The Asymptotic Behavior of Frobenius Direct Images of Rings of Invariants

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Abstract

We define the Frobenius limit of a module over a ring of prime characteristic to be the limit of the normalized Frobenius direct images in a certain Grothendieck group. When a finite group acts on a polynomial ring, we calculate this limit for all the modules over the twisted group algebra that are free over the polynomial ring; we also calculate the Frobenius limit for the restriction of these to the ring of invariants. As an application, we generalize the description of the generalized F-signature of a ring of invariants by the second author and Nakajima to the modular case.

1. Introduction

(1.1) In commutative algebra, the study of the asymptotic behavior of the Frobenius direct images of a ring of prime characteristic p (or a module over it) has been very fruitful. This includes the study of invariants such as the Hilbert–Kunz multiplicity [Mon] and the F-signature [HL] and its variants [San, HasN].

These invariants have been studied for the ring of invariants of a finite group acting on a ring, see [WY, (2.7), (5.4)], [HL, Example 18], [WY2, (4.2)], [HasN, (3.9)], and [Nak].

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(1.2) Let $T = \bigoplus_{n\geq 0} T_n$ be a graded Noetherian commutative ring, where T_0 is a finite direct product of Henselian local rings. Let $S = \bigoplus_{n\geq 0} S_n$ be a finite graded T-algebra, which might not be commutative.

Let $\Theta^*(S)$ denote the Grothendieck group of the commutative monoid of finitely generated $\mathbb{Q}$-graded S-modules under direct sum, but tensored with $\mathbb{R}$; this means that $\Theta^*(S)$ is the $\mathbb{R}$-space generated by the finitely generated $\mathbb{Q}$-graded S-modules subject to the relations $[M] = [M_1] + [M_2]$ whenever $M \cong M_1 \oplus M_2$. We define $\Theta^\circ(S)$ to be the quotient of $\Theta^*(S)$ by the relation $[M] = [M[\lambda]]$ for a finitely generated $\mathbb{Q}$-graded S-module M and $\lambda \in \mathbb{Q}$, where $?[\lambda]$ denotes shift of degree by λ.

Because of our hypotheses on S, the Krull–Schmidt property holds and so the finitely generated indecomposable $\mathbb{Q}$-graded modules form a basis for $\Theta^*(S)$. Thus $\mathrm{Ind}^\circ(S)$, the set of indecomposable $\mathbb{Q}$-graded modules modulo shift of degree, forms a basis for $\Theta^\circ(S)$. For $\alpha \in \Theta^\circ(S)$ we can write

$$\alpha = \sum_{M\in \mathrm{Ind}^\circ S} c_M[M] \qquad (c_M \in \mathbb{R})$$

uniquely. We define $\|\alpha\|_S := \sum_M |c_M| u_S(M)$, where $u_S(M)$ denotes $\ell_S(M/\mathfrak{m}_S M)$, where $\mathfrak{m}_S = S_+ + J(S_0)$ is the graded Jacobson radical of S and ℓ_S denotes the length function. It is easy to see that $(\Theta(S), \|\cdot\|)_S$ is a normed space.

(1.3) Now let k be an F-finite (that is, $[k : k^p] < \infty$) field of characteristic p, and $R = \bigoplus_{n\geq 0} R_n$ a graded Noetherian commutative ring such that R_0 is an F-finite Henselian local ring. We assume that $\dim_k R_0/J(R_0) < \infty$. Let G be a finite group acting on R as k-algebra automorphisms. Let $S = R * G$ and $T = R^G$. Then T and S are as in (1.2).

Let d be the Krull dimension of R. Set $\mathfrak{d} := \log_p[k : k^p]$, and $\delta := d + \mathfrak{d}$. For any finitely generated S-module M, we define the *Frobenius limit* of M to be

$$\mathrm{FL}(M) = \lim_{e\to\infty} \frac{1}{p^{\delta e}}[{}^e M]$$

in $\Theta(S)$, provided that this limit exists, where ${}^e M$ is the eth Frobenius direct image of M. Note that $\mathrm{FL}(M)$ is considered to be the limit of the modules themselves, rather than of some numerical invariant. If the ring is commutative and $\mathrm{FL}(M)$ exists, the Hilbert–Kunz multiplicity and the (generalized) F-signature can be read off from it; see section 3.

(1.4) Suppose that R be commutative. The group $\Theta(R)$ is larger than the Grothendieck group $G_0(R)_{\mathbb{R}}$, where the relations come from short exact sequences. The latter is isomorphic to $A_*(R)_{\mathbb{R}}$, the Chow group of R (tensored with $\mathbb{R}$) through the Riemann–Roch isomorphism τ_R, see [Ful]. Let us write $\tau_R([R]) = c_d + c_{d-1} + \cdots + c_0$, where c_i is the component of dimension i. Then $\tau_R(\mathrm{FL}[R])$ is just c_d, which plays an important role in the intersection theory of commutative algebra, see [Kur, (2.2)] and [KurO].

Bruns gave a formula for $\mathrm{FL}(R)$ for a normal affine semigroup ring (although he did not define Frobenius limits, he proved a theorem [Bru, Theorem 3.1] giving some more information than $\mathrm{FL}(R)$, see Example 3.23).

(1.5) Now suppose that a finite group G acts faithfully on a graded polynomial ring B, so we can form the twisted group algebra $B * G$. The generators of B must be in positive degrees, but not necessarily all the same. Let $A = B^G$.

Theorem ((4.13), (4.16))**.** *Suppose that F is a $\mathbb{Q}$-graded $B * G$-module that is free of rank f over B. Then the F-limits of $[F]$ and $[F^G]$ exist and*

$$\mathrm{FL}(F) = \frac{f}{|G|}[B * G]$$

in $\Theta^\circ(B * G)$ and

$$\mathrm{FL}(F^G) = \frac{f}{|G|}[B]$$

in $\Theta^\circ(A)$. Analogous formulas hold after completion at the irrelevant ideal.

As a consequence we obtain the following theorem.

Theorem ((5.1))**.** *Let $k = V_0, V_1, \ldots, V_n$ be the simple kG-modules. For each i, let $P_i \to V_i$ be the projective cover, and $M_i := (B \otimes_k P_i)^G$. Suppose that F is a $\mathbb{Q}$-graded $B * G$-module that is free of rank f over B. Then the F-limit of $[F^G]$ exists, and*

$$\mathrm{FL}([F^G]) = \frac{f}{|G|}[B] = \frac{f}{|G|}\sum_{i=0}^{n} \frac{\dim_k V_i}{\dim_k \mathrm{End}_{kG}(V_i)}[\hat{M}_i]$$

in $\Theta^\circ(A)$. The analogous formula holds after completion at the irrelevant ideal.

In particular, we have a formula for FL$[A]$ and FL$([\hat{A}])$: see Corollary 5.2.

Using this theorem, we generalize a result on the generalized F-signature [HasN, (3.9)] to the modular case (Corollary 5.7). We also get a new proof of the theorem of Broer [Bro] and Yasuda [Yas] which says that if G does not have a pseudo-reflection and p divides the order $|G|$ of G, then A is not weakly F-regular.

For another application of this work to invariant theory, see [Has2].

In section 2, we fix our notation for Frobenius direct images. In section 3, we study the group $\Theta(S)$ and define the Frobenius limits. In section 4, we prove the main theorems and in section 5 we derive some consequences.

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2. Rings, modules and Frobenius direct image

(2.1) Let k be a field. By a module over a ring we mean a left module, unless otherwise specified. A graded ring means a ring graded by the semigroup of non-negative integers. Modules will be graded by $\mathbb{Q}$; since we only consider finitely generated modules, the graded pieces are only non-zero on a discrete subgroup, which is contained in $\frac{1}{r}\mathbb{Z}$ for some $r \in \mathbb{N}$. The morphisms are degree preserving. Let G be a finite group acting on a ring R. By an (R, G)-module M, we mean an R-module that is also a kG-module in such a way that $g(rm) = (gr)(gm)$, $g \in G, r \in R, m \in M$. If M is an (R, G)-module and V a G-module, then $M \otimes_k V$ is an (R, G)-module by $r(m \otimes v) = rm \otimes v$ and $g(m \otimes v) = gm \otimes gv$ for $r \in R$, $m \in M$, $v \in V$, and $g \in G$.

(2.2) By a virtually commutative ring we mean a ring S that contains some central subalgebra T such that S is finite over T. The example we have in mind is when G acts on a commutative ring R and S is the twisted group algebra $R * G$. That is, $R * G = \bigoplus_{g \in G} Rg$ as an R-module, and the product is given by $(rg)(r'g') = (r(gr'))(gg')$. The ring $R * G$ is finite over the ring of invariants $T = R^G$ in many cases. For example, assume that R is a commutative Noetherian k-algebra and the action of G is by k-algebra automorphisms. If R is of finite type over k; R is complete with residue field k; the characteristic of k is $p > 0$ and R is F-finite (see 2.10) [Fog], [Has, (9.6)]; or the order of G is not divisible by the characteristic of k, then R and $S = R * G$ are finite over $T = R^G$.

An $R*G$-module is an (R,G)-module in an obvious way, and vice versa. We identify these two objects.

(2.3) Note that the (G,R)-module $R\otimes_k kG$ as an $R*G$-module is identified with the rank-one free module $R*G$ by the obvious map $r\otimes g\mapsto rg$.

(2.4) Let k be of characteristic $p>0$. For a commutative k-algebra R, the Frobenius homomorphism $F:R\to R$ is defined by $F(a)=a^p$. For $r\in\mathbb{Z}$, let ${}^rR$ be a copy of the ring R, except that, in the graded case, the values of the grading are divided by p^r (here we briefly suspend our convention that all rings are integer graded). For any $e\geq 0$, we regard ${}^{r+e}R$ an ${}^rR$-algebra through the Frobenius map $F^e: {}^rR=R\to R={}^{r+e}R$. An R-module M, viewed as an ${}^rR$-module is denoted by ${}^rM$; $m\in M$ is denoted by ${}^rm$ when it is viewed as an element of ${}^rM$. When $e\geq 0$, we can regard ${}^eM$ as an R-module by $a({}^em)=F^e(a){}^em={}^e(a^{p^e}m)$. Then $F^e({}^ra)={}^{r+e}(a^{p^e})=({}^{r+e}a)^{p^e}$. The R-module ${}^eM$ is sometimes written as F^e_*M, and is called the eth Frobenius direct image (also called Frobenius pushforward) of M. If R is graded, M is $\mathbb{Q}$-graded, and m is a homogeneous element of degree λ, then letting ${}^rm$ of degree λ/p^r, we have that ${}^rM$ is a $\mathbb{Q}$-graded ${}^rR$-module. If $e\geq 0$, ${}^eM$ is a $\mathbb{Q}$-graded R-module via $F^e:R={}^0R\to{}^eR$.

(2.5) If V is a k-vector space then ${}^eV$ is considered to be a k-vector space through the map F^e for $e\geq 0$: more explicitly, ${}^ev+{}^ev'={}^e(v+v')$ and $\alpha\cdot{}^ev={}^e(\alpha^{p^e}v)$ for $\alpha\in k$ and $v,v'\in V$. When k is perfect, ${}^rV$ has a meaning for $r\in\mathbb{Z}$, and it has the same dimension as V. Note that ${}^eA$ is again a k-algebra, and $F^e:{}^{e'}A\to{}^{e'+e}A$ is a k-algebra map for $e,e'\geq 0$.

(2.6) In the notation above, ${}^0R$, ${}^0M$, ${}^0m$, and so on, are sometimes written as R, M, m, and so on.

(2.7) Slightly more generally, for a commutative k-algebra R and a finite group G acting on R, we define the Frobenius map $F=F_S$ of $S=R*G$ by $F_S(\sum_{g\in G}r_gg)=\sum_g r_g^pg$. If G is trivial, then $R=S$, and F_S is the usual Frobenius map. Thus for an $R*G$-module M, ${}^eM$ is again an $R*G$-module.

(2.8) Applying this to the group ring kG (the case that $R=k$), we find that ${}^eV$ is a kG-module by $g\cdot{}^ev={}^e(gv)$ for $g\in G$ and $v\in V$.

If V is n-dimensional, let $v_1,\ldots,v_n$ be a basis of V; then we can write $gv_j=\sum_i c_{ij}v_i$. If k is perfect, then $g\cdot{}^ev_j={}^e(gv_j)={}^e(\sum_i c_{ij}v_i)=\sum_i c_{ij}^{p^{-e}}{}^ev_i$. Namely, ${}^eV$, as a matrix representation, is obtained by taking the p^eth root of each matrix entry.

Lemma 2.9. *Let k and G be as above.*

1 *Let V be a finite dimensional G-module. If V is defined over $\mathbb{F}_q$, the field with $q = p^e$ elements, and $\mathfrak{d} := \log_p[k : k^p] < \infty$, then ${}^eV \cong V^{p^{\mathfrak{d}e}}$.*

2 ${}^e(kG) \cong (kG)^{p^{\mathfrak{d}e}}$ *for any $e \geq 0$.*

Proof. **1**. We set $r := [{}^ek : k] = p^{\mathfrak{d}e}$. Let V_0 be the finite dimensional $\mathbb{F}_q$-module such that $k \otimes_{\mathbb{F}_q} V_0 \cong V$. Then

$$ {}^eV \cong {}^ek \otimes_{\mathbb{F}_q} {}^eV_0 \cong k^r \otimes_{\mathbb{F}_q} V_0 \cong V^r. $$

2. Since kG is defined over $\mathbb{F}_p$, the assertion follows from **1**. □

(2.10) S is said to be F-finite if ${}^1S$ is a finite S-module. If so, then $F^e : {}^rS \to {}^{r+e}S$ is finite for any $r \in \mathbb{Z}$ and $e \geq 0$.

3. The Grothendieck group $\Theta(S)$

(3.1) Let $\mathcal{C}$ be an additive category. We define its (additive) Grothendieck group to be

$$ [\mathcal{C}] := (\bigoplus_{M \in \operatorname{Iso} \mathcal{C}} \mathbb{Z} \cdot M)/(M - M_1 - M_2 \mid M \cong M_1 \oplus M_2), $$

where $\operatorname{Iso}\mathcal{C}$ is the set of isomorphism classes of objects in $\mathcal{C}$. The class of M in the group $[\mathcal{C}]$ is denoted by $[M]$. We define $[\mathcal{C}]_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} [\mathcal{C}]$. Note that we only have relations for split exact sequences, not all exact sequences, even if $\mathcal{C}$ is abelian.

(3.2) The group $[\mathcal{C}]$ is universal for additive maps from $\mathcal{C}$ to abelian groups, i.e. given an abelian group Γ and an additive map $f : \mathcal{C} \to \Gamma$ (that is, f is a map $\mathcal{C} \to \Gamma$ such that $f(M) = f(M_1) + f(M_2)$ for every M, M_1, M_2 such that $M \cong M_1 \oplus M_2$), f extends to a unique homomorphism of abelian groups $f_* : [\mathcal{C}] \to \Gamma$. Thus $[\mathcal{C}]_{\mathbb{R}}$ is universal for additive maps to $\mathbb{R}$-spaces. It follows that an additive functor $h : \mathcal{C} \to \mathcal{D}$ yields a homomorphism $h_* : [\mathcal{C}] \to [\mathcal{D}]$ which maps $[M]$ to $[hM]$.

Example 3.3. Let S be a k-algebra. Let $S\,\mathrm{mod}$ denote the category of finitely generated S-modules. Let $J(S)$ denote the Jacobson radical of S and assume that $S/J(S)$ is finite dimensional over k. Then $u_{k,S}(M) := \dim_k(M/J(S)M)$ defines an additive function on $S\,\mathrm{mod}$, which extends to $[S\,\mathrm{mod}]_{\mathbb{R}}$.

If S is a commutative integral domain and we let $Q(S)$ denote the field of fractions of S, then $\mathrm{rank}_S(M) = \dim_{Q(S)} Q(S) \otimes_S M$ is also additive and extends to $[S\,\mathrm{mod}]_{\mathbb{R}}$.

(3.4) An additive category $\mathcal{C}$ is said to have the Krull–Schmidt property if the endomorphism ring of any object is semiperfect. If so, the endomorphism ring of an indecomposable object is local, and hence the Krull–Schmidt theorem holds, see [Pop, (5.1.3)]. Thus $[\mathcal{C}]$ is a $\mathbb{Z}$-free module with $\mathrm{Ind}\,\mathcal{C}$ as free basis, where $\mathrm{Ind}\,\mathcal{C}$ is the set of isomorphism classes of indecomposable objects of $\mathcal{C}$ and $\mathrm{Ind}\,\mathcal{C}$ is an $\mathbb{R}$-basis of $[\mathcal{C}]_{\mathbb{R}}$.

(3.5) Let $T = \bigoplus_{n\geq 0} T_n$ be a commutative non-negatively graded Noetherian ring (which might not be a k-algebra) such that T_0 is a finite direct product of Henselian local rings. Let $S = \bigoplus_{n\geq 0} S_n$ be a graded T-algebra that is a finite T-module. For any finite graded S-module M, $\mathrm{End}_{S\,\mathrm{Gr\,mod}}\,M = (\mathrm{End}_S\,M)_0$ is a finite T_0-algebra and is semiperfect [Fac, (3.8)], where $S\,\mathrm{Gr\,mod}$ is the category of graded finite S-modules. Thus the Krull–Schmidt theorem holds for the category $S\,\mathrm{Gr\,mod}$; see [Pop][(5.1.3)]. Let $\mathfrak{m}_S$ denote the graded Jacobson radical $S_+ + J(S_0)$, where $S_+ = \bigoplus_{n>0} S_n$ is the irrelevant ideal. We denote by $\hat{?}$ the $\mathfrak{m}_S$-adic completion, which agrees with the $\mathfrak{m}_T$-adic completion, where $\mathfrak{m}_T$ is the graded Jacobson radical of T.

(3.6) We write $\Theta^*(S) := [S\,\mathrm{Gr\,mod}]_{\mathbb{R}}$, where $S\,\mathrm{Gr\,mod}$ is the category of S-finite $\mathbb{Q}$-graded modules. It will be convenient to consider the quotient of this where we identify any two indecomposable modules that differ only by a shift in degree, which we denote by $\Theta^\circ(S)$ or $\Theta(S)$. We write $\Theta^\wedge(S) := [S\,\mathrm{mod}]_{\mathbb{R}}$, where $S\,\mathrm{mod}$ is the category of S-finite ungraded modules.

(3.7) There is a sequence of natural maps $\Theta^*(S) \to \Theta^\circ(S) \to \Theta^\wedge(S) \to \Theta^\wedge(\hat{S})$.

(3.8) It is easy to see that if S is concentrated in degree zero, then $\Theta^\circ = \Theta^\wedge$, and the theory of $\Theta^\wedge$ for ungraded S is contained in that of Θ°.

(3.9) From now on we will assume that all our rings are of the type just described. If $f : S' \to S$ is a finite degree-preserving map, there is a natural restriction map $f^* : \Theta(S) \to \Theta(S')$ and the inflation map $f_* : \Theta(S') \to \Theta(S)$.

If I is an ideal in S and $q : S \to S/I$ is the quotient map then we sometimes write $\alpha/I\alpha$ for $q_*(\alpha)$.

(3.10) For $\alpha \in \Theta^\circ(S)$, we can write

$$\alpha = \sum_{[M] \in \mathrm{Ind}^\circ S} c_M[M]$$

uniquely, where $\mathrm{Ind}^\circ(S)$ denotes $\mathrm{Ind}(S\,\mathrm{Gr\,mod})/\!\sim$, where $M \sim M'$ if $M \cong M'[\lambda]$ for some $\lambda \in \mathbb{Q}$ (?$[\lambda]$ denotes shift of degree). We define $\|\alpha\|_S := \sum_M |c_M| u_S(M)$, where $u_S(M) = \ell_S(M/\mathfrak{m}_S M)$. Then $(\Theta(S), \|\cdot\|_S)$ is a normed space. Thus $\Theta(S)$ becomes a metric space with the distance function d given by $d(\alpha, \beta) := \|\alpha - \beta\|_S$.

Lemma 3.11. *Let S be as above.*

1 *Let J be any ideal of S such that there exists some $n \geq 1$ such that $\mathfrak{m}_S^n \subset J \subset \mathfrak{m}_S$. Define a norm $\|\cdot\|_S^J$ on $\Theta(S)$ by $\|\alpha\|_S^J = \sum_M |c_M| \ell_S(M/JM)$, where $\ell_S(-)$ denotes the length of an S-module. Then $\|\cdot\|_S^J$ is equivalent to $\|\cdot\|_S$.*

2 *Let $f : S' \to S$ be a degree-preserving ring homomorphism such that $\mathfrak{m}_{S'} S \supset \mathfrak{m}_S^n$ for some $n \geq 1$ and $\mathfrak{m}_{S'}^m S \subset \mathfrak{m}_S$ for some $m \geq 1$ (e.g. S is S'-finite). Define $\|\cdot\|_{S'}^S$ by $\|\alpha\|_{S'}^S = \sum_M |c_M| \ell_{S'}(M/\mathfrak{m}_{S'} M)$. Then $\|\cdot\|_{S'}^S$ is equivalent to $\|\cdot\|_S$.*

3 *Let k be a field, and assume that S is a k-algebra and $\dim_k S/\mathfrak{m}_S < \infty$. Define $\|\alpha\|_{k,S} = \sum_M |c_M| \dim_k M/\mathfrak{m}_S M$. Then $\|\cdot\|_{k,S}$ is equivalent to $\|\cdot\|_S$.*

Proof. **1**. For $M \in S\,\mathrm{Gr\,mod}$ we have $\ell_S(M/JM) \geq \ell_S(M/\mathfrak{m}_S M)$ and $\|\alpha\|_S^J \geq \|\alpha\|_S$ follows easily. There is a surjective map of graded S-modules $F \to M$, with F free of rank $\ell_S(M/\mathfrak{m}_S M)$, which induces a surjection $F/\mathfrak{m}_S^n F \to M/\mathfrak{m}_S^n M$. Setting $r := \ell_S(S/\mathfrak{m}_S^n)$, we obtain $\ell_S(M/JM) \leq \ell_S(M/\mathfrak{m}_S^n M) \leq \ell_S(F/\mathfrak{m}_S^n F) = r\ell_S(M/\mathfrak{m}_S M)$, and $\|\alpha\|_S^J \leq r\|\alpha_S\|$ follows easily. It follows that $\|\cdot\|_S^J$ is equivalent to $\|\cdot\|_S$, as required.

2. Let T' be the center of S'.

First we assume that S is S'-finite (or equivalently, T'-finite) and show that the hypothesis on f is satisfied. If $\mathfrak{m}_{T'}S \not\subset \mathfrak{m}_S$, then there exists some $a \in \mathfrak{m}_{T'}$ such that the ideal $a(S/\mathfrak{m}_S)$ of $S/\mathfrak{m}_S$ is nonzero. As $S/\mathfrak{m}_S$ has finite length, $a^n(S/\mathfrak{m}_S) = a^{n+1}(S/\mathfrak{m}_S)$ for some $n \geq 1$, then by the graded Nakayama's lemma $a^n(S/\mathfrak{m}_S) = 0$. Since $S/\mathfrak{m}_S$ is semisimple, $a(S/\mathfrak{m}_S)$ is an idempotent ideal and so $a^n(S/\mathfrak{m}_S) \neq 0$, a contradiction. Therefore $\mathfrak{m}_{T'}S \subset \mathfrak{m}_S$. Note that $S/\mathfrak{m}_{T'}S$ is a finite $T'/\mathfrak{m}_{T'}$-algebra and is an Artinian algebra, so its radical $\mathfrak{m}_S/\mathfrak{m}_{T'}S$ is nilpotent, and $\mathfrak{m}_S^n \subset \mathfrak{m}_{T'}S$ for some $n \geq 1$. If S is T'-finite, then $\mathfrak{m}_S^n \subset \mathfrak{m}_{T'}S \subset \mathfrak{m}_S$ for some n. Similarly, $\mathfrak{m}_{S'}^m \subset \mathfrak{m}_{T'}S' \subset \mathfrak{m}_{S'}$ for some m. So $\mathfrak{m}_{S'}^m S \subset \mathfrak{m}_S$ and $\mathfrak{m}_S^n \subset \mathfrak{m}_{S'}S$, and the hypothesis is satisfied.

Now we prove the assertion. Let $M \in S\,\mathrm{Gr\,mod}$. Then

$$\begin{aligned} u_S(M) = \ell_S(M/\mathfrak{m}_S M) \leq \ell_{S'}(M/\mathfrak{m}_S M) &\leq \ell_{S'}(M/\mathfrak{m}_{S'}^m M) \\ &\leq \ell_{S'}(S'/\mathfrak{m}_{S'}^m S') \cdot \ell_{S'}(M/\mathfrak{m}_{S'} M). \end{aligned}$$

That $\|\alpha\|_S \leq \ell_{S'}(S'/\mathfrak{m}_{S'}^m S')\|\alpha\|_{S'}^S$ follows easily. On the other hand, we have

$$\ell_{S'}(M/\mathfrak{m}_{S'}M) \leq \ell_{S'}(M/\mathfrak{m}_S^n M) \leq \ell_{S'}(S/\mathfrak{m}_S^n S) \cdot u_S(M),$$

and $\|\alpha\|_{S'}^S \leq \ell_{S'}(S/\mathfrak{m}_S^n S)\|\alpha\|_S$ follows easily. Hence $\|\alpha\|_{S'}^S$ is equivalent to $\|\alpha\|_S$.

3. This is because

$$\ell_S(M/\mathfrak{m}_S M) \leq \dim_k M/\mathfrak{m}_S M \leq \dim_k S/\mathfrak{m}_S \cdot \ell_S(M/\mathfrak{m}_S M).$$

□

Lemma 3.12. *The following $\mathbb{R}$-linear maps are continuous:*

1 $\Theta^*(S) \to \Theta^\circ(S)$;

2 $\Theta^\circ(S) \to \Theta^\wedge(\hat{S})$;

3 $f^* : \Theta(S) \to \Theta(S')$, *for* $f : S' \to S$, *finite;*

4 $f_* : \Theta(S') \to \Theta(S)$ *given by* $f_*(M) = S \otimes_{S'} M$, *for* $f : S' \to S$, *finite;*

5 $\ell_S : \Theta(S) \to \mathbb{R}$, *when* $\ell_S(S) < \infty$.

6 $\mathrm{rank}_R := \dim_{Q(R)}(Q(R) \otimes_R -) : \Theta(R) \to \mathbb{R}$, *where R is a domain (graded or not) and $Q(R)$ is its (ungraded) field of fractions.*

Proof. We only prove **3** and leave the routine verifications of the others to the reader.

Let $\|\cdot\|_{S'}^{S}$ be as in Lemma 3.11. By Lemma 3.11, there exists some $r > 0$ such that $\|\cdot\|_{S'}^{S} \leq r \cdot \|\cdot\|_{S}$. For $\alpha = \sum_M c_M[M]$ as a sum of indecomposable modules in $\Theta(S)$, we have

$$\|f^*\alpha\|_{S'} = \|\sum_M c_M[M]\|_{S'} \leq \sum_M |c_M| \|M\|_{S'} = \sum_M |c_M| \|M\|_{S'}^{S} \leq r \cdot \sum_M |c_M| \|M\|_{S} = r \cdot \|\alpha\|_{S},$$

and continuity follows. □

(3.13) Define $\Theta_+(S)$ to be the subset of $\Theta(S)$ consisting of the $\alpha = \sum c_M[M]$ with all the $c_M \geq 0$.

Lemma 3.14. *Suppose that $f : S' \to S$ is finite and let $\{\alpha_i\}_{i \in \mathbb{N}}$ be a sequence of elements of $\Theta(S)$ such that each α_i is in $\Theta_+(S)$ or $-\Theta_+(S)$. Then $\|\alpha_i\|_S \to 0$ if and only if $u_{S'}(f^*\alpha_i) \to 0$.*

Proof. Note that $\|\alpha_i\|_S \to 0$ if and only if $\|\alpha_i\|_{S'}^{S} \to 0$ by Lemma 3.11. As $\alpha_i \in \pm\Theta_+(S)$, we have that $\|\alpha_i\|_{S'}^{S} = |u_{S'}(f^*(\alpha_i))|$, and we are done. □

(3.15) For $M, N \in S \operatorname{Gr} \operatorname{mod}$, we define

$$\operatorname{sum}_N M := \max\{n \in \mathbb{Z}_{\geq 0} \mid \bigoplus_{i=1}^{n} N[\lambda_i] \text{ is a direct summand of } M \text{ for some } \lambda_1, \ldots, \lambda_n \in \mathbb{Q}\}.$$

For $N \in \operatorname{Ind}^\circ S$, $\operatorname{sum}_N : S \operatorname{Gr} \operatorname{mod} \to \mathbb{Z}$ is an additive function, and hence induces a linear map $\operatorname{sum}_N : \Theta(S) \to \mathbb{R}$. More precisely, sum_N is given by $\operatorname{sum}_N(\sum_M c_M[M]) = c_N$, thus sum_N is continuous.

(3.16) Let k be a field of prime characteristic p. Let $R = \bigoplus_{n \geq 0} R_n$ be a commutative graded k-algebra such that R_0 is an F-finite Henselian local ring. Let $\mathfrak{m}_R$ be the graded maximal ideal of R, and assume that $R/\mathfrak{m}_R$ is a finite-dimensional k-vector space. Let G be a finite group acting on R as degree-preserving k-algebra automorphisms (the case that G is trivial is also important in what follows). Let $S := R * G$. Note that T is central in R and S. Note also that $R/\mathfrak{m}_R$ and k are F-finite, and R and S are finite over $T := R^G$ [Has, (9.6)]. It is easy to see that T is F-finite and Henselian.

Let $d = \dim R$, $\mathfrak{d} := \log_p [k : k^p]$, and set $\delta = d + \mathfrak{d}$.

(3.17) For $\alpha = \sum_{M \in \operatorname{Ind}^\circ S} c_M[M] \in \Theta(S)$, define

$$ {}^e\alpha = \sum_{M \in \operatorname{Ind}^\circ S} c_M[{}^eM], $$

and call it the eth Frobenius direct image of α. We define $\operatorname{NF}_e(\alpha) = \frac{1}{p^{\delta e}} {}^e\alpha$.

Definition 3.18. Let

$$ \operatorname{FL}(\alpha) := \lim_{e \to \infty} \frac{1}{p^{\delta e}} {}^e\alpha = \lim_{e \to \infty} \operatorname{NF}_e(\alpha) $$

in $\Theta(S)$, provided the limit exists. We call $\operatorname{FL}(\alpha)$ the *Frobenius limit* of α.

(3.19) Assume that R is a domain. As we have $\log_p[Q(R) : Q(R)^p] = \delta$ by [Kun, (2.3)], $\operatorname{rank}_R {}^eM = p^{\delta e} \operatorname{rank}_{{}^eR} {}^eM = p^{\delta e} \operatorname{rank}_R M$. It follows that $\operatorname{rank}_R \operatorname{NF}_e(\alpha) = \operatorname{rank}_R \alpha$ for $\alpha \in \Theta(S)$. If $\operatorname{FL}(\alpha)$ exists, then $\operatorname{rank}_R \operatorname{FL}(\alpha) = \operatorname{rank}_R \alpha$.

(3.20) When I is a G-ideal in R, we sometimes write $\alpha/I\alpha$ for $R/I \otimes_R \alpha$. Note that ${}^e\alpha/I({}^e\alpha) = {}^e(\alpha/I^{[p^e]}\alpha)$, where $I^{[p^e]}$ is the ideal generated by $\{a^{p^e} \mid a \in I\}$, which is a G-ideal.

(3.21) If $\mathfrak{q}$ is a homogeneous $\mathfrak{m}_T$-primary ideal of T, the Hilbert–Kunz multiplicity of $M \in T \operatorname{Gr} \operatorname{mod}$ [Mon] is defined by

$$ e_{\mathrm{HK}}(\mathfrak{q}, M) := \lim_{e \to \infty} \frac{\ell_T(M/\mathfrak{q}^{[p^e]}M)}{p^{de}} = \lim_{e \to \infty} \frac{\ell_T(T/\mathfrak{q} \otimes_T {}^eM)}{p^{\delta e}}. $$

This is an additive function, so it induces a function on $\Theta(T)$:

$$ e_{\mathrm{HK}}(\mathfrak{q}, \alpha) = \lim_{e \to \infty} \frac{\ell_T(T/\mathfrak{q} \otimes_T {}^e\alpha)}{p^{\delta e}} = \lim_{e \to \infty} \ell_T(T/\mathfrak{q} \otimes_T \operatorname{NF}_e(\alpha)). $$

By Lemma 3.12, $e_{\mathrm{HK}}(\mathfrak{q}, \alpha) = \ell_T(T/\mathfrak{q} \otimes_T \operatorname{FL}(\alpha))$, provided $\operatorname{FL}(\alpha)$ exists. Note that if T is a domain then $e_{\mathrm{HK}}(\mathfrak{q}, M) = \operatorname{rank}_T M \cdot e_{\mathrm{HK}}(\mathfrak{q}, T)$.

(3.22) Let $N \in \operatorname{Ind}^\circ S$. We define

$$ \operatorname{FS}_N(\alpha) := \lim_{e \to \infty} \operatorname{sum}_N(\operatorname{NF}_e(\alpha)), $$

provided the limit exists. We call it the generalized F-signature of M with respect to N, see [HasN]. If $\operatorname{FL}(\alpha)$ exists, then $\operatorname{FS}_N(\alpha) = \operatorname{sum}_N(\operatorname{FL}(\alpha))$, since sum_N is continuous.

Example 3.23. In [Bru], Bruns studied the asymptotic behavior of the Frobenius direct images of normal affine semigroup rings; we follow the notation used there. In [Bru, Theorem 3.1], assume for simplicity that M is positive in the sense that there is a rational hyperplane H of $\mathbb{R}^d$ through the origin such that $H \cap M = \{0\}$. Let $h : \mathbb{R}^d \to \mathbb{R}$ be a defining equation of H (that is, $h^{-1}(0) = H$) such that $h(\mathbb{Z}^d) \subset \mathbb{Z}$ and $h(M) \subset \mathbb{Z}_{\geq 0}$. Then $R = \bigoplus_{n\in\mathbb{Z}} R_n$ is positively graded (that is, $R_n = 0$ for $n < 0$ and $R_0 = K$), where $R_n = \bigoplus_{x\in h^{-1}(n)\cap M} Kx$. Let $\mathfrak{m} = \bigoplus_{n>0} R_n$. By [Bru, Theorem 3.1], we immediately have that

$$\mathrm{FL}(R) = \sum_{\gamma} \mathrm{vol}(\gamma)[\mathcal{C}_\gamma]$$

in $\Theta^\circ(R)$.

4. The Frobenius limit for a group acting on a polynomial ring

(4.1) Let k be a field, and let B be a graded polynomial ring over k with the degrees of the generators all positive integers, but *not* necessarily the same. Let G be a finite group that acts *faithfully* on B as a graded k-algebra. We can form the twisted group algebra $B * G$ and we define the Frobenius operator on it as in (2.7).

Let $A = B^G$, the ring of invariants. Let $\mathfrak{m}_A$ and $\mathfrak{m}_B$ denote the irrelevant maximal ideals of A and B, respectively. Let $\hat{A}$ be the $\mathfrak{m}_A$-adic completion of A and let $\hat{B}$ be the $\mathfrak{m}_B$-adic completion of B (it is also the $\mathfrak{m}_A$-adic completion).

Let $\mathcal{V}$ be the category of $\mathbb{Q}$-graded kG-modules and let $\mathcal{M}$ be the category of $\mathbb{Q}$-graded $B * G$-modules.

Let $\mathcal{F}$ denote the full subcategory of $\mathcal{M}$ consisting of $F \in \mathcal{M}$ such that F is B-finite and B-free. In other words, F is a $\mathbb{Q}$-graded $B * G$-lattice.

(4.2) Let $V = \bigoplus_\lambda V_\lambda$ be an object of $\mathcal{V}$. Then V is a projective object of $\mathcal{V}$ if and only if it is so as a kG-module, since $\mathrm{Hom}_{\mathcal{V}}(V, W) = \prod_\lambda \mathrm{Hom}_{kG}(V_\lambda, W_\lambda)$. We denote the category of finite dimensional projective objects of $\mathcal{V}$ by $\mathcal{P}_0$. Then clearly $\mathcal{P}_0 = \mathrm{add}\{kG[\lambda] \mid \lambda \in \mathbb{Q}\}$, where $[\lambda]$ denotes shift of degree by λ.

Lemma 4.3. *Let $R = \bigoplus_{i\geq 0} R_i$ be a commutative positively-graded (that is, $R_0 = k$) k-algebra. Let F and F' be graded R-finite R-free modules, and $h : F \to F'$ a graded R-homomorphism. Then the following are equivalent:*

1 *h is injective, and $C := \operatorname{Coker} h$ is R-free;*

2 *$1 \otimes h : R/\mathfrak{m} \otimes_R F \to R/\mathfrak{m} \otimes_R F'$ is injective;*

where $\mathfrak{m} = \bigoplus_{i>0} R_i$ is the irrelevant ideal.

Proof. **1⇒2**. As the sequence

$$0 \to F \xrightarrow{h} F' \to C \to 0$$

is exact,

$$0 = \operatorname{Tor}_1^R(R/\mathfrak{m}, C) \to R/\mathfrak{m} \otimes_R F \xrightarrow{1\otimes h} R/\mathfrak{m} \otimes_R F'$$

is exact.

2⇒1. Take a homogeneous free basis $f_1, \ldots, f_r$ of F, and take homogeneous elements $f'_1, \ldots, f'_s$ of F' such that their images in C form a minimal set of generators for C. As

$$0 \to R/\mathfrak{m} \otimes_R F \to R/\mathfrak{m} \otimes_R F' \to R/\mathfrak{m} \otimes_R C \to 0$$

is exact, we have that $\operatorname{rank} F' = r+s$, and $h(f_1), \ldots, h(f_r), f'_1, \ldots, f'_s$ generate F' by the graded version of Nakayama's lemma (this applies since the grading on the modules must be discrete). Thus it is easy to see that this set of elements forms a free basis for F'. In particular, $h(f_1), \ldots, h(f_r)$ are linearly independent and hence h is injective. Also, $C = F'/F$ is a free module with basis $f'_1, \ldots, f'_s$. □

Lemma 4.4. **1** *$P := \{(B \otimes kG)[\lambda] \mid \lambda \in \mathbb{Q}\}$ is a set of Noetherian projective objects that generate $\mathcal{M}$. In particular, $\mathcal{P} := \operatorname{add} P$ is the full subcategory of Noetherian projective objects of $\mathcal{M}$.*

2 *For $M \in \mathcal{M}$, the following are equivalent.*

- **a** *$M \in \mathcal{P}$;*
- **b** *$M \cong B \otimes_k V$ as graded modules, for some $V \in \mathcal{P}_0$;*
- **c** *$M \in \mathcal{F}$, and $M/\mathfrak{m}_B M \in \mathcal{P}_0$.*

If these conditions are satisfied, then $M \cong B \otimes_k M/\mathfrak{m}_B M$ as graded modules.

3 *$\mathcal{F}$ is a Frobenius category with respect to all short exact sequences (see* [Hap] *for definition), and $\mathcal{P}$ is its full subcategory of projective and injective objects.*

Proof. **1** Obviously, each $(B \otimes_k kG)[\lambda]$ is a Noetherian object. On the other hand,

$$\operatorname{Hom}_{\mathcal{M}}(B \otimes kG[\lambda], N) \cong \operatorname{Hom}_{\mathcal{V}}(kG[\lambda], N) \cong \operatorname{Hom}_{\operatorname{Gr Mod} k}(k[\lambda], N) \cong N_{-\lambda},$$

and each object of P is a projective object, and P generates $\mathcal{M}$, where $\operatorname{Gr Mod} k$ denotes the category of graded k-vector spaces.

2. **a**⇔**b**⇒**c** is trivial. We show the last assertion, assuming **c**. This also proves **c**⇒**b**. As $M/\mathfrak{m}_B M$ is projective in $\mathcal{V}$, the canonical map $M \to M/\mathfrak{m}_B M$ has a splitting $j : M/\mathfrak{m}_B M \to M$ in $\mathcal{V}$. Then, defining $\varphi : B \otimes_k M/\mathfrak{m}_B M \to M$ by $\varphi(b \otimes v) = bj(v)$, φ is $B * G$-linear. By Lemma 4.3, it is easy to see that φ is an isomorphism.

3. By **1**, $\mathcal{P}$ is the category of the projectives of $\mathcal{F}$, and $\mathcal{F}$ has enough projectives. On the other hand, $\operatorname{Hom}_B(?, B)$ is a dualizing functor on the exact category $\mathcal{F}$ and $\mathcal{P}$ is mapped to itself by it. Thus $\mathcal{P}$ is also the category of injectives of $\mathcal{F}$, and $\mathcal{F}$ has enough injectives. □

Lemma 4.5. *Let $F \in \mathcal{F}$. Then there is a filtration*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_n = F$$

in $\mathcal{M}$ such that for each $i = 1, \dots, n$, there exist $\lambda_i \in \mathbb{Q}$ and $V_i \in kG \operatorname{mod}$ such that $F_i/F_{i-1} \cong B \otimes_k V_i[-\lambda_i]$ (so F_i and F_i/F_{i-1} are in $\mathcal{F}$), where $kG \operatorname{mod}$ denotes the category of finite dimensional kG-modules, and each object of $kG \operatorname{mod}$ is viewed as an object of $\mathcal{V}$ of degree zero.

Proof. We use induction on $\operatorname{rank}_B F$. If $\operatorname{rank}_B F = 0$, there is nothing to prove. Assume that $\operatorname{rank} F > 0$ and take the smallest $\lambda \in \mathbb{Q}$ such that $F_\lambda \neq 0$. Set $V_1 = F_\lambda[\lambda]$, $\lambda_1 = \lambda$, and $F_1 = B \otimes_k V_1[-\lambda]$. There is a canonical map

$$q : F_1 = B \otimes_k V_1[-\lambda] = B \otimes_k F_\lambda \xrightarrow{a} F,$$

where $a(b \otimes f) = bf$. Then, by Lemma 4.3, q is injective, and $C \in \mathcal{F}$, where $C = \operatorname{Coker} q$. Applying the induction hypothesis to C, we are done. □

Lemma 4.6. *Let $F \in \mathcal{F}$ and $f \geq 0$. Then the following are equivalent.*

1 $F \cong B \otimes_k F_0$ *for some* $\mathbb{Q}$*-graded* G*-module* F_0 *such that* $F_0 \cong (kG)^f$ *as* G*-modules.*

2 $F \cong (B \otimes_k kG)^f$ *as a* $B * G$*-module.*

3 $F/\mathfrak{m}_B F \cong (kG)^f$ *as a* G*-module.*

Proof. **1**⇒**2**⇒**3** is trivial. **3**⇒**1** follows from Lemma 4.4, **2**. □

(4.7) We denote the full subcategory of $\mathcal{F}$ with objects the $F \in \mathcal{F}$ satisfying the equivalent conditions in Lemma 4.6 by $\mathcal{G}$. Note that $\mathcal{G}$ is closed under extensions and shift of degree.

Lemma 4.8. *Let* V *be a* kG*-module. Let* V' *be the* k*-vector space* V *with the trivial* G*-action. Then* $kG \otimes V \cong kG \otimes V'$*. Hence* $kG \otimes V$ *is a direct sum of copies of* kG*.*

Proof. The map $g \otimes v \mapsto g \otimes g^{-1}v$ gives a kG-isomorphism $kG \otimes V \cong kG \otimes V'$. □

(4.9) From now on, we assume that k is of characteristic p, and is F-finite. We set $\mathfrak{d} := \log_p[k : k^p]$ and $\delta := d + \mathfrak{d}$.

Lemma 4.10. *If* $F \in \mathcal{G}$*, then* ${}^eF \in \mathcal{G}$*.*

Proof. We can write $F = B \otimes_k F_0$ with $F_0 \cong (kG)^f$ as a kG-module for some f. We have ${}^eF \in \mathcal{F}$ and

$$ {}^eF/\mathfrak{m}_B {}^eF \cong {}^e(B/\mathfrak{m}_B^{[p^e]} \otimes_B (B \otimes_k F_0)) \cong {}^e(B/\mathfrak{m}_B^{[p^e]} \otimes_k F_0). $$

As $F_0 \cong (kG)^f$, we have that $B/\mathfrak{m}_B^{[p^e]} \otimes_k F_0 \cong (kG)^{fp^{de}}$ by Lemma 4.8. Hence ${}^eF/\mathfrak{m}_B {}^eF \cong {}^e((kG)^{fp^{de}}) = (kG)^{fp^{\delta e}}$ by Lemma 2.9. By Lemma 4.6, we have that ${}^eF \in \mathcal{G}$. □

Lemma 4.11. *There exists some* $e_0 \geq 1$ *such that for each* $F \in \mathcal{F}$ *of rank* f*, there exists some direct summand* F' *of* ${}^{e_0}F$ *in* $\mathcal{F}$ *such that* $F' \cong (B \otimes_k kG)^{fp^{\mathfrak{d}e_0}}$ *as* $B * G$*-modules.*

Proof. Let $Q(A)$ and $Q(B)$ denote the fields of fractions of A and B respectively. Then $Q(B)$ is a Galois extension of $Q(A)$ with Galois group G (here we use the assumption G acts faithfully on B). So $u : Q(B) \otimes_{Q(A)} Q(B)' \to kG \otimes_k Q(B)'$ given by $u(x \otimes y) = \sum_{g \in G} g^{-1} \otimes (gx)y$ is an isomorphism of

$(G, Q(B)')$-modules, where $Q(B)'$ is the field $Q(B)$ with the trivial G-action. So $Q(B)$ as a G-module is a direct sum of copies of kG. Thus there is at least one injective kG-map $kG \to Q(B)$. Multiplying by an appropriate element of $A \setminus \{0\}$, we get an injective G-linear map $kG \to B$. Its image is in $B_0 \oplus B_1 \oplus \cdots \oplus B_r$ for some $r \geq 1$, and it is a direct summand, since kG is an injective module. Then by the Krull–Schmidt theorem, there is a graded kG-direct summand E_0 of B which is isomorphic to kG as a G-module. The argument so far, which we have given for the convenience of the reader, can be found in [Sym].

We can take e_0 sufficiently large that $E_0 \cap \mathfrak{m}_B^{[p^{e_0}]} = 0$ for degree reasons, so $E_0 \to B/\mathfrak{m}_B^{[p^{e_0}]}$ is injective. We claim that this choice of e_0 has the required property.

Let V be any finite-dimensional kG-module. Then the inclusion $E_0 \hookrightarrow B$ induces a split monomorphism $\phi : {}^{e_0}(E_0 \otimes_k V) \to {}^{e_0}(B \otimes_k V)$. Note that the composite

$$ {}^{e_0}(E_0 \otimes_k V) \xrightarrow{\phi} {}^{e_0}(B \otimes_k V) \to B/\mathfrak{m}_B \otimes_B {}^{e_0}(B \otimes_k V) \cong {}^{e_0}(B/\mathfrak{m}_B^{[p^{e_0}]} \otimes_k V) $$

is injective, since ${}^{e_0}(? \otimes_k V)$ is an exact functor. Note that ${}^{e_0}(E_0 \otimes_k V) \cong (kG)^{p^{\mathfrak{d} e_0} \dim_k V}$ as G-modules. By Lemma 4.3, it is easy to see that

$$ B \otimes_k {}^{e_0}(E_0 \otimes_k V) \to {}^{e_0}(B \otimes_k V) $$

given by $b \otimes m \mapsto b\phi(m)$ is an injective map of $\mathcal{F}$ whose cokernel D_V lies in $\mathcal{F}$. As $B \otimes_k {}^{e_0}(E_0 \otimes_k V) \in \mathcal{G} \subset \mathcal{P}$, we have a decomposition

$$ {}^{e_0}(B \otimes_k V[\lambda]) = B \otimes_k {}^{e_0}(E_0 \otimes_k V)[\lambda/p^{e_0}] \oplus D_V[\lambda/p^{e_0}]. $$

So if $F \cong B \otimes_k V[\lambda]$ for some finite-dimensional kG-module V and $\lambda \in \mathbb{Q}$, the lemma holds.

Now let

$$ 0 \to E \to F \to H \to 0 $$

be a short exact sequence in $\mathcal{F}$ such that the assertion of the lemma (for our e_0) is satisfied for E and H. That is, ${}^{e_0}E$ has a direct summand E' such that $E' \cong (B \otimes_k kG)^{\oplus p^{\mathfrak{d} e_0} \operatorname{rank} E}$ as a $B * G$-module, and ${}^{e_0}H$ has a direct summand H' such that $H' \cong (B \otimes_k kG)^{\oplus p^{\mathfrak{d} e_0} \operatorname{rank} H}$ as a (G, B)-module. As H' is a projective object of $\mathcal{F}$, the inclusion $H' \hookrightarrow H$ lifts to $H' \hookrightarrow F$. So we have a commutative diagram of $B * G$-modules, with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E' & \longrightarrow & E' \oplus H' & \longrightarrow & H' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & H & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E'' & \longrightarrow & F'' & \longrightarrow & H'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}.$$

As E' and H' are direct summands of E and H, respectively, we have that $E'' \in \mathcal{F}$ and $H'' \in \mathcal{F}$. So $F'' \in \mathcal{F}$, and hence $E' \oplus H'$ is a direct summand of F by Lemma 4.4. As $E' \oplus H' \cong (B \otimes_k kG)^{\oplus(p^{\delta e_0}(\operatorname{rank}_B E + \operatorname{rank}_B H))}$ and $\operatorname{rank}_B E + \operatorname{rank}_B H = \operatorname{rank}_B F$, we conclude that the assertion of the lemma is also true for F.

Now by Lemma 4.5, we are done. $\square$

Proposition 4.12. *There exists some* $c > 0$ *and* $0 \leq \alpha < 1$ *such that for any* $F \in \mathcal{F}$ *of rank* f *and any* $e \geq 0$, *there exists some decomposition*

$$ {}^eF \cong F_{0,e} \oplus F_{1,e} \tag{1}$$

such that $F_{1,e} \in \mathcal{G}$ *and* $\operatorname{rank}_B F_{0,e} \leq c\alpha^e f p^{\delta e}$.

Proof. If the dimension $d = 0$, then $A = B = k$ and G is trivial, and this case is obvious, since we may set $c = 1$, $\alpha = 0$, $F_{0,e} = 0$ and $F_{1,e} = {}^eF$ for each e.

So we may assume that $d \geq 1$. Take e_0 as in Lemma 4.11, and set $\alpha := (1 - |G| \cdot p^{-de_0})^{1/e_0}$ so that $0 \leq \alpha < 1$. Set $c = \alpha^{-e_0} > 0$.

We prove the existence of a decomposition by induction on $e \geq 0$.

If $0 \leq e < e_0$, then we set $F_{0,e} = {}^eF$ and $F_{1,e} = 0$. As we have $\operatorname{rank}_B F_{0,e} = fp^{\delta e}$ and $c\alpha^e = \alpha^{e-e_0} > 1$, we are done.

Now assume that $e \geq e_0$. By the induction hypothesis, we have a decomposition

$$ {}^{e-e_0}F \cong F_{0,e-e_0} \oplus F_{1,e-e_0}$$

such that $F_{1,e-e_0} \in \mathcal{G}$ and $\operatorname{rank}_B F_{0,e-e_0} \leq c\alpha^{e-e_0} f p^{\delta(e-e_0)}$. Then

$$ {}^{e}F \cong {}^{e_0}F_{0,e-e_0} \oplus {}^{e_0}F_{1,e-e_0}. $$

By Lemma 4.10, that ${}^{e_0}F_{1,e-e_0} \in \mathcal{G}$. Moreover,

$$ \operatorname{rank}_B {}^{e_0}F_{0,e-e_0} = p^{\delta e_0} \operatorname{rank}_B F_{0,e-e_0}. $$

By the choice of e_0, there is a decomposition

$$ {}^{e_0}F_{0,e-e_0} \cong F' \oplus F'' $$

such that $F' \in \mathcal{G}$ and $\operatorname{rank}_B F' = |G| \cdot p^{\partial e_0} \operatorname{rank}_B F_{0,e-e_0}$.

Now let $F_{0,e} := F''$ and $F_{1,e} := {}^{e_0}F_{1,e-e_0} \oplus F'$. As ${}^{e_0}F_{1,e-e_0} \in \mathcal{G}$ and $F' \in \mathcal{G}$, we have $F_{1,e} \in \mathcal{G}$. On the other hand,

$$ \begin{aligned} \operatorname{rank}_B F_{0,e} = \operatorname{rank}_B {}^{e_0}F_{0,e-e_0} - \operatorname{rank}_B F' &= (p^{\delta e_0} - |G| \cdot p^{\partial e_0}) \operatorname{rank}_B F_{0,e-e_0} \\ &\leq \alpha^{e_0} p^{\delta e_0} c\alpha^{e-e_0} f p^{\delta(e-e_0)} = c\alpha^e f p^{\delta e}, \end{aligned} $$

and we are done. □

Theorem 4.13. *For any $B * G$-module F that is free of rank f over B we have*

$$ \operatorname{FL}(F) = \frac{f}{|G|}[B * G] $$

in $\Theta^\circ(B * G)$ and the analogous formula

$$ \operatorname{FL}(\hat{F}) = \frac{f}{|G|}[\hat{B} * G] $$

in $\Theta^\wedge(\hat{B} * G)$.

Proof. From Proposition 4.12, we have

$$ (\dagger) \qquad \frac{[{}^eF]}{p^{\delta e}} - \frac{f}{|G|}[B * G] = \left(\frac{[F_{1,e}]}{p^{\delta e}} - \frac{f}{|G|}[B * G] \right) + \frac{[F_{0,e}]}{p^{\delta e}}. $$

Notice that $[F_{0,e}]/p^{\delta e} \in \Theta^\circ_+(B*G)$ and $\lim_{e\to\infty} \operatorname{rank}_B([F_{0,e}]/p^{\delta e}) = 0$. But $F_{0,e}$ is free as a B-module, so $u_B(F_{0,e}) = \operatorname{rank}_B(F_{0,e})$. It follows from Lemma 3.14 that $\lim_{e\to\infty} \|[F_{0,e}]/p^{\delta e}\|_{B*G} = 0$.

By Lemma 4.6, the term $[F_{1,e}]/p^{\delta e}$ is of the form $a_e[B*G]$ for some number a_e; taking ranks shows that $\lim_{e\to\infty} a_e = f/|G|$. Thus

$$\lim_{e\to\infty}\left(\frac{[F_{1,e}]}{p^{\delta e}} - \frac{f}{|G|}[B*G]\right) = 0$$

and the first part of the theorem is proved.

The second part follows from Lemma 3.12. □

Lemma 4.14. *$B \cong (B \otimes_k kG)^G$ as graded A-modules. More explicitly, $b \mapsto \sum_g gb \otimes g$ gives a graded A-isomorphism. The inverse is given by $\sum_g b_g \otimes g \mapsto b_e$.*

Proof. Easy. □

Lemma 4.15. *For any $B*G$-module M, $\operatorname{rank}_A M^G = \operatorname{rank}_B M$.*

Proof. It is well known that $Q(B)*G$ is isomorphic to a matrix ring over $Q(A)$ ([CR, 28.3]), hence $Q(B)$ is its only indecomposable module. Thus

$$Q(A)\otimes_A M^G \cong (Q(A)\otimes_A M)^G \cong (Q(B)\otimes_B M)^G \cong (Q(B)^m)^G \cong Q(A)^m,$$

where $m = \operatorname{rank}_B M$. □

Theorem 4.16. *For any $B*G$-module F that is free of rank f over B we have*

$$\operatorname{FL}(F^G) = \frac{f}{|G|}[B]$$

in $\Theta^\circ(A)$ and

$$\operatorname{FL}(\hat{F}^G) = \frac{f}{|G|}[\hat{B}]$$

$\Theta^\wedge(\hat{A})$*, where $A = B^G$.*

Proof. From the proof of Theorem 4.13 we have $[{}^eF]/p^{\delta e} = a_e[B\otimes_k kG] + [F_{0,e}]/p^{\delta e}$, where $\lim_{e\to\infty} a_e = f/|G|$. Applying the fixed point functor and using Lemma 4.14 yields

$$[{}^eF^G]/p^{\delta e} = a_e[B] + [F_{0,e}^G]/p^{\delta e}.$$

The theorem will follow once we can show that $\lim_{e\to\infty} u_A([F_{0,e}^G]/p^{\delta e}) = 0$, since this takes place in $\Theta_+(A)$.

Applying u_A gives

$$u_A([{}^eF^G]/p^{\delta e}) = u_A(a_e[B]) + u_A([F^G_{0,e}]/p^{\delta e}).$$

Clearly,

$$\lim_{e\to\infty} u_A(a_e[B]) = (f/|G|)u_A(B) = (f/|G|)\dim_k B/\mathfrak{m}_A B.$$

Now we use the Hilbert-Kunz multiplicity (see (3.21)).

$$\lim_{e\to\infty} u_A\left(\frac{[{}^eF^G]}{p^{\delta e}}\right) = e_{\mathrm{HK}}(\mathfrak{m}_A, F^G) = \mathrm{rank}_A(F^G)\cdot e_{\mathrm{HK}}(\mathfrak{m}_A, A).$$

But $\mathrm{rank}_A(F^G) = \mathrm{rank}_B(F) = f$, by Lemma 4.15.

It was shown by Watanabe and Yoshida [WY, 2.7] that $e_{\mathrm{HK}}(\mathfrak{m}_A, A) = \frac{1}{|G|}\ell_B(B/\mathfrak{m}_A B)$, and this right hand side is equal to $\frac{1}{|G|}\dim_k B/\mathfrak{m}_A B$. Combining these, we see that $\lim_{e\to\infty} u_A([F^G_{0,e}]/p^{\delta e}) = 0$, as required. □

Remark 4.17. When p does not divide $|G|$ it is easy to see that the map induced by the fixed point functor $\Theta^\circ(B * G) \to \Theta^\circ(A)$ is continuous, so Theorem 4.16 follows immediately from Theorem 4.13.

5. Applications

We continue to use the notation of (4.1).

Theorem 5.1. *Let k be a field of characteristic $p > 0$ such that $[k : k^p] < \infty$, and let V be a faithful G-module. Let $k = V_0, V_1, \dots, V_n$ be the simple kG-modules. For each i, let $P_i \to V_i$ be the projective cover, and set $M_i := (B \otimes_k P_i)^G$. Let F be a $\mathbb{Q}$-graded B-finite B-free $B * G$-module. Then the F-limit of $[F^G]$ exists in $\Theta^\circ(A)$, where $A = B^G$, and*

$$\mathrm{FL}([F^G]) = \frac{f}{|G|}[B] = \frac{f}{|G|}\sum_{i=0}^{n}\frac{\dim_k V_i}{\dim_k \mathrm{End}_{kG}(V_i)}[M_i],$$

where $f = \mathrm{rank}_B F$. An analogous formula holds for $\mathrm{FL}([\hat{F}^G])$ in $\Theta^\wedge(\hat{A})$.

Proof. The first equality is just Theorem 4.16.

We can write $kG = \bigoplus_{i=0}^n P_i^{\oplus u_i}$ for some $u_i \ge 0$, so $B \cong (B \otimes_k kG)^G \cong \bigoplus_{i=0}^n M_i^{\oplus u_i}$. Applying $\dim_k \mathrm{Hom}_{kG}(-, V_i)$ to the first equality shows that $u_i = \dim_k(V_i)/\dim_k \mathrm{End}_{kG}(V_i)$. □

Corollary 5.2. *Under the conditions of* Theorem 5.1*, we have*

$$\mathrm{FL}([A]) = \frac{1}{|G|}[B] = \frac{1}{|G|}\sum_{i=0}^{n} \frac{\dim_k V_i}{\dim_k \mathrm{End}_{kG}(V_i)}[M_i]$$

in $\Theta^\circ(A)$ *and similarly after completion.* □

(5.3) Let the notation be as in Theorem 5.1. We say that the action of G on B (or on $X := \operatorname{Spec} B$) is *small* if there is a G-stable open subset U of X such that the action of G on U is free, and the codimension of $X \setminus U$ in X is at least two.

For $g \in G$, let X_g be the locus in X that the action of g and the identity map agree. Note that X_g is a closed subscheme of X. If all the generators of B are in degree one, then X_g is nothing but the eigenspace in V with eigenvalue 1 of the action of g on V, where $V = B_1$. We say that g is a pseudo-reflection if the codimension of X_g in X is one. The action of G on B is small if and only if G does not have a pseudo-reflection.

Now assume further that the action of G on B is small.

Theorem 5.4. *Let the notation be as in* (5.3)*. Then* $(B\otimes_A ?) : \mathrm{Ref}(A) \to \mathrm{Ref}(G, B)$ *is an equivalence with quasi-inverse* $(?)^G : \mathrm{Ref}(G, B) \to \mathrm{Ref}(A)$*, where* $\mathrm{Ref}(A)$ *denotes the category of reflexive* A*-modules, and* $\mathrm{Ref}(G, B)$ *denotes the full subcategory of* $(G, B)\,\mathrm{mod}$ *consisting of* (G, B)*-modules which are reflexive as* B*-modules. A similar assertion for* $\hat{A} \to \hat{B}$ *also holds.*

Proof. This is a special case of [Has, (14.24)]. See also [HasN, (2.4)]. □

Using Theorem 5.4, we can obtain the following equivalences.

Corollary 5.5. *Let the notation be as in* (5.3)*. For* $V \in kG\,\mathrm{mod}$*, define* $M_V := (B \otimes_k V)^G$*.*

1 *For* $V \in G\,\mathrm{mod}$*, the following are equivalent.*

- **a** V *is an indecomposable* kG*-module.*
- **b** $B \otimes_k V$ *is an indecomposable object in* $(B * G)\,\mathrm{mod}$*.*
- $\hat{\mathbf{b}}$ $\hat{B} \otimes_k V$ *is an indecomposable object in* $(\hat{B} * G)\,\mathrm{mod}$*.*
- **c** M_V *is an indecomposable* A*-module.*
- $\hat{\mathbf{c}}$ $\hat{M}_V$ *is an indecomposable* $\hat{A}$*-module.*

2 *Let $V, V' \in G\,\mathrm{mod}$. Then the following are equivalent.*

- **a** $V \cong V'$ *in* $G\,\mathrm{mod}$.
- **b** $B \otimes_k V \cong B \otimes_k V'$ *in* $(B * G)\,\mathrm{mod}$.
- $\hat{\mathbf{b}}$ $\hat{B} \otimes_k V \cong \hat{B} \otimes_k V'$ *in* $(\hat{B} * G)\,\mathrm{mod}$.
- **c** $M_V \cong M_{V'}$ *as A-modules.*
- $\hat{\mathbf{c}}$ $\hat{M}_V \cong \hat{M}_{V'}$.

Proof. We only prove **1**.

b⇒**a**. This is because $B \otimes_k ?$ is a faithful exact functor from $G\,\mathrm{mod}$ to $B * G\,\mathrm{mod}$.

a⇒**b**. This is because $B/\mathfrak{m}_B \otimes_B ?$ is an additive functor from the category of B-finite B-free $B * G$-modules to $kG\,\mathrm{mod}$, which sends a nonzero object to a nonzero object.

a⇔$\hat{\mathbf{b}}$ is similar. **b**⇔**c** and $\hat{\mathbf{b}}$⇔$\hat{\mathbf{c}}$ are by Theorem 5.4. □

Theorem 5.6. *Let the notation be as in* (5.3), *so in particular the action of G on B is small. Then for each $0 \le i, j \le n$, $\mathrm{FS}_{M_j}(M_i)$ exists, and*

$$\mathrm{FS}_{M_j}(M_i) = \frac{(\dim_k P_i)(\dim_k V_j)}{|G| \dim_k \mathrm{End}_{kG}(V_i)}.$$

A similar formula holds in the complete case.

Proof. By Theorem 5.1, $\mathrm{FS}_{M_j}(M_i)$ exists and

$$\mathrm{FS}_{M_j}(M_i) = \mathrm{sum}_{M_j}(\mathrm{FL}(M_i)) = \frac{\mathrm{rank}_B(B \otimes_k P_i)}{|G|} \sum_{l=0}^{n} \frac{\dim_k V_l}{\dim_k \mathrm{End}_{kG}(V_i)} \mathrm{sum}_{M_j}[M_l].$$

Because each P_l is indecomposable and $P_l \cong P_j$ if and only if $l = j$, it follows from Corollary 5.5 that each M_l is indecomposable and $M_j \cong M_l$ (after shift of degree) if and only if $l = j$. This shows that $\mathrm{sum}_{M_j}[M_l] = \delta_{jl}$ (Kronecker's delta). The theorem follows. □

Corollary 5.7 ([HasN, (3.9)])**.** *Let the notation be as in* (5.3) *and assume that k is algebraically closed and that $|G|$ is not divisible by the characteristic of k. Then, for each $0 \le i, j \le n$, $\mathrm{FS}_{\hat{M}_j}(\hat{M}_i)$ exists, and*

$$\mathrm{FS}_{\hat{M}_j}(\hat{M}_i) = \frac{(\dim_k V_i)(\dim_k V_j)}{|G|}.$$

Proof. This is because $P_i \cong V_i$, by Maschke's theorem. $\square$

Corollary 5.8 ([Bro, Corollary 2], [Yas, Corollary 3.3])**.** *Let the notation be as in* (5.3)*. If p divides $|G|$, then none of $\hat{A}$, $A_{\mathfrak{m}_A}$, nor A is weakly F-regular.*

Proof. By Corollary 5.5, **1**, $\hat{M}_j$ is indecomposable for $j = 0, 1, \ldots, n$. By Corollary 5.5, **2**, $\hat{M}_j = \hat{M}_{P_j} \cong \hat{M}_k = \hat{A}$ if and only if $P_j \cong k$. This happens if and only if $j = 0$ and $P_0 \to k$ is an isomorphism. This is equivalent to saying that p does not divide $|G|$ and $j = 0$. By our assumption, $\operatorname{sum}_{\hat{A}}(\hat{M}_j) = 0$ for $j = 0, \ldots, n$. So by Theorem 5.1,

$$\operatorname{FS}_{\hat{A}}(\hat{A}) = \operatorname{sum}_{\hat{A}}(\operatorname{FL}(\hat{A})) = \sum_{j=0}^{n} \frac{\dim_k V_j}{\dim_k \operatorname{End}_{kG}(V_i)} \operatorname{sum}_{\hat{A}}(\hat{M}_j) = 0.$$

Since $\operatorname{FS}_{\hat{A}}(\hat{A})$ is just the F-signature of $\hat{A}$ of Huneke–Leuschke [HL], we see that $\hat{A}$ is not strongly F-regular, by the theorem of Aberbach and Leuschke [AL]. So $\hat{A}$ cannot be a direct summand subring of the regular local ring $\hat{B}$. As a weakly F-regular ring is a splinter [HH, (5.17)], $\hat{A}$ is not weakly F-regular. By smooth base change [HH2, (7.3)], $A_{\mathfrak{m}_A}$ is not weakly F-regular. It follows that A is not weakly F-regular. $\square$

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