I}) \cong R(p_\bullet)_* R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X'_\bullet}}(p_\bullet^*\mathbb{F}, \mathbb{I}),$$

$p_\bullet$ is affine by assumption, and $R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X'_\bullet}}(p_\bullet^*\mathbb{F}, \mathbb{I})$ has quasi-coherent cohomology groups, we may assume that $X_\bullet = \mathrm{Nerve}(f)|_{\Delta_M}$ for some faithfully flat affine morphism $f : X \to Y$ between S-schemes with Y locally noetherian (but we may lose the assumption that X_0 is locally noetherian). For each l, we have that

$$\begin{aligned}(?)_l R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathbb{G}) &\cong R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_l}}((?)_l\mathbb{A}\mathbb{D}\mathbb{F}, (?)_l\mathbb{A}\mathbb{D}\mathbb{G}) \cong \\ &R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_l}}(e(l)^*\mathbb{D}\mathbb{F}, e(l)^*\mathbb{D}\mathbb{G}) \cong e(l)^* R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_Y}(\mathbb{D}\mathbb{F}, \mathbb{D}\mathbb{G})\end{aligned}$$

by [17, (II.5.8)] (note that $\mathbb{D}\mathbb{F}$ is coherent). As $\mathbb{D} : \mathrm{Qch}(X_\bullet) \to \mathrm{Qch}(Y)$ is an equivalence, $\mathbb{D}\mathbb{G}$ is an injective object of $\mathrm{Qch}(Y)$. Hence it is also an injective object of $\mathrm{Mod}(Y)$ by [17, (II.7)]. Hence $\underline{\mathrm{Ext}}^i_{\mathcal{O}_Y}(\mathbb{D}\mathbb{F}, \mathbb{D}\mathbb{G}) = 0$ for $i > 0$, as desired. □

The following is a generalization of [19, Theorem II.1.1.12].

15.3 Corollary. *Let the notation be as in the lemma. Then* $\mathbb{G}_0$ *is* $\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_0}}(\mathbb{F}_0, ?)$*-acyclic as a complex of* $\mathcal{O}_{X_0}$*-modules.*

Proof. Let $\mathbb{G} \to \mathbb{I}$ be a K-injective resolution in $\mathrm{Mod}(X_\bullet)$ such that $\mathbb{I}$ is bounded below. Let $\mathbb{C}$ be the mapping cone of this. Since $(?)_0$ has an exact left adjoint, $\mathbb{G}_0 \to \mathbb{I}_0$ is a K-injective resolution in $K(\mathrm{Mod}(X_0))$. So it suffices to show that $\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_0}}(\mathbb{F}_0, \mathbb{C}_0)$ is exact. As each term of $\mathbb{F}$ is equivariant, this complex is isomorphic to $\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathbb{C})_0$, which is exact by the lemma. $\square$

16 The Composition of two almost-pseudofunctors

16.1 Definition. We say that $\mathcal{C} = (\mathcal{A}, \mathcal{F}, \mathcal{P}, \mathcal{I}, \mathcal{D}, \mathcal{D}^+, (?)^\#, (?)^\flat, \zeta)$ is a *composition data* of contravariant almost-pseudofunctors if the following eighteen conditions are satisfied:

1 $\mathcal{A}$ is a category with fiber products.

2 $\mathcal{P}$ and $\mathcal{I}$ are sets of morphisms of $\mathcal{A}$.

3 Any isomorphism in $\mathcal{A}$ is in $\mathcal{P} \cap \mathcal{I}$.

4 The composite of two morphisms in $\mathcal{P}$ is again a morphism in $\mathcal{P}$.

5 The composite of two morphisms in $\mathcal{I}$ is again a morphism in $\mathcal{I}$.

6 A base change of a morphism in $\mathcal{P}$ is again a morphism in $\mathcal{P}$.

7 Any $f \in \mathrm{Mor}(\mathcal{A})$ admits a factorization $f = pi$ such that $p \in \mathcal{P}$ and $i \in \mathcal{I}$.

Before stating the remaining conditions, we give some definitions for convenience.

i Let $\mathcal{C}$ be a set of morphisms in $\mathcal{A}$ containing all identity maps and being closed under compositions. We define $\mathcal{A}_\mathcal{C}$ by $\mathrm{ob}(\mathcal{A}_\mathcal{C}) := \mathrm{ob}(\mathcal{A})$ and $\mathrm{Mor}(\mathcal{A}_\mathcal{C}) := \mathcal{C}$. In particular, the subcategories $\mathcal{A}_\mathcal{P}$ and $\mathcal{A}_\mathcal{I}$ of $\mathcal{A}$ are defined.

ii We call a commutative diagram of the form $p \circ i = i' \circ p'$ with $p, p' \in \mathcal{P}$ and $i, i' \in \mathcal{I}$ a *pi-square*. We denote the set of all pi-squares by Π.

8 $\mathcal{D} = (\mathcal{D}(X))_{X \in \mathrm{ob}(\mathcal{A})}$ is a family of categories.

9 $(?)^\#$ is a contravariant almost-pseudofunctor on $\mathcal{A}_\mathcal{P}$, $(?)^\flat$ is a contravariant almost-pseudofunctor on $\mathcal{A}_\mathcal{I}$, and we have $X^\# = X^\flat = \mathcal{D}(X)$ for each $X \in \mathrm{ob}(\mathcal{A})$.

10 $\zeta = (\zeta(\sigma))_{\sigma=(pi=jq)\in\Pi}$ is a family of natural transformations

$$\zeta(\sigma) : i^\flat p^\# \to q^\# j^\flat.$$

11 If

$$\begin{array}{ccccc} U_1 & \xrightarrow{j_1} & V_1 & \xrightarrow{i_1} & X_1 \\ \downarrow p_U & \sigma' & \downarrow p_V & \sigma & \downarrow p_X \\ U & \xrightarrow{j} & V & \xrightarrow{i} & X \end{array}$$

is a commutative diagram in $\mathcal{A}$ such that $\sigma, \sigma' \in \Pi$, then the composite map

$$(i_1 j_1)^\flat p_X^\# \xrightarrow{d^{-1}} j_1^\flat i_1^\flat p_X^\# \xrightarrow{\zeta(\sigma)} j_1^\flat p_V^\# i^\flat \xrightarrow{\zeta(\sigma')} p_U^\# j^\flat i^\flat \xrightarrow{d} p_U^\# (ij)^\flat$$

agrees with $\zeta(\sigma'\sigma)$, where $\sigma'\sigma$ is the pi-square $p_X(i_1 j_1) = (ij)p_U$.

12 For any morphism $p : X \to Y$ in $\mathcal{P}$, the composite

$$p^\# \xrightarrow{\mathfrak{f}^{-1}} 1_X^\flat p^\# \xrightarrow{\zeta(p1_X=1_Y p)} p^\# 1_Y^\flat \xrightarrow{\mathfrak{f}} p^\#$$

is the identity.

13 If

$$\begin{array}{ccc} U_2 & \xrightarrow{i_2} & X_2 \\ \downarrow q_U & \sigma' & \downarrow q_X \\ U_1 & \xrightarrow{i_1} & X_1 \\ \downarrow p_U & \sigma & \downarrow p_X \\ U & \xrightarrow{i} & X \end{array}$$

is a commutative diagram in $\mathcal{A}$ such that $\sigma, \sigma' \in \Pi$, then the composite

$$i_2^\flat (p_X q_X)^\# \xrightarrow{d^{-1}} i_2^\flat q_X^\# p_X^\# \xrightarrow{\zeta(\sigma')} q_U^\# i_1^\flat p_X^\# \xrightarrow{\zeta(\sigma)} q_U^\# p_U^\# i^\flat \xrightarrow{d} (p_U q_U)^\# i^\flat$$

agrees with $\zeta((p_X q_X) i_2 = i(p_U q_U))$.

14 For any morphism $i : U \to X$ in $\mathcal{I}$, the composite

$$i^\flat \xrightarrow{\mathfrak{f}^{-1}} i^\flat 1_X^\# \xrightarrow{\zeta} 1_U^\# i^\flat \xrightarrow{\mathfrak{f}} i^\flat$$

is the identity.

15 $\mathcal{F}$ is a subcategory of $\mathcal{A}$, and any isomorphism in $\mathcal{A}$ between objects of $\mathcal{F}$ is in $\mathrm{Mor}(\mathcal{F})$.

16 $\mathcal{D}^+ = (\mathcal{D}^+(X))_{X\in \mathrm{ob}(\mathcal{F})}$ is a family of categories such that $\mathcal{D}^+(X)$ is a full subcategory of $\mathcal{D}(X)$ for each $X \in \mathrm{ob}(\mathcal{F})$.

17 If $f : X \to Y$ is a morphism in $\mathcal{F}$, $f = p \circ i$, $p \in \mathcal{P}$ and $i \in \mathcal{I}$, then we have $i^\flat p^!(\mathcal{D}^+(Y)) \subset \mathcal{D}^+(X)$.

18 If

$$\begin{array}{ccccc} V & \xrightarrow{j} & U_1 & \xrightarrow{i_1} & X_1 & & \\ & & \downarrow p_U & \sigma & \downarrow p_X & & \\ & & U & \xrightarrow{i} & X & \xrightarrow{q} & Y \end{array}$$

is a diagram in $\mathcal{A}$ such that $\sigma \in \Pi$, $V, U, Y \in \mathrm{ob}(\mathcal{F})$, $p_U j \in \mathrm{Mor}(\mathcal{F})$ and $qi \in \mathrm{Mor}(\mathcal{F})$, then

$$j^\flat \zeta(\sigma) q^\# : j^\flat i_1^\flat p_X^\# q^\# \to j^\flat p_U^\# i^\flat q^\#$$

is an isomorphism between functors from $\mathcal{D}^+(Y)$ to $\mathcal{D}^+(V)$.

(16.2) Let $\mathcal{C} = (\mathcal{A}, \mathcal{F}, \mathcal{P}, \mathcal{I}, \mathcal{D}, \mathcal{D}^+, (?)^\#, (?)^\flat, \zeta)$ be a composition data of contravariant almost-pseudofunctors. We call a commutative diagram of the form $f = pi$ with $p \in \mathcal{P}$, $i \in \mathcal{I}$ and $f \in \mathrm{Mor}(\mathcal{A})$ a *compactification*. We call a commutative diagram of the form $pi = qj$ with $p, q \in \mathcal{P}$, $i, j \in \mathcal{I}$ and $pi = qj \in \mathrm{Mor}(\mathcal{F})$ an *independence diagram.*

16.3 Lemma. *Let*

$$\begin{array}{ccc} U & \xrightarrow{i_1} & X_1 \\ \downarrow i & \tau & \downarrow p_1 \\ X & \xrightarrow{p} & Y \end{array}$$

be an independence diagram. Then the following hold:

1 *There is a diagram of the form*

$$\begin{array}{ccccc} U & \xrightarrow{j} & Z & \xrightarrow{q_1} & X_1 \\ & & \downarrow q & & \downarrow p_1 \\ & & X & \xrightarrow{p} & Y \end{array}$$

such that $qj = i$, $q_1 j = i_1$, $pq = p_1 q_1$, $q, q_1 \in \mathcal{P}$, and $j \in \mathcal{I}$.

2 *$\zeta(qj = i1_U)p^\# : j^\flat q^\# p^\# \to i^\flat p^\#$ is an isomorphism between functors from $\mathcal{D}^+(Y)$ to $\mathcal{D}^+(U)$.*

3 *$\zeta(q_1 j_1 = i_1 1_U)p_1^\#$ is also an isomorphism between functors from $\mathcal{D}^+(Y)$ to $\mathcal{D}^+(U)$.*

4 *The composite isomorphism*

$$\Upsilon(\tau) : i^\flat p^\# \xrightarrow{\mathfrak{f}^{-1}} 1_U^\# i^\flat p^\# \xrightarrow{(\zeta(qj=i1_U)p^\#)^{-1}} j^\flat q^\# p^\#$$
$$\xrightarrow{d} j^\flat q_1^\# p_1^\# \xrightarrow{\zeta(q_1 j_1 = i_1 1_U)p_1^\#} 1_U^\# i_1^\flat p_1^\# \xrightarrow{\mathfrak{f}} i_1^\flat p_1^\#$$

(between functors defined over $\mathcal{D}^+(Y)$, not over $\mathcal{D}(Y)$) depends only on τ.

5 *If $\tau' = (p_1 i_1 = p_2 i_2)$ is an independence diagram, then we have*

$$\Upsilon(\tau') \circ \Upsilon(\tau) = \Upsilon(pi = p_2 i_2).$$

The proof is left to the reader. We call $\Upsilon(\tau)$ the *independence isomorphism* of τ.

(16.4) Any $f \in \mathrm{Mor}(\mathcal{A})$ has a compactification by assumption. We fix a family of compactifications $\mathcal{T} := (\tau(f) : (f = p(f) \circ i(f)))_{f \in \mathrm{Mor}(\mathcal{A})}$.

For $X \in \mathrm{ob}(\mathcal{F})$, we define $X^! := \mathcal{D}^+(X)$. For a morphism $f : X \to Y$ in $\mathcal{F}$, we define $f^! := i(f)^\flat p(f)^\#$, which is a functor from $Y^!$ to $X^!$ by assumption.

Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathcal{F}$. Let $i(g)p(f) = qj$ be a compactification of $i(g)p(f)$. Then by **18** in Definition 16.1 and Lemma 16.3, the composite map

$$(gf)^! = i(gf)^\flat p(gf)^\# \xrightarrow{\Upsilon(p(gf)i(gf)=(p(g)q)(ji(f)))} (j \circ i(f))^\flat (p(g) \circ q)^\#$$
$$\cong i(f)^\flat j^\flat q^\# p(g)^\# \xrightarrow{i(f)^\flat \zeta(qj=i(g)p(f))p(g)^\#} i(f)^\flat p(f)^\# i(g)^\flat p(g)^\# = f^! g^!$$

is an isomorphism. We define $d_{f,g} : f^! g^! \to (gf)^!$ to be the inverse of this composite.

16.5 Lemma. *The definition of $d_{f,g}$ is independent of choice of q and j above.*

The proof of the lemma is left to the reader.

For $X \in \mathrm{ob}(\mathcal{F})$, we define $\mathfrak{f}_X : \mathrm{id}_X^! \to \mathrm{Id}_{X^!}$ to be the composite

$$\mathrm{id}_X^! = i(\mathrm{id}_X)^\flat p(\mathrm{id}_X)^\# \xrightarrow{\zeta} \mathrm{id}_X^\# \mathrm{id}_X^\flat \xrightarrow{\mathfrak{f}} \mathrm{id}_X^\flat \xrightarrow{\mathfrak{f}} \mathrm{Id}_{X^!}.$$

16.6 Proposition. *Let the notation be as above.*

1 $(?)^!$ *together with* $(d_{f,g})$ *and* $(\mathfrak{f}_X)$ *form a contravariant almost-pseudofunctor on* $\mathcal{F}$.

2 *For* $j \in \mathcal{I} \cap \mathrm{Mor}(\mathcal{F})$, *define* $\psi : j^! \to j^\flat$ *to be the composite*

$$j^! = i(j)^\flat p(j)^\# \xrightarrow{\Upsilon} j^\flat \mathrm{id}^\# \xrightarrow{\mathfrak{f}} j^\flat.$$

Then $\psi : (?)^! \to (?)^\flat$ *is an isomorphism of contravariant almost-pseudofunctors on* $\mathcal{A}_\mathcal{I} \cap \mathcal{F}$.

3 *For* $q \in \mathcal{P} \cap \mathrm{Mor}(\mathcal{F})$, *define* $\psi : q^! \to q^\#$ *to be the composite*

$$q^! = i(q)^\flat p(q)^\# \xrightarrow{\Upsilon} \mathrm{id}^\flat q^\# \xrightarrow{\mathfrak{f}} q^\#.$$

Then $\psi : (?)^! \to (?)^\#$ *is an isomorphism of contravariant almost-pseudofunctors on* $\mathcal{A}_\mathcal{P} \cap \mathcal{F}$.

4 *Let us take another family of compactifications* $(f = p_1(f)i_1(f))_{f \in \mathrm{Mor}(\mathcal{F})}$, *and let* $(?)^\star$ *be the resulting contravariant almost-pseudofunctor defined by* $f^\star = i_1(f)^\flat p_1(f)^\#$. *Then* $\Upsilon : f^! \to f^\star$ *induces an isomorphism of contravariant almost-pseudofunctors* $(?)^! \cong (?)^\star$.

The proof is left to the reader. We call $(?)^!$ the *composite* of $(?)^\#$ and $(?)^\flat$. The composite is uniquely defined up to isomorphisms of almost-pseudofunctors on $\mathcal{F}$. The discussion above has an obvious triangulated version. Composition data of contravariant triangulated almost-pseudofunctors are defined appropriately, and the composition of two contravariant triangulated almost-pseudofunctors is obtained as a contravariant triangulated almost-pseudofunctor.

(16.7) Let S be a scheme. Let $\mathcal{A}$ be the category whose objects are noetherian S-schemes and morphisms are morphisms separated of finite type. Set $\mathcal{F} = \mathcal{A}$. Let $\mathcal{I}$ be the class of open immersions. Let $\mathcal{P}$ be the class of proper morphisms. Set $\mathcal{D}(X) = \mathcal{D}^+(X) = D^+_{\mathrm{Qch}}(X)$ for $X \in \mathcal{A}$. Let $(?)^\flat := (?)^*$, the (derived) inverse image almost-pseudofunctor for morphisms in $\mathcal{I}$, where $X^\flat := \mathcal{D}(X)$. Let $(?)^\# := (?)^\times$, the *twisted inverse* almost-pseudofunctor (see [26, Chapter 4]) for morphisms in $\mathcal{P}$, where $X^\# := \mathcal{D}(X)$ again. Note that the left adjoint $R(?)_*$ is way-out left for morphisms in $\mathcal{P}$ so that $(?)^\times$ is way-out right by Lemma 14.13, and $(?)^\#$ is well-defined. The conditions **1–9** in Definition 16.1 hold. Note that **7** is nothing but Nagata's compactification theorem [34]. The conditions **15–17** are trivial.

Let $\sigma_0 : pi = jq$ be a pi-diagram, which is also a fiber square. Then the canonical map

$$\theta : j^*(Rp_*) \to (Rq_*)i^*$$

is an isomorphism of triangulated functors, see [26, (3.9.5)]. Hence, taking the inverse of the conjugate, we have an isomorphism

$$\xi = \xi(\sigma_0) : (Ri_*)q^\times \cong p^\times(Rj_*). \tag{16.8}$$

So we have a morphism of triangulated functors

$$\zeta_0(\sigma_0) : i^*p^\times \xrightarrow{\text{via } u} i^*p^\times(Rj_*)j^* \xrightarrow{\text{via } \xi^{-1}} i^*(Ri_*)q^\times j^* \xrightarrow{\text{via } \varepsilon} q^\times j^*,$$

which is an *isomorphism*, see [41]. The statements **10, 12** and **14**, and corresponding statements to **11, 13** only for *fiber square pi-diagrams*, are readily proved.

In particular, for a closed open immersion $\eta : U \to X$, we have an isomorphism

$$v(\eta) : \eta^* \xrightarrow{\mathfrak{f}^{-1}} 1_U^\times\eta^* \xrightarrow{\zeta_0(\eta 1_U = \eta 1_U)^{-1}} 1_U^*\eta^\times \xrightarrow{\mathfrak{f}} \eta^\times.$$

Let $\sigma = (pi = qj)$ be an arbitrary pi-diagram. Let j_1 be the base change of j by p, and let p_1 be the base change of p by j. Let η be the unique morphism such that $q = p_1\eta$ and $i = j_1\eta$. Note that η is a closed open immersion. Define $\zeta(\sigma)$ to be the composite isomorphism

$$i^*p^\times \cong \eta^*j_1^*p^\times \xrightarrow{\text{via } \zeta_0} \eta^*p_1^\times j^* \xrightarrow{\text{via } v(\eta)} \eta^\times p_1^\times j^* \cong q^\times j^*.$$

Now the proof of conditions **11, 13** consists in diagram chasing arguments, while **18** is trivial, since $\zeta(\sigma)$ is always an isomorphism. Thus the twisted inverse triangulated almost-pseudofunctor $(?)^!$ on $\mathcal{A}$ is defined to be the composite of $(?)^\times$ and $(?)^*$.

17 The right adjoint of the derived direct image functor of a morphism of diagrams

Let I be a small category, S a scheme, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$.

17.1 Lemma. *Assume that $X_\bullet$ is concentrated. That is, X_i is concentrated for each $i \in I$. Let C_i be a small set of compact generators of $D_{\mathrm{Qch}}(X_i)$, which exists by* Theorem 14.14*. Then*

$$C := \{LL_i c \mid i \in I,\ c \in C_i\}$$

is a small set of compact generators of $D_{\mathrm{Lqc}}(X_\bullet)$. In particular, the category $D_{\mathrm{Lqc}}(X_\bullet)$ is compactly generated.

Proof. Let $t \in D_{\mathrm{Lqc}}(X_\bullet)$ and assume that

$$\mathrm{Hom}_{D(X_\bullet)}(LL_i c, t) \cong \mathrm{Hom}_{D(X_i)}(c, t_i) = 0$$

for any $i \in I$ and any $c \in C_i$. Then, $t_i = 0$ for all i. This shows $t = 0$. It is easy to see that $LL_i c$ is compact, and C is small. So C is a small set of compact generators. As $D_{\mathrm{Lqc}}(X_\bullet)$ has coproducts, it is compactly generated. $\square$

17.2 Lemma. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a concentrated morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Then*

$$R(f_\bullet)_* : D_{\mathrm{Lqc}}(X_\bullet) \to D_{\mathrm{Lqc}}(Y_\bullet)$$

preserves coproducts.

Proof. Let (t_λ) be a small family of objects in $D_{\mathrm{Lqc}}(X_\bullet)$, and consider the canonical map

$$\bigoplus_\lambda R(f_\bullet)_* t_\lambda \to R(f_\bullet)_*(\bigoplus_\lambda t_\lambda).$$

For each $i \in I$, apply $(?)_i$ to the map. As $(?)_i$ obviously preserves coproducts and we have a canonical isomorphism

$$(?)_i R(f_\bullet)_* \cong R(f_i)_*(?)_i,$$

the result is the canonical map

$$\bigoplus_\lambda R(f_i)_*(t_\lambda)_i \to R(f_i)_*(\bigoplus_\lambda (t_\lambda)_i),$$

which is an isomorphism by Lemma 14.15. Hence, $R(f_\bullet)_*$ preserves coproducts. $\square$

By Theorem 14.10, we have the first (original) theorem of these notes:

17.3 Theorem. *Let I be a small category, S a scheme, and $f_\bullet : X_\bullet \to Y_\bullet$ a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. If $X_\bullet$ and $f_\bullet$ are concentrated, then*

$$R(f_\bullet)_* : D_{\mathrm{Lqc}}(X_\bullet) \to D_{\mathrm{Lqc}}(Y_\bullet)$$

has a right adjoint $f_\bullet^\times$.

(17.4) Let I be a small category. Let $\mathcal{S}$ be the category of concentrated I^{op} diagrams of schemes. Note that any morphism of $\mathcal{S}$ is concentrated (follows easily from [15, (1.2.3), (1.2.4)]). For $X_\bullet \in \mathcal{S}$, set $R(X_\bullet)_* := D_{\mathrm{Lqc}}(X_\bullet)$. For a morphism $f_\bullet$ of $\mathcal{S}$, set $R(f_\bullet)_*$ be the derived direct image. Then $R(?)_*$ is a covariant almost-pseudofunctor on $\mathcal{S}$. Thus its right adjoint $(?)^\times$ is a contravariant almost-pseudofunctor on $\mathcal{S}$, and $(R(?)_*, (?)^\times)$ is an opposite adjoint pair of Δ-pseudofunctors on $\mathcal{S}$, see (1.18).

For composable morphisms $f_\bullet$ and $g_\bullet$, $d_{f_\bullet, g_\bullet} : f_\bullet^\times g_\bullet^\times \to (g_\bullet f_\bullet)^\times$ is the composite

$$f_\bullet^\times g_\bullet^\times \xrightarrow{u} (g_\bullet f_\bullet)^\times R(g_\bullet f_\bullet)_* f_\bullet^\times g_\bullet^\times \xrightarrow{c} (g_\bullet f_\bullet)^\times R(g_\bullet)_* R(f_\bullet)_* f_\bullet^\times g_\bullet^\times \xrightarrow{\varepsilon} (g_\bullet f_\bullet)^\times R(g_\bullet)_* g_\bullet^\times \xrightarrow{\varepsilon} (g_\bullet f_\bullet)^\times.$$

For $X_\bullet \in \mathcal{S}$, $\mathfrak{f} : \mathrm{id}_{X_\bullet}^\times \to \mathrm{Id}_{X_\bullet^\times}$ is the composite

$$\mathrm{id}_{X_\bullet}^\times \xrightarrow{\mathfrak{e}} R(\mathrm{id}_{X_\bullet})_* \mathrm{id}_{X_\bullet}^\times \xrightarrow{\varepsilon} \mathrm{Id}\,.$$

17.5 Lemma. *Let S be a scheme, I a small category, and $f_\bullet : X_\bullet \to Y_\bullet$ and $g_\bullet : Y_\bullet' \to Y_\bullet$ be morphisms in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Set $X_\bullet' := Y_\bullet' \times_{Y_\bullet} X_\bullet$. Let $f_\bullet' : X_\bullet' \to Y_\bullet'$ be the first projection, and $g_\bullet' : X_\bullet' \to X_\bullet$ the second projection. Assume that $f_\bullet$ is concentrated, and $g_\bullet$ is flat. Then the canonical map*

$$\theta(g_\bullet, f_\bullet) : (g_\bullet)^* R(f_\bullet)_* \to R(f_\bullet')_* (g_\bullet')^*$$

is an isomorphism of functors from $D_{\mathrm{Lqc}}(X_\bullet)$ to $D_{\mathrm{Lqc}}(Y_\bullet')$.

Proof. It suffices to show that for each $i \in I$,

$$(?)_i\theta : (?)_i (g_\bullet)^* R(f_\bullet)_* \to (?)_i R(f_\bullet')_* (g_\bullet')^*$$

is an isomorphism. By Lemma 1.22, it is easy to verify that the diagram

$$\begin{tikzcd} g_i^* R(f_i)_*(?)_i \arrow[r, "c^{-1}"] \arrow[d, "\theta(g_i, f_i)"] & g_i^*(?)_i R(f_\bullet)_* \arrow[r, "\theta"] & (?)_i g_\bullet^* R(f_\bullet)_* \arrow[d, "(?)_i\theta(g_\bullet, f_\bullet)"] \\ R(f_i')_*(g_i')^*(?)_i \arrow[r, "\theta"] & R(f_i')_*(?)_i(g_\bullet')^* \arrow[r, "c^{-1}"] & (?)_i R(f_\bullet')_*(g_\bullet')^* \end{tikzcd} \tag{17.6}$$

is commutative. As the horizontal maps and $\theta(g_i, f_i)$ are isomorphisms by [26, (3.9.5)], $(?)_i\theta(g_\bullet, f_\bullet)$ is also an isomorphism. $\square$

18 Commutativity of twisted inverse with restrictions

(18.1) Let S be a scheme, I a small category, and $f_\bullet : X_\bullet \to Y_\bullet$ a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Let J be an admissible subcategory of I. Assume that $f_\bullet$ is concentrated. Then there is a natural map

$$\theta(J, f_\bullet) : LL_J \circ R(f_\bullet|_J)_* \to R(f_\bullet)_* \circ LL_J \tag{18.2}$$

between functors from $D_{\mathrm{Lqc}}(X_\bullet|_J)$ to $D_{\mathrm{Lqc}}(Y_\bullet)$, see [26, (3.7.2)].

(18.3) Let S, I and $f_\bullet$ be as in (18.1). We assume that $X_\bullet$ and $f_\bullet$ are concentrated, so that the right adjoint functor

$$f_\bullet^\times : D_{\mathrm{Lqc}}(Y_\bullet) \to D_{\mathrm{Lqc}}(X_\bullet)$$

of

$$R(f_\bullet)_* : D_{\mathrm{Lqc}}(X_\bullet) \to D_{\mathrm{Lqc}}(Y_\bullet)$$

exists. Let J be a subcategory of I which may not be admissible.

We define the natural transformation

$$\xi(J, f_\bullet) : (?)_J \circ f_\bullet^\times \to (f_\bullet|_J)^\times \circ (?)_J$$

to be the composite

$$(?)_J f_\bullet^\times \xrightarrow{u} (f_\bullet|_J)^\times R(f_\bullet|_J)_*(?)_J f_\bullet^\times \xrightarrow{c^{-1}} (f_\bullet|_J)^\times (?)_J R(f_\bullet)_* f_\bullet^\times \xrightarrow{\varepsilon} (f_\bullet|_J)^\times (?)_J.$$

By definition, ξ is the conjugate map of $\theta(J, f_\bullet)$ in (18.2) if J is admissible. Do not confuse $\xi(J, f_\bullet)$ with $\xi(f_\bullet, J)$ (see Corollary 6.26).

18.4 Lemma. *Let $J_2 \subset J_1 \subset I$ be subcategories of I. Let S be a scheme, $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Assume both $X_\bullet$ and $f_\bullet$ are concentrated. Then the composite map*

$$
\begin{aligned}
(?)_{J_2} f_\bullet^\times \xrightarrow{c} (?)_{J_2}(?)_{J_1} f_\bullet^\times &\xrightarrow{\xi(J_1,f_\bullet)} (?)_{J_2}(f_\bullet|_{J_1})^\times (?)_{J_1} \\
&\xrightarrow{\xi(J_2,f_\bullet|_{J_1})} (f_\bullet|_{J_2})^\times (?)_{J_2}(?)_{J_1} \xrightarrow{c^{-1}} (f_\bullet|_{J_2})^\times (?)_{J_2}
\end{aligned}
$$

is equal to $\xi(J_2, f_\bullet)$.

Proof. Straightforward (and tedious) diagram drawing. □

18.5 Lemma. *Let S and $f_\bullet : X_\bullet \to Y_\bullet$ be as in* Lemma 18.4. *Let J be a subcategory of I. Assume that $Y_\bullet$ has flat arrows and $f_\bullet$ is cartesian. Then $\xi(J, f_\bullet) : (?)_J f_\bullet^\times \to (f_\bullet|_J)^\times (?)_J$ is an isomorphism between functors from $D_{\mathrm{Lqc}}(Y_\bullet)$ to $D_{\mathrm{Lqc}}(X_\bullet|_J)$.*

Proof. In view of Lemma 18.4, we may assume that $J = i$ for an object i of I. Then, as i is an admissible subcategory of I and $\xi(i, f_\bullet)$ is a conjugate map of $\theta(i, f_\bullet)$, it suffices to show that $(?)_j\theta(i, f_\bullet)$ is an isomorphism for any $j \in \mathrm{ob}(I)$. As $Y_\bullet$ has flat arrows, $L_i : \mathrm{Mod}(Y_i) \to \mathrm{Mod}(Y_\bullet)$ is exact. As $f_\bullet$ is cartesian, $L_i : \mathrm{Mod}(X_i) \to \mathrm{Mod}(X_\bullet)$ is also exact.

By Proposition 6.23, the composite

$$
(?)_j L_i R(f_i)_* \xrightarrow{(?)_j\theta} (?)_j R(f_\bullet)_* L_i \xrightarrow{c} R(f_j)_* (?)_j L_i
$$

agrees with the composite

$$
\begin{aligned}
(?)_j L_i R(f_i)_* \xrightarrow{\lambda_{i,j}} \bigoplus_{\phi \in I(i,j)} Y_\phi^* R(f_i)_* &\xrightarrow{\oplus\theta} \bigoplus_\phi R(f_j)_* X_\phi^* \\
&\xrightarrow{\mathrm{C}} R(f_j)_* \left(\bigoplus_\phi X_\phi^* \right) \xrightarrow{\lambda_{i,j}^{-1}} R(f_j)_* (?)_j L_i,
\end{aligned}
$$

where C is the canonical map. By Lemma 14.15, C is an isomorphism. As $f_\bullet$ is cartesian and $Y_\bullet$ has flat arrows, $\theta : Y_\phi^* R(f_i)_* \to R(f_j)_* X_\phi^*$ is an isomorphism for each $\phi \in I(i,j)$ by [26, (3.9.5)]. Hence the second composite is an isomorphism. As the first composite is an isomorphism and c is also an isomorphism, we have that $(?)_j\theta(i, f_\bullet)$ is an isomorphism. □

(18.6) Let I be a small category, S a scheme, and $f_\bullet : X_\bullet \to Y_\bullet$ a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Assume that $X_\bullet$ and $f_\bullet$ are concentrated.

18.7 Lemma. *Let J be a subcategory of I. Then the following hold:*

1 *The composite map*

$$(?)_J \xrightarrow{u} (?)_J f_\bullet^\times R(f_\bullet)_* \xrightarrow{\xi(J,f_\bullet)} (f_\bullet|_J)^\times (?)_J R(f_\bullet)_* \xrightarrow{c_{J,f_\bullet}} (f_\bullet|_J)^\times R(f_\bullet|_J)_*(?)_J$$

agrees with u.

2 *The composite map*

$$(?)_J R(f_\bullet)_* f_\bullet^\times \xrightarrow{c_{J,f_\bullet}} R(f_\bullet|_J)_*(?)_J f_\bullet^\times \xrightarrow{\xi(J,f_\bullet)} R(f_\bullet|_J)_*(f_\bullet|_J)^\times (?)_J \xrightarrow{\varepsilon} (?)_J$$

agrees with ε.

Proof. The proof consists in straightforward diagram drawings. □

(18.8) Let I be a small category. For $i, j \in \mathrm{ob}(I)$, we say that $i \leq j$ if $I(i,j) \neq \emptyset$. This definition makes $\mathrm{ob}(I)$ a pseudo-ordered set. We say that I is *ordered* if $\mathrm{ob}(I)$ is an ordered set with the pseudo-order structure above, and $I(i,i) = \{\mathrm{id}\}$ for $i \in I$.

18.9 Lemma. *Let I be an ordered small category. Let J_0 and J_1 be full subcategories of I, such that $\mathrm{ob}(J_0) \cup \mathrm{ob}(J_1) = \mathrm{ob}(I)$, $\mathrm{ob}(J_0) \cap \mathrm{ob}(J_1) = \emptyset$, and $I(j_1, j_0) = \emptyset$ for $j_1 \in J_1$ and $j_0 \in J_0$. Let $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Then, we have the following.*

1 *The unit of adjunction $u : \mathrm{Id}_{\mathrm{Mod}(X_\bullet|_{J_1})} \to (?)_{J_1} \circ L_{J_1}$ is an isomorphism.*

2 *$(?)_{J_0} \circ L_{J_1}$ is zero.*

3 *L_{J_1} is exact, and J_1 is an admissible subcategory of I.*

4 *For $\mathcal{M} \in \mathrm{Mod}(X_\bullet)$, $\mathcal{M}_{J_0} = 0$ if and only if $\varepsilon : L_{J_1}\mathcal{M}_{J_1} \to \mathcal{M}$ is an isomorphism.*

5 *The counit of adjunction $(?)_{J_0} \circ R_{J_0} \to \mathrm{Id}_{\mathrm{Mod}(X_\bullet|_{J_0})}$ is an isomorphism.*

6 *$(?)_{J_1} \circ R_{J_0}$ is zero.*

7 *R_{J_0} is exact and preserves local-quasi-coherence.*

8 *For* $\mathcal{M} \in \mathrm{Mod}(X_\bullet)$*,* $\mathcal{M}_{J_1} = 0$ *if and only if* $u : \mathcal{M} \to R_{J_0}\mathcal{M}_{J_0}$ *is an isomorphism.*

9 *The sequence*

$$0 \to L_{J_1}(?)_{J_1} \xrightarrow{\varepsilon} \mathrm{Id} \xrightarrow{u} R_{J_0}(?)_{J_0} \to 0$$

is exact, and induces a distinguished triangle in $D(X_\bullet)$*.*

Proof. **1** This is obvious by Lemma 6.15.

2 The category $I_{j_0}^{(J_1^{\mathrm{op}} \hookrightarrow I^{\mathrm{op}})}$ is empty, if $j_0 \in J_0$, since $I^{\mathrm{op}}(j_0, j_1) = \emptyset$ if $j_1 \in J_1$ and $j_0 \in J_0$. It follows that $(?)_{j_0} \circ L_{J_1} = 0$ if $j_0 \in J_0$. Hence, $(?)_{J_0} \circ L_{J_1} = 0$.

3 This is trivial by **1**,**2** and their proof.

4 The 'if' part is trivial by **2**. We prove the 'only if' part. By assumption and **2**, both $\mathcal{M}_{J_0}$ and $(?)_{J_0}L_{J_1}\mathcal{M}_{J_1}$ are zero, and $(?)_{J_0}\varepsilon$ is an isomorphism. On the other hand, $((?)_{J_1}\varepsilon)(u(?)_{J_1}) = \mathrm{id}$, and u is an isomorphism by **1**. Hence,

$$(?)_{J_1}\varepsilon : (?)_{J_1}L_{J_1}\mathcal{M}_{J_1} \to (?)_{J_1}\mathcal{M}$$

is an isomorphism. Hence, ε is an isomorphism.

The assertions **5**,**6**,**7**,**8**,**9** are similar, and we omit the proof. □

18.10 Lemma. *Let* I*,* S*,* J_1 *and* J_0 *be as in* Lemma 18.9*. Let* $f_\bullet : X_\bullet \to Y_\bullet$ *be a morphism in* $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$*. Assume that* $X_\bullet$ *and* $f_\bullet$ *are concentrated. Then we have that* $\theta(J_1, f_\bullet)$ *and* $\xi(J_1, f_\bullet)$ *are isomorphisms.*

Proof. Note that J_1 is admissible by Lemma 18.9, **3**, and hence $\theta(J_1, f_\bullet)$ and $\xi(J_1, f_\bullet)$ are defined. Since we have $\xi(J_1, f_\bullet)$ is the conjugate of $\theta(J_1, f_\bullet)$ by definition, it suffices to show that $\theta(J_1, f_\bullet)$ is an isomorphism. It suffices to show that

$$(?)_i LL_{J_1} R(f_\bullet|_{J_1})_* \xrightarrow{(?)_i\theta} (?)_i R(f_\bullet)_* LL_{J_1} \cong R(f_i)_*(?)_i LL_{J_1}$$

is an isomorphism for any $i \in I$.

If $i \in J_0$, then both hand sides are zero functors, and it is an isomorphism. On the other hand, if $i \in J_1$, then the map in question is equal to the composite isomorphism

$$(?)_i LL_{J_1} R(f_\bullet|_{J_1})_* \cong (?)_i R(f_\bullet|_{J_1})_* \xrightarrow{c_{i,f_\bullet|_{J_1}}} R(f_i)_*(?)_i \cong R(f_i)_*(?)_i LL_{J_1}$$

by Proposition 6.23. Hence $\theta(J_1, f_\bullet)$ is an isomorphism, as desired. □

(18.11) Let S be a scheme, I an ordered small category, $i \in I$, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Let J_1 be a filter of $\mathrm{ob}(I)$ such that i is a minimal element of J_1 (e.g., $[i, \infty)$), and set $\Gamma_i := L_{I,J_1} \circ R_{J_1,i}$. Then we have $(?)_j\Gamma_i = 0$ if $j \neq i$ and $(?)_i\Gamma_i = \mathrm{Id}$. Hence Γ_i does not depend on the choice of J_1, and depends only on i. Note that Γ_i preserves arbitrary limits and colimits (hence is exact). Assume that X_i is concentrated. Then $D_{\mathrm{Qch}}(X_i)$ is compactly generated, and the derived functor

$$\Gamma_i : D_{\mathrm{Qch}}(X_i) \to D_{\mathrm{Lqc}}(X_\bullet)$$

preserves coproducts. It follows that there is a right adjoint

$$\Sigma_i : D_{\mathrm{Lqc}}(X_\bullet) \to D_{\mathrm{Qch}}(X_i).$$

As Γ_i is obviously way-out left, we have Σ_i is way-out right by Lemma 14.13.

(18.12) Let S be a scheme, I a small category, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. We define $\mathcal{D}^+(X_\bullet)$ (resp. $\mathcal{D}^-(X_\bullet)$) to be the full subcategory of $D(X_\bullet)$ consisting of $\mathbb{F} \in D(X_\bullet)$ such that $\mathbb{F}_i$ is bounded below (resp. above) and has quasi-coherent cohomology groups for each $i \in I$. For a plump full subcategory $\mathcal{A}$ of $\mathrm{Lqc}(X_\bullet)$, we denote the triangulated subcategory of $\mathcal{D}^+(X_\bullet)$ (resp. $\mathcal{D}^-(X_\bullet)$) consisting of objects all of whose cohomology groups belong to $\mathcal{A}$ by $\mathcal{D}^+_{\mathcal{A}}(X_\bullet)$ (resp. $\mathcal{D}^-_{\mathcal{A}}(X_\bullet)$).

(18.13) Let P be an ordered set. We say that P is *upper Jordan-Dedekind* (*UJD* for short) if for any $p \in P$, the subset

$$[p, \infty) := \{q \in P \mid q \geq p\}$$

is finite. We say that an ordered small category I is UJD if the ordered set $\mathrm{ob}(I)$ is UJD, and $I(i, j)$ is finite for $i, j \in I$.

18.14 Proposition. *Let I be an ordered UJD small category. Let S be a scheme, and $g_\bullet : U_\bullet \to X_\bullet$ and $f_\bullet : X_\bullet \to Y_\bullet$ be morphisms in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Assume that $Y_\bullet$ is noetherian with flat arrows, $f_\bullet$ is proper, $g_\bullet$ is an open immersion such that $g_i(U_i)$ is dense in X_i for each $i \in I$, and $f_\bullet \circ g_\bullet$ is cartesian. Then $g_\bullet$ is cartesian, and for any $i \in I$ the composite natural map*

$$(?)_i g_\bullet^* f_\bullet^\times \xrightarrow{\text{via } \theta^{-1}} g_i^*(?)_i f_\bullet^\times \xrightarrow{\text{via } \xi(i)} g_i^* f_i^\times (?)_i$$

is an isomorphism between functors $\mathcal{D}^+(Y_\bullet) \to D_{\mathrm{Qch}}(U_i)$, where $\theta : g_i^*(?)_i \to (?)_i g_\bullet^*$ is the canonical isomorphism.

Proof. Note that $U_\bullet$ has flat arrows, since $Y_\bullet$ has flat arrows and $f_\bullet \circ g_\bullet$ is cartesian.

We prove that $g_\bullet$ is cartesian. Let $\phi : i \to j$ be a morphism in I. Then, the canonical map $(U_\phi, f_j g_j) : U_j \to U_i \times_{Y_i} Y_j$ is an isomorphism by assumption. This map factors through $(U_\phi, g_j) : U_j \to U_i \times_{X_i} X_j$, and it is easy to see that (U_ϕ, g_j) is a closed immersion. On the other hand, it is an image dense open immersion, as can be seen easily, and hence it is an isomorphism. So $g_\bullet$ is cartesian.

Set $J_1 := [i, \infty)$ and $J_0 := \mathrm{ob}(I) \setminus J_1$. By Lemma 18.10, $\xi(J_1, f_\bullet)$ is an isomorphism. By Lemma 18.4, we may replace I by J_1, and we may assume that I is an ordered finite category, and i is a minimal element of $\mathrm{ob}(I)$. Now we have $\mathcal{D}^+(Y_\bullet)$ agrees with $D^+_{\mathrm{Lqc}}(Y_\bullet)$. Since we have $\mathrm{ob}(I)$ is finite, it is easy to see that $R(f_\bullet)_*$ is way-out in both directions. It follows that $f_\bullet^\times$ is way-out right by Lemma 14.13.

It suffices to show that

$$g_i^* \xi(i) : g_i^*(?)_i f_\bullet^\times \to g_i^* f_i^\times (?)_i$$

is an isomorphism of functors from $D^+_{\mathrm{Lqc}}(Y_\bullet)$ to $D^+_{\mathrm{Qch}}(U_i)$. As $g_i^* R(g_i)_* \cong \mathrm{Id}$, it suffices to show that $R(g_i)_* g_i^* \xi(i)$ is an isomorphism. This is equivalent to say that for any perfect complex $\mathbb{P} \in C(\mathrm{Qch}(X_i))$, we have

$$R(g_i)_* g_i^* \xi(i) : \mathrm{Hom}_{D(X_i)}(\mathbb{P}, R(g_i)_* g_i^*(?)_i f_\bullet^\times) \\ \to \mathrm{Hom}_{D(X_i)}(\mathbb{P}, R(g_i)_* g_i^* f_i^\times (?)_i)$$

is an isomorphism. By [41, Lemma 2], this is equivalent to say that the canonical map

$$\varinjlim \mathrm{Hom}_{D(Y_\bullet)}(R(f_\bullet)_* LL_i(\mathbb{P} \otimes^{\bullet,L}_{\mathcal{O}_{X_i}} \mathcal{J}^n), ?) \\ \to \varinjlim \mathrm{Hom}_{D(Y_\bullet)}(LL_i R(f_i)_*(\mathbb{P} \otimes^{\bullet,L}_{\mathcal{O}_{X_i}} \mathcal{J}^n), ?)$$

induced by the conjugate $\theta(i, f_\bullet)$ of $\xi(i)$ is an isomorphism, where $\mathcal{J}$ is a defining ideal sheaf of the closed subset $X_i \setminus U_i$ in X_i.

As I is ordered and $\mathrm{ob}(I)$ is finite, we may label

$$\mathrm{ob}(I) = \{i = i(0), i(1), i(2), \ldots\}$$

so that $I(i(s), i(t)) \neq \emptyset$ implies that $s \leq t$. Let $J(r)$ denote the full subcategory of I whose object set is $\{i(r), i(r+1), \ldots\}$.

By descending induction on t, we prove that the map

$$\text{via } \theta(i, f_\bullet) : \varinjlim \operatorname{Hom}_{D(Y_\bullet)}(L_{J(t)}(?)_{J(t)} R(f_\bullet)_* LL_i(\mathbb{P} \otimes^{\bullet,L}_{\mathcal{O}_{X_i}} \mathcal{J}^n), ?)$$
$$\to \varinjlim \operatorname{Hom}_{D(Y_\bullet)}(L_{J(t)}(?)_{J(t)} LL_i R(f_i)_*(\mathbb{P} \otimes^{\bullet,L}_{\mathcal{O}_{X_i}} \mathcal{J}^n), ?)$$

is an isomorphism. This is enough to prove the proposition, since $L_{J(1)}(?)_{J(1)} = \mathrm{Id}$.

Since the sequence

$$0 \to L_{J(t+1)}(?)_{J(t+1)} \xrightarrow{\text{via } \varepsilon} L_{J(t)}(?)_{J(t)} \to \Gamma_{i(t)}(?)_{i(t)} \to 0$$

is an exact sequence of exact functors, it suffices to prove that the map

$$\text{via } \theta(i, f_\bullet) : \varinjlim \operatorname{Hom}_{D(Y_\bullet)}(\Gamma_{i(t)}(?)_{i(t)} R(f_\bullet)_* LL_i(\mathbb{P} \otimes^{\bullet,L}_{\mathcal{O}_{X_i}} \mathcal{J}^n), ?)$$
$$\to \varinjlim \operatorname{Hom}_{D(Y_\bullet)}(\Gamma_{i(t)}(?)_{i(t)} LL_i R(f_i)_*(\mathbb{P} \otimes^{\bullet,L}_{\mathcal{O}_{X_i}} \mathcal{J}^n), ?)$$

is an isomorphism by induction assumption and the five lemma. By Proposition 6.23, this is equivalent to say that the map

$$\text{via } \theta : \varinjlim \operatorname{Hom}_{D(Y_{i(t)})}(R(f_{i(t)})_* \bigoplus_\phi LX_\phi^*(\mathbb{P} \otimes^{\bullet,L}_{\mathcal{O}_{X_i}} \mathcal{J}^n), \Sigma_i(?))$$
$$\to \varinjlim \operatorname{Hom}_{D(Y_{i(t)})}(\bigoplus_\phi Y_\phi^* R(f_i)_*(\mathbb{P} \otimes^{\bullet,L}_{\mathcal{O}_{X_i}} \mathcal{J}^n), \Sigma_i(?))$$

is an isomorphism, where the sum is taken over the *finite* set $I(i, i(t))$. It suffices to prove that the map

$$\text{via } \xi : \varinjlim \operatorname{Hom}_{D(X_i)}(\mathbb{P} \otimes^{\bullet,L}_{\mathcal{O}_{X_i}} \mathcal{J}^n, R(X_\phi)_* f_{i(t)}^\times \Sigma_i(?))$$
$$\to \varinjlim \operatorname{Hom}_{D(X_i)}(\mathbb{P} \otimes^{\bullet,L}_{\mathcal{O}_{X_i}} \mathcal{J}^n, f_i^\times R(Y_\phi)_* \Sigma_i(?))$$

induced by the map $\xi : R(X_\phi)_* f_{i(t)}^\times \to f_i^\times R(Y_\phi)_*$, which is conjugate to

$$\theta : Y_\phi^* R(f_i)_* \to R(f_{i(t)})_* LX_\phi^*,$$

is an isomorphism for $\phi \in I(i, i(t))$. Since Σ_i, $R(X_\phi)_* f_{i(t)}^\times \Sigma_i$, and $f_i^\times R(Y_\phi)_* \Sigma_i$ are way-out right, it suffices to show the canonical map

$$g_i^* \xi(Y_\phi f_{i(t)} = f_i X_\phi) : g_i^* R(X_\phi)_* f_{i(t)}^\times \to g_i^* f_i^\times R(Y_\phi)_*$$

is an isomorphism between functors from $D^+_{\mathrm{Qch}}(Y_{i(t)})$ to $D^+_{\mathrm{Qch}}(U_i)$. Let $X' := X_i \times_{Y_i} Y_{i(t)}$, $p_1 : X' \to X_i$ be the first projection, $p_2 : X' \to Y_{i(t)}$ the second projection, and $\pi : X_{i(t)} \to X'$ be the map $(X_\phi, f_{i(t)})$. It is easy to see that $\xi(Y_\phi f_{i(t)} = f_i X_\phi)$ equals the composite map

$$R(X_\phi)_* f^\times_{i(t)} \cong R(p_1)_* R\pi_*\pi^\times p_2^\times \xrightarrow{\varepsilon} R(p_1)_* p_2^\times \cong f_i^\times R(Y_\phi)_*.$$

Note that the last map is an isomorphism since Y_ϕ is flat. As we have

$$U_{i(t)} \to U_i \times_{Y_i} Y_{i(t)} \cong U_i \times_{X_i} X'$$

is an isomorphism and g_i is an open immersion by assumption, the canonical map

$$g_i^* R(p_1)_* \to R(U_\phi)_*(\pi \circ g_{i(t)})^*$$

is an isomorphism. So it suffices to prove that

$$(\pi \circ g_{i(t)})^* R\pi_*\pi^\times \xrightarrow{\text{via } \varepsilon} (\pi \circ g_{i(t)})^*$$

is an isomorphism.

Consider the fiber square

$$\begin{array}{ccc} U_{i(t)} & \xrightarrow{g_{i(t)}} & X_{i(t)} \\ \downarrow \mathrm{id} & \sigma & \downarrow \pi \\ U_{i(t)} & \xrightarrow{\pi \circ g_{i(t)}} & X'. \end{array}$$

By [41, Theorem 2], $\zeta_0(\sigma) : g^*_{i(t)}\pi^\times \to (\pi \circ g_{i(t)})^*$ is an isomorphism (in [41], schemes are assumed to have finite Krull dimension, but this assumption is not used in the proof there and unnecessary). By definition (16.7), ζ_0 is the composite map

$$g^*_{i(t)}\pi^\times \xrightarrow{u} \mathrm{id}^\times R\mathrm{id}_* g^*_{i(t)}\pi^\times \xrightarrow{\cong} \mathrm{id}^\times (\pi \circ g_{i(t)})^* R\pi_*\pi^\times \xrightarrow{\varepsilon} (\pi \circ g_{i(t)})^*.$$

Since the first and the second maps are isomorphisms, the third map is an isomorphism. This was what we wanted to prove. □

18.15 Corollary. *Under the same assumption as in the proposition, we have* $g_\bullet^* f_\bullet^\times(\mathcal{D}^+(Y_\bullet)) \subset \mathcal{D}^+(U_\bullet)$.

Proof. This is because $g_i^* f_i^\times (D^+_{\mathrm{Qch}}(Y_i)) \subset D^+_{\mathrm{Qch}}(U_i)$ for each $i \in I$.

19 Open immersion base change

(19.1) Let S be a scheme, I a small category, and

$$\begin{array}{ccc} X'_\bullet & \xrightarrow{g'_\bullet} & X_\bullet \\ \downarrow f'_\bullet & \sigma & \downarrow f_\bullet \\ Y'_\bullet & \xrightarrow{g_\bullet} & Y_\bullet \end{array}$$

a fiber square in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Assume that $X_\bullet$ and $f_\bullet$ are concentrated, and $g_\bullet$ is flat. By Lemma 17.5, the canonical map

$$\theta(g_\bullet, f_\bullet) : g_\bullet^* R(f_\bullet)_* \to R(f'_\bullet)_*(g'_\bullet)^*$$

is an isomorphism of functors from $D_{\mathrm{Lqc}}(X_\bullet)$ to $D_{\mathrm{Lqc}}(Y'_\bullet)$. We define $\zeta(\sigma) = \zeta(g_\bullet, f_\bullet)$ to be the composite map

$$\zeta(\sigma) : (g'_\bullet)^* f_\bullet^\times \xrightarrow{u} (f'_\bullet)^\times R(f'_\bullet)_*(g'_\bullet)^* f_\bullet^\times \xrightarrow{\theta^{-1}} (f'_\bullet)^\times g_\bullet^* R(f_\bullet)_* f_\bullet^\times \xrightarrow{\varepsilon} (f'_\bullet)^\times g_\bullet^*.$$

19.2 Lemma. *Let σ be as above, and J a subcategory of I. Then the diagram*

$$\begin{array}{ccccc} (?)_J(g'_\bullet)^* f_\bullet^\times & \xrightarrow{\theta^{-1}} & (g'_\bullet|_J)^*(?)_J f_\bullet^\times & \xrightarrow{\xi} & (g'_\bullet|_J)^*(f_\bullet|_J)^\times(?)_J \\ \downarrow \zeta & & & & \downarrow \zeta \\ (?)_J(f'_\bullet)^\times g_\bullet^* & \xrightarrow{\xi} & (f'_\bullet|_J)^\times(?)_J g_\bullet^* & \xrightarrow{\theta^{-1}} & (f'_\bullet|_J)^\times(g_\bullet|_J)^*(?)_J \end{array}$$

is commutative.

Proof. Follows immediately from Lemma 18.7 and the commutativity of (17.6). □

19.3 Lemma. *Let σ be as above. Then the composite*

$$(g'_\bullet)^* \xrightarrow{u} (g'_\bullet)^* f_\bullet^\times R(f_\bullet)_* \xrightarrow{\zeta} (f'_\bullet)^\times g_\bullet^* R(f_\bullet)_* \xrightarrow{\theta} (f'_\bullet)^\times R(f'_\bullet)_*(g'_\bullet)^*$$

is u.

Proof. Follows from the commutativity of the diagram

$$\begin{tikzcd}
(g'_\bullet)^* \arrow[r, "u"] \arrow[d, "u"] & (g'_\bullet)^* f_\bullet^\times R(f_\bullet)_* \arrow[d, "u"] \\
(f'_\bullet)^\times R(f'_\bullet)_*(g'_\bullet)^* \arrow[r, "u"] \arrow[d, "\theta^{-1}"] \arrow[ddd, bend right=60, "\mathrm{id}"'] & (f'_\bullet)^\times R(f'_\bullet)_*(g'_\bullet)^*(f_\bullet)^\times R(f_\bullet)_* \arrow[d, "\theta^{-1}"] \\
(f'_\bullet)^\times g_\bullet^* R(f_\bullet)_* \arrow[r, "u"] \arrow[dr, "\mathrm{id}"] & (f'_\bullet)^\times g_\bullet^* R(f_\bullet)_* f_\bullet^\times R(f_\bullet)_* \arrow[d, "\varepsilon"] \\
(f'_\bullet)^\times R(f'_\bullet)_*(g'_\bullet)^* & (f'_\bullet)^\times g_\bullet^* R(f_\bullet)_* \arrow[l, "\theta"]
\end{tikzcd} .$$

□

19.4 Theorem. *Let S be a scheme, I an ordered UJD small category, and*

$$\begin{array}{ccccccc}
V_\bullet & \xrightarrow{j_\bullet} & U'_\bullet & \xrightarrow{i'_\bullet} & X'_\bullet & & \\
 & & \downarrow p^U_\bullet & \sigma & \downarrow p^X_\bullet & & \\
 & & U_\bullet & \xrightarrow{i_\bullet} & X_\bullet & \xrightarrow{q_\bullet} & Y_\bullet
\end{array}$$

be a diagram in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Assume the following.

1 *$Y_\bullet$ is noetherian with flat arrows.*

2 *$j_\bullet$, $i'_\bullet$ and $i_\bullet$ are image dense open immersions.*

3 *$q_\bullet$, $p^X_\bullet$ and $p^U_\bullet$ are proper.*

4 *$p^X_\bullet i'_\bullet = i_\bullet p^U_\bullet$.*

5 *$q_\bullet i_\bullet$ and $p^U_\bullet j_\bullet$ are cartesian.*

Then σ is a fiber square, and

$$j_\bullet^* \zeta(\sigma) q_\bullet^\times : j_\bullet^*(i'_\bullet)^*(p^X_\bullet)^\times q_\bullet^\times \to j_\bullet^*(p^U_\bullet)^\times i_\bullet^* q_\bullet^\times$$

is an isomorphism of functors from $\mathcal{D}^+(Y_\bullet)$ to $\mathcal{D}^+(V_\bullet)$.

Proof. The square σ is a fiber square, since the canonical map $U'_\bullet \to U_\bullet \times_{X_\bullet} X'_\bullet$ is an image dense closed open immersion, and is an isomorphism.

To prove the theorem, it suffices to show that the map in question is an isomorphism after applying $(?)_i$ for any $i \in I$. By Proposition 18.14, Lemma 19.2, and [26, (3.7.2), (iii)], the problem is reduced to the flat base change theorem (in fact open immersion base change theorem is enough) for schemes [41, Theorem 2], and we are done. $\square$

20 The existence of compactification and composition data for diagrams of schemes over an ordered finite category

(20.1) Let I be an ordered finite category which is non-empty. Let $\mathcal{A}$ denote the category of noetherian I^{op}-diagrams of schemes as its objects and morphisms separated of finite type as its morphisms. Let $\mathcal{P}$ denote the class of proper morphisms in $\mathrm{Mor}(\mathcal{A})$. Let $\mathcal{I}$ denote the class of image dense open immersions in $\mathrm{Mor}(\mathcal{A})$.

Define $\mathcal{D}(X_\bullet) := D_{\mathrm{Lqc}}(X_\bullet)$ for $X_\bullet \in \mathrm{ob}(\mathcal{A})$. Define a pseudofunctor $(?)^\#$ on $\mathcal{A}_\mathcal{P}$ to be $(?)^\times$, where $X_\bullet^\# = \mathcal{D}(X_\bullet)$ for $X_\bullet \in \mathrm{ob}(\mathcal{A}_\mathcal{P})$. Define a pseudofunctor $(?)^\flat$ on $\mathcal{A}_\mathcal{I}$ to be $(?)^*$, where $X_\bullet^\flat = \mathcal{D}(X_\bullet)$ for $X_\bullet \in \mathrm{ob}(\mathcal{A}_\mathcal{I})$.

For a pi-square σ, define $\zeta(\sigma)$ to be the natural map defined in (19.1).

20.2 Lemma. *Let the notation be as above. Conditions* **1–6** *and* **8–14** *in* Definition 16.1 *are satisfied. Moreover, any pi-square is a fiber square.*

Proof. This is easy. $\square$

20.3 Proposition. *Let the notation be as in* (20.1). *Then the condition* **7** *in* Definition 16.1 *is satisfied. That is, for any morphism $f_\bullet$ in $\mathcal{A}$, there is a factorization $f_\bullet = p_\bullet j_\bullet$ with $p_\bullet \in \mathcal{P}$ and $j_\bullet \in \mathcal{I}$.*

Proof. Label the object set $\mathrm{ob}(I)$ of I as $\{i(1), \cdots, i(n)\}$ so that $I(i(s), i(t)) = \emptyset$ if $s > t$. Set $J(r)$ to be the full subcategory of I with $\mathrm{ob}(J(r)) = \{i(1), \ldots, i(r)\}$. By induction on r, we construct morphisms $j_\bullet(r) : X_\bullet|_{J(r)} \to Z_\bullet(r)$ and $p_\bullet(r) : Z_\bullet(r) \to Y_\bullet|_{J(r)}$ such that

1 $j_\bullet(r)$ is an open immersion whose scheme theoretic image is $Z_\bullet(r)$ (i.e., for any j, $j_\bullet(r)_j$ is an open immersion whose scheme theoretic image is $Z_\bullet(r)_j$). In particular, $j_\bullet(r)$ is an image dense open immersion.

2 $p_\bullet(r)$ is proper.

3 $p_\bullet(r)j_\bullet(r) = f_\bullet|_{J(r)}$.

4 $Z_\bullet(r)|_{J(j)} = Z_\bullet(j)$, $j_\bullet(r)|_{J(j)} = j_\bullet(j)$, $p_\bullet(r)|_{J(j)} = p_\bullet(j)$ for $j < r$.

The proposition follows from this construction for $r = n$. We may assume that the construction is done for $j < r$.

First, consider the case where $i(r)$ is a minimal element in $\mathrm{ob}(I)$. By Nagata's compactification theorem [34] (see also [27]), there is a factorization

$$X_{i(r)} \xrightarrow{k} Z \xrightarrow{p} Y_{i(r)}$$

such that p is proper, k is an open immersion whose scheme theoretic image is Z, and $pk = f_{i(r)}$. Now define $Z_\bullet(r)_{i(r)} := Z$ and $Z_\bullet(r)_{\mathrm{id}_{i(r)}} = \mathrm{id}_Z$. Defining the other structures after **4**, we get $Z_\bullet(r)$, since $I(i(r), i(s)) = \emptyset = I(i(s), i(r))$ for $s < r$ and $I(i(r), i(r)) = \{\mathrm{id}\}$ by assumption. Define $j_\bullet(r)_{i(r)} := k$ and by **4**, we get a morphism $j_\bullet(r) : X_\bullet|_{J(r)} \to Z_\bullet(r)$ by the same reason. Similarly, $p_\bullet(r)_{i(r)} := p$ and **4** define $p_\bullet(r) : Z_\bullet(r) \to Y_\bullet|_{J(r)}$. and **1**–**4** are satisfied by the induction assumption. So this case is OK.

Now assume that $i(r)$ is not minimal so that $\bigcup_{j<r} I(i(j), i(r)) \neq \emptyset$. For simplicity, set $Y_j = Y_{i(j)}$, $X_j = X_{i(j)}$ for $1 \leq j \leq n$ and $Z_j = Z_\bullet(j)_j$ for $j < r$. Similarly, $f_j := f_{i(j)}$ $(1 \leq j \leq n)$, $j_j := j_\bullet(j)_{i(j)}$, and $p_j := p_\bullet(j)_{i(j)}$ $(1 \leq j < r)$.

Consider the direct product of Y_r-schemes

$$W := \prod_{j<r} \prod_{\phi \in I(i(j), i(r))} Y_r \,{}_{Y_\phi}\!\times_{Y_j} Z_j.$$

Note that each $Y_r \,{}_{Y_\phi}\!\times_{Y_j} Z_j$ is proper over Y_r, and hence W is proper over Y_r. There is a unique Y_r-morphism $h : X_r \to W$ induced by

$$(f_r, j_j \circ X_\phi) : X_r \to Y_r \,{}_{Y_\phi}\!\times_{Y_j} Z_j.$$

Since h is separated of finite type, there is a factorization

$$X_r \xrightarrow{k} Z \xrightarrow{p} W$$

such that k is an open immersion whose scheme theoretic image is Z, p is proper, and $pk = h$.

Now define $Z_{i(r)} = Z$, $Z_{\mathrm{id}_{i(r)}} = \mathrm{id}_Z$, and Z_ϕ to be the composite

$$Z \xrightarrow{p} W \xrightarrow{\text{projection}} Z_j$$

for $j < r$ and $\phi \in I(i(j), i(r))$. We define $Z_\bullet(r)$ by these data and by **4**. Set $j_r = j_\bullet(r)_r = k$, and $p_r = p_\bullet(r)_r$ to be the composite

$$Z \xrightarrow{p} W \to Y_r,$$

where the second map is the structure map of W as a Y_r-scheme.

Note that $Z_\psi j_j = j_{j'} X_\psi$ and $Y_\psi p_j = p_{j'} Z_\psi$ hold for $1 \le j' \le j \le r$ and $\psi \in I(i(j'), i(j))$. Indeed, this is trivial by induction assumption if $j < r$, also trivial if $\psi = \mathrm{id}_{i(r)}$, and follows easily from the construction if $j' < j = r$. In particular, $j_\bullet(r)$ and $p_\bullet(r)$ are defined so that **4** is satisfied and are morphisms of diagrams of schemes provided that $Z_\bullet(r)$ is a diagram of schemes.

We need to check that $Z_\bullet(r)$ is certainly a diagram of schemes. To verify this, it suffices to show that, for any $j' < j < r$ and any $\phi \in I(i(j), i(r))$ and $\psi \in I(i(j'), i(j))$, $Z_{\phi\psi} = Z_\psi Z_\phi$ holds. Let A be the locus in Z such that $Z_{\phi\psi}$ and $Z_\psi Z_\phi$ agree. Note that the diagrams

$$\begin{tikzcd} X_r \arrow[r, "X_\phi"] \arrow[d, "j_r"] \arrow[dr, phantom, "\text{(a)}"] & X_j \arrow[r, "X_\psi"] \arrow[d, "j_j"] \arrow[dr, phantom, "\text{(b)}"] & X_{j'} \arrow[d, "j_{j'}"] \\ Z_r \arrow[r, "Z_\phi"] \arrow[d, "p_r"] \arrow[dr, phantom, "\text{(c)}"] & Z_j \arrow[r, "Z_\psi"] \arrow[d, "p_j"] \arrow[dr, phantom, "\text{(d)}"] & Z_{j'} \arrow[d, "p_{j'}"] \\ Y_r \arrow[r, "Y_\phi"] & Y_j \arrow[r, "Y_\psi"] & Y_{j'} \end{tikzcd} \qquad \begin{tikzcd} X_r \arrow[r, "X_{\phi\psi}"] \arrow[d, "j_r"] \arrow[dr, phantom, "\text{(e)}"] & X_{j'} \arrow[d, "j_{j'}"] \\ Z_r \arrow[r, "Z_{\phi\psi}"] \arrow[d, "p_r"] \arrow[dr, phantom, "\text{(f)}"] & Z_{j'} \arrow[d, "p_{j'}"] \\ Y_r \arrow[r, "Y_{\phi\psi}"] & Y_{j'} \end{tikzcd}$$

are commutative. Since (c), (d) and (f) are commutative and $Y_{\phi\psi} = Y_\psi Y_\phi$, we have that $p_{j'} Z_{\phi\psi} = p_{j'} Z_\psi Z_\phi$. Since there is a cartesian square

$$\begin{array}{ccc} A & \longrightarrow & Z_{j'} \\ \downarrow & & \downarrow \Delta \\ Z_r & \xrightarrow{(Z_{\phi\psi}, Z_\psi Z_\phi)} & Z_{j'} \times_{Y_{j'}} Z_{j'} \end{array}$$

and $p_{j'} : Z_{j'} \to Y_{j'}$ is separated, we have that A is a closed subscheme of Z_r. Since the scheme theoretic image of the open immersion $j_r : X_r \hookrightarrow Z_r$ is Z_r, it suffices to show that j_r factors through A. That is, it suffices to show that

$j_r Z_{\phi\psi} = j_r Z_\psi Z_\phi$. But this is trivial by the commutativity of (a), (b), and (e), and the fact $X_{\phi\psi} = X_\psi X_\phi$. So $Z_\bullet(r)$ is a diagram of schemes, and $j_\bullet(r)$ and $p_\bullet(r)$ are morphisms of diagrams of schemes.

The conditions **1–4** are now easy to verify, and the proof is complete. $\square$

20.4 Theorem. *Let the notation be as in* (20.1). *Set $\mathcal{F}$ to be the subcategory of $\mathcal{A}$ whose objects are objects of $\mathcal{A}$ with flat arrows, and whose morphisms are cartesian morphisms in $\mathcal{A}$. Define $\mathcal{D}^+$ by $\mathcal{D}^+(X_\bullet) := D^+_{\mathrm{Lqc}}(X_\bullet)$. Then $(\mathcal{A}, \mathcal{F}, \mathcal{P}, \mathcal{I}, \mathcal{D}, \mathcal{D}^+, (?)^\#, (?)^\flat, \zeta)$ is a composition data of contravariant almost-pseudofunctors.*

Proof. Conditions **1–14** in Definition 16.1 have already been checked. **15** follows from Lemma 7.16. **16** is trivial. Since I is finite, the definition of $\mathcal{D}^+(X_\bullet)$ is consistent with that in (18.12). Hence **17** is Corollary 18.15. **18** is Theorem 19.4. $\square$

(20.5) We call the composite of $(?)^\#$ and $(?)^\flat$ defined by the composition data in the theorem the *equivariant twisted inverse* almost-pseudofunctor, and denote it by $(?)^!$.

21 Flat base change

Let the notation be as in Theorem 20.4. Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{F}$, and J a subcategory of I. Let $f_\bullet = p_\bullet i_\bullet$ be a compactification.

21.1 Lemma. *The composite map*

$$(?)_J f_\bullet^! \xrightarrow{\text{via } \Upsilon} (?)_J i_\bullet^* p_\bullet^\times \xrightarrow{\theta^{-1}} i_J^* (?)_J p_\bullet^\times \xrightarrow{\xi} i_J^* p_J^\times (?)_J \xrightarrow{\text{via } \Upsilon} f_\bullet|_J^! (?)_J$$

is independent of choice of compactification $f_\bullet = p_\bullet i_\bullet$, where Υ's are the independence isomorphisms.

The proof utilizes Lemma 16.3, and left to the reader. We denote by $\bar{\xi} = \bar{\xi}(J, f_\bullet)$ the composite map in the lemma.

21.2 Lemma. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{F}$, and $K \subset J \subset I$ be subcategories. Then the composite map*

$$(?)_K f_\bullet^! \cong (?)_K (?)_J f_\bullet^! \xrightarrow{\bar{\xi}} (?)_K f_\bullet|_J^! (?)_J \xrightarrow{\bar{\xi}} f_\bullet|_K^! (?)_K (?)_J \cong f_K^! (?)_K$$

agrees with $\bar{\xi}(K, f_\bullet)$.

Proof. Follows easily from Lemma 18.4. □

21.3 Lemma. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{F}$, and J a subcategory of I. Then $\bar{\xi}(J, f_\bullet)$ is an isomorphism.*

Proof. It suffices to show that $(?)_i\bar{\xi}(J, f_\bullet)$ is an isomorphism for any $i \in \mathrm{ob}(J)$. By Lemma 21.2, we have

$$\bar{\xi}(i, f_\bullet|_J) \circ ((?)_i\bar{\xi}(J, f_\bullet)) = \bar{\xi}(i, f_\bullet).$$

By Proposition 18.14, we have $\bar{\xi}(i, f_\bullet|_J)$ and $\bar{\xi}(i, f_\bullet)$ are isomorphisms. Hence the natural map $(?)_i\bar{\xi}(J, f_\bullet)$ is also an isomorphism. □

21.4 Lemma. *Let $f : X \to Y$ be a flat morphism of locally noetherian schemes, and U a dense open subset of Y. Then $f^{-1}(U)$ is a dense open subset of X.*

Proof. The question is local both on Y and X, and hence we may assume that both $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$ are affine. Let I be the radical ideal of A defining the closed subset $Y \setminus U$. By assumption, I is not contained in any minimal prime of A. Assume that $f^{-1}(U)$ is not dense in X. Then, there is a minimal prime P of B which contains IB. As we have $I \subset IB \cap A \subset P \cap A$ and $P \cap A$ is minimal by the going-down theorem (see [30, Theorem 9.5]), this is a contradiction. □

(21.5) Let the notation be as in Theorem 20.4. Let

$$\begin{array}{ccc} X'_\bullet & \xrightarrow{f'_\bullet} & Y'_\bullet \\ \downarrow g^X_\bullet & \sigma & \downarrow g_\bullet \\ X_\bullet & \xrightarrow{f_\bullet} & Y_\bullet \end{array}$$

be a diagram in $\mathcal{P}(I, \underline{\mathrm{Sch}})$ such that

1 All objects lie in $\mathcal{F}$;

2 $f_\bullet$ and $f'_\bullet$ are morphisms in $\mathcal{F}$;

3 σ is a fiber square;

4 $g_\bullet$ is flat (not necessarily a morphism of $\mathcal{A}$).

By assumption, there is a diagram

$$\begin{array}{ccccc} X'_\bullet & \xrightarrow{i'_\bullet} & Z'_\bullet & \xrightarrow{p'_\bullet} & Y'_\bullet \\ \downarrow g^X_\bullet & \sigma_1 & \downarrow g^Z_\bullet & \sigma_2 & \downarrow g_\bullet \\ X_\bullet & \xrightarrow{i_\bullet} & Z_\bullet & \xrightarrow{p_\bullet} & Y_\bullet \end{array} \tag{21.6}$$

such that $f_\bullet = p_\bullet i_\bullet$ is a compactification, σ_1 and σ_2 are fiber squares, and the whole rectangle $\sigma_1\sigma_2$ equals σ. By Lemma 21.4, we have that $f'_\bullet = p'_\bullet i'_\bullet$ is a compactification.

21.7 Lemma. *The composite map*

$$(g^X_\bullet)^* f^!_\bullet \xrightarrow{\Upsilon} (g^X_\bullet)^* i^*_\bullet p^\times_\bullet \xrightarrow{d} (i'_\bullet)^*(g^Z_\bullet)^* p^\times_\bullet \xrightarrow{\zeta} (i'_\bullet)^*(p'_\bullet)^\times g^*_\bullet \xrightarrow{\Upsilon} (f'_\bullet)^! g^*_\bullet$$

is independent of choice of the diagram (21.6)*, and depends only on σ, where Υ's are independence isomorphisms.*

Proof. Obvious by Lemma 16.3. □

We denote the composite map in the lemma by $\bar{\zeta} = \bar{\zeta}(\sigma)$.

21.8 Theorem. *Let the notation be as above. Then we have:*

1 *Let J be a subcategory of I. Then the diagram*

$$\begin{array}{ccccc} (?)_J(g^X_\bullet)^* f^!_\bullet & \xrightarrow{\theta^{-1}} & (g^X_J)^*(?)_J f^!_\bullet & \xrightarrow{\bar{\xi}} & (g^X_J)^* f^!_J (?)_J \\ \downarrow (?)_J\bar{\zeta}(\sigma) & & & & \downarrow \bar{\zeta}(\sigma_J)(?)_J \\ (?)_J(f'_\bullet)^!(g_\bullet)^* & \xrightarrow{\bar{\xi}} & (f'_J)^!(?)_J(g_\bullet)^* & \xrightarrow{\theta^{-1}} & (f'_J)^!(g_J)^*(?)_J \end{array} \tag{21.9}$$

is commutative.

2 $\bar{\zeta}(\sigma)$ *is an isomorphism.*

Proof. **1** is an immediate consequence of Lemma 19.2 and [26, (3.7.2)].

2 Let i be an object of I. By Lemma 21.3, the horizontal arrows in the diagram (21.9) for $J = i$ are isomorphisms. By Verdier's flat base change theorem [41, Theorem 2], we have that $\bar{\zeta}(\sigma_i)$ is an isomorphism. Hence, we have that $(?)_i\bar{\zeta}(\sigma)$ is an isomorphism for any $i \in I$ by **1** applied to $J = i$, and the assertion follows. □

22 Preservation of Quasi-coherent cohomology

(22.1) Let the notation be as in (20.1). Let $\mathcal{F}$ be as in Theorem 20.4.

22.2 Lemma. *Let $f : X \to Y$ be a separated morphism of finite type between noetherian schemes. If $\mathbb{F} \in \mathcal{D}^+_{\mathrm{Coh}(Y)}(\mathrm{Mod}(Y))$, then $f^!\mathbb{F} \in \mathcal{D}^+_{\mathrm{Coh}(X)}(\mathrm{Mod}(X))$.*

Proof. We may assume that both Y and X are affine. So we may assume that f is either smooth or a closed immersion. The case where f is smooth is obvious by [41, Theorem 3]. The case where f is a closed immersion is also obvious by Proposition III.6.1 and Theorem III.6.7 in [17]. $\square$

22.3 Proposition. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{F}$, and $\phi : i \to j$ a morphism in I. Then the composite map*

$$X_\phi^*(?)_i f_\bullet^! \xrightarrow{\alpha_\phi} (?)_j f_\bullet^! \xrightarrow{\bar{\xi}} f_j^!(?)_j$$

agrees with the composite map

$$X_\phi^*(?)_i f_\bullet^! \xrightarrow{\bar{\xi}} X_\phi^* f_i^!(?)_i \xrightarrow{\bar{\zeta}} f_j^! Y_\phi^*(?)_i \xrightarrow{\alpha_\phi} f_j^!(?)_j.$$

Proof. By Lemma 21.2, we may assume that I is the ordered category given by $\mathrm{ob}(I) = \{i, j\}$ and $I(i,j) = \{\phi\}$.

Then it is easy to see that there is a compactification

$$X_\bullet \xrightarrow{i_\bullet} Z_\bullet \xrightarrow{p_\bullet} Y_\bullet$$

of $f_\bullet$ such that $p_\bullet$ is cartesian. Note that $i_\bullet$ is cartesian, and $Z_\bullet$ has flat arrows.

By the definition of $\bar{\xi}$ and $\bar{\zeta}$, it suffices to prove that the composite map

$$X_\phi^*(?)_i i_\bullet^* p_\bullet^\times \xrightarrow{\theta^{-1}} X_\phi^* i_i^*(?)_i p_\bullet^\times \xrightarrow{d} i_j^* Z_\phi^*(?)_i p_\bullet^\times \xrightarrow{\xi} i_j^* Z_\phi^* p_i^\times(?)_i \xrightarrow{\zeta} i_j^* p_j^\times Y_\phi^*(?)_i \xrightarrow{\alpha_\phi} i_j^* p_j^\times(?)_j$$

agrees with

$$X_\phi^*(?)_i i_\bullet^* p_\bullet^\times \xrightarrow{\alpha_\phi} (?)_j i_\bullet^* p_\bullet^\times \xrightarrow{\theta^{-1}} i_j^*(?)_j p_\bullet^\times \xrightarrow{\xi} i_j^* p_j^\times(?)_j.$$

By the "derived version" of (6.31), the composite map

$$X_\phi^*(?)_i i_\bullet^* p_\bullet^\times \xrightarrow{\theta^{-1}} X_\phi^* i_i^*(?)_i p_\bullet^\times \xrightarrow{d} i_j^* Y_\phi^*(?)_i p_\bullet^\times \xrightarrow{\alpha_\phi} i_j^*(?)_j p_\bullet^\times$$

agrees with

$$X_\phi^*(?)_i i_\bullet^* p_\bullet^\times \xrightarrow{\alpha_\phi} (?)_j i_\bullet^* p_\bullet^\times \xrightarrow{\theta^{-1}} i_j^*(?)_j p_\bullet^\times.$$

Hence it suffices to prove the map

$$Z_\phi^*(?)_i p_\bullet^\times \xrightarrow{\xi} Z_\phi^* p_i^\times (?)_i \xrightarrow{\zeta} p_j^\times Y_\phi^*(?)_i \xrightarrow{\alpha_\phi} p_j^\times (?)_j$$

agrees with

$$Z_\phi^*(?)_i p_\bullet^\times \xrightarrow{\alpha_\phi} (?)_j p_\bullet^\times \xrightarrow{\xi} p_j^\times (?)_j.$$

Now the proof consists in a straightforward diagram drawing utilizing Lemma 19.3 and the derived version of Lemma 6.20. □

22.4 Corollary. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{F}$. Then we have $f_\bullet^!(\mathcal{D}^+_{\mathrm{Qch}}(Y_\bullet)) \subset \mathcal{D}^+_{\mathrm{Qch}}(X_\bullet)$ and $f_\bullet^!(\mathcal{D}^+_{\mathrm{Coh}}(Y_\bullet)) \subset \mathcal{D}^+_{\mathrm{Coh}}(X_\bullet)$.*

Proof. Let $\phi : i \to j$ be a morphism in I. By the flat base change theorem, Lemma 21.3, and the proposition, we have $\alpha_\phi : X_\phi^*(?)_i f_\bullet^! \to (?)_j f_\bullet^!$ is an isomorphism if $\alpha_\phi : Y_\phi^*(?)_i \to (?)_j$ is an isomorphism. So $f_\bullet^!$ preserves equivariance of cohomology groups, and the first assertion follows.

On the other hand, by Lemma 22.2 and Proposition 18.14, $f^!$ preserves local coherence of cohomology groups. Hence it also preserves the coherence of cohomology groups, by the first paragraph. □

23 Compatibility with derived direct images

(23.1) Let the notation be as in (21.5). Consider the diagram (21.6). Lipman's theta $\theta(\sigma_2) : g_\bullet^* R(p_\bullet)_* \to R(p'_\bullet)_*(g_\bullet^Z)^*$ induces the conjugate map

$$\xi(\sigma_2) : R(g_\bullet^Z)_*(p'_\bullet)^\times \to p_\bullet^\times R(g_\bullet)_*.$$

As σ_2 is a fiber square, $\theta(\sigma_2)$ is an isomorphism. Hence $\xi(\sigma_2)$ is also an isomorphism. Note that

$$\theta : i_\bullet^* R(g_\bullet^Z)_* \to R(g_\bullet^X)_*(i'_\bullet)^*$$

is an isomorphism, since σ_1 is a fiber square. We define $\bar\xi : R(g_\bullet^X)_*(f'_\bullet)^! \to f_\bullet^! R(g_\bullet)_*$ to be the composite

$$R(g_\bullet^X)_*(f'_\bullet)^! \xrightarrow{\Upsilon} R(g_\bullet^X)_*(i'_\bullet)^*(p'_\bullet)^\times \xrightarrow{\theta^{-1}} i_\bullet^* R(g_\bullet^Z)_*(p'_\bullet)^\times \xrightarrow{\xi} i_\bullet^* p_\bullet^\times R(g_\bullet)_*. \xrightarrow{\Upsilon} f_\bullet^! R(g_\bullet)_*.$$

As the all maps in the composition are isomorphisms, we have

23.2 Lemma. *$\bar{\xi}$ is an isomorphism.*

23.3 Lemma. *For any subcategory J, the composite*

$$(?)_J R(g^Z_\bullet)_*(p'_\bullet)^\times \xrightarrow{\xi} (?)_J p^\times_\bullet R(g_\bullet)_* \xrightarrow{\xi} (p_\bullet|_J)^\times (?)_J R(g_\bullet)_* \xrightarrow{c} (p_\bullet|_J)^\times R(g_\bullet|_J)_*(?)_J$$

agrees with the composite

$$(?)_J R(g^Z_\bullet)_*(p'_\bullet)^\times \xrightarrow{c} R(g^Z_\bullet|_J)_*(?)_J(p'_\bullet)^\times \xrightarrow{\xi}$$
$$R(g^Z_\bullet|_J)_*(p'_\bullet|_J)^\times(?)_J \xrightarrow{\xi} (p_\bullet|_J)^\times R(g_\bullet|_J)_*(?)_J.$$

Proof. Follows from Lemma 18.7. □

24 Compatibility with derived right inductions

(24.1) Let I be a finite category, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}})$. Assume that $X_\bullet$ has concentrated arrows. Let J be a subcategory of I. For $i \in I$, $I_i^{(J\to I)}$ is finite, since I is finite. So for any $\mathcal{M} \in \mathrm{Lqc}(X_\bullet|_J)$, we have $R_J\mathcal{M} \in \mathrm{Lqc}(X_\bullet)$ by (6.14). So $R_J : \mathrm{Lqc}(X_\bullet|_J) \to \mathrm{Lqc}(X_\bullet)$ is a right adjoint of $(?)_J : \mathrm{Lqc}(X_\bullet) \to \mathrm{Lqc}(X_\bullet|_J)$.

24.2 Lemma. *Let I be a finite category, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}})$. Assume that $X_\bullet$ is noetherian. If $\mathcal{I} \in \mathrm{Lqc}(X_\bullet)$ is an injective object, then it is injective as an object of* $\mathrm{Mod}(X_\bullet)$.

Proof. Let J be the discrete subcategory of I such that $\mathrm{ob}(J) = \mathrm{ob}(I)$. Let $\mathcal{I}_J \hookrightarrow \mathcal{J}$ be the injective hull in $\mathrm{Lqc}(X_\bullet|_J)$. Since $(?)_J$ is faithful, the composite $\mathcal{I} \to R_J(?)_J\mathcal{I} \hookrightarrow R_J\mathcal{J}$ is a monomorphism, and hence it splits. For each $i \in I$, $\mathcal{J}_i$ is injective as an object of $\mathrm{Qch}(X_i)$, since $\mathrm{Lqc}(X_\bullet|_J) \cong \prod_i \mathrm{Qch}(X_i)$ in a natural way. So it is also injective as an object of $\mathrm{Mod}(X_i)$ by [17, (II.7)]. So $\mathcal{J}$ is injective as an object of $\mathrm{Mod}(X_\bullet|_J)$. Since R_J preserves injectives, $R_J\mathcal{J}$ is an injective object of $\mathrm{Mod}(X_\bullet)$. Hence its direct summand $\mathcal{I}$ is also injective in $\mathrm{Mod}(X_\bullet)$. □

24.3 Corollary. *Let I and $X_\bullet$ be as in the lemma. Then for any subcategory $J \subset I$,*

$$RR_J : D(X_\bullet|_J) \to D(X_\bullet)$$

takes $D^+_{\mathrm{Lqc}}(X_\bullet|_J)$ *to* $D^+_{\mathrm{Lqc}}(X_\bullet)$*, and* RR_J *is right adjoint to* $(?)_J : D^+_{\mathrm{Lqc}}(X_\bullet) \to D^+_{\mathrm{Lqc}}(X_\bullet|_J)$*.*

Proof. By the way-out lemma, it suffices to prove that for a single object $\mathcal{M} \in \mathrm{Lqc}(X_\bullet|_J)$, $R^n R_J \mathcal{M} \in \mathrm{Lqc}(X_\bullet)$. Let $\mathcal{M} \to \mathbb{I}$ be an injective resolution in the category $\mathrm{Lqc}(X_\bullet|_J)$, which exists by Lemma 11.9. Then $\mathbb{I}$ is also an injective resolution in $\mathrm{Mod}(X_\bullet|_J)$ by the lemma. So $R^n R_J \mathcal{M} \cong H^n(R_J \mathbb{I})$ lies in $\mathrm{Lqc}(X_\bullet)$. $\square$

(24.4) Let the notation be as in Theorem 20.4. Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{A}$. Let J be a subcategory of I. As

$$c : (?)_J R(f_\bullet)_* \to R(f_\bullet|_J)_*(?)_J$$

is an isomorphism of functors from $D^+_{\mathrm{Lqc}}(X_\bullet)$ to $D^+_{\mathrm{Lqc}}(X_\bullet|_J)$, its conjugate map

$$c' : RR_J(f_\bullet|_J)^\times \to f_\bullet^\times RR_J$$

is also an isomorphism.

Let $g_\bullet : U_\bullet \to X_\bullet$ be a cartesian image dense open immersion in $\mathcal{A}$. Let $\mu = \mu(g_\bullet, J)$ be the canonical map

$$g_\bullet^* RR_J \xrightarrow{u} g_\bullet^* RR_J R(g_\bullet|_J)_*(g_\bullet|_J)^* \xrightarrow{\xi^{-1}} g_\bullet^* R(g_\bullet)_* RR_J(g_\bullet|_J)^* \xrightarrow{\varepsilon} RR_J(g_\bullet|_J)^*,$$

where $\xi : R(g_\bullet)_* RR_J \to RR_J R(g_\bullet|_J)_*$ is the conjugate of the isomorphism $\theta : (g_\bullet)|_J^*(?)_J \to (?)_J g_\bullet^*$.

24.5 Lemma. *Let the notation be as above. Then* $\mu : g_\bullet^* RR_J \to RR_J(g_\bullet|_J)^*$ *is an isomorphism of functors from* $D(X_\bullet|_J)$ *to* $D(U_\bullet)$*.*

Proof. As $g_\bullet^*$, $(g_\bullet|_J)^*$, R_J, $(g_\bullet)_*$, and $(g_\bullet|_J)_*$ have exact left adjoints, it suffices to show that

$$(?)_i\mu : (?)_i g_\bullet^* R_J \mathbb{I} \to (?)_i R_J (g_\bullet|_J)^* \mathbb{I}$$

is an isomorphism for any K-injective complex in $C(\mathrm{Mod}(X_\bullet|_J))$ and $i \in \mathrm{ob}(I)$. This map agrees with

$$\begin{aligned}(?)_i g_\bullet^* R_J \mathbb{I} &\xrightarrow{\theta^{-1}} g_i^*(?)_i R_J \mathbb{I} \cong g_i^* \varprojlim (X_\phi)_* \mathbb{I}_j \cong \varprojlim g_i^*(X_\phi)_* \mathbb{I}_j \\ &\xrightarrow{\theta} \varprojlim (U_\phi)_* g_j^* \mathbb{I}_j \xrightarrow{\theta} \varprojlim (U_\phi)_*(?)_j (g_\bullet|_J)^* \mathbb{I} \cong (?)_i R_J (g_\bullet|_J)^* \mathbb{I},\end{aligned}$$

which is obviously an isomorphism, where the limit is taken over $\phi \in I_i^{(J \to I)}$. $\square$

(24.6) Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{F}$, and $f_\bullet = p_\bullet i_\bullet$ a compactification. We define $\bar{c} : f_\bullet^! RR_J \to RR_J(f_\bullet|_J)^!$ to be the composite

$$f_\bullet^! RR_J \xrightarrow{\Upsilon} i_\bullet^* p_\bullet^\times RR_J \xrightarrow{c'} i_\bullet^* RR_J (p_\bullet|_J)^\times \xrightarrow{\mu} RR_J (i_\bullet|_J)^* (p_\bullet|_J)^\times \xrightarrow{\Upsilon} RR_J(f_\bullet|_J)^!.$$

By Lemma 24.5, we have

24.7 Lemma. *$\bar{c} : f_\bullet^! RR_J \to RR_J(f_\bullet|_J)^!$ is an isomorphism of functors from $D^+_{\mathrm{Lqc}}(Y_\bullet|_J)$ to $D^+_{\mathrm{Lqc}}(X_\bullet)$.*

25 Equivariant Grothendieck's duality

25.1 Theorem (Grothendieck's duality). *Let $f : X \to Y$ be a proper morphism of noetherian schemes. For $\mathbb{F} \in D_{\mathrm{Qch}}(X)$ and $\mathbb{G} \in D^+_{\mathrm{Qch}}(Y)$, The canonical map*

$$\Theta(f) : Rf_* R\, \underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_X}(\mathbb{F}, f^\times \mathbb{G}) \xrightarrow{H} R\, \underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_Y}(Rf_*\mathbb{F}, Rf_* f^\times \mathbb{G}) \xrightarrow{\varepsilon} R\, \underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_Y}(Rf_*\mathbb{F}, \mathbb{G})$$

is an isomorphism.

Proof. As pointed out in [35, section 6], this is an immediate consequence of the open immersion base change [41, Theorem 2]. □

25.2 Theorem (Equivariant Grothendieck's duality). *Let I be a small category, and $f_\bullet : X_\bullet \to Y_\bullet$ a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}}/\mathbb{Z})$. If $Y_\bullet$ is noetherian with flat arrows and $f_\bullet$ is proper cartesian, then the composite*

$$\Theta(f_\bullet) : R(f_\bullet)_* R\, \underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(X_\bullet)}(\mathbb{F}, f_\bullet^\times \mathbb{G}) \xrightarrow{H} R\, \underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(Y_\bullet)}(R(f_\bullet)_*\mathbb{F}, R(f_\bullet)_* f_\bullet^\times \mathbb{G}) \xrightarrow{\varepsilon} R\, \underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(Y_\bullet)}(R(f_\bullet)_*\mathbb{F}, \mathbb{G})$$

is an isomorphism for $\mathbb{F} \in D_{\mathrm{Qch}}(X_\bullet)$ and $\mathbb{G} \in \mathcal{D}^+(Y_\bullet)$.

Proof. It suffices to show that $(?)_i \Theta(f_\bullet)$ is an isomorphism for $i \in \mathrm{ob}(I)$. By Lemma 1.39 and Lemma 18.7, **2**, it is easy to see that the composite

$$(?)_i R(f_\bullet)_* R\, \underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(X_\bullet)}(\mathbb{F}, f_\bullet^\times \mathbb{G}) \xrightarrow{c} R(f_i)_* (?)_i R\, \underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(X_\bullet)}(\mathbb{F}, f_\bullet^\times \mathbb{G}) \xrightarrow{H_i} R(f_i)_* R\, \underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(X_i)}(\mathbb{F}_i, (?)_i f_\bullet^\times \mathbb{G}) \xrightarrow{\xi} R(f_i)_* R\, \underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(X_i)}(\mathbb{F}_i, f_i^\times \mathbb{G}_i) \xrightarrow{\Theta(f_i)} R\, \underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(Y_i)}(R(f_i)_*\mathbb{F}_i, \mathbb{G}_i)$$

agrees with the composite

$$
\begin{aligned}
(?)_i R(f_\bullet)_* R\,\underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(X_\bullet)}(\mathbb{F}, f_\bullet^\times \mathbb{G}) &\xrightarrow{(?)_i\Theta(f_\bullet)} (?)_i R\,\underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(Y_\bullet)}(R(f_\bullet)_*\mathbb{F}, \mathbb{G}) \\
&\xrightarrow{H_i} R\,\underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(Y_i)}((?)_i R(f_\bullet)_*\mathbb{F}, \mathbb{G}_i) \xrightarrow{c} R\,\underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(Y_i)}(R(f_i)_*\mathbb{F}_i, \mathbb{G}_i).
\end{aligned}
$$

Consider the first composite map. By Lemma 13.9, H_i is an isomorphism. By Lemma 18.5, ξ is an isomorphism. By Theorem 25.1, $\Theta(f_i)$ is an isomorphism. Hence the first composite is an isomorphism, and so is the second.

Consider the second composite map. By Lemma 8.7, $R(f_\bullet)_*\mathbb{F} \in D_{\mathrm{Qch}}(Y_\bullet)$. So the second map H_i is an isomorphism by Lemma 13.9. So the first map $(?)_i\Theta(f_\bullet)$ must be an isomorphism. This is what we wanted to prove. $\square$

26 Morphisms of finite flat dimension

(26.1) Let $((?)^*, (?)_*)$ be a monoidal adjoint pair of almost-pseudofunctors over a category $\mathcal{S}$. For a morphism $f : X \to Y$ in $\mathcal{S}$, we define the *projection morphism* $\Pi = \Pi(f)$ to be the composite

$$
f_*a \otimes b \xrightarrow{u} f_*f^*(f_*a \otimes b) \xrightarrow{\Delta} f_*(f^*f_*a \otimes f^*b) \xrightarrow{\varepsilon} f_*(a \otimes f^*b),
$$

where $a \in X_*$ and $b \in Y_*$ (see Chapter 1 for the notation).

26.2 Lemma. *Let the notation be as above, and $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathcal{S}$. For $x \in X_*$ and $z \in Z_*$, the composite*

$$
\begin{aligned}
(gf)_*x \otimes z \xrightarrow{c} g_*(f_*x) \otimes z &\xrightarrow{\Pi(g)} g_*(f_*x \otimes g^*z) \xrightarrow{\Pi(f)} g_*f_*(x \otimes f^*g^*z) \\
&\xrightarrow{c^{-1}} (gf)_*(x \otimes f^*g^*z) \xrightarrow{d} (gf)_*(x \otimes (gf)^*z)
\end{aligned}
$$

agrees with $\Pi(gf)$.

Proof. Left to the reader. $\square$

(26.3) Let I be a small category, S a scheme, and $f_\bullet : X_\bullet \to Y_\bullet$ a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$.

26.4 Lemma (Projection Formula). *Assume that $f_\bullet$ is concentrated. Then the natural map*

$$\Pi = \Pi(f_\bullet) : (Rf_\bullet)_*\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{G} \to (Rf_\bullet)_*(\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} Lf_\bullet^*\mathbb{G})$$

is an isomorphism for $\mathbb{F} \in D_{\mathrm{Lqc}}(X_\bullet)$ and $\mathbb{G} \in D_{\mathrm{Lqc}}(Y_\bullet)$.

Proof. For each $i \in \mathrm{ob}(I)$, the composite

$$\begin{aligned} R(f_i)_*\mathbb{F}_i \otimes^{\bullet,L}_{\mathcal{O}_{Y_i}} \mathbb{G}_i \xrightarrow{\Pi(f_i)} R(f_i)_*(\mathbb{F}_i \otimes^{\bullet,L}_{\mathcal{O}_{X_i}} Lf_i^*\mathbb{G}_i) \xrightarrow{\theta(f_\bullet,i)} R(f_i)_*(\mathbb{F}_i \otimes^{\bullet,L}_{\mathcal{O}_{X_i}} (?)_i Lf_\bullet^*\mathbb{G}) \\ \xrightarrow{m} R(f_i)_*(?)_i(\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} Lf_\bullet^*\mathbb{G}) \xrightarrow{c} (?)_i R(f_\bullet)_*(\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} Lf_\bullet^*\mathbb{G}) \end{aligned}$$

is an isomorphism by [26, (3.9.4)] and Lemma 8.13, **1**. On the other hand, it is straightforward to check that this composite isomorphism agrees with the composite

$$\begin{aligned} R(f_i)_*\mathbb{F}_i \otimes^{L}_{\mathcal{O}_{Y_i}} \mathbb{G}_i \xrightarrow{c^{-1}} (?)_i R(f_\bullet)_*\mathbb{F} \otimes^{L}_{\mathcal{O}_{Y_i}} \mathbb{G}_i \xrightarrow{m} (?)_i (R(f_\bullet)_*\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{G}) \\ \xrightarrow{(?)_i\Pi(f_\bullet)} (?)_i R(f_\bullet)_*(\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} Lf_\bullet^*\mathbb{G}). \end{aligned}$$

It follows that $(?)_i\Pi(f_\bullet)$ is an isomorphism for any $i \in \mathrm{ob}(I)$. Hence, $\Pi(f_\bullet)$ is an isomorphism. $\square$

(26.5) Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$, and assume that both $X_\bullet$ and $f_\bullet$ are concentrated. Define $\chi = \chi(f_\bullet)$ to be the composite

$$\begin{aligned} f_\bullet^\times\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} Lf_\bullet^*\mathbb{G} \xrightarrow{u} f_\bullet^\times R(f_\bullet)_*(f_\bullet^\times\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} Lf_\bullet^*\mathbb{G}) \\ \xrightarrow{\Pi(f_\bullet)^{-1}} f_\bullet^\times(R(f_\bullet)_* f_\bullet^\times\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{G}) \xrightarrow{\varepsilon} f_\bullet^\times(\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{G}), \end{aligned}$$

where $\mathbb{F}, \mathbb{G} \in D_{\mathrm{Lqc}}(Y_\bullet)$.

Utilizing the commutativity as in the proof of Lemma 26.4 and Lemma 18.7, it is not so difficult to show the following.

26.6 Lemma. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be as in* (26.5)*. For a subcategory J of I, the composite*

$$\begin{aligned} (?)_J f_\bullet^\times\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet|_J}} (?)_J Lf_\bullet^*\mathbb{G} \xrightarrow{\xi\otimes\theta^{-1}} (f_\bullet|_J)^\times\mathbb{F}_J \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet|_J}} L(f_\bullet|_J)^*\mathbb{G}_J \\ \xrightarrow{\chi(f_\bullet|_J)} (f_\bullet|_J)^\times(\mathbb{F}_J \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet|_J}} \mathbb{G}_J) \xrightarrow{m} (f_\bullet|_J)^\times(?)_J(\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{G}) \end{aligned}$$

agrees with

$$(?)_J f_\bullet^\times \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet|_J}} (?)_J Lf_\bullet^* \mathbb{G} \xrightarrow{m} (?)_J (f_\bullet^\times \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} Lf_\bullet^* \mathbb{G})$$
$$\xrightarrow{\chi(f_\bullet)} (?)_J f_\bullet^\times (\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{G}) \xrightarrow{\xi} (f_\bullet|_J)^\times (?)_J (\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{G}).$$

26.7 Lemma. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be as in* (26.5)*. The composite*

$$f_\bullet^\times \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} Lf_\bullet^* (\mathbb{G} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{H}) \xrightarrow{\Delta} f_\bullet^\times \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} (Lf_\bullet^* \mathbb{G} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} Lf_\bullet^* \mathbb{H}) \xrightarrow{\alpha^{-1}}$$
$$(f_\bullet^\times \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} Lf_\bullet^* \mathbb{G}) \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} Lf_\bullet^* \mathbb{H} \xrightarrow{\chi} f_\bullet^\times (\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{G}) \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} Lf_\bullet^* \mathbb{H} \xrightarrow{\chi}$$
$$f_\bullet^\times ((\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{G}) \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{H}) \xrightarrow{\alpha} f_\bullet^\times (\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} (\mathbb{G} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{H}))$$

agrees with χ.

26.8 Lemma. *Let S, I and σ be as in* (19.1)*. For $\mathbb{F} \in D_{\mathrm{Lqc}}(Y_\bullet)$, the composite*

$$(g'_\bullet)^* f_\bullet^\times \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X'_\bullet}} (g'_\bullet)^* Lf_\bullet^* \mathbb{G} \xrightarrow{\Delta^{-1}} (g'_\bullet)^* (f_\bullet^\times \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} Lf_\bullet^* \mathbb{G})$$
$$\xrightarrow{\chi} (g'_\bullet)^* f_\bullet^\times (\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{G}) \xrightarrow{\zeta(\sigma)} (f'_\bullet)^\times g_\bullet^* (\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{G})$$

agrees with

$$(g'_\bullet)^* f_\bullet^\times \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X'_\bullet}} (g'_\bullet)^* Lf_\bullet^* \mathbb{G} \xrightarrow{\zeta(\sigma)\otimes d} (f'_\bullet)^\times g_\bullet^* \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X'_\bullet}} L(f'_\bullet)^* g_\bullet^* \mathbb{G}$$
$$\xrightarrow{\chi} (f'_\bullet)^\times (g_\bullet^* \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y'_\bullet}} g_\bullet^* \mathbb{G}) \xrightarrow{\Delta^{-1}} (f'_\bullet)^\times g_\bullet^* (\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{G}).$$

26.9 Lemma. *Let $f_\bullet : X_\bullet \to Y_\bullet$ and $g_\bullet : Y_\bullet \to Z_\bullet$ be morphisms in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Assume that $X_\bullet$, $Y_\bullet$ and $g_\bullet$ are concentrated. Then the composite*

$$(g_\bullet f_\bullet)^\times \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} (g_\bullet f_\bullet)^* \mathbb{G} \cong f_\bullet^\times g_\bullet^\times \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} f_\bullet^* g_\bullet^* \mathbb{G} \xrightarrow{\chi(f_\bullet)} f_\bullet^\times (g_\bullet^\times \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} g_\bullet^* \mathbb{G})$$
$$\xrightarrow{\chi(g_\bullet)} f_\bullet^\times g_\bullet^\times (\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Z_\bullet}} \mathbb{G}) \cong (g_\bullet f_\bullet)^\times (\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Z_\bullet}} \mathbb{G})$$

agrees with $\chi(g_\bullet f_\bullet)$.

The proof of the lemmas above are left to the reader.

(26.10) Let the notation be as in Theorem 20.4. Let $i_\bullet : U_\bullet \to X_\bullet$ be a morphism in $\mathcal{I}$, and $p_\bullet : X_\bullet \to Y_\bullet$ a morphism in $\mathcal{P}$.

We define $\bar{\chi} = \bar{\chi}(p_\bullet, i_\bullet)$ to be the composite

$$\begin{aligned}\bar{\chi} : i_\bullet^* p_\bullet^\times \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{U_\bullet}} L(p_\bullet i_\bullet)^* \mathbb{G} &\xrightarrow{d^{-1}} i_\bullet^* p_\bullet^\times \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{U_\bullet}} i_\bullet^* L p_\bullet^* \mathbb{G} \\ &\xrightarrow{\Delta^{-1}} i_\bullet^*(p_\bullet^\times \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} L p_\bullet^* \mathbb{G}) \xrightarrow{i_\bullet^* \chi} i_\bullet^* p_\bullet^\times (\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{G}).\end{aligned}$$

26.11 Lemma. *Let $f_\bullet$ be a morphism in $\mathcal{F}$, and $f_\bullet = p_\bullet i_\bullet = q_\bullet j_\bullet$ an independence square. Then the composite*

$$i_\bullet^* p_\bullet^\times \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{U_\bullet}} L(p_\bullet i_\bullet)^* \mathbb{G} \xrightarrow{\bar{\chi}} i_\bullet^* p_\bullet^\times (\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{G}) \xrightarrow{\Upsilon(p_\bullet i_\bullet = q_\bullet j_\bullet)} j_\bullet^* q_\bullet^\times (\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{G})$$

agrees with

$$i_\bullet^* p_\bullet^\times \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{U_\bullet}} L(p_\bullet i_\bullet)^* \mathbb{G} \xrightarrow{\Upsilon \otimes 1} j_\bullet^* q_\bullet^\times \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{U_\bullet}} L(q_\bullet j_\bullet)^* \mathbb{G} \xrightarrow{\bar{\chi}} j_\bullet^* q_\bullet^\times (\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} \mathbb{G}).$$

Proof. As Υ is constructed from ζ and d by definition, the assertion follows easily from Lemma 26.8 and Lemma 26.9. □

(26.12) Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{F}$. We define $\bar{\chi}(f_\bullet)$ to be $\bar{\chi}(p_\bullet, i_\bullet)$, where $f_\bullet = p_\bullet i_\bullet$ is the (fixed) compactification of $f_\bullet$.

By Lemma 26.11, $\bar{\chi}(f_\bullet)$ is an isomorphism if and only if there exists some compactification $f_\bullet = q_\bullet j_\bullet$ such that $\bar{\chi}(q_\bullet, j_\bullet)$ is an isomorphism.

26.13 Lemma. *Let the notation be as in* Theorem 20.4*, and $f_\bullet : X_\bullet \to Y_\bullet$ and $g_\bullet : Y_\bullet \to Z_\bullet$ morphisms in $\mathcal{F}$. Then the composite*

$$\begin{aligned}(g_\bullet f_\bullet)^! \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} L(g_\bullet f_\bullet)^* \mathbb{G} \cong f_\bullet^! g_\bullet^! \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} L f_\bullet^* L g_\bullet^* \mathbb{G} &\xrightarrow{\bar{\chi}(f_\bullet)} f_\bullet^! (g_\bullet^! \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} L g_\bullet^* \mathbb{G}) \\ &\xrightarrow{\bar{\chi}(g_\bullet)} f_\bullet^! g_\bullet^! (\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Z_\bullet}} \mathbb{G}) \cong (g_\bullet f_\bullet)^! (\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Z_\bullet}} \mathbb{G})\end{aligned}$$

agrees with $\bar{\chi}(g_\bullet f_\bullet)$.

26.14 Theorem. *Let the notation be as in* Theorem 20.4*, and $f_\bullet : X_\bullet \to Y_\bullet$ a morphism in $\mathcal{F}$. If $f_\bullet$ is of finite flat dimension, then*

$$\bar{\chi}(f_\bullet) : f_\bullet^! \mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{Y_\bullet}} L f_\bullet^* \mathbb{G} \to f_\bullet^! (\mathbb{F} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} \mathbb{G})$$

is an isomorphism for $\mathbb{F}, \mathbb{G} \in D^+_{\mathrm{Lqc}}(Y_\bullet)$.

Proof. Let $f_\bullet = p_\bullet i_\bullet$ be a compactification of $f_\bullet$. It suffices to show that $\bar{\chi}(p_\bullet, i_\bullet)$ is an isomorphism. In view of Lemma 26.7, we may assume that $\mathbb{F} = \mathcal{O}_{Y_\bullet}$. Then in view of Proposition 18.14 and Lemma 26.6, it suffices to show that

$$\bar{\chi}(f_j) : i_j^* p_j^\times \mathcal{O}_{Y_j} \otimes^{\bullet,L}_{\mathcal{O}_{X_j}} Lf_j^* \mathbb{G}_i \to i_j^* p_j^\times (\mathcal{O}_{Y_j} \otimes^{\bullet,L}_{\mathcal{O}_{Y_j}} \mathbb{G}_j)$$

is an isomorphism for any $j \in \mathrm{ob}(I)$. So we may assume that $I = j$.

By the flat base change theorem and Lemma 26.8, the question is local on Y_j. Clearly, the question is local on X_j. Hence we may assume that Y_j and X_j are affine. Set $f = f_j$, $Y = Y_j$, and $X = X_j$. Note that f is a closed immersion defined by an ideal of finite projective dimension, followed by an affine n-space.

By Lemma 26.13, it suffices to prove that $\bar{\chi}(f)$ is an isomorphism if f is a closed immersion defined by an ideal of finite projective dimension or an affine n-space. Both cases are proved easily, using [35, Theorem 5.4] (note that an affine n-space is an open subscheme of a projective n-space). □

27 Cartesian finite morphisms

(27.1) Let I be a small category, S a scheme, and $f_\bullet : X_\bullet \to Y_\bullet$ a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Let Z denote the ringed site $(\mathrm{Zar}(Y_\bullet), (f_\bullet)_*(\mathcal{O}_{X_\bullet}))$. Assume that $Y_\bullet$ is locally noetherian. There are obvious admissible ringed continuous functors $i : \mathrm{Zar}(Y_\bullet) \to Z$ and $g : Z \to \mathrm{Zar}(X_\bullet)$ such that $gi = f_\bullet^{-1}$. If $f_\bullet$ is affine, then $g_\# : \mathrm{Mod}(Z) \to \mathrm{Mod}(X_\bullet)$ is an exact functor, as can be seen easily.

27.2 Lemma. *If $f_\bullet$ is affine, then the counit*

$$\varepsilon : g_\# R g^\# \mathbb{F} \to \mathbb{F}$$

is an isomorphism for $\mathbb{F} \in D^+_{\mathrm{Lqc}}(X_\bullet)$.

Proof. The construction of ε is compatible with restrictions. So we may assume that $f_\bullet = f : X \to Y$ is an affine morphism of single schemes. Further, the question is local on Y, and hence we may assume that $Y = \operatorname{Spec} A$ is affine. As f is affine, $X = \operatorname{Spec} B$ is affine. By Lemma 14.3, we may assume that $\mathbb{F} = F_X \mathbb{G}$ for some $\mathbb{G} \in D(\mathrm{Qch}(X))$. In view of Lemma 14.6, it suffices to show that $\varepsilon : g_\# g^\# \mathbb{G} \to \mathbb{G}$ is an isomorphism if $\mathbb{G}$ is a K-injective complex in $\mathrm{Qch}(X)$.

To verify this, it suffices to show that $\varepsilon : g_\# g^\# \mathcal{M} \to \mathcal{M}$ is an isomorphism for $\mathcal{M} \in \mathrm{Qch}(X)$. By Lemma 7.19, $f_* = i^\# g^\#$ on $\mathrm{Qch}(X)$ respects coproducts and is exact. Since $i^\#$ respects coproducts and is faithful exact, $g^\#$ respects coproducts and is exact. So $g_\# g^\# : \mathrm{Qch}(X) \to \mathrm{Qch}(X)$ respects coproducts and is exact.

Since X is affine, there is an exact sequence of the form

$$\mathcal{O}_X^{(J)} \to \mathcal{O}_X^{(I)} \to \mathcal{M} \to 0.$$

So we may assume that $\mathcal{M} = \mathcal{O}_X$. But this case is trivial. □

(27.3) Let I, S, $f_\bullet : X_\bullet \to Y_\bullet$, Z, g, and i be as in (27.1). Assume that $f_\bullet$ is finite cartesian.

We say that an $\mathcal{O}_Z$-module $\mathcal{M}$ is locally quasi-coherent (resp. quasi-coherent, coherent) if $i^\# \mathcal{M}$ is. The corresponding full subcategory of $\mathrm{Mod}(Z)$ is denoted by $\mathrm{Lqc}(Z)$ (resp. $\mathrm{Qch}(Z)$, $\mathrm{Coh}(Z)$).

27.4 Lemma. *Let the notation be as above. Then an $\mathcal{O}_Z$-module $\mathcal{M}$ is locally quasi-coherent if and only if for any $j \in \mathrm{ob}(I)$ and any affine open subscheme U of Y_j, there exists an exact sequence of $((\mathcal{O}_Z)_j)|_U$-modules*

$$(((\mathcal{O}_Z)_j)|_U)^{(T)} \to (((\mathcal{O}_Z)_j)|_U)^{(\Sigma)} \to \mathcal{M}_j|_U \to 0.$$

Proof. As we assume that $f_\bullet$ is finite cartesian, $\mathcal{O}_Z$ is coherent. Hence the existence of such exact sequences implies that $\mathcal{M}$ is locally quasi-coherent.

We prove the converse. Let $j \in \mathrm{ob}(I)$ and U an affine open subset of Y_j. Set $C := \Gamma(U, (\mathcal{O}_Z)_j) = \Gamma(f_j^{-1}(U), \mathcal{O}_{X_j})$ and $M := \Gamma(U, \mathcal{M}_j)$. There is a canonical map $(g_j|_{f_j^{-1}(U)})^\#(\tilde{M}) \to \mathcal{M}_j|_U$, where $\tilde{M}$ is the quasi-coherent sheaf over $\operatorname{Spec} C \subset X_j$ associated with the C-module M. When we apply $(i_j|_U)^\#$ to this map, we get $\tilde{M_0} \to ((i^\# \mathcal{M})_j)|_U$, where M_0 is M viewed as a $\Gamma(U, \mathcal{O}_{Y_j})$-module. This is an isomorphism, since $(i^\# \mathcal{M})_j|_U$ is quasi-coherent and U is affine. As $(i_j|_U)^\#$ is faithful and exact, we have $(g_j|_{f_j^{-1}(U)})^\#(\tilde{M}) \cong \mathcal{M}_j|_U$.

Take an exact sequence of the form

$$C^{(T)} \to C^{(\Sigma)} \to M \to 0.$$

Applying the exact functor $(g_j|_{f_j^{-1}(U)})^\# \circ \tilde{?}$, we get an exact sequence of the desired type. □

27.5 Corollary. *Under the same assumption as in the lemma, the functor $g_\#$ preserves local quasi-coherence.*

Proof. As $g_\#$ is compatible with restrictions, we may assume that I consists of one object and one morphism. Further, as the question is local, we may assume that $Y_\bullet = Y$ is an affine scheme. By the lemma, it suffices to show that $g_\#\mathcal{O}_Z$ is quasi-coherent, since $g_\#$ is exact and preserves direct sums. As $g_\#\mathcal{O}_Z = g_\# g^\#\mathcal{O}_X \cong \mathcal{O}_X$, we are done. $\square$

27.6 Lemma. *Let the notation be as above. The unit of adjunction $u : \mathbb{F} \to Rg^\# g_\#\mathbb{F}$ is an isomorphism for $\mathbb{F} \in D^+_{\mathrm{Lqc}}(Z)$.*

Proof. We may assume that I consists of one object and one morphism, and $Y_\bullet = Y$ is affine. By the way-out lemma, we may assume that $\mathbb{F}$ is a single quasi-coherent sheaf. Then by Corollary 27.5, $g_\#\mathbb{F}$ is quasi-coherent, and is $g^\#$-acyclic. So it suffices to show that $u : \mathcal{M} \to g^\# g_\#\mathcal{M}$ is an isomorphism for a quasi-coherent sheaf $\mathcal{M}$ on Z. Note that $g^\# g_\# : \mathrm{Qch}(Z) \to \mathrm{Qch}(Z)$ respects coproducts. By Lemma 27.4 and the five lemma, we may assume that $\mathbb{F} = \mathcal{O}_Z = g^\#\mathcal{O}_X$ and it suffices to prove that $ug^\# : g^\#\mathcal{O}_X \to g^\# g_\# g^\#\mathcal{O}_X$ is an isomorphism. As $\mathrm{id} = (g^\#\varepsilon)(ug^\#)$ and ε is an isomorphism, we are done. $\square$

For $\mathcal{N} \in \mathrm{Mod}(Y_\bullet)$ and $\mathcal{M} \in \mathrm{Mod}(Z)$, the sheaf $\underline{\mathrm{Hom}}_{\mathcal{O}_{Y_\bullet}}(\mathcal{M}, \mathcal{N})$ on $Y_\bullet$ has a structure of $\mathcal{O}_Z$-module, and it belongs to $\mathrm{Mod}(Z)$. There is an obvious isomorphism of functors

$$\kappa : i^\# \underline{\mathrm{Hom}}_{\mathcal{O}_{Y_\bullet}}(\mathcal{M}, \mathcal{N}) \cong \underline{\mathrm{Hom}}_{\mathcal{O}_{Y_\bullet}}(i^\#\mathcal{M}, \mathcal{N}).$$

For $\mathcal{M}, \mathcal{M}' \in \mathrm{Mod}(Z)$, there is a natural map

$$\upsilon : \underline{\mathrm{Hom}}_{\mathrm{Mod}(Z)}(\mathcal{M}, \mathcal{M}') \to \underline{\mathrm{Hom}}_{\mathcal{O}_{Y_\bullet}}(\mathcal{M}, i^\#\mathcal{M}').$$

Note that the composite

$$i^\# \underline{\mathrm{Hom}}_{\mathrm{Mod}(Z)}(\mathcal{M}, \mathcal{M}') \xrightarrow{i^\#\upsilon} i^\# \underline{\mathrm{Hom}}_{\mathcal{O}_{Y_\bullet}}(\mathcal{M}, i^\#\mathcal{M}') \xrightarrow{\kappa} \underline{\mathrm{Hom}}_{\mathcal{O}_{Y_\bullet}}(i^\#\mathcal{M}, i^\#\mathcal{M}')$$

agrees with H.

(27.7) Let I, S, $f_\bullet : X_\bullet \to Y_\bullet$, Z, g, and i be as in (27.1). Assume that $f_\bullet$ is finite cartesian, and $Y_\bullet$ has flat arrows. Define $f_\bullet^\natural : \mathcal{D}^+(Y_\bullet) \to D(X_\bullet)$ by

$$f_\bullet^\natural(\mathbb{F}) := g_\# R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{Y_\bullet}}(\mathcal{O}_Z, \mathbb{F}).$$

As $f_\bullet$ is finite cartesian, $\mathcal{O}_Z$ is coherent. By Lemma 13.10, $i^\# R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{Y_\bullet}}(\mathcal{O}_Z, \mathbb{F}) \in \mathcal{D}^+(Y_\bullet)$. It follows that $f_\bullet^\natural(\mathbb{F}) \in \mathcal{D}^+(X_\bullet)$, and $f_\bullet^\natural$ is a functor from $\mathcal{D}^+(Y_\bullet)$ to $\mathcal{D}^+(X_\bullet)$.

Define $\varepsilon : R(f_\bullet)_* f_\bullet^\natural \to \mathrm{Id}_{\mathcal{D}^+(Y_\bullet)}$ by

$$\begin{aligned} &R(f_\bullet)_* f_\bullet^\natural \mathbb{F} = i^\# R g^\# g_\# R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{Y_\bullet}}(\mathcal{O}_Z, \mathbb{F}) \xrightarrow{u^{-1}} i^\# R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{Y_\bullet}}(\mathcal{O}_Z, \mathbb{F}) \\ &\xrightarrow{\kappa} R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{Y_\bullet}}(i^\#\mathcal{O}_Z, \mathbb{F}) = R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{Y_\bullet}}((f_\bullet)_*\mathcal{O}_{X_\bullet}, \mathbb{F}) \xrightarrow{\eta} R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{Y_\bullet}}(\mathcal{O}_{Y_\bullet}, \mathbb{F}) \cong \mathbb{F}. \end{aligned}$$

Define $u : \mathrm{Id}_{\mathcal{D}^+(X_\bullet)} \to f_\bullet^\natural R(f_\bullet)_*$ by

$$\begin{aligned} &\mathbb{F} \cong R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathcal{O}_{X_\bullet}, \mathbb{F}) \xrightarrow{\varepsilon^{-1}} g_\# R g^\# R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathcal{O}_{X_\bullet}, \mathbb{F}) \\ &\xrightarrow{H} g_\# R\,\underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(Z)}(g^\#\mathcal{O}_{X_\bullet}, R g^\# \mathbb{F}) \xrightarrow{v} g_\# R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{Y_\bullet}}(\mathcal{O}_Z, R(f_\bullet)_*\mathbb{F}) = f_\bullet^\natural R(f_\bullet)_*\mathbb{F}. \end{aligned}$$

27.8 Theorem. *Let the notation be as above. Then $f_\bullet^\natural$ is right adjoint to $R(f_\bullet)_*$, and ε and u defined above are the counit and unit of adjunction, respectively. In particular, if, moreover, $X_\bullet$ is quasi-compact, then $f_\bullet^\natural$ is isomorphic to $f_\bullet^\times$.*

Proof. It is easy to see that the composite

$$\begin{aligned} &R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{Y_\bullet}}(\mathcal{O}_Z, \mathbb{F}) \cong R\,\underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(Z)}(\mathcal{O}_Z, R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{Y_\bullet}}(\mathcal{O}_Z, \mathbb{F})) \\ &\xrightarrow{v} R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{Y_\bullet}}(\mathcal{O}_Z, i^\# R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{Y_\bullet}}(\mathcal{O}_Z, \mathbb{F})) \xrightarrow{\kappa} R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{Y_\bullet}}(\mathcal{O}_Z, R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{Y_\bullet}}(i^\#\mathcal{O}_Z, \mathbb{F})) \\ &\qquad \xrightarrow{\eta} R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{Y_\bullet}}(\mathcal{O}_Z, R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{Y_\bullet}}(\mathcal{O}_{Y_\bullet}, \mathbb{F})) \cong R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{Y_\bullet}}(\mathcal{O}_Z, \mathbb{F}) \end{aligned}$$

is the identity. Utilizing this and Lemma 1.47, $(f_\bullet^\natural \varepsilon) \circ (u f_\bullet^\natural) = \mathrm{id}$ and $(\varepsilon R(f_\bullet)_*) \circ (R(f_\bullet)_* u) = \mathrm{id}$ are checked directly.

The last assertion is obvious, as the right adjoint functor is unique. □

28 Cartesian regular embeddings and cartesian smooth morphisms

(28.1) Let I be a small category, S a scheme, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. An $\mathcal{O}_{X_\bullet}$-module sheaf $\mathcal{M} \in \mathrm{Mod}(X_\bullet)$ is said to be *locally free* (resp. *invertible*) if $\mathcal{M}$ is coherent and $\mathcal{M}_i$ is locally free (resp. invertible) for any $i \in \mathrm{ob}(I)$. A *perfect complex* of $X_\bullet$ is a bounded complex in $C^b(\mathrm{Mod}(X_\bullet))$ each of whose terms is locally free.

A point of $X_\bullet$ is a pair (i, x) such that $i \in \mathrm{ob}(I)$ and $x \in X_i$. A *stalk* of a sheaf $\mathcal{M} \in \mathrm{AB}(X_\bullet)$ at the point (i, x) is defined to be $(\mathcal{M}_i)_x$, and we denote it by $\mathcal{M}_{i,x}$.

A *connected component* of $X_\bullet$ is an equivalence class with respect to the equivalence relation of the set of points of $X_\bullet$ generated by the following relations.

1 (i, x) and (i', x') are equivalent if $i = i'$ and x and x' belong to the same connected component of X_i.

2 (i, x) and (i', x') are equivalent if there exists some $\phi : i \to i'$ such that $X_\phi(x') = x$.

We say that $X_\bullet$ is *d-connected* if $X_\bullet$ consists of one connected component (note that the word 'connected' is reserved for componentwise connectedness). If $X_\bullet$ is locally noetherian, then a connected component of $X_\bullet$ is a closed open subdiagram of schemes in a natural way. If this is the case, the rank function $(i, x) \mapsto \mathrm{rank}_{\mathcal{O}_{X_{i,x}}} \mathcal{F}_{i,x}$ of a locally free sheaf $\mathcal{F}$ is constant on a connected component of $X_\bullet$.

28.2 Lemma. *Let I be a small category, S a scheme, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Let $\mathbb{F}$ be a perfect complex of $X_\bullet$. Then we have*

1 *The canonical map*

$$H_J : (?)_J R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathbb{G}) \to R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet|_J}}(\mathbb{F}_J, \mathbb{G}_J)$$

is an isomorphism for $\mathbb{G} \in D(X_\bullet)$.

2 *The canonical map*

$$R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathbb{G}) \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} \mathbb{H} \to R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathbb{G} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} \mathbb{H})$$

is an isomorphism for $\mathbb{G}, \mathbb{H} \in D(X_\bullet)$.

Proof. **1** It suffices to show that

$$H_J : (?)_J\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathbb{G}) \to \underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet|_J}}(\mathbb{F}_J, \mathbb{G}_J)$$

is an isomorphism of complexes if $\mathbb{G}$ is a K-injective complex in $C(\mathrm{Mod}(X_\bullet))$, since $\mathbb{F}_J$ is K-flat and $\mathbb{G}_J$ is weakly K-injective. The assertion follows immediately by Lemma 6.36.

2 We may assume that $\mathbb{F}$ is a single locally free sheaf. By **1**, we may assume that $X = X_\bullet$ is a single scheme. We may assume that X is affine and $\mathbb{F} = \mathcal{O}_X^n$ for some n. This case is trivial. $\square$

(28.3) Let I be a small category, S a scheme, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. An $\mathcal{O}_{X_\bullet}$-module $\mathcal{M}$ is said to be *locally of finite projective dimension* if $\mathcal{M}_{i,x}$ is of finite projective dimension as an $\mathcal{O}_{X_{i,x}}$-module for any point (i,x) of $X_\bullet$. We say that $\mathcal{M}$ has *finite projective dimension* if there exists some non-negative integer d such that $\mathrm{proj.dim}_{\mathcal{O}_{X_{i,x}}} \mathcal{M}_{i,x} \leq d$ for any point (i,x) of $X_\bullet$.

28.4 Lemma. *Let I be a small category, S a scheme, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Assume that $X_\bullet$ has flat arrows and is locally noetherian. If $\mathbb{F}$ is a complex in $C(\mathrm{Mod}(X_\bullet))$ with bounded coherent cohomology groups which have finite projective dimension, then the canonical map*

$$R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathbb{G}) \otimes^{\bullet, L}_{\mathcal{O}_{X_\bullet}} \mathbb{H} \to R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathbb{G} \otimes^{\bullet, L}_{\mathcal{O}_{X_\bullet}} \mathbb{H})$$

is an isomorphism for $\mathbb{G}, \mathbb{H} \in D(X_\bullet)$.

Proof. We may assume that $\mathbb{G} = \mathcal{O}_{X_\bullet}$. By the way-out lemma, we may assume that $\mathbb{F}$ is a single coherent sheaf which has finite projective dimension, say d. By Lemma 13.9, it is easy to see that $\underline{\mathrm{Ext}}^i_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, G) = 0$ $(i > d)$ for $G \in \mathrm{Mod}(X_\bullet)$. In particular, $R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, ?)$ is way-out in both directions. On the other hand, as $R\,\underline{\mathrm{Hom}}_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathcal{O}_{X_\bullet})$ has finite flat dimension, and hence $R\,\underline{\mathrm{Hom}}_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathcal{O}_{X_\bullet}) \otimes^{\bullet, L}_{\mathcal{O}_{X_\bullet}} ?$ is also way-out in both directions. By the way-out lemma, we may assume that $\mathbb{H}$ is a single $\mathcal{O}_{X_\bullet}$-module. By Lemma 13.9, we may assume that $X = X_\bullet$ is a single scheme. The question is local, and we may assume that $X = \operatorname{Spec} A$ is affine. Moreover, we may assume that $\mathbb{F}$ is a complex of sheaves associated with a finite projective resolution of a single finitely generated module. As $\mathbb{F}$ is perfect, the result follows from Lemma 28.2. $\square$

(28.5) Let S, I, and $X_\bullet$ be as above. For a locally free sheaf $\mathcal{F}$ over $X_\bullet$, we denote $\underline{\mathrm{Hom}}_{\mathcal{O}_{X_\bullet}}(\mathcal{F}, \mathcal{O}_{X_\bullet})$ by $\mathcal{F}^\vee$. It is easy to see that $\mathcal{F}^\vee$ is again locally free. If $\mathcal{L}$ is an invertible sheaf, then

$$\mathcal{O}_{X_\bullet} \xrightarrow{\mathrm{tr}} \underline{\mathrm{Hom}}_{\mathcal{O}_{X_\bullet}}(\mathcal{L}, \mathcal{L}) \cong \mathcal{L}^\vee \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{L}$$

are isomorphisms.

(28.6) Let I be a small category, S a scheme, and $i_\bullet : Y_\bullet \to X_\bullet$ a closed immersion in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Then the canonical map $\eta : \mathcal{O}_{X_\bullet} \to (i_\bullet)_*\mathcal{O}_{Y_\bullet}$ is an epimorphism in $\mathrm{Lqc}(X_\bullet)$. Set $\mathcal{I} := \operatorname{Ker} \eta$. Then $\mathcal{I}$ is a locally quasi-coherent ideal of $\mathcal{O}_{X_\bullet}$. Conversely, if $\mathcal{I}$ is a given locally quasi-coherent ideal of $\mathcal{O}_{X_\bullet}$, then

$$Y_\bullet := \underline{\mathrm{Spec}}_\bullet\, \mathcal{O}_{X_\bullet}/\mathcal{I} \xrightarrow{i_\bullet} X_\bullet$$

is defined appropriately, and $i_\bullet$ is a closed immersion. Thus the isomorphism classes of closed immersions to $X_\bullet$ in the category $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)/X_\bullet$ and locally quasi-coherent ideals of $\mathcal{O}_{X_\bullet}$ are in one-to-one correspondence. We call $\mathcal{I}$ the *defining ideal sheaf* of $Y_\bullet$.

Note that $i_\bullet$ is cartesian if and only if $(i_\bullet)_*\mathcal{O}_{Y_\bullet}$ is equivariant. If $X_\bullet$ has flat arrows, this is equivalent to say that $\mathcal{I}$ is equivariant.

(28.7) Let $X_\bullet$ be locally noetherian. A morphism $i_\bullet : Y_\bullet \to X_\bullet$ is said to be a *regular embedding*, if $i_\bullet$ is a closed immersion such that $i_j : Y_j \to X_j$ is a regular embedding for each $j \in \mathrm{ob}(I)$, or equivalently, $\mathcal{I}$ is locally coherent and $\mathcal{I}_{j,x}$ is a complete intersection ideal of $\mathcal{O}_{X_j,x}$ for any $j \in \mathrm{ob}(I)$ and $x \in X_j$. If this is the case, we say that $\mathcal{I}$ is a *local complete intersection* ideal sheaf.

A cartesian closed immersion $i_\bullet : Y_\bullet \to X_\bullet$ with $X_\bullet$ locally noetherian with flat arrows is a cartesian regular embedding if and only if $i_\bullet^*\mathcal{I}$ is locally free and $\mathcal{I}_{i,x}$ is of finite projective dimension as an $\mathcal{O}_{X_i,x}$-module for any $i \in I$ and $x \in X_i$.

Note that $i_\bullet^*\mathcal{I} \cong \mathcal{I}/\mathcal{I}^2$, and we have

$$\mathrm{ht}_{\mathcal{O}_{X_i,x}} \mathcal{I}_x = \mathrm{rank}_{\mathcal{O}_{Y_i,y}} (i_\bullet^*\mathcal{I})_{i,y}$$

for any point (i, y) of $Y_\bullet$, where $x = i_i(y)$. We call these numbers the codimension of $\mathcal{I}$ at (i, y).

28.8 Proposition. *Let I be a small category, S a scheme, and $i_\bullet : Y_\bullet \to X_\bullet$ a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Assume that $X_\bullet$ is locally noetherian with flat arrows and $i_\bullet$ is a cartesian regular embedding. Let $\mathcal{I}$ be the defining ideal of $Y_\bullet$, and assume that $Y_\bullet$ has a constant codimension d. Then we have the following.*

1 $\underline{\mathrm{Ext}}^i_{\mathcal{O}_{X_\bullet}}((i_\bullet)_*\mathcal{O}_{Y_\bullet}, \mathcal{O}_{X_\bullet}) = 0$ *for $i \neq d$.*

2 *The canonical map*

$$\underline{\mathrm{Ext}}^d_{\mathcal{O}_{X_\bullet}}((i_\bullet)_*\mathcal{O}_{Y_\bullet}, \mathcal{O}_{X_\bullet}) \to \underline{\mathrm{Ext}}^d_{\mathcal{O}_{X_\bullet}}((i_\bullet)_*\mathcal{O}_{Y_\bullet}, (i_\bullet)_*\mathcal{O}_{Y_\bullet})$$

is an isomorphism.

3 *The Yoneda algebra*

$$\underline{\mathrm{Ext}}^{\bullet}_{\mathcal{O}_{Y_\bullet}}((i_\bullet)_*\mathcal{O}_{Y_\bullet}, (i_\bullet)_*\mathcal{O}_{Y_\bullet}) := \bigoplus_{j\geq 0} \underline{\mathrm{Ext}}^{j}_{\mathcal{O}_{Y_\bullet}}((i_\bullet)_*\mathcal{O}_{Y_\bullet}, (i_\bullet)_*\mathcal{O}_{Y_\bullet})$$

is isomorphic to the exterior algebra $(i_\bullet)_* \bigwedge^\bullet (i_\bullet^*\mathcal{I})^\vee$ *as graded* $\mathcal{O}_{X_\bullet}$*-algebras.*

4 *There is an isomorphism*

$$i_\bullet^\natural \mathcal{O}_{X_\bullet} \cong \bigwedge^d (i_\bullet^*\mathcal{I})^\vee[-d].$$

5 *For* $\mathbb{F} \in \mathcal{D}^+(X_\bullet)$*, there is a functorial isomorphism*

$$i_\bullet^\natural \mathbb{F} \cong \bigwedge^d (i_\bullet^*\mathcal{I})^\vee \otimes^{\bullet, L}_{\mathcal{O}_{Y_\bullet}} Li_\bullet^*\mathbb{F}[-d].$$

Proof. **1** is trivial, since $\mathcal{I}_{i,x}$ is a complete intersection ideal of the local ring $\mathcal{O}_{X_{i,x}}$ of codimension d for any point (i,x) of $X_\bullet$.

2 Note that $(i_\bullet)_*\mathcal{O}_{Y_\bullet} \cong \mathcal{O}_{X_\bullet}/\mathcal{I}$. From the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X_\bullet} \to \mathcal{O}_{X_\bullet}/\mathcal{I} \to 0,$$

we get an isomorphism

$$\underline{\mathrm{Ext}}^1_{\mathcal{O}_{X_\bullet}}((i_\bullet)_*\mathcal{O}_{Y_\bullet}, (i_\bullet)_*\mathcal{O}_{Y_\bullet}) \cong \underline{\mathrm{Hom}}_{\mathcal{O}_{X_\bullet}}(\mathcal{I}, \mathcal{O}_{X_\bullet}/\mathcal{I}) \cong (i_\bullet)_*(i_\bullet^*\mathcal{I})^\vee.$$

The canonical map

$$(i_\bullet)_*(i_\bullet^*\mathcal{I})^\vee \cong \underline{\mathrm{Ext}}^1_{\mathcal{O}_{X_\bullet}}((i_\bullet)_*\mathcal{O}_{Y_\bullet}, (i_\bullet)_*\mathcal{O}_{Y_\bullet}) \hookrightarrow \underline{\mathrm{Ext}}^\bullet_{\mathcal{O}_{X_\bullet}}((i_\bullet)_*\mathcal{O}_{Y_\bullet}, (i_\bullet)_*\mathcal{O}_{Y_\bullet})$$

is uniquely extended to an $\mathcal{O}_{X_\bullet}$-algebra map

$$T_\bullet((i_\bullet)_*(i_\bullet^*\mathcal{I})^\vee) \to \underline{\mathrm{Ext}}^\bullet_{\mathcal{O}_{X_\bullet}}((i_\bullet)_*\mathcal{O}_{Y_\bullet}, (i_\bullet)_*\mathcal{O}_{Y_\bullet}),$$

where $T_\bullet$ denotes the tensor algebra. It suffices to prove that this map is an epimorphism, which induces an isomorphism

$$\bigwedge^\bullet((i_\bullet)_*(i_\bullet^*\mathcal{I})^\vee) \to \underline{\mathrm{Ext}}^\bullet_{\mathcal{O}_{X_\bullet}}((i_\bullet)_*\mathcal{O}_{Y_\bullet}, (i_\bullet)_*\mathcal{O}_{Y_\bullet}).$$

In fact, the exterior algebra is compatible with base change, and

$$\begin{aligned}\textstyle\bigwedge^\bullet((i_\bullet)_*(i_\bullet^*\mathcal{I})^\vee) &\cong (i_\bullet)_* i_\bullet^* \textstyle\bigwedge^\bullet((i_\bullet)_*(i_\bullet^*\mathcal{I})^\vee) \\ &\cong (i_\bullet)_* \textstyle\bigwedge^\bullet((i_\bullet^*(i_\bullet)_*)(i_\bullet^*\mathcal{I})^\vee) \cong (i_\bullet)_* \textstyle\bigwedge^\bullet(i_\bullet^*\mathcal{I})^\vee.\end{aligned}$$

To verify this, we may assume that $i_\bullet : Y_\bullet \to X_\bullet$ is a morphism of single schemes, $X_\bullet = \operatorname{Spec} A$ affine, and $\mathcal{I} = \tilde{I}$ generated by an A-sequence. The proof for this case is essentially the same as [19, Lemma IV.1.1.8], and we omit it.

4 Let Z denote the ringed site $(\operatorname{Zar}(X_\bullet), (i_\bullet)_*\mathcal{O}_{Y_\bullet})$, and $g : Z \to \operatorname{Zar}(Y_\bullet)$ the associated admissible ringed continuous functor. By **2–3**, there is a sequence of isomorphisms in $\operatorname{Coh}(X_\bullet)$

$$(i_\bullet)_* \textstyle\bigwedge^d(i_\bullet^*\mathcal{I})^\vee \cong \underline{\operatorname{Ext}}^d_{\mathcal{O}_{Y_\bullet}}((i_\bullet)_*\mathcal{O}_{Y_\bullet}, (i_\bullet)_*\mathcal{O}_{Y_\bullet}) \cong \underline{\operatorname{Ext}}^d_{\mathcal{O}_{X_\bullet}}((i_\bullet)_*\mathcal{O}_{Y_\bullet}, \mathcal{O}_{X_\bullet}).$$

In view of **1**, there is an isomorphism

$$Rg^\# \textstyle\bigwedge^d(i_\bullet^*\mathcal{I})^\vee \cong R\,\underline{\operatorname{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathcal{O}_Z, \mathcal{O}_{X_\bullet})[d]$$

in $D^b_{\mathrm{Coh}}(Z)$. Applying $g_\#$ to both sides, we get

$$\textstyle\bigwedge^d(i_\bullet^*\mathcal{I})^\vee \cong i_\bullet^\natural\mathcal{O}_{X_\bullet}[d].$$

5 is an immediate consequence of **4** and Lemma 28.4. □

(28.9) Let I and S be as in (28.6). Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Assume that $f_\bullet$ is separated so that the diagonal $\Delta_{X_\bullet/Y_\bullet} : X_\bullet \to X_\bullet \times_{Y_\bullet} X_\bullet$ is a closed immersion. Define $\Omega_{X_\bullet/Y_\bullet} := i_\bullet^*\mathcal{I}$, where $\mathcal{I} := \operatorname{Ker}(\eta : \mathcal{O}_{X_\bullet\times_{Y_\bullet}X_\bullet} \to (\Delta_{X_\bullet/Y_\bullet})_*\mathcal{O}_{X_\bullet})$. Note that $(\Delta_{X_\bullet/Y_\bullet})_*\Omega_{X_\bullet/Y_\bullet} \cong \mathcal{I}/\mathcal{I}^2$.

28.10 Lemma. *Let the notation be as above. Then we have*

1 $\Omega_{X_\bullet/Y_\bullet}$ *is locally quasi-coherent.*

2 *If $f_\bullet$ is cartesian, then $\Omega_{X_\bullet/Y_\bullet}$ is quasi-coherent.*

3 *For $i \in \operatorname{ob}(I)$, there is a canonical isomorphism $\Omega_{X_i/Y_i} \cong (\Omega_{X_\bullet/Y_\bullet})_i$.*

Proof. Easy. □

28.11 Theorem. *Let I be a finite ordered category, and $f_\bullet : X_\bullet \to Y_\bullet$ a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}})$. Assume that $Y_\bullet$ is noetherian with flat arrows, and $f_\bullet$ is separated cartesian smooth of finite type. Assume that $f_\bullet$ has a constant relative dimension d. Then for any $\mathbb{F} \in D^+_{\mathrm{Lqc}}(Y_\bullet)$, there is a functorial isomorphism*

$$\bigwedge^d \Omega_{X_\bullet/Y_\bullet}[d] \otimes^\bullet_{\mathcal{O}_{X_\bullet}} f_\bullet^* \mathbb{F} \cong f_\bullet^! \mathbb{F},$$

where $[d]$ denotes the shift of degree.

Proof. In view of Theorem 26.14, it suffices to show that there is an isomorphism $f_\bullet^! \mathcal{O}_{Y_\bullet} \cong \bigwedge^d \Omega_{X_\bullet/Y_\bullet}[d]$. Consider the commutative diagram

$$\begin{tikzcd}
X_\bullet \arrow[r, "\Delta"] \arrow[dr, "\mathrm{id}"'] & X_\bullet \times_{Y_\bullet} X_\bullet \arrow[r, "p_2"] \arrow[d, "p_1"] & X_\bullet \arrow[d, "f_\bullet"] \\
& X_\bullet \arrow[r, "f_\bullet"] & Y_\bullet .
\end{tikzcd}$$

By Lemma 7.17 and Lemma 7.16, the all morphisms in the diagrams are cartesian. As p_1 is smooth of finite type of relative dimension d, Δ is a cartesian regular embedding of the constant codimension d.

By Theorem 21.8, Theorem 27.8, and Proposition 28.8, we have

$$\mathcal{O}_{X_\bullet} \cong f_\bullet^* \mathcal{O}_{Y_\bullet} \cong \Delta^! p_1^! f_\bullet^* \mathcal{O}_{Y_\bullet} \cong \Delta^\natural p_2^* f_\bullet^! \mathcal{O}_{Y_\bullet} \cong$$
$$\bigwedge^d \Omega^\vee_{X_\bullet/Y_\bullet}[-d] \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} L\Delta^* p_2^* f_\bullet^! \mathcal{O}_{Y_\bullet} \cong \bigwedge^d \Omega^\vee_{X_\bullet/Y_\bullet}[-d] \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} f_\bullet^! \mathcal{O}_{Y_\bullet}.$$

As $\bigwedge^d \Omega_{X_\bullet/Y_\bullet}$ is an invertible sheaf, we are done. $\square$

29 Group schemes flat of finite type

(29.1) Let S be a scheme.

(29.2) Let $\mathcal{F}$ (resp. $\mathcal{F}_M$) denote the subcategory of $\mathcal{P}((\Delta), \underline{\mathrm{Sch}}/S)$ (resp. subcategory of $\mathcal{P}(\Delta_M, \underline{\mathrm{Sch}}/S)$) consisting of noetherian objects with flat arrows and cartesian morphisms separated of finite type.

Let G be a flat S-group scheme of finite type. Note that G is faithfully flat over S. A *G-scheme* is an S-scheme with a left G-action by definition. Set $\mathcal{A}_G$ to be the category of noetherian G-schemes and G-morphisms separated of finite type.

For $X \in \mathcal{A}_G$, we associate a simplicial scheme $B_G(X)$ by $B_G(X)_n = G^n \times X$. For $n \geq 1$, $d_i(n) : G^n \times X \to G^{n-1} \times X$ is the projection $p \times 1_{G^{n-1}\times X}$ if $i = n$, where $p : G \to S$ is the structure morphism. While $d_i(n) = 1_{G^{n-1}} \times a$ if $i = 0$, where $a : G \times X \to X$ is the action. If $0 < i < n$, then $d_i(n) = 1_{G^{n-i-1}} \times \mu \times 1_{G^{i-1}\times X}$, where $\mu : G \times G \to G$ is the product. For $n \geq 0$, $s_i(n) : G^n \times X \to G^{n+1} \times X$ is given by

$$s_i(n)(g_n, \ldots, g_1, x) = (g_n, \ldots, g_{i+1}, e, g_i, \ldots, g_1, x),$$

where $e : S \to G$ is the unit element. Indeed, $B_G(X)$ satisfies the relations (11^{op}), (12^{op}), and (13^{op}) in [29, (VII.5)].

Note that $(B_G(X)')|_{(\Delta)}$ is canonically isomorphic to $B_G(G \times X)$, where $G \times X$ is viewed as a principal G-action. Note also that there is an isomorphism from $B_G(X)'$ to $\mathrm{Nerve}(p_2 : G \times X \to X)$ given by

$$\begin{aligned} B_G(X)'_n = G^{n+1} \times X &\to (G \times X) \times_X \cdots \times_X (G \times X) = \mathrm{Nerve}(p_2)_n \\ (g_n, \ldots, g_0, x) &\mapsto ((g_n \cdots g_0, x), (g_{n-1} \cdots g_0, x), \ldots, (g_0, x)). \end{aligned}$$

Hence we have

29.3 Lemma. *Let G and X be as above. Then $B_G(X)$ is a simplicial S-groupoid with $d_0(1)$ and $d_1(1)$ faithfully flat of finite type.*

We denote the restriction $B_G(X)|_{\Delta_M}$ by $B_G^M(X)$. Obviously, $B_G^M(X)$ is an S-groupoid with $d_0(1)$ and $d_1(1)$ faithfully flat of finite type.

For a morphism $f : X \to Y$ in $\mathcal{A}_G$, we define $B_G(f) : B_G(X) \to B_G(Y)$ by $(B_G(f))_n = 1_{G^n} \times f$. It is easy to check that B_G is a functor from $\mathcal{A}_G$ to $\mathcal{F}$. Thus B_G^M is a functor from $\mathcal{A}_G$ to $\mathcal{F}_M$.

We define a $(G, \mathcal{O}_X)$-module to be an $\mathcal{O}_{B_G^M(X)}$-module. That is, an object of $\mathrm{Mod}(B_G^M(X))$. So an equivariant (resp. locally quasi-coherent, quasi-coherent, coherent) $(G, \mathcal{O}_X)$-module is an equivariant (resp. locally quasi-coherent, quasi-coherent, coherent) object of $\mathrm{Mod}(B_G^M(X))$. The category of G-linearized $\mathcal{O}_X$-modules in [32] is equivalent to that of our equivariant $(G, \mathcal{O}_X)$-modules. See also [6] and [19].

We denote the category of $(G, \mathcal{O}_X)$-modules by $\mathrm{Mod}(G, X)$. The category of equivariant (resp. locally quasi-coherent, quasi-coherent, coherent) $(G, \mathcal{O}_X)$-modules is denoted by $\mathrm{EM}(G, X)$ (resp. $\mathrm{Lqc}(G, X)$, $\mathrm{Qch}(G, X)$, $\mathrm{Coh}(G, X)$).

Note that $\mathrm{EM}(B_G(X))$ (resp. $\mathrm{Qch}(B_G(X))$, $\mathrm{Coh}(B_G(X))$) is equivalent to $\mathrm{EM}(G, \mathcal{O}_X)$ (resp. $\mathrm{Qch}(G, X)$, $\mathrm{Coh}(G, X)$) (Lemma 9.4). However, the author does not know whether $\operatorname{Mod}(B_G(X))$ is equivalent to $\operatorname{Mod}(B_G^M(X))$. From our point of view, it seems that it is more convenient to work over Δ_M, which is a finite ordered category, than (Δ).

The discussion on derived categories of categories of sheaves over diagrams of schemes are interpreted to the derived categories of the categories of $(G, \mathcal{O}_X)$-modules.

By Lemma 29.3 and Lemma 12.8, we have

29.4 Lemma. *Let $X \in \mathcal{A}_G$. Then $\operatorname{Qch}(G, X)$ is a locally noetherian abelian category. $\mathcal{M} \in \operatorname{Qch}(G, X)$ is a noetherian object if and only if $\mathcal{M}_0$ is coherent if and only if $\mathcal{M} \in \operatorname{Coh}(G, X)$.*

Let $\mathcal{M}$ be a $(G, \mathcal{O}_X)$-module. If there is no danger of confusion, we may write $\mathcal{M}_0$ instead of $\mathcal{M}$. For example, $\mathcal{O}_X$ sometimes means $\mathcal{O}_{B_G^M(X)}$, since $(\mathcal{O}_{B_G^M(X)})_0 = \mathcal{O}_X$. This abuse of notation is what we always do when $S = X = \operatorname{Spec} k$ and G is an affine algebraic group over k. A G-module and its underlying vector space are denoted by the same symbol. Similarly, an object of $D(B_G^M(X))$ and its restriction to $D(B_G^M(X)_0) = D(X)$ are sometimes denoted by the same symbol. Moreover, for a morphism f in $\mathcal{A}_G$, we denote for example $R(B_G^M(f))_*$ by Rf_*, and $B_G^M(f)^!$ by $f^!$.

$D(B_G^M(X))$, or $D(\operatorname{Mod}(G, X))$, is denoted by $D(G, X)$ for short. Thus for example, $D^+_{\operatorname{Qch}(G,X)}(\operatorname{Mod}(G, X))$ is denoted by $D^+_{\operatorname{Qch}}(G, X)$.

Thus, as a corollary to Theorem 25.2, we have

29.5 Theorem (G-Grothendieck's duality). *Let S be a scheme, and G a flat S-group scheme of finite type. Let X and Y be noetherian S-schemes with G-actions, and $f : X \to Y$ a proper G-morphism. Then the composite*

$$\begin{aligned}\Theta(f) : Rf_* R\,\underline{\operatorname{Hom}}^\bullet_{\operatorname{Mod}(G,X)}(\mathbb{F}, f^\times \mathbb{G}) &\xrightarrow{H} R\,\underline{\operatorname{Hom}}^\bullet_{\operatorname{Mod}(G,Y)}(Rf_*\mathbb{F}, Rf_* f^\times \mathbb{G}) \\ &\xrightarrow{\varepsilon} R\,\underline{\operatorname{Hom}}^\bullet_{\operatorname{Mod}(G,Y)}(Rf_*\mathbb{F}, \mathbb{G})\end{aligned}$$

is an isomorphism in $D(G, Y)$ for $\mathbb{F} \in D_{\operatorname{Qch}}(G, X)$ and $\mathbb{G} \in D^+_{\operatorname{Qch}}(G, Y)$.

30 Compatibility with derived G-invariance

(30.1) Let S be a scheme, and G a flat S-group scheme. Let X be an S-scheme with a trivial G-action. That is, $a : G \times X \to X$ agrees with the second projection p_2. In other words, $d_0(1) = d_1(1)$ in $B_G^M(X)$.

For an object $\mathcal{M}$ of $\mathrm{Mod}(G, X)$, we define the *G-invariance* of $\mathcal{M}$ to be the kernel of the natural map

$$\beta_{d_0(1)} - \beta_{d_1(1)} : \mathcal{M}_0 \to d_0(1)_*\mathcal{M}_1 = d_1(1)_*\mathcal{M}_1,$$

and we denote it by $\mathcal{M}^G$.

(30.2) Let X be as in (30.1). Define $\tilde{B}_G^M(X)$ to be the augmented diagram

$$G \times_S G \times_S X \;\underset{p_{23}}{\overset{\substack{1_G \times a \\ \longrightarrow}}{\xrightarrow{\mu \times 1_X}}}\; G \times_S X \;\underset{p_2}{\overset{a}{\rightrightarrows}}\; X \xrightarrow{\mathrm{id}} X.$$

Note that $\tilde{B}_G^M(X)$ is an object of $\mathcal{P}(\Delta_M^+, \underline{\mathrm{Sch}}/S)$. For an S-morphism $f : X \to Y$ between S-schemes with trivial G-actions, $\tilde{B}_G^M(f) : \tilde{B}_G^M(X) \to \tilde{B}_G^M(Y)$ is defined by $\tilde{B}_G^M(f)_n = 1_{G^n} \times f$ for $n \geq 0$ and $\tilde{B}_G^M(f)_{-1} = f$. Thus $\tilde{B}_G^M$ is a functor from the category of S-schemes (with trivial G-actions) to the category $\mathcal{P}(\Delta_M^+, \underline{\mathrm{Sch}}/S)$ such that $(?)|_{\Delta_M}\tilde{B}_G^M = B_G^M$ and $(?)|_{-1}\tilde{B}_G^M = \mathrm{Id}$.

30.3 Lemma. *The functor* $(?)^G : \mathrm{Mod}(G, X) \to \mathrm{Mod}(X)$ *agrees with* $(?)_{-1}R_{\Delta_M}$.

Proof. Follows easily from (6.14). □

(30.4) We say that an object $\mathcal{M}$ of $\mathrm{Mod}(G, X)$ is G-trivial if $\mathcal{M}$ is equivariant, and the canonical inclusion $\mathcal{M}^G \to \mathcal{M}_0$ is an isomorphism. Note that $(?)_{\Delta_M}L_{-1}$ is the exact left adjoint of $(?)^G$. Note also that $\mathcal{M}$ is G-trivial if and only if the counit of adjunction $\varepsilon : (?)_{\Delta_M}L_{-1}\mathcal{M}^G \to \mathcal{M}$ is an isomorphism if and only if $\mathcal{M} \cong \mathcal{N}_{\Delta_M}$ for some $\mathcal{N} \in \mathrm{EM}(\tilde{B}_G^M(X))$.

Let $\mathrm{triv}(G, X)$ denote the full subcategory of $\mathrm{Mod}(B_G^M(X))$ consisting of G-trivial objects. Note that $(?)^G : \mathrm{triv}(G, X) \to \mathrm{Mod}(X)$ is an equivalence, whose quasi-inverse is $(?)_{\Delta_M}L_{-1}$.

Assume that G is concentrated over S. If $\mathcal{M}$ is locally quasi-coherent, then $\mathcal{M}^G$ is quasi-coherent. Thus we get a derived functor $R(?)^G : D^+_{\mathrm{Lqc}}(G, X) \to D^+_{\mathrm{Qch}}(X)$.

30.5 Proposition. *Let G be of finite type over S. Let X and Y be noetherian S-schemes with trivial G-actions, and $f : X \to Y$ an S-morphism, which is automatically a G-morphism, separated of finite type. Then there is a canonical isomorphism*

$$f^! R(?)^G \cong R(?)^G f^!$$

between functors from $D^+_{\mathrm{Lqc}}(G, Y)$ to $D^+_{\mathrm{Qch}}(X)$.

Proof. As $(?)_{-1}$ is exact, we have $R(?)^G \cong (?)_{-1}RR_{\Delta_M}$ by Lemma 30.3. Thus, we have a composite isomorphism

$$f^!R(?)^G \cong f^!(?)_{-1}RR_{\Delta_M} \xrightarrow{\bar{\xi}^{-1}} (?)_{-1}(\tilde{B}^G_M(f))^! RR_{\Delta_M} \xrightarrow{\bar{c}} (?)_{-1}RR_{\Delta_M}f^! \cong R(?)^G f^!$$

by Lemma 21.3 and Lemma 24.7. □

31 Equivariant dualizing complexes and canonical modules

(31.1) Let $\mathcal{A}$ be a Grothendieck category, and $\mathbb{I} \in D(\mathcal{A})$. We say that $\mathbb{I}$ has a finite injective dimension if $R\operatorname{Hom}_{\mathcal{A}}(?, \mathbb{I})$ is way-out in both directions, see [17, (I.7)]. By definition, an object of $C(\mathcal{A})$ or $K(\mathcal{A})$ has a finite injective dimension if it does in $D(\mathcal{A})$. $\mathbb{F} \in C(\mathcal{A})$ has a finite injective dimension if and only if there is a bounded complex $\mathbb{J}$ of injective objects in $\mathcal{A}$ and a quasi-isomorphism $\mathbb{F} \to \mathbb{J}$.

(31.2) Let I be a finite ordered category, S a scheme, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$.

31.3 Lemma. *Assume that $X_\bullet$ has flat arrows. Let $\mathbb{I} \in D(X_\bullet)$. Then $\mathbb{I}$ has a finite injective dimension if and only if $\mathbb{I}_i$ has a finite injective dimension for any $i \in \operatorname{ob}(I)$.*

Proof. We prove the 'only if' part. Since $(?)_i$ is exact and has an exact left adjoint L_i, and $\mathbb{I}$ has a finite injective dimension, $\mathbb{I}_i$ has a finite injective dimension for $i \in \operatorname{ob}(I)$.

We prove the converse by induction on the number of objects of I. We may assume that I has at least two objects.

Let i be a maximal element of $\operatorname{ob}(I)$. There is a triangle of the form

$$\mathbb{I} \xrightarrow{u} RR_i(?)_i\mathbb{I} \to \mathbb{C} \to \mathbb{I}[1].$$

Since $\mathbb{I}_i$ has a finite injective dimension and R_i has an exact left adjoint $(?)_i$, it is easy to see that $RR_i(?)_i\mathbb{I}$ has a finite injective dimension. So it suffices to show that $\mathbb{C}$ has a finite injective dimension. Applying $(?)_i$ to the triangle above, it is easy to see that $\mathbb{C}_i = 0$. Let J be the full subcategory of I such that $\operatorname{ob}(J) = \operatorname{ob}(I) \setminus \{i\}$. Then $u : \mathbb{C} \to R_J\mathbb{C}_J$ is an isomorphism by Lemma 18.9. On the other hand, by the only if part, which has already been proved, it is easy to see that $\mathbb{C}_j$ has a finite injective dimension for

$j \in \mathrm{ob}(J)$. By induction assumption, $\mathbb{C}_J$ has a finite injective dimension. So $\mathbb{C} \cong R_J\mathbb{C}_J$ has a finite injective dimension, since R_J is exact and has an exact left adjoint $(?)_J$. $\square$

(31.4) Let the notation be as in Theorem 20.4. Let $X_\bullet$ be an object of $\mathcal{F}$ (i.e., an I^{op}-diagram of noetherian S-schemes with flat arrows). We say that $\mathbb{F} \in D(X_\bullet)$ is a *dualizing complex* of $X_\bullet$ if $\mathbb{F} \in D_{\mathrm{Coh}}(X_\bullet)$, $\mathbb{F}$ has a finite injective dimension, and the canonical map

$$\mathcal{O}_{X_\bullet} \xrightarrow{\mathrm{tr}} R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathbb{F})$$

is an isomorphism. A complex $\mathbb{F} \in C(\mathrm{Mod}(X_\bullet))$ is said to be a dualizing complex if it is as an object of $D(X_\bullet)$.

(31.5) If there is a dualizing complex of $X_\bullet$, then it is represented by a bounded injective complex $\mathbb{F} \in C(\mathrm{Mod}(X_\bullet))$ with coherent cohomology groups such that

$$\mathcal{O}_{X_\bullet} \xrightarrow{\mathrm{tr}} \underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathbb{F})$$

is a quasi-isomorphism.

More is true. We may further assume that $\mathbb{F} \in C(\mathrm{Lqc}(X_\bullet))$. Indeed, we may replace $\mathbb{F}$ above by $\mathrm{lqc}\,\mathbb{F}$. Since $\mathbb{F}$ has coherent cohomology groups, it is easy to see that the canonical map $\mathrm{lqc}\,\mathbb{F} \to \mathbb{F}$ is a quasi-isomorphism by Lemma 14.3, **3**. Each term of $\mathrm{lqc}\,\mathbb{F}$ is an injective object of $\mathrm{Lqc}(X_\bullet)$, since lqc has an exact left adjoint. Note that each term of $\mathrm{lqc}\,\mathbb{F}$ is still injective in $\mathrm{Mod}(X_\bullet)$ by Lemma 24.2.

31.6 Lemma. *Let the notation be as in* (31.4)*. An object $\mathbb{F} \in D(X_\bullet)$ is a dualizing complex of $X_\bullet$ if and only if $\mathbb{F}$ has equivariant cohomology groups and $\mathbb{F}_i \in D(X_i)$ is a dualizing complex of X_i for any $i \in \mathrm{ob}(I)$.*

Proof. This is obvious by Lemma 13.9 and Lemma 31.3. $\square$

31.7 Corollary. *Let the notation be as in* (31.4)*. If $X_\bullet$ is Gorenstein with finite Krull dimension, then $\mathcal{O}_{X_\bullet}$ is a dualizing complex of $X_\bullet$.*

Proof. This is clear by the lemma and [17, (V.10)]. $\square$

31.8 Lemma. *Let the notation be as in* (31.4)*. If $X_\bullet$ has a dualizing complex $\mathbb{F}$, then $X_\bullet$ has finite Krull dimensions, and $X_\bullet$ has Gorenstein arrows.*

Proof. As $\mathbb{F}_i$ is a dualizing complex of X_i for each $i \in \mathrm{ob}(I)$, $X_\bullet$ has finite Krull dimensions by [17, Corollary V.7.2].

Let $\phi : i \to j$ be a morphism of I. As X_ϕ is flat, $\alpha_\phi : X_\phi^* \mathbb{F}_i \to \mathbb{F}_j$ is an isomorphism of $D(X_j)$. As $X_\phi^* \mathbb{F}_i$ is a dualizing complex of X_j, X_ϕ is Gorenstein by [4, (5.1)]. $\square$

31.9 Proposition. *Let the notation be as above, and $\mathbb{I}$ a dualizing complex of $X_\bullet$. Let $\mathbb{F} \in D_{\mathrm{Coh}}(X_\bullet)$. Then we have*

1 $R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathbb{I}) \in D_{\mathrm{Coh}}(X_\bullet)$.

2 *The canonical map*

$$\mathbb{F} \to R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathbb{I}), \mathbb{I})$$

is an isomorphism for $\mathbb{F} \in D_{\mathrm{Coh}}(X_\bullet)$.

Proof. **1** As $\mathbb{I}$ has a finite injective dimension, $R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(?, \mathbb{I})$ is way-out in both directions. Hence by [17, Proposition I.7.3], we may assume that $\mathbb{F}$ is bounded. This case is trivial by Lemma 13.10.

2 Using Lemma 13.9 twice, we may assume that $X_\bullet$ is a single scheme. This case is [17, Proposition V.2.1]. $\square$

31.10 Lemma. *Let X be a noetherian scheme, and $\mathfrak{U} = (U_i)$ a finite open covering of X. Let $\mathbb{I} \in D(X)$. Then $\mathbb{I}$ is dualizing if and only if $\mathbb{I}|_{U_i}$ is dualizing for each i.*

Proof. It is obvious that $\mathbb{I}$ has coherent cohomology groups if and only if $\mathbb{I}|_{U_i}$ has coherent cohomology groups for each i.

Assume that $\mathbb{I}$ is a bounded injective complex. Then $\mathbb{I}|_{U_i}$ is a bounded injective complex, since $(?)|_{U_i}$ preserves injectives. Conversely, assume that $\mathbb{I}$ has coherent cohomology groups and $\mathbb{I}|_{U_i}$ has a finite injective dimension for each i. Then by [17, (II.7.20)], there is an integer n_0 such that for any i, any $G \in \mathrm{Coh}(U_i)$, and any $j > n_0$, we have $\underline{\mathrm{Ext}}^j_{\mathcal{O}_{U_i}}(G, \mathbb{I}|_{U_i}) = 0$. This shows that for any $G \in \mathrm{Coh}(X)$ and any $j > n_0$, $\underline{\mathrm{Ext}}^j_{\mathcal{O}_X}(G, \mathbb{I}) = 0$, and again by [17, (II.7.20)], we have that $\mathbb{I}$ has a finite injective dimension.

Let $\mathbb{I}$ be a bounded injective complex. Let $\mathbb{C}$ be the mapping cone of $\mathrm{tr} : \mathcal{O}_X \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{I}, \mathbb{I})$. $\mathbb{C}$ is exact (i.e., tr is a quasi-isomorphism) if and only if $\mathbb{C}|_{U_i}$ is exact for each i. On the other hand, $\mathbb{C}|_{U_i}$ is isomorphic to the mapping cone of the trace map $\mathcal{O}_{U_i} \to \underline{\mathrm{Hom}}_{\mathcal{O}_{U_i}}(\mathbb{I}|_{U_i}, \mathbb{I}|_{U_i})$. Thus $\mathrm{tr} : \mathcal{O}_X \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{I}, \mathbb{I})$ is a quasi-isomorphism if and only if $\mathcal{O}_{U_i} \to \underline{\mathrm{Hom}}_{\mathcal{O}_{U_i}}(\mathbb{I}|_{U_i}, \mathbb{I}|_{U_i})$ is a quasi-isomorphism for each i. Thus the lemma is obvious now. $\square$

31.11 Lemma. *Let the notation be as in* (31.4). *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{F}$, and let $\mathbb{I}$ be a dualizing complex of $Y_\bullet$. Then $f_\bullet^!(\mathbb{I})$ is a dualizing complex of $X_\bullet$.*

Proof. By Corollary 22.4, $f_\bullet^!(\mathbb{I})$ has coherent cohomology groups.

By Lemma 31.6 and Proposition 18.14, we may assume that $f : X \to Y$ is a morphism of single schemes. By Lemma 31.10, the question is local both on Y and X, So we may assume that both Y and X are affine, and f is either an affine n-space or a closed immersion. These cases are done in [17, Chapter V]. □

31.12 Lemma. *Let the notation be as in* (31.4), *and $\mathbb{I}$ and $\mathbb{J}$ dualizing complexes on $X_\bullet$. If $X_\bullet$ is d-connected and X_i is non-empty for some $i \in \mathrm{ob}(I)$, then there exist a unique invertible sheaf $\mathcal{L}$ and a unique integer n such that*

$$\mathbb{J} \cong \mathbb{I} \otimes^{\bullet,L}_{\mathcal{O}_{X_\bullet}} \mathcal{L}[n].$$

Such $\mathcal{L}$ and n are determined by

$$\mathcal{L}[n] \cong R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_\bullet}}(\mathbb{I}, \mathbb{J}).$$

Proof. Use [17, Theorem V.3.1].

31.13 Definition. Let the notation be as in (31.4), and $\mathbb{I}$ a fixed dualizing complex of $X_\bullet$. For any object $f_\bullet : Y_\bullet \to X_\bullet$ of $\mathcal{F}/X_\bullet$, we define *the* dualizing complex of $Y_\bullet$ (or better, of $f_\bullet$) to be $f_\bullet^!\mathbb{I}$. It is certainly a dualizing complex of $Y_\bullet$ by Lemma 31.11. If $Y_\bullet$ is d-connected and Y_i is non-empty for some $i \in \mathrm{ob}(I)$, then we define *the canonical sheaf* $\omega_{Y_\bullet}$ of $Y_\bullet$ (or better, $f_\bullet$) to be $H^s(f_\bullet^!\mathbb{I})$, where s is the smallest i such that $H^i(f_\bullet^!\mathbb{I}) \neq 0$. If $Y_\bullet$ is not d-connected, then we define $\omega_{Y_\bullet}$ componentwise.

31.14 Lemma. *Let S be a noetherian scheme, and G a flat S-group scheme of finite type. Then $G \to S$ is a* (*flat*) *local complete intersection morphism. That is,* (*it is flat and*) *all fibers are locally complete intersections.*

Proof. We may assume that $S = \operatorname{Spec} k$, with k a field. Then by [3, Theorem 1], we may assume that k is algebraically closed.

First assume that the characteristic is $p > 0$. Then there is some $r \gg 0$ such that the scheme theoretic image of the Frobenius map $F^r : G \to G^{(r)}$ is reduced (or equivalently, k-smooth) and agrees with $G^{(r)}_{\mathrm{red}}$. Note that the induced morphism $G \to G_{\mathrm{red}}^{(r)}$ is flat, since the flat locus is a G-stable open subset of G [19, Lemma 2.1.10], and the morphism is flat at the generic point.

As the group G_{red} acts on G transitively, it suffices to show that G is locally a complete intersection at the unit element e. So by [3, Theorem 2], it suffices to show that the rth Frobenius kernel G_r is a complete intersection. As G_r is finite connected, this is well known [46, (14.4)].

Now consider the case that G is of characteristic zero. We are to prove that G is k-smooth. Take a finitely generated $\mathbb{Z}$-subalgebra R of k such that G is defined. We may take R so that G_R is R-flat of finite type. Set $H := (G_R)_{\mathrm{red}}$. We may take R so that H is also R-flat. Then H is a closed subgroup scheme over R, since $\operatorname{Spec} R$ and $H \times_R H$ are reduced. As a reduced group scheme over a field of characteristic zero is smooth, we may localize R if necessary, and we may assume that H is R-smooth.

Let $\mathcal{J}$ be the defining ideal sheaf of H in G_R. There exists some $s \geq 0$ such that $\mathcal{J}^{s+1} = 0$. Note that $\mathcal{G} := \bigoplus_{i=0}^{s} \mathcal{J}^i/\mathcal{J}^{i+1}$ is a coherent $(H, \mathcal{O}_H)$-module. Applying Corollary 10.15 to the case that $Y = \operatorname{Spec} R$ and $X_\bullet = B_H^M(H)$, the coherent $(H, \mathcal{O}_H)$-module $\mathcal{G}$ is of the form $f^*(\tilde{V})$, where V is a finite R-module, and $f : H \to \operatorname{Spec} R$ is the structure map. Replacing R if necessary, we may assume that $V \cong R^u$. Now we want to prove that $u = 1$ so that $H = G_R$, which implies G is k-smooth.

There exists some prime number $p > u$ and a maximal ideal $\mathfrak{m}$ of R such that $R/\mathfrak{m}$ is a finite field of characteristic p. Let κ be the algebraic closure of $R/\mathfrak{m}$, and consider the base change $(\bar{?}) := ? \otimes_R \kappa$. Note that $\bar{\mathcal{G}} = \bigoplus_i \overline{\mathcal{J}^i}/\overline{\mathcal{J}^{i+1}}$ (recall that R is $\mathbb{Z}$-flat and H is R-flat). Let $\mathcal{I}^{[p^r]}$ denote the defining ideal of the rth Frobenius kernel of $\bar{G}_R$. By [46, (14.4)] again, $\dim_k(\mathcal{O}_{\bar{G}_R}/\mathcal{I}^{[p^r]})_e$ is a power of p, say $p^{v(r)}$. Similarly, the k-dimension of the coordinate ring $(\mathcal{O}_{\bar{G}_R}/(\bar{\mathcal{J}} + \mathcal{I}^{[p^r]}))_e$ of the rth Frobenius kernel of $\bar{H}$ is a power of p, say $p^{w(r)}$. Note that

$$p^{w(r)} \leq p^{v(r)} \leq \dim_k(\mathcal{O}_{\bar{G}_R}/\mathcal{I}^{[p^r]} \otimes_{\mathcal{O}_{\bar{G}_R}} \bar{\mathcal{G}})_e = p^{w(r)}u < p^{w(r)+1}.$$

Hence $w(r) \leq v(r) < w(r) + 1$, and we have $\bar{\mathcal{J}}_e \subset \mathcal{I}_e^{[p^r]}$ for any r. By Krull's intersection theorem, $\bar{\mathcal{J}}_e \subset \bigcap_r (\mathcal{I}_e)^{p^r} = 0$. This shows that $\bar{G}_R$ is reduced at e, which shows that $\bar{G}_R$ is κ-smooth everywhere. So the nilpotent ideal $\bar{\mathcal{J}}$ must be zero, and this shows $u = 1$. $\square$

(31.15) Let S be a scheme, G a flat S-group scheme of finite type, and X a noetherian G-scheme. By definition, a *G-dualizing complex* of X is a dualizing complex of $B_G^M(X)$. Let us fix X and a G-dualizing complex $\mathbb{I}$. For $(f : Y \to X) \in \mathcal{A}_G/X$, we define *the* G-dualizing complex of Y (or better, of f) to be $f^!(\mathbb{I})$. It is certainly a G-dualizing complex of Y. The canonical sheaf of $B_G^M(Y)$ is called *the G-canonical sheaf* of Y, and is denoted by ω_Y.

31.16 Lemma. *Let $f : X \to Y$ be a Gorenstein flat morphism of finite type between noetherian schemes. If $\mathbb{I}$ is a dualizing complex of Y, then $f^*(\mathbb{I})$ is a dualizing complex of X.*

Proof. Since Y has a dualizing complex, Y has finite Krull dimension [17, Corollary V.7.2]. Since X is of finite type over Y, X has finite Krull dimension. By [17, Proposition V.8.2], it suffices to show that $f^*(\mathbb{I})$ is pointwise dualizing. So we may assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine, both A and B are local, and f is induced by a local homomorphism from A to B. Then the assertion follows from [4, (5.1)]. $\square$

31.17 Lemma. *Let S, G, and X be as in* (31.15)*. Then $\mathbb{I} \in D(G, X)$ is a G-dualizing complex of X if and only if $\mathbb{I}$ has equivariant cohomology groups and $\mathbb{I}_0 \in D(X)$ is a dualizing complex of X.*

Proof. The 'only if' part is obvious by Lemma 31.6.

To prove the converse, it suffices to show that $\mathbb{I}_i \in D(G^i \times X)$ is dualizing for $i = 1, 2$ by the same lemma. Since $B_G^M(X)$ has flat arrows and $\mathbb{I}$ has equivariant cohomology groups, $\alpha_{\rho_i(i)} : r_i(i)^*\mathbb{I}_0 \to \mathbb{I}_i$ is an isomorphism in $D(G^i \times X)$. Since $r_i(i)$ is Gorenstein flat of finite type, the assertion follows from Lemma 31.16. $\square$

31.18 Lemma. *Let R be a Gorenstein local ring of dimension d, and $S =$* $\operatorname{Spec} R$*. Then $B_G^M(S)$ is Gorenstein of finite Krull dimension. In particular, $\mathcal{O}_S[d]$ is a G-dualizing complex of S (i.e., $\mathcal{O}_{B_G^M(S)}[d]$ is a dualizing complex of $B_G^M(S)$).*

Proof. As $S = \operatorname{Spec} R$ is Gorenstein by assumption and G is Gorenstein over S by Lemma 31.14, the assertions are trivial. $\square$

(31.19) When R, S and d are as in the lemma, then we usually choose and fix the G-dualizing complex $\mathcal{O}_S[d]$ of S. Thus for an object $X \in \mathcal{A}_G = \mathcal{A}_G/S$, the G-dualizing complex of X is $f^!(\mathcal{O}_S[d])$, where f is the structure morphism $X \to S$ of X. The G-canonical sheaf is defined accordingly.

31.20 Lemma. *Let R, S and d be as in* Lemma 31.18. *Let $X \in \mathcal{A}_G$, and assume that G acts on X trivially. Then the dualizing complex $\mathbb{I}_X := f^!(\mathcal{O}_S[d])$ has G-trivial cohomology groups, where $f : X \to S$ is the structure map. In particular, ω_X is G-trivial.*

Proof. By Proposition 18.14,

$$f^!(\mathcal{O}_S[d]) \cong f^!((\mathcal{O}_{\tilde{B}_G^M(S)})_{\Delta_M})[d] \cong (?)_{\Delta_M}(\tilde{B}_G^M(f)^!(\mathcal{O}_{\tilde{B}_G^M(S)}))[d].$$

By Corollary 22.4, $\tilde{B}_G^M(f)^!(\mathcal{O}_{\tilde{B}_G^M(S)})$ has coherent cohomology groups. Hence, $f^!(\mathcal{O}_S[d])$ has G-trivial cohomology groups. □

32 A generalization of Watanabe's theorem

32.1 Lemma. *Let R be a noetherian commutative ring, and G a finite group which acts on R. Set $A = R^G$, and assume that* $\operatorname{Spec} A$ *is connected. Then G permutes the connected components of* $\operatorname{Spec} R$ *transitively.*

Proof. Since $\operatorname{Spec} R$ is a noetherian space, $\operatorname{Spec} R$ has only finitely many connected components, say $X_1, \ldots, X_n$. Then $R = R_1 \times \cdots \times R_n$, and each R_i is of the form Re_i, where e_i is a primitive idempotent. Note that $E := \{e_1, \ldots, e_n\}$ is the set of primitive idempotents of R, and G acts on E. Let E_1 be an orbit of this action. Then $e = \sum_{e_i \in E_1} e_i$ is in A. As A does not have any nontrivial idempotent, $e = 1$. This shows that G acts on E transitively, and we are done. □

32.2 Lemma. *Let R be a noetherian commutative ring, and G a finite group which acts on R. Set $A = R^G$, and assume that the inclusion $A \hookrightarrow R$ is finite. If $\mathfrak{p} \in$* $\operatorname{Spec} A$*, then G acts transitively on the set of primes of R lying over $\mathfrak{p}$. Moreover, the going-down theorem holds for the ring extension $A \hookrightarrow R$.*

Proof. Note that A is noetherian by Eakin-Nagata theorem [30, Theorem 3.7]. Let A' be the $\mathfrak{p}A_\mathfrak{p}$-adic completion of $A_\mathfrak{p}$, and set $R' := A' \otimes_A R$. As A' is A-flat, $A' = (R')^G$. It suffices to prove that G acts transitively on the maximal ideals of R'. But R' is the direct product $\prod_i R'_i$ of complete local rings R'_i. Consider the corresponding primitive idempotents. Since A' is a local ring, G permutes these idempotents transitively by Lemma 32.1. It is obvious that this action induces a transitive action on the maximal ideals of R'.

We prove the last assertion. Let $\mathfrak{p} \supset \mathfrak{q}$ be prime ideals of A, and P be a prime ideal of R such that $P \cap A = \mathfrak{p}$. By the lying over theorem [30, Theorem 9.3], there exists some prime ideal Q' of R such that $Q' \cap A = \mathfrak{q}$. By the going-up theorem [30, Theorem 9.4], there exists some prime $P' \supset Q'$ such that $P' \cap A = \mathfrak{p}$. Then there exists some $g \in G$ such that $gP' = P$. Letting $Q := gQ'$, we have that $Q \subset P$, and $Q \cap A = \mathfrak{q}$. $\square$

(32.3) Let k be a field, and G a finite k-group scheme. Let $S = \operatorname{Spec} R$ be an affine k-scheme of finite type with a left G-action. It gives a k-algebra automorphism action of G on R. Let $A := R^G$ be the ring of invariants.

32.4 Proposition. *Assume that G is linearly reductive (i.e., any G-module is semisimple). Then the following hold.*

1 *If R satisfies Serre's (S_r) condition, then the A-module R satisfies (S_r), and A satisfies (S_r).*

2 *If R is Cohen-Macaulay, then R is a maximal Cohen-Macaulay A-module, and A is also a Cohen-Macaulay ring.*

3 *If R is Cohen-Macaulay, then $\omega_R^G \cong \omega_A$ as A-modules.*

4 *Assume that R is Gorenstein and $\omega_R \cong R$ as (G, R)-modules. Then $A = R^G$ is Gorenstein and $\omega_A \cong A$.*

Proof. Note that the associated morphism $\pi : S = \operatorname{Spec} R \to \operatorname{Spec} A$ is finite surjective.

To prove the proposition, we may assume that $\operatorname{Spec} A$ is connected.

Set $\bar{G} := G \otimes_k \bar{k}$, and $\bar{R} = R \otimes_k \bar{k}$, where $\bar{k}$ is the algebraic closure of k. Let G_0 be the identity component (or the Frobenius kernel for sufficiently high Frobenius maps, if the characteristic is nonzero) of $\bar{G}$, which is a normal subgroup scheme of $\bar{G}$. Note that $\operatorname{Spec} \bar{R} \to \operatorname{Spec} \bar{R}^{G_0}$ is finite and is a homeomorphism, since G_0 is trivial if the characteristic is zero, and $\bar{R}^{G_0}$ contains some sufficiently high Frobenius power of $\bar{R}$, if the characteristic is positive. On the other hand, the finite group $\bar{G}(\bar{k}) = (\bar{G}/G_0)(\bar{k})$ acts on $\bar{R}^{G_0}$, and the ring of invariants under this action is $A \otimes_k \bar{k}$. By Lemma 32.2, for any prime ideal $\mathfrak{p}$ of $A \otimes_k \bar{k}$, $\bar{G}(\bar{k})$ acts transitively on the set of prime ideals of $\bar{R}$ (or $\bar{R}^{G_0}$) lying over $\mathfrak{p}$. It follows that for any prime ideal $\mathfrak{p}$ of A and a prime ideal $\mathfrak{P}$ of R lying over $\mathfrak{p}$, we have $\operatorname{ht} \mathfrak{p} = \operatorname{ht} \mathfrak{P}$.

Let M be the sum of all non-trivial simple G-submodules of R. As G is linearly reductive, R is the direct sum of M and A as a G-module. It is easy to see that $R = M \oplus A$ is a direct sum decomposition as a (G, A)-module.

1 Since A is a direct summand of R as an A-module, it suffices to prove that the A-module R satisfies the (S_r)-condition. Let $\mathfrak{p} \in \operatorname{Spec} A$ and assume that $\operatorname{depth}_{A_\mathfrak{p}} R_\mathfrak{p} < r$. Note that $\operatorname{depth}_{A_\mathfrak{p}} R_\mathfrak{p} = \inf_{\mathfrak{P}} \operatorname{depth} R_{\mathfrak{P}}$, where $\mathfrak{P}$ runs through the prime ideals lying over $\mathfrak{p}$. So there exists some $\mathfrak{P}$ such that $\operatorname{depth} R_{\mathfrak{P}} \leq \operatorname{depth}_{A_\mathfrak{p}} R_\mathfrak{p} < r$. As R satisfies Serre's (S_r)-condition, we have that $R_{\mathfrak{P}}$ is Cohen-Macaulay. So

$$\operatorname{ht} \mathfrak{p} = \operatorname{ht} \mathfrak{P} = \operatorname{depth} R_{\mathfrak{P}} \leq \operatorname{depth}_{A_\mathfrak{p}} R_\mathfrak{p} \leq \operatorname{depth} A_\mathfrak{p} \leq \operatorname{ht} \mathfrak{p},$$

and all $\leq$ must be $=$. In particular, $R_\mathfrak{p}$ is a maximal Cohen-Macaulay $A_\mathfrak{p}$-module. This shows that the A-module R satisfies Serre's (S_r)-condition.

2 is obvious by **1**.

We prove **3**. We may assume that $\operatorname{Spec} A$ is connected. Note that $\pi : S = \operatorname{Spec} R \to \operatorname{Spec} A$ is a finite G-morphism. Set $d = \dim R = \dim A$. As A is Cohen-Macaulay and $\operatorname{Spec} A$ is connected, A is equidimensional of dimension d. So $\operatorname{ht} \mathfrak{m} = d$ for all maximal ideals of A. The same is true of R, and hence R is also equidimensional. So $\omega_R[d]$ and $\omega_A[d]$ are the equivariant dualizing complexes of R and A, respectively. In particular, we have $\pi^! \omega_A \cong \omega_R$. By Lemma 31.20, ω_A is G-trivial. By Theorem 29.5, we have isomorphisms in $D(G, \operatorname{Spec} A)$

$$\omega_R \cong R\pi_* R \underline{\operatorname{Hom}}_{\mathcal{O}_{\operatorname{Spec} R}}(\mathcal{O}_{\operatorname{Spec} R}, \pi^! \omega_A) \cong R \underline{\operatorname{Hom}}_{\mathcal{O}_{\operatorname{Spec} A}}(R\pi_* \mathcal{O}_{\operatorname{Spec} R}, \omega_A).$$

As π is affine, $R\pi_* \mathcal{O}_{\operatorname{Spec} R} = R$. As R is a maximal Cohen-Macaulay A-module and ω_A is a finitely generated A-module which is of finite injective dimension, we have that $\operatorname{Ext}^i_A(R, \omega_A) = 0$ $(i > 0)$. Hence

$$\omega_R \cong R \underline{\operatorname{Hom}}_{\mathcal{O}_{\operatorname{Spec} A}}(R, \omega_A) \cong \operatorname{Hom}_A(R, \omega_A)$$

in $D(G, \operatorname{Spec} A)$. As G is linearly reductive, there is a canonical direct sum decomposition $R \cong R^G \oplus U_R$ (as an (G, A)-module), where U_R is the sum of all non-trivial simple G-submodules of R. As ω_A is G-trivial, $\operatorname{Hom}_G(U_R, \omega_A) = 0$. In particular, $\operatorname{Hom}_A(U_R, \omega_A)^G = 0$.

On the other hand, we have that

$$\operatorname{Hom}_A(R^G, \omega_A)^G = \operatorname{Hom}_A(A, \omega_A)^G = \omega_A^G = \omega_A.$$

Hence

$$\omega_R^G \cong \operatorname{Hom}_A(R, \omega_A)^G \cong \operatorname{Hom}_A(U_R, \omega_A)^G \oplus \operatorname{Hom}_A(R^G, \omega_A)^G \cong \omega_A.$$

4 follows from **2** and **3** immediately. $\square$

32.5 Corollary. *Let k be a field, G a linearly reductive finite k-group scheme, and V a finite dimensional G-module. Assume that the representation $G \to GL(V)$ factors through $SL(V)$. Then the ring of invariants $A := (\operatorname{Sym} V)^G$ is Gorenstein, and $\omega_A \cong A$.*

Proof. Set $R := \operatorname{Sym} V$. As R is k-smooth, we have that $\omega_R \cong \bigwedge^n \Omega_{R/k} \cong R \otimes \bigwedge^n V$, where $n = \dim_k V$. By assumption, $\bigwedge^n V \cong k$, and we have that $\omega_R \cong R$, as (G, R)-modules. By the proposition, A is Gorenstein and $\omega_A \cong A$. $\square$

Although it has nothing to do with the twisted inverse, we give some normality results on invariant subrings under the action of group schemes. For a ring R, let $R^\star$ denote the set of nonzerodivisors of R.

32.6 Lemma. *Let S be a finite direct product of normal domains, R a commutative ring, and F a set of ring homomorphisms from S to R. Assume that $f(s) \in R^\star$ for any $f \in F$ and $s \in S^\star$. Then*

$$A := \{a \in S \mid f(a) = f'(a) \text{ for } f, f' \in F\}$$

is a subring of S, and is a finite direct product of normal domains.

Proof. We may assume that F has at least two elements. It is obvious that A is closed under subtraction and multiplication, and $1 \in A$. So A is a subring of S.

We prove that A is a finite direct product of normal domains. Let $h : A \to R$ be the restriction of $f \in F$ to A, which is independent of choice of f. Let $e_1, \ldots, e_r$ be the primitive idempotents of A. Replacing S by Se_i, A by Ae_i, R by $R(h(e_i))$, and F by

$$\{f|_{Se_i} \mid f \in F\},$$

we may assume that $A \neq 0$ and that A does not have a nontrivial idempotent. Indeed, if $se_i \in (Se_i)^\star$, then $se_i + (1 - e_i) \in S^\star$ as can be seen easily. So we have $f(se_i) + (1 - f(e_i)) \in R^\star$, and hence $h(e_i)f(se_i) = f(se_i) \in R(h(e_i))^\star$.

Assume that $a \in A \setminus \{0\}$ is a zerodivisor of S. Then there is a nontrivial idempotent e of S such that $ae = a$ and $1 - e + a \in S^\star$. Then for $f \in F$, $h(a)f(e) = h(a)$, and $1 - f(e) + h(a) \in R^\star$. So for any $f, f' \in F$, $f(e)(1 - f'(e)) = 0$, since

$$(1 - f(e) + h(a))f(e)(1 - f'(e)) = h(a)(1 - f'(e)) = h(a) - h(a) = 0.$$

Similarly we have $f'(e)(1 - f(e)) = 0$, and hence

$$f(e) = f(e)(1 - f'(e) + f'(e)) = f(e)f'(e) = (1 - f(e) + f(e))f'(e) = f'(e).$$

This shows that $e \in A$, and this contradicts our additional assumption. Hence any nonzero element a of A is a nonzerodivisor of S. In particular, A is an integral domain, since the product of two nonzero elements of A is a nonzerodivisor of S and cannot be zero.

Let $K = Q(A)$ be the field of fractions of A, and $L = Q(S)$ be the total quotient ring of S. By the argument above, $A \hookrightarrow S \hookrightarrow L$ can be extended to a unique injective homomorphism $K \hookrightarrow L$. We regard K as a subring of L. As $f(S^\star) \subset R^\star$, $f \in F$ is extended to the map $Q(f) : L = Q(S) \to Q(R)$.

Set

$$B := \{\alpha \in L \mid Q(f)(\alpha) = Q(f')(\alpha) \text{ in } Q(R) \text{ for } f, f' \in F\}.$$

Then B is a subring of L. Note that $K \subset B$. As $R \to Q(R)$ is injective, $A = B \cap S$.

If $\alpha \in K$ is integral over A, then it is an element of $B \subset L$ which is integral over S. This shows that $\alpha \in B \cap S = A$, and we are done. $\square$

32.7 Corollary. *Let Γ be an abstract group acting on a finite direct product S of normal domains. Then S^Γ is a finite direct product of normal domains.*

Proof. Set $R = S$ and $F = \Gamma$, and apply the lemma. $\square$

32.8 Corollary. *Let H be an affine algebraic k-group scheme, and S an H-algebra which is a finite direct product of normal domains. Then S^H is also a finite direct product of normal domains.*

Proof. Set $R = S \otimes k[H]$, and $F = \{i, \omega\}$, where $i : S \to R$ is given by $i(s) = s \otimes 1$, and $\omega : S \to R$ is the coaction. Since both i and ω are flat, the lemma is applicable. $\square$

33 Other examples of diagrams of schemes

(33.1) We define an ordered finite category $\mathcal{K}$ by $\mathrm{ob}(\mathcal{K}) = \{s, t\}$, and $\mathcal{K}(s,t) = \{u, v\}$. Pictorially, $\mathcal{K}$ looks like $t \underset{v}{\overset{u}{\leftleftarrows}} s$.

Let p be a prime number, and X an $\mathbb{F}_p$-scheme. We define the *Lyubeznik diagram* $\mathrm{Ly}(X)$ of X to be an object of $\mathcal{P}(\mathcal{K}, \underline{\mathrm{Sch}}/\mathbb{F}_p)$ given by $(\mathrm{Ly}(X))_s = (\mathrm{Ly}(X))_t = X$, $\mathrm{Ly}(X)_u = \mathrm{id}_X$, and $\mathrm{Ly}(X)_v = F_X$, where F_X denotes the absolute Frobenius morphism of X. Thus $\mathrm{Ly}(X)$ looks like

$$X \underset{F_X}{\overset{\mathrm{id}_X}{\rightrightarrows}} X\ .$$

We define an *F-sheaf* of X to be a quasi-coherent sheaf over $\mathrm{Ly}(X)$. It can be identified with a pair $(\mathcal{M}, \phi)$ such that $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module, and $\phi : \mathcal{M} \to \mathcal{F}_X^*\mathcal{M}$ is an isomorphism of $\mathcal{O}_X$-modules. Indeed, if $\mathcal{N} \in \mathrm{Qch}(\mathrm{Ly}(X))$, then letting $\mathcal{M} := \mathcal{N}_s$ and setting ϕ to be the composite

$$\mathcal{M} = \mathcal{N}_s \cong \mathrm{id}_X^*\mathcal{N}_s = \mathrm{Ly}(X)_u^*\mathcal{N}_s \xrightarrow{\alpha_u} \mathcal{N}_t \xrightarrow{\alpha_v^{-1}} \mathrm{Ly}(X)_v^*\mathcal{N}_s = F_X^*\mathcal{N}_s = F_X^*\mathcal{M},$$

$(\mathcal{M}, \phi)$ is such a pair. Thus if $X = \operatorname{Spec} R$ is affine, then the category $\mathrm{Qch}(\mathrm{Ly}(X))$ of F-sheaves of X is equivalent to the category of F-modules defined by Lyubeznik [28].

Note that $\mathrm{Ly}(X)$ is noetherian with flat arrows if and only if X is a noetherian regular scheme by Kunz's theorem [24]. Let $f\colon X \to Y$ be a morphism of noetherian $\mathbb{F}_p$-schemes. Then $\mathrm{Ly}(f) : \mathrm{Ly}(X) \to \mathrm{Ly}(Y)$ is defined in an obvious way.

(33.2) For a ring A of characteristic p, the Frobenius map $A \to A$ ($a \mapsto a^p$) is denoted by $F = F_A$. So $F_A^e(a) = a^{p^e}$. Let k be a perfect field of characteristic p. For a k-algebra $u : k \to A$, we define a k algebra $A^{(r)}$ as follows. As a ring, $A^{(r)} = A$, but the k-algebra structure of $A^{(r)}$ is given by

$$k \xrightarrow{F_k^{-r}} k \xrightarrow{u} A.$$

For $e \geq 0$, $F_A^e\colon A^{(r+e)} \to A^{(r)}$ is a k-algebra map. We sometimes denote an element $a \in A$, viewed as an element of $A^{(r)}$, by $a^{(r)}$. Thus $F^e(a^{(r)}) = (a^{p^e})^{(r-e)}$. For a k-scheme X, the k-scheme $X^{(r)}$ is defined similarly, and the Frobenius morphism $F_X^e\colon X^{(r)} \to X^{(r+e)}$ is a k-morphism. This notation is used for $k = \mathbb{F}_p$ for all rings of characteristic p.

33.3 Lemma. *Let k be a field of characteristic p, and K a finitely generated extension field of k. Then the canonical map $\Phi^{\mathrm{RA}}\colon k \otimes_{k^{(1)}} K^{(1)} \to K$ (the* Radu-André homomorphism) *given by $\Phi^{\mathrm{RA}}(\alpha \otimes \beta^{(1)}) = \alpha\beta^p$ is an isomorphism if and only if K is a separable algebraic extension of k.*

Proof. We prove the 'if' part. Note that $k \otimes_{k^{(1)}} K^{(1)}$ is a field. If $d = [K : k]$, then both $k \otimes_{k^{(1)}} K^{(1)}$ and K have the same k-dimension d. Since Φ^{RA} is an injective k-algebra map, it is an isomorphism.

We prove the 'only if' part. Since $k \otimes_{k^{(1)}} K^{(1)}$ is isomorphic to K, it is a field. So K/k is separable.

Let $x_1, \dots, x_n$ be a separable basis of K over k. Then K, which is the image of Φ^{RA}, is a finite separable extension of $k(x_1^p, \dots, x_n^p)$. If $n \geq 1$, then x_1 is both separable and purely inseparable over $k(x_1^p, \dots, x_n^p)$. Namely, $x_1 \in k(x_1^p, \dots, x_n^p)$, which is a contradiction. So $n = 0$, that is, K is separable algebraic over k. $\square$

33.4 Lemma. *Let A be a noetherian ring, and $\varphi : F \to F'$ an A-linear map between A-flat modules. Then φ is an isomorphism if and only if $\varphi \otimes 1_{\kappa(\mathfrak{p})} : F \otimes_A \kappa(\mathfrak{p}) \to F' \otimes_A \kappa(\mathfrak{p})$ is an isomorphism for any $\mathfrak{p} \in \operatorname{Spec} A$.*

Proof. Follows easily from [19, (I.2.1.4) and (I.2.1.5)]. $\square$

33.5 Lemma. *Let $f\colon X \to Y$ be a morphism locally of finite type between locally noetherian $\mathbb{F}_p$-schemes. Then the diagram*

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "f"] & X^{(1)} \arrow[d, "f^{(1)}"] \\ Y \arrow[r, "F_Y"] & Y^{(1)} \end{tikzcd} \tag{33.6}$$

is cartesian if and only if f is étale.

Proof. Obviously, the question is local on both X and Y, so we may assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine.

We prove the 'only if' part. By Radu's theorem [38, Corollaire 6], $A \to B$ is regular. In particular, B is A-flat. The canonical map $A \otimes_{A^{(1)}} B^{(1)} \to B$ is an isomorphism. So for any $P \in \operatorname{Spec} B$, $\kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})^{(1)}} (\kappa(\mathfrak{p}) \otimes_A B_P)^{(1)} \to \kappa(\mathfrak{p}) \otimes_A B_P$ is an isomorphism, where $\mathfrak{p} = P \cap A$. Let K be the field of fractions of the regular local ring $\kappa(\mathfrak{p}) \otimes_A B_P$. Then $\kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})^{(1)}} K^{(1)} \to K$ is an isomorphism. By Lemma 33.3, K is a separable algebraic extension of

$\kappa(\mathfrak{p})$. Since $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{p}) \otimes_A B_P \subset K$, we have that $\kappa(\mathfrak{p}) \otimes_A B_P$ is a separable algebraic extension field of $\kappa(\mathfrak{p})$. So f is étale at P. As P is arbitrary, B is étale over A.

We prove the 'if' part. By Lemma 33.3, $\kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})^{(1)}} (\kappa(\mathfrak{p}) \otimes_A B_P)^{(1)} \to \kappa(\mathfrak{p}) \otimes_A B_P$ is an isomorphism for $P \in \operatorname{Spec} B$, where $\mathfrak{p} = P \cap A$. Then it is easy to see that $\kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})^{(1)}} (\kappa(\mathfrak{p}) \otimes_A B)^{(1)} \to \kappa(\mathfrak{p}) \otimes_A B$ is a isomorphism for $\mathfrak{p} \in \operatorname{Spec} A$. By Lemma 33.4, $A \otimes_{A^{(1)}} B^{(1)} \to B$ is an isomorphism, as desired. $\square$

By the lemma, for a morphism $f\colon X \to Y$ of noetherian $\mathbb{F}_p$-schemes, $\operatorname{Ly}(f)$ is cartesian of finite type if and only if f is étale.

(33.7) Let I be a small category, and $R_\bullet$ a covariant functor from I to the category of (non-commutative) rings. A left $R_\bullet$*-module* is a collection $\mathcal{M} = ((M_i)_{i\in \mathrm{ob}(I)}, (\beta_\phi)_{\phi\in \mathrm{Mor}(I)})$ such that M_i is a left R_i-module for each $i \in \mathrm{ob}(I)$, and for $\phi \in I(i,j)$, $\beta_\phi\colon M_i \to M_j$ is an R_i-linear map, where M_j is viewed as an R_i-module through the ring homomorphism $R_\phi : R_i \to R_j$. Moreover, we require the following conditions.

1 For $i \in \mathrm{ob}(I)$, $\beta_{\mathrm{id}_i} = \mathrm{id}_{M_i}$.

2 For $\phi, \psi \in \mathrm{Mor}(I)$ such that $\psi\phi$ is defined, $\beta_\psi\beta_\phi = \beta_{\psi\phi}$.

For $\phi \in \mathrm{Mor}(I)$, let $s(\phi) = i$ and $t(\phi) = j$ if $\phi \in I(i,j)$. i (resp. j) is the source (resp. target) of ϕ. Set $\mathcal{A}(R_\bullet) := \bigoplus_{\phi\in\mathrm{Mor}(I)} R_{t(\phi)}\phi$. We define $(b\psi)(a\phi) = (b \cdot R_\psi a)(\psi\phi)$ if $\psi\phi$ is defined, and $(b\psi)(a\phi) = 0$ otherwise. Then $\mathcal{A}(R_\bullet)$ is a ring possibly without the identity element. If $\mathrm{ob}(I)$ is finite, then $\sum_{i\in\mathrm{ob}(I)} \mathrm{id}_i$ is the identity element of $\mathcal{A}(R_\bullet)$. We call $\mathcal{A}(R_\bullet)$ the *total ring* of $R_\bullet$.

Let $\mathcal{M} = ((M_i)_{i\in\mathrm{ob}(I)}, (\beta_\phi)_{\phi\in\mathrm{Mor}(I)})$ be an $R_\bullet$-module. Then $M = \bigoplus_i M_i$ is an $\mathcal{A}(R_\bullet)$-module by $(a\phi)(\sum_j m_j) = a\beta_\phi m_{s(\phi)}$ for $\phi \in \mathrm{Mor}(I)$, $a \in R_{t(\phi)}$, and $m_j \in M_j$. It is a unitary module if $\mathrm{ob}(I)$ is finite.

From now on, assume that $\mathrm{ob}(I)$ is finite. Then a (unitary) $\mathcal{A}(R_\bullet)$-module M yields an $R_\bullet$-module. Set $M_i = \mathrm{id}_i M$. Then M_i is an R_i-module via $r(\mathrm{id}_i m) = (r\mathrm{id}_i m) = \mathrm{id}_i(r\mathrm{id}_i m)$. For $\phi \in I(i,j)$, $\beta_\phi : M_i \to M_j$ is defined by $\beta_\phi(m) = \phi m$. Thus an $R_\bullet$-module $((M_i), (\beta_\phi))$ is obtained. Note that the category of $R_\bullet$-modules and the category of $\mathcal{A}(R_\bullet)$-modules are equivalent.

Now consider the case that each R_i is commutative. Then $R_\bullet$ yields $X_\bullet = \operatorname{Spec}_\bullet R_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}})$. By (4.10), the category of $R_\bullet$-modules is equivalent to $\mathrm{Lqc}(X_\bullet)$.

33.8 Lemma. *Let I be a finite ordered category, and $R_\bullet$ a covariant functor from I to the category of commutative rings. If R_i is regular with finite Krull dimension for each $i \in \mathrm{ob}(I)$, and R_ϕ is flat for each $\phi \in \mathrm{Mor}(I)$, then $\mathcal{A}(R_\bullet)$ has a finite global dimension.*

Proof. Follows easily from Lemma 31.3. $\square$

Glossary

$[?, -]$	the internal hom, 12
$\heartsuit$	stands for either PA, AB, PM, or Mod, 31
$(?)\|_J$	the pull-back associated with the inclusion $J \hookrightarrow I$, 61
$(?)_J^{\heartsuit}$	the abbreviation for $Q(X_\bullet, J)^{\#}_{\heartsuit}$, 63
$(?)_{J_1,J}^{\heartsuit}$	the restriction $\heartsuit(X_\bullet\|_J) \to \heartsuit(X_\bullet\|_{J_1})$, 63
$(?)_J^{\mathrm{AB}}$	the abbreviation for $Q(X_\bullet, J)^{\#}_{\mathrm{AB}}$, 61
$(?)_J^{\mathrm{PA}}$	the abbreviation for $Q(X_\bullet, J)^{\#}_{\mathrm{PA}}$, 61
$\otimes$	the product structure, 12
$\otimes_{\mathcal{O}_{\mathbb{X}}}$	the sheaf tensor product, 27
$\otimes^p_{\mathcal{O}_{\mathbb{X}}}$	the presheaf tensor product, 27
$(?)^!$	the equivariant twisted inverse, 148
$(?)\|_x^{\heartsuit}$	the restriction functor, 32
$\mathbb{A}$	the ascent functor, 104
$\mathcal{A}$	the category of noetherian I^{op}-diagrams of schemes and morphisms separated of finite type, 145
$\underline{\mathrm{Ab}}$	the category of abelian groups, 23
$\mathrm{AB}(\mathbb{X})$	the category of sheaves of abelian groups on $\mathbb{X}$, 23
$\mathcal{A}_G$	the category of noetherian G-schemes and G-morphisms separated of finite type, 169
α	the associativity isomorphism, 12
(α)	the canonical map $(d_0)^* \to (?)_{(\Delta)} \circ (?)'$, 100
(α^+)	the canonical map $(d_0^+)^* \to (?)' \circ (?)_{(\Delta)}$, 100
$\alpha_\phi^{\heartsuit}$	the translation map, 61
$\mathcal{A}(R_\bullet)$	the total ring of $R_\bullet$, 186
$a(\mathbb{X}, \mathrm{AB})$	the sheafification functor $\mathrm{PA}(\mathbb{X}) \to \mathrm{AB}(\mathbb{X})$, 23
$a(\mathbb{X}, \mathrm{Mod})$	the sheafification $\mathrm{PM}(\mathbb{X}) \to \mathrm{Mod}(\mathbb{X})$, 25

$\mathcal{D}^{-}(X_\bullet)$	locally bounded above derived category of $X_\bullet$, 139
$\mathcal{D}^{+}(X_\bullet)$	locally bounded below derived category of $X_\bullet$, 139
$\mathfrak{D}_{\heartsuit}(X_\bullet)$	the category of structure data, 67
$\mathrm{EM}(G, X)$	the category of equivariant $(G, \mathcal{O}_X)$-modules, 170
$\mathrm{EM}(X_\bullet)$	the category of equivariant sheaves of $\mathcal{O}_{X_\bullet}$-modules, 63
ε	the counit map of adjunction, 8
$\eta = \eta(f)$	the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$, 14
ev	the evaluation map, 13
$\mathfrak{e}_X$	the isomorphism $\mathrm{Id}_{X_\#} \to (\mathrm{id}_X)_\#$, 5
$\mathcal{F}$	the subcategory of $\mathcal{A}$ consisting of objects with flat arrows and cartesian morphisms, 148
$f_\bullet^\natural$	the twisted inverse for a cartesian finite morphism $f_\bullet$, 163
$(f_\bullet)^*_\heartsuit$	the inverse image functor, 65
$(f_\bullet)_*^\heartsuit$	the direct image functor, 65
$f_\bullet^\times$	the right adjoint of $R(f_\bullet)_*$, 134
$f^\#$	the pull-back associated with f, 26
$f^\times$	the right adjoint of $Rf_* : D_{\mathrm{Qch}}(X) \to D(Y)$, 125
$f^\#_{\mathrm{AB}}$	the pull-back $\mathrm{AB}(\mathbb{X}) \to \mathrm{AB}(\mathbb{Y})$, 26
$f_\#^{\mathrm{AB}}$	the left adjoint of $f^\#_{\mathrm{AB}}$, 26
$f_\#^{\mathcal{C}}$	the left adjoint of $f^\#_{\mathcal{C}}$, 26
$(f_\bullet)_*^{\mathrm{Lqc}}$	the direct image functor for Lqc, 86
$\mathcal{F}_M$	see page, 169
$f^\#_{\mathrm{Mod}}$	the pull-back $\mathrm{Mod}(\mathbb{X}) \to \mathrm{Mod}(\mathbb{Y})$ for a ringed continuous functor $f\colon (\mathbb{Y}, \mathcal{O}_{\mathbb{Y}}) \to (\mathbb{X}, \mathcal{O}_{\mathbb{X}})$, 31
$f_\#^{\mathrm{Mod}}$	the left adjoint of $f^\#_{\mathrm{Mod}}$, 31
$f^\#_{\mathrm{PA}}$	the pull-back $\mathrm{PA}(\mathbb{X}) \to \mathrm{PA}(\mathbb{Y})$ for $f : \mathbb{Y} \to \mathbb{X}$, 25
$f_\#^{\mathrm{PA}}$	the left adjoint of $f^\#_{\mathrm{PA}}$, 26

$f_\flat^{\mathrm{PM}}$	the right adjoint of $f^\#_{\mathrm{PM}}$, 31
$f^\#_{\mathrm{PM}}$	the pull-back $\mathrm{PM}(\mathbb{X}) \to \mathrm{PM}(\mathbb{Y})$ for a ringed functor $f\colon (\mathbb{Y}, \mathcal{O}_{\mathbb{Y}}) \to (\mathbb{X}, \mathcal{O}_{\mathbb{X}})$, 30
$f_\#^{\mathrm{PM}}$	the left adjoint of $f^\#_{\mathrm{PM}}$, 30
$F(\mathbb{X})$	the forgetful functor $\mathrm{Mod}(\mathbb{X}) \to \mathrm{AB}(\mathbb{X})$, 25
$\mathfrak{f}_X$	the isomorphism $\mathrm{id}_X^\# \to \mathrm{Id}_{X^\#}$, 6
γ	the twisting (symmetry) isomorphism, 12
Γ_i	$L_{I,J_1} \circ R_{J_1,i}$, 139
H	see page, 14
hocolim	the homotopy colimit, 114
$\operatorname{holim} t_i$	the homotopy limit of (t_i), 114
$\underline{\mathrm{Hom}}_{\heartsuit(\mathbb{X})}(\mathcal{M}, \mathcal{N})$	the sheaf Hom functor, 32
ι	the inclusion $(\Delta) \hookrightarrow (\Delta^+)$, 98
I_x^f	see page, 26
$K^?(\mathcal{A})$	the homotopy category of $\mathcal{A}$ with the boundedness ?, 49
$K^?_{\mathcal{A}'}(\mathcal{A})$	the full subcategory of $K^?(\mathcal{A})$ consisting of complexes whose cohomology groups lie in $\mathcal{A}'$, 49
λ	the left unit isomorphism, 12
$\lambda_{J,i}$	the canonical isomorphism $(L_J^\heartsuit(\mathcal{M}))_i^\heartsuit \cong \varinjlim (X_\phi)^*_\heartsuit(\mathcal{M}_j)$, 70
Lch	the category of locally coherent sheaves, 117
$L_J^\heartsuit$	the left induction functor, 65
$L_{J,J_1}^\heartsuit$	the left adjoint of $(?)^\heartsuit_{J_1,J}$, 65
lqc	the local quasi-coherator for a diagram of schemes, 118
$\mathrm{Lqc}(G, X)$	the category of locally quasi-coherent $(G, \mathcal{O}_X)$-modules, 170
$\mathrm{Lqc}(X_\bullet)$	the full subcategory of locally quasi-coherent sheaves in $\mathrm{Mod}(X_\bullet)$, 83

$\mathcal{P}(I,\mathcal{C})$	the category of presheaves over the category I with values in $\mathcal{C}$, 23
$\Pi(f)$	the canonical map (projection morphism) $f_*a\otimes b \to f_*(a\otimes f^*b)$, 156
$\mathrm{PM}(\mathbb{X})$	the category of presheaves of $\mathcal{O}_{\mathbb{X}}$-modules, 25
$\mathrm{PM}(X_\bullet)$	the abbreviation for $\mathrm{PM}(\mathrm{Zar}(X_\bullet))$, 63
$\mathcal{P}(X_\bullet,\mathcal{C})$	the abbreviation for $\mathcal{P}(\mathrm{Zar}(X_\bullet),\mathcal{C})$, 61
Q	the localization $K^?(\mathcal{A}) \to D^?(\mathcal{A})$, 49
$\mathfrak{Q}$	the full subcategory of $C(\mathrm{Mod}(\mathbb{X}))$ consisting of bounded above complexes whose terms are direct sums of copies of $\mathcal{O}_x$, 55
$\mathrm{Qch}(G,X)$	the category of quasi-coherent $(G,\mathcal{O}_X)$-modules, 170
$\mathrm{qch}(X)$	the quasi-coherator on a scheme X, 118
$\mathrm{Qch}(X_\bullet)$	the full subcategory of $\mathrm{Mod}(X_\bullet)$ consisting of quasi-coherent modules, 83
$\mathrm{Qch}(Z)$	the category of quasi-coherent $\mathcal{O}_Z$-modules of a scheme Z, 1
$q(\mathbb{X},\mathrm{AB})$	the inclusion $\mathrm{AB}(\mathbb{X}) \to \mathrm{PA}(\mathbb{X})$, 23
$Q(X_\bullet,J)$	the inclusion $\mathrm{Zar}((X_\bullet)\vert_J) \hookrightarrow \mathrm{Zar}(X_\bullet)$, 61
$q(\mathbb{X},\mathrm{Mod})$	the inclusion $\mathrm{Mod}(\mathbb{X}) \to \mathrm{PM}(\mathbb{X})$, 25
$R^\star$	the set of nonzerodivisors of R, 182
ρ	the right unit isomorphism, 18
$\rho^{J,i}$	the canonical isomorphism $(R_J^\heartsuit(\mathcal{M}))_i^\heartsuit \cong \varprojlim (X_\phi)_*^\heartsuit(\mathcal{M}_j)$, 73
$R_J^\heartsuit$	the right induction functor, 65
$R_{J,J_1}^\heartsuit$	the right adjoint of $(?)_{J_1,J}^\heartsuit$, 65
$\mathfrak{R}_\phi$	the canonical functor $\mathbb{X}/x \to \mathbb{X}/y$ for $\phi : x \to y$, 32
$\mathfrak{R}_x$	the canonical functor $\mathbb{X}/x \to \mathbb{X}$, 32
$\underline{\mathrm{Sch}}$	the category of schemes, 60

$\underline{\mathrm{Sch}}/S$	the category of S-schemes, 60
$\underline{\mathrm{Set}}$	the category of small sets, 23
shift	the standard shifting functor $(\Delta^+) \to (\Delta)$, 98
Σ	the suspension of a triangulated category, 49
Σ_i	the right adjoint of Γ_i, 139
$\Sigma(X_\bullet)$	the simplicial S-scheme associated with $X_\bullet$, 110
$\mathcal{S}(\mathbb{X}, \mathcal{C})$	the category of sheaves over $\mathbb{X}$ with values in $\mathcal{C}$, 23
$\mathcal{S}(X_\bullet, \mathcal{C})$	the abbreviation for $\mathcal{S}(\mathrm{Zar}(X_\bullet), \mathcal{C})$, 61
$\tau_{\geq n}\mathbb{F}$	the truncation of a complex, 56
$\tau_{\leq n}\mathbb{F}$	the truncation of a complex, 56
$\bar{\theta}$	the canonical map $af^\# \to f^\# a$, 33
$\Theta(f)$	the duality isomorphism for schemes, 155
$\Theta(f_\bullet)$	the duality isomorphism, 155
$\theta_\heartsuit(f_\bullet, J)$	the canonical isomorphism $((f_\bullet)\|_J)^*_\heartsuit \circ (?)_J \to (?)_J \circ (f_\bullet)^*_\heartsuit$, 77
$\theta(J, f_\bullet)$	the canonical map $L_J(f_\bullet\|_J)_* \to (f_\bullet)_* L_J$, 76
$\theta(\sigma)$	Lipman's theta, 10
tr	the trace map, 13
u	the unit map of adjunction, 8
Υ	the independence isomorphism, 130
$X'_\bullet$	the augmented simplicial scheme $\mathrm{shift}^\#(X_\bullet) = X_\bullet\,\mathrm{shift}$, 98
Ξ	the canonical map $QF \to (RF)Q$, 50
$\bar{\xi}$	the canonical map $R(g^X_\bullet)_*(f'_\bullet)^! \to f^!_\bullet R(g_\bullet)_*$, 152
$\xi_\heartsuit(f_\bullet, J)$	the isomorphism $(f_\bullet)^\heartsuit_* R_J \to R_J(f_\bullet\|_J)^\heartsuit_*$, 77
$\xi(J, f_\bullet)$	the natural map $(?)_J \circ f^\times_\bullet \to f^\times_J \circ (?)_J$, 135
$\bar{\xi}(J, f_\bullet)$	the canonical map $(?)_J f^!_\bullet \to f_\bullet\|^!_J (?)_J$, 148

References

[1] L. Alonso Tarrío, A. Jeremías López, and M. J. Souto Salorio, Localization in categories of complexes and unbounded resolutions, *Canad. J. Math.* **52** (2000), 225–247.

[2] M. Artin, *Grothendieck Topology,* Mimeographed notes, Harvard University (1962).

[3] L. L. Avramov, Flat morphisms of complete intersections, *Dokl. Akad. Nauk SSSR* **225** (1975), *Soviet Math. Dokl.* **16** (1975), 1413–1417.

[4] L. L. Avramov and H.-B. Foxby, Locally Gorenstein homomorphisms, *Amer. J. Math.* **114** (1992), 1007–1047.

[5] A. A. Beilinson, J. Bernstein, and P. Deligne, Faisceaux pervers, Astérisque **100** (1982), 5–171.

[6] J. Bernstein and V. Lunts, *Equivariant sheaves and functors*, *Lect. Notes Math.* **1578**, Springer Verlag (1994).

[7] M. Bökstedt and A. Neeman, Homotopy limits in triangulated categories, *Compositio Math.* **86** (1993), 209–234.

[8] A. Bondal and M. van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, *Mosc. Math. J.* **3** (2003), 1–36, 258.

[9] J. Franke, On the Brown representability theorem for triangulated categories, *Topology* **40** (2001), 667–680.

[10] E. M. Friedlander, *Etale homotopy of simplicial schemes,* Princeton (1982).

[11] P. Gabriel, Des categories abeliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.

[12] P. Gabriel, Construction de préschémas quotient, in *Schémas en Groupes I* (*SGA3*), *Lect. Notes Math.* **151**, Springer Verlag (1970), pp. 251–286.

[13] A. Grothendieck, Sur quelques points d'algèbre homologique, *Tôhoku Math. J.* **9** (1957), 119–221.

[14] A. Grothendieck, *Eléments de Géométrie Algébrique I,* IHES Publ. Math. **4** (1960).

[15] A. Grothendieck, *Eléments de Géométrie Algébrique IV,* IHES Publ. Math. **20** (1964), **24** (1965), **28** (1966), **32** (1967).

[16] A. Grothendieck et J.-L. Verdier, Préfaisceaux, in *Théorie des Topos et Cohomologie Etale des Schémas, SGA 4, Lect. Notes Math.* **269**, Springer Verlag (1972), pp. 1–217.

[17] R. Hartshorne, *Residues and Duality, Lect. Notes Math.* **20**, Springer Verlag, (1966).

[18] R. Hartshorne, *Algebraic Geometry, Graduate Texts in Math.* **52**, Springer Verlag (1977).

[19] M. Hashimoto, *Auslander-Buchweitz Approximations of Equivariant Modules, London Mathematical Society Lecture Note Series* **282**, Cambridge (2000).

[20] L. Illusie, Existence de résolutions globales, in *Théorie des Intersections et Théorème de Riemann-Roch* (*SGA 6*)*, Lect. Notes Math.* **225**, Springer Verlag, pp. 160–221.

[21] J. C. Jantzen, *Representations of algebraic groups,* Second edition, AMS (2003).

[22] B. Keller and D. Vossieck, Sous les catégories dérivées, *C. R. Acad. Sci. Paris Sér. I Math.* **305** (1987), 225–228.

[23] G. R. Kempf, Some elementary proofs of basic theorems in the cohomology of quasi-coherent sheaves, *Rocky Mountain J. Math.* **10** (1980), 637–645.

[24] E. Kunz, Characterizations of regular local rings of characteristic p, *Amer. J. Math.* **91** (1969), 772–784.

[25] G. Lewis, Coherence for a closed functor, in *Coherence in Categories, Lect. Notes Math.* **281**, Springer Verlag (1972), pp. 148–195.

[26] J. Lipman, Notes on Derived Functors and Grothendieck Duality, *Foundations of Grothendieck Duality for Diagrams of Schemes, Lect. Notes Math.,* Springer Verlag (in this volume).

[27] W. Lütkebohmert, On compactification of schemes, *Manuscripta Math.* **80** (1993), 95–111.

[28] G. Lyubeznik, F-modules: applications to local cohomology and D-modules in characteristic $p > 0$, *J. reine angew. Math.* **491** (1997), 65–130.

[29] S. Mac Lane, *Categories for the Working Mathematician,* 2nd ed. *Graduate Texts in Math.* **52**, Springer Verlag (1998).

[30] H. Matsumura, *Commutative Ring Theory,* First paperback edition, Cambridge (1989).

[31] J. S. Milne, *Étale cohomology,* Princeton (1980).

[32] D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant Theory,* third edition, Springer (1994).

[33] J. P. Murre, *Lectures on an introduction to Grothendieck's theory of the fundamental group,* Tata Institute, Bombay (1967).

[34] M. Nagata, A generalization of the imbedding problem of an abstract variety in a complete variety, *J. Math. Kyoto Univ.* **3** (1963), 89–102.

[35] A. Neeman, The Grothendieck duality theorem via Bousfield's techniques and Brown representability, *J. Amer. Math. Soc.* **9** (1996), 205–236.

[36] A. Neeman, *Triangulated Categories,* Princeton (2001).

[37] N. Popescu, *Abelian Categories with Applications to Rings and Modules,* Academic Press (1973).

[38] N. Radu, Une classe d'anneaux noethériens, *Rev. Roumanie Math. Pures Appl.* **37** (1992), 79–82.

[39] N. Spaltenstein, Resolutions of unbounded complexes, *Compositio Math.* **65** (1988), 121–154.

[40] H. Sumihiro, Equivariant completion. II, *J. Math. Kyoto Univ.* **15** (1975), 573–605.

[41] J.-L. Verdier, Base change for twisted inverse images of coherent sheaves, in *Algebraic Geometry* (*Internat. Colloq.*), Tata Inst. Fund. Res., Bombay (1968), pp. 393–408.

[42] J.-L. Verdier, Topologies et faisceaux, in *Théorie des Topos et Cohomologie Etale des Schémas, SGA 4, Lect. Notes Math.* **269**, Springer Verlag (1972), pp. 219–263.

[43] J.-L. Verdier, Fonctorialité des catégories de faisceaux, in *Théorie des Topos et Cohomologie Etale des Schémas, SGA 4, Lect. Notes Math.* **269**, Springer Verlag (1972), pp. 265–297.

[44] J.-L. Verdier, Catégories dérivées, quelques résultats (etat 0), in *Cohomologie Etale, SGA* $4\frac{1}{2}$*, Lect. Notes Math.* **569**, Springer Verlag (1977), pp. 262–311.

[45] K.-i. Watanabe, Certain invariant subrings are Gorenstein I, *Osaka J. Math.* **11** (1974), 1–8.

[46] W. C. Waterhouse, *Introduction to Affine Group Schemes, Graduate Texts in Math.* **66**, Springer Verlag (1979).

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