F-pure homomorphisms, strong F-regularity, and F-injectivity

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Abstract

We discuss Matijevic–Roberts type theorem on strong F-regularity, F-purity, and Cohen–Macaulay F-injective (CMFI for short) property. Related to this problem, we also discuss the base change problem and the openness of loci of these properties. In particular, we define the notion of F-purity of homomorphisms using Radu–André homomorphisms, and prove basic properties of it. We also discuss a strong version of strong F-regularity (very strong F-regularity), and compare these two versions of strong F-regularity. As a result, strong F-regularity and very strong F-regularity agree for local rings, F-finite rings, and essentially finite-type algebras over an excellent local rings. We prove the F-pure base change of strong F-regularity.

1. Introduction

Throughout this article, p denotes a prime number.

The main objective of this paper is to prove Matijevic–Roberts type theorem on strong F-regularity, F-purity, and Cohen–Macaulay F-injective (CMFI, for short) properties. F-purity was defined and discussed by M. Hochster

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and J. Roberts in 1970's [21], [22]. It turned out that the F-purity is deeply connected with the notion of log canonical singularities [30]. Strong F-regularity was defined by Hochster and Huneke [18] for F-finite rings. F-injectivity was first defined by Fedder [8]. Recently, Schwede [28] proved that singularities of dense F-injective type in characteristic zero are Du Bois.

We prove

Theorem 5.2 Let y be a point of X, and Y the integral closed subscheme of X whose generic point is y. Let η be the generic point of an irreducible component of Y^*, where Y^* is the smallest G-stable closed subscheme of X containing Y. Assume either that the second projection $p_2 : G \times X \to X$ is smooth, or that $S = \operatorname{Spec} k$ with k a perfect field and G is of finite type over S. Assume that $\mathcal{O}_{X,\eta}$ is of characteristic p. Then $\mathcal{O}_{X,y}$ is of characteristic p. Moreover,

1 If $\mathcal{O}_{X,\eta}$ is F-pure, then $\mathcal{O}_{X,y}$ is F-pure.

2 If $\mathcal{O}_{X,\eta}$ is excellent and strongly F-regular, then $\mathcal{O}_{X,y}$ is strongly F-regular.

3 If $\mathcal{O}_{X,\eta}$ is CMFI, then $\mathcal{O}_{X,y}$ is CMFI.

Matijevic–Roberts type theorems were originally conjectured and proved for graded rings, see the introduction of [13] for a short history. In [13], roughly speaking, it is proved that if a ring theoretic property $\mathbb{P}$ enjoys 'smooth base change' and 'flat descent,' then Matijevic–Roberts type theorem for $\mathbb{P}$ under the action of a smooth group scheme holds. Applying this principle, Matijevic–Roberts type theorems for (weak) F-regularity and F-rationality were proved in [13].

Smooth base change of F-purity is not so difficult. In order to discuss this problem, we define the notion of F-purity of homomorphism between (noetherian) commutative rings of characteristic p. We use Radu–André homomorphism to do so. This map is used to characterize the regularity of homomorphism between noetherian commutative rings of characteristic p. Radu [27] and André [2] proved that a homomorphism $f : A \to B$ between noetherian commutative rings of characteristic p is regular (i.e., flat with geometrically regular fibers) if and only if the Radu–André homomorphism $\Phi_e(A, B) : B^{(e)} \otimes_{A^{(e)}} A \to B$ given by $\Phi_e(A, B)(b^{(e)} \otimes a) = b^{p^e} a$ is flat for some (or equivalently, any) $e > 0$. After that, the Radu–André homomorphisms were used to study the reducedness of homomorphisms by Dumitrescu, and Cohen–Macaulay F-injective property by Enescu [6] and the author [11].

In this article, we define a homomorphism $f : A \to B$ between commutative rings of characteristic p is F-pure if $\Phi_e(A, B)$ is pure for any $e > 0$. This property behaves well under composition, localization, and base change, and this notion is a canonical generalization of the notion of F-purity of a ring.

Many properties of rings are promoted to properties of homomorphisms, using the properties of (geometric) fibers. For example, Grothendieck [10, (6.8.1)] defined that a ring homomorphism $f : A \to B$ is Cohen–Macaulay if f is flat with Cohen–Macaulay fibers. However, it seems that it is not appropriate to define an F-pure homomorphism to be a flat homomorphism with geometrically F-pure fibers, because Singh [29] constructed an example of homomorphism $f : A \to B$ of noetherian rings of characteristic p such that A is a DVR (in particular, F-pure), f is flat with geometrically F-pure fibers, but B is not F-pure.

We also define the notion corresponding to the reduced property. We say that a homomorphism $f : A \to B$ between commutative rings of characteristic p is Dumitrescu if $\Phi_e(A, B)$ is A-pure for any $e > 0$, see (2.7).

Dumitrescu proved that if $f : A \to B$ is a flat homomorphism between noetherian rings of characteristic p, then f is Dumitrescu if and only if f is a reduced homomorphism (that is, a flat homomorphism with geometrically reduced fibers).

It is natural to ask whether a Dumitrescu homomorphism is flat. We prove that if A is regular, then a Dumitrescu homomorphism between noetherian commutative rings of characteristic p $f : A \to B$ is flat. A homomorphism $f : A \to B$ is said to be almost quasi-finite if f has finite fibers. We prove that an almost quasi-finite Dumitrescu homomorphism is flat (Theorem 2.19).

In the late 1980's, Hochster and Huneke defined F-regularity using tight closure [19]. They also defined strong F-regularity using Frobenius splittings for F-finite rings of characteristic p [18]. Hochster and Huneke defined the strong F-regularity for non-F-finite homomorphisms [20, (5.3)]. Recently, Hochster [17] gave another definition of strong F-regularity. In this paper, we call Hochster–Huneke's definition the very strong F-regularity (Definition 3.4), and Hochster's definition the strong F-regularity. We compare these two definitions. Obviously, very strong F-regularity implies the strong F-regularity. They agree for local rings, F-finite rings, and essentially finite-type algebras over excellent local rings. We give a sufficient condition for the strong F-regular locus to be open (Proposition 3.33). We prove the F-pure base change of the strong F-regularity (Theorem 3.37).

We also discuss some basic properties of Cohen–Macaulay F-injectivity.

The base change of F-injectivity was first proved by Aberbach–Enescu [1], see [7]. We give another proof using Radu–André homomorphism (Proposition 4.16). This is a slight modification of Enescu's base change theorem on F-rationality [6]. We also prove the openness of CMFI locus (Corollary 4.18).

In section 2, we discuss F-purity of homomorphisms. In section 3, we prove basic properties of strong F-regularity and very strong F-regularity, and prove the F-pure base change of strong F-regularity. In section 4, we discuss some properties of Cohen–Macaulay F-injectivity. In section 5, we prove Matijevic–Roberts type theorem for F-purity, strong F-regularity, and CMFI property.

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2. F-pure homomorphism

(2.1) Let p be a prime number. Let k be a perfect field of characteristic p. For a k-algebra A of k and $r \in \mathbb{Z}$, we define a new k-algebra $A^{(r)}$ as follows. $A^{(r)}$ as a ring is A, and the structure map of $A^{(r)}$ as a k-algebra is the composite

$$k \xrightarrow{F_k^{-r}} k \xrightarrow{\varphi} A,$$

where φ is the original structure map of the k-algebra A, and $F_k^{-r}(\alpha) = \alpha^{p^{-r}}$ for $\alpha \in k$. The element $a \in A$, viewed as an element of $A^{(r)}$, is denoted by $a^{(r)}$. Note that for $e \geq 0$, the eth Frobenius map $F^e : A^{(e+r)} \to A^{(r)}$ given by $F^e(a^{(e+r)}) = (a^{p^e})^{(r)}$ is a k-algebra map. This notation is consistent with that of Frobenius twisting in representation theory [23, (I.9.10)]. Note that $A^{(-r)}$ is also written as ${}^rA$, $A^{(r)}$, $A^{(p^r)}$, or $A^{p^{-r}}$ by some authors. For an A-module M, the module M viewed as an $A^{(r)}$-module (because $A^{(r)} = A$ as a ring) is denoted by $M^{(r)}$. An element $m \in M$ viewed as an element of $M^{(r)}$ is denoted by $m^{(r)}$. If $e \geq 0$, then $M^{(-e)}$ is also an A-module through $F_A : A \to A^{(-e)}$. The action of A on $M^{(-e)}$ is given by $a \cdot m^{(-e)} = (a^{p^e} m)^{(-e)}$.

For a k-algebra map $f : A \to B$, the map f, viewed as a map $A^{(r)} \to B^{(r)}$

is denoted by $f^{(r)}$. Note that $f^{(r)}$ is a k-algebra map, and the diagram

$$\begin{array}{ccc} A^{(e)} & \xrightarrow{f^{(e)}} & B^{(e)} \\ \downarrow F_A^e & & \downarrow F_B^e \\ A & \xrightarrow{f} & B \end{array}$$

is commutative for $e \geq 0$. Thus, the eth Radu-André homomorphism

$$\Phi_e(A,B) : B^{(e)} \otimes_{A^{(e)}} A \to B \qquad (b^{(e)} \otimes a \mapsto b^{p^e}(fa))$$

is induced. It is a k-algebra map for $e \geq 0$. The $(-e)$-shift $\Phi_e(A,B)^{(-e)}$: $B \otimes_A A^{(-e)} \to B^{(-e)}$ is denoted by $\Psi_e(A,B)$.

2.2 Lemma. *Let $A \to B$ be a ring homomorphism, and $\varphi : M \to M'$ a B-linear map between B-modules. Then φ is B-pure if and only if $\varphi_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ is $B_{\mathfrak{m}}$-pure for any maximal ideal $\mathfrak{m}$ of A.*

Proof. If W is a $B_{\mathfrak{m}}$-module, then $1_W \otimes \varphi_{\mathfrak{m}} : W \otimes_{B_{\mathfrak{m}}} M_{\mathfrak{m}} \to W \otimes_{B_{\mathfrak{m}}} M'_{\mathfrak{m}}$ is identified with $1_W \otimes \varphi : W \otimes_B M \to W \otimes_B M'$. So the 'only if' part is clear.

We prove the 'if' part. Let N be any B-module, and let $K := \operatorname{Ker}(1_N \otimes \varphi)$. Then

$$1_{A_{\mathfrak{m}}} \otimes (1_N \otimes \varphi) : A_{\mathfrak{m}} \otimes_A (N \otimes_B M) \to A_{\mathfrak{m}} \otimes_A (N \otimes_B M)$$

is identified with

$$1_{N_{\mathfrak{m}}} \otimes \varphi_{\mathfrak{m}} : N_{\mathfrak{m}} \otimes_{B_{\mathfrak{m}}} M_{\mathfrak{m}} \to N_{\mathfrak{m}} \otimes_{B_{\mathfrak{m}}} M'_{\mathfrak{m}}.$$

So $K_{\mathfrak{m}} = 0$ for any $\mathfrak{m}$, and thus $K = 0$ for any N. This shows that φ is B-pure. □

(2.3) Consider that $k = \mathbb{F}_p$ in (2.1). We say that a k-algebra map $f : A \to B$ is F-pure if for any $e \geq 1$, $\Phi_e(A,B)$ is a pure ring homomorphism.

2.4 Proposition. *Let $f : A \to B$ and $g : B \to C$ be $\mathbb{F}_p$-algebra maps.*

1 *If f and g are F-pure homomorphisms, then so is gf.*

2 *If gf is F-pure and g is pure, then f is F-pure.*

3 *An $\mathbb{F}_p$-algebra A is F-pure if and only if the unique map $\mathbb{F}_p \to A$ is F-pure.*

4 *If A is F-pure and f is an F-pure homomorphism, then B is F-pure.*

5 *A pure subring of an F-pure ring of characteristic p is again F-pure.*

6 *A regular homomorphism of noetherian rings of characteristic p is F-pure.*

7 *A base change of an F-pure ring homomorphism is again F-pure. Namely, if $f : A \to B$ is an F-pure homomorphism, A' an A-algebra, and $B' := B \otimes_A A'$, then $f \otimes 1 : A' \to B'$ is F-pure.*

8 *If $A \to A'$ is a pure ring homomorphism and $f \otimes 1 : A' \to B' := B \otimes_A A'$ is F-pure, then f is F-pure.*

9 *The following are equivalent.*

- **(i)** *f is F-pure.*
- **(ii)** *For any prime ideal $\mathfrak{p}$ of A, $f_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is F-pure.*
- **(iii)** *For any maximal ideal $\mathfrak{m}$ of A, $f_{\mathfrak{m}} : A_{\mathfrak{m}} \to B_{\mathfrak{m}}$ is F-pure.*
- **(iv)** *For any prime ideal $\mathfrak{q}$ of B, $A \to B_{\mathfrak{q}}$ is F-pure.*
- **(v)** *For any maximal ideal $\mathfrak{n}$ of B, $A \to B_{\mathfrak{n}}$ is F-pure.*
- **(vi)** *For any prime ideal $\mathfrak{q}$ of B, $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is F-pure, where $\mathfrak{p} := \mathfrak{q} \cap A$.*
- **(vii)** *For any maximal ideal $\mathfrak{n}$ of B, $A_{\mathfrak{p}} \to B_{\mathfrak{n}}$ is F-pure, where $\mathfrak{p} := \mathfrak{n} \cap A$.*

Proof. **1** and **2** are obvious by [12, Lemma 4.1, **1**]. **3** follows from [12, Lemma 4.1, **5**]. **4** follows immediately by **1** and **3**. **5** follows immediately by **2** and **3**. **6** follows from a theorem of Radu [27] and André [2] which states that a homomorphism of noetherian rings of characteristic p $A \to B$ is regular if and only if there exists some $e > 0$ such that $\Phi_e(A, B)$ is flat if and only if $\Phi_e(A, B)$ is faithfully flat for any $e > 0$, see also [4]. **7** and **8** follow easily from [12, Lemma 4.1, **4**].

9 **(ii)**$\Rightarrow$**(iii)**, **(iv)**$\Rightarrow$**(v)**, and **(vi)**$\Rightarrow$**(vii)** are trivial. **(i)**$\Rightarrow$**(ii)** is a consequence of **7**. $\Phi_e(A, B)$ is pure if and only if $\Phi_e(A, B)_{\mathfrak{m}}$ is pure for every $\mathfrak{m} \in \mathrm{Max}(A)$ by Lemma 2.2, where $\mathrm{Max}(A)$ denotes the set of maximal ideals of A. Now **(iii)**$\Rightarrow$**(i)** follows from [12, Lemma 4.1, **4**]. Applying [12, Lemma 4.1, **1**] to $A \to B$ and $B \to B_{\mathfrak{q}}$, we have that $\Phi_e(A, B)_{\mathfrak{q}^{(e)}}$ is pure if and only if $\Phi_e(A, B_{\mathfrak{q}})$ is pure, since $\Phi_e(B, B_{\mathfrak{q}})$ is an isomorphism. In view of Lemma 2.2 again, this proves **(i)**$\Rightarrow$**(iv)** and **(v)**$\Rightarrow$**(i)**. Combining **(i)**$\Rightarrow$**(ii)**

and **(i)⇒(iv)**, we get **(i)⇒(vi)** easily. We prove **(vii)⇒(v)**. Applying [12, Lemma 4.1, **1**] to $A \to A_{\mathfrak{p}}$ and $A_{\mathfrak{p}} \to B_{\mathfrak{n}}$, $\Phi_e(A, B_{\mathfrak{n}})$ is pure if and only if $\Phi_e(A_{\mathfrak{p}}, B_{\mathfrak{n}})$ is pure, since $\Phi_e(A, A_{\mathfrak{p}})$ is an isomorphism. The assertion is now clear. □

2.5 Lemma. *Let $f : A \to B$ be an $\mathbb{F}_p$-algebra map.*

1 *If $\Phi_1(A, B)$ is pure, then f is F-pure.*

2 *If $\Phi_e(A, B)$ is pure for some $e > 0$ and A is F-pure, then f is F-pure, and $B^{(e')} \otimes_{A^{(e')}} A$ is F-pure for any $e' > 0$.*

3 *If $e > 0$ and $B^{(e)} \otimes_{A^{(e)}} A$ is F-pure, then $\Phi_e(A, B)$ is pure. In particular, if $B^{(1)} \otimes_{A^{(1)}} A$ is F-pure, then f is F-pure.*

Proof. **1** is an immediate consequence of [12, Lemma 4.1, **2**]. **2** f is F-pure by **1** and [12, Lemma 4.1, **2**]. So $\Phi_{e'}(A, B)$ is F-pure, and A is F-pure. Now applying [12, Lemma 4.1, **7**], we have that $B^{(e')} \otimes_{A^{(e')}} A$ is F-pure. **3** By [12, Lemma 4.1, **7**], $\Phi_e(A, B)$ is pure. The last assertion follows from **1**. □

2.6 Question. Is a homomorphism $f : A \to B$ between noetherian rings of characteristic p F-pure if $\Phi_e(A, B)$ is pure for some $e > 0$?

(2.7) We say that a ring homomorphism $f : A \to B$ between rings of characteristic p is *Dumitrescu* if $\Phi_e(A, B)$ is pure as an A-linear map for every $e > 0$. By definition, an F-pure homomorphism is Dumitrescu.

If $\Phi_1(A, B)$ is A-pure, then f is Dumitrescu by [12, Lemma 4.1, **2**].

Dumitrescu [5] proved that a *flat* ring homomorphism $f : A \to B$ between noetherian rings of characteristic p is Dumitrescu if and only if f is reduced, that is, f is flat with geometrically reduced fibers.

2.8 Proposition. *Let $f : A \to B$ and $g : B \to C$ be $\mathbb{F}_p$-algebra maps.*

1 *If f is F-pure and g is Dumitrescu, then gf is Dumitrescu.*

1' *If f and g are Dumitrescu, and g is flat, then gf is Dumitrescu.*

2 *If gf is Dumitrescu and g is A-pure, then f is Dumitrescu.*

3 *An $\mathbb{F}_p$-algebra A is reduced if and only if the unique map $\mathbb{F}_p \to A$ is Dumitrescu.*

4 *If A is F-pure and f is Dumitrescu, then B is reduced.*

5 *A subring of a reduced ring (of characteristic p) is reduced.*

6 *A reduced homomorphism of noetherian rings of characteristic p is Dumitrescu.*

7 *A base change of a Dumitrescu homomorphism is again Dumitrescu. Namely, if $f : A \to B$ is a Dumitrescu homomorphism, A' an A-algebra, and $B' := B \otimes_A A'$, then $f \otimes 1 : A' \to B'$ is Dumitrescu.*

8 *If $A \to A'$ is a pure ring homomorphism and $f \otimes 1 : A' \to B' := B \otimes_A A'$ is Dumitrescu, then f is Dumitrescu.*

9 *The following are equivalent.*

- **(i)** *f is Dumitrescu.*
- **(ii)** *For any prime ideal $\mathfrak{p}$ of A, $f_\mathfrak{p} : A_\mathfrak{p} \to B_\mathfrak{p}$ is Dumitrescu.*
- **(iii)** *For any maximal ideal $\mathfrak{m}$ of A, $f_\mathfrak{m} : A_\mathfrak{m} \to B_\mathfrak{m}$ is Dumitrescu.*
- **(iv)** *For any prime ideal $\mathfrak{q}$ of B, $A \to B_\mathfrak{q}$ is Dumitrescu.*
- **(v)** *For any maximal ideal $\mathfrak{n}$ of B, $A \to B_\mathfrak{n}$ is Dumitrescu.*
- **(vi)** *For any prime ideal $\mathfrak{q}$ of B, $A_\mathfrak{p} \to B_\mathfrak{q}$ is Dumitrescu, where $\mathfrak{p} := \mathfrak{q} \cap A$.*
- **(vii)** *For any maximal ideal $\mathfrak{n}$ of B, $A_\mathfrak{p} \to B_\mathfrak{n}$ is Dumitrescu, where $\mathfrak{p} := \mathfrak{n} \cap A$.*

Proof. Similar to Proposition 2.4. □

2.9 Lemma. *Let $f : A \to B$ be an $\mathbb{F}_p$-algebra map.*

1 *If $\Phi_1(A, B)$ is A-pure, then f is Dumitrescu.*

2 *If $\Phi_e(A, B)$ is A-pure for some $e > 0$ and A is F-pure, then f is Dumitrescu, and $B^{(e')} \otimes_{A^{(e')}} A$ is reduced for any $e' > 0$.*

Proof. Similar to Lemma 2.5. □

2.10 Question. Is an F-pure homomorphism between noetherian rings of characteristic p flat? More generally, is a Dumitrescu homomorphism between noetherian rings of characteristic p flat?

2.11 Lemma. *Let $f : A \to B$ be a ring homomorphism between rings of characteristic p. Then for $e \geq 0$, the composite*

$$A \cong A^{(e)} \otimes_{A^{(e)}} A \xrightarrow{f^{(e)}\otimes 1} B^{(e)} \otimes_{A^{(e)}} A \xrightarrow{\Phi_e(A,B)} B$$

is f.

Proof. Clear. □

2.12 Lemma. *Let $f : A \to B$ be a ring homomorphism between rings of characteristic p. Assume that A is noetherian, and the image of the associated map ${}^a f : \operatorname{Spec} B \to \operatorname{Spec} A$ contains $\operatorname{Max}(A)$, the set of maximal ideals of A. If f is Dumitrescu, then f is pure.*

Proof. We may assume that $(A, \mathfrak{m})$ is local. Let E be the injective hull of the residue field $A/\mathfrak{m}$ of A. Set $E_n := 0 :_E \mathfrak{m}^n$. Let $A_n := A/\mathfrak{m}^n$ and $B_n := A_n \otimes_A B$.

It suffices to show that $f_n : A_n \to B_n$ is pure. Indeed, if so, $E_n = E_n \otimes_A A \to E_n \otimes_A B$ is injective, and hence taking the inductive limit, $E = E \otimes_A A \to E \otimes_A B$ is injective, and hence f is pure, see [21].

So we may and shall assume that $(A, \mathfrak{m})$ is artinian local. Take $e > 0$ sufficiently large so that $\mathfrak{m}^{[p^e]} = 0$. Namely, $a^{p^e} = 0$ for every $a \in \mathfrak{m}$. Then the Frobenius map $F^e : A^{(e)} \to A$ factors through $(A/\mathfrak{m})^{(e)}$. Thus

$$A = A^{(e)} \otimes_{A^{(e)}} A \xrightarrow{f^{(e)}\otimes 1} B^{(e)} \otimes_{A^{(e)}} A$$

is identified with the map

$$A = (A/\mathfrak{m})^{(e)} \otimes_{(A/\mathfrak{m})^{(e)}} A \xrightarrow{f_1^{(e)}\otimes 1} (B/\mathfrak{m}B)^{(e)} \otimes_{(A/\mathfrak{m})^{(e)}} A.$$

Since $B/\mathfrak{m}B \neq 0$, this map is faithfully flat and hence is pure.

By the assumption and Lemma 2.11, f is pure. □

2.13 Corollary. *A local homomorphism between noetherian local rings of characteristic p is pure, if it is Dumitrescu.*

2.14 Lemma. *If $f : A \to B$ is F-pure and B is noetherian, then $B^{(e)} \otimes_{A^{(e)}} A$ is noetherian for $e \geq 0$.*

Proof. This is because $\Phi_e(A, B)$ is pure. □

2.15 Lemma. *Let K be a field of characteristic p, and B a K-algebra. Then the following are equivalent.*

1 *$K \to B$ is F-pure, and B is noetherian.*

2 *For any $e > 0$, $B \otimes_K K^{(-e)}$ is noetherian and F-pure.*

3 *There exists some $e > 0$ such that $B \otimes_K K^{(-e)}$ is noetherian and F-pure.*

4 *B is noetherian, and B is geometrically F-pure over K, that is to say, for any finite algebraic extension L of K, $B \otimes_K L$ is F-pure.*

Proof. Note that $(B \otimes_K K^{(-e)})^{(e)} \cong B^{(e)} \otimes_{K^{(e)}} K$. **1⇒2** Let $e > 0$. $B \otimes_K K^{(-e)}$ is noetherian by Lemma 2.14. $B \otimes_K K^{(-e)}$ is F-pure by Lemma 2.5, **2**.

2⇒3 is trivial.

3⇒1 As

$$B = B \otimes_K K \xrightarrow{1_B \otimes F^e} B \otimes_K K^{(-e)}$$

is faithfully flat and $B \otimes_K K^{(-e)}$ is noetherian, B is noetherian. $K \to B$ is F-pure by Lemma 2.5, **3**.

1,2,3⇒4 B is noetherian, as assumed. As the field L is F-pure and $L \to B \otimes_K L$ is F-pure (as it is the base change of the F-pure homomorphism $K \to B$), $B \otimes_K L$ is also F-pure by Proposition 2.4, **4**.

4⇒ 1 Let L be a finite extension field of K such that $L \subset K^{(-1)}$. Then $F : B^{(1)} \otimes_{K^{(1)}} L^{(1)} \to B \otimes_K L$ is pure. As F factors through B, the map $B^{(1)} \otimes_{K^{(1)}} L^{(1)} \to B$ is pure. Taking the inductive limit on L, $\Phi_1(K, B) : B^{(1)} \otimes_{K^{(1)}} K \to B$ is also pure. So $K \to B$ is F-pure. □

2.16 Corollary. *If $f : A \to B$ is F-pure homomorphism between noetherian rings of characteristic p, then f has geometrically F-pure fibers. That is, for any $\mathfrak{p} \in \operatorname{Spec} A$ and any finite algebraic extension L of $\kappa(\mathfrak{p})$, $B \otimes_A L$ is F-pure.*

2.17 Remark. The converse is not true in general. Indeed, a flat homomorphism with geometrically F-pure fibers need not be an F-pure homomorphism. Singh's example [29, section 6] shows that for $p > 2$, there is an example of a homomorphism $f : A \to B$ such that $A = \mathbb{F}_p[t]_{(t)}$, f is flat with geometrically F-pure fibers, but B is not F-pure. If f were F-pure, then B must have been F-pure by Proposition 2.4, **4**.

(2.18) Let $f : A \to B$ be a homomorphism between noetherian rings. We say that f is almost quasi-finite if for any $P \in \operatorname{Spec} A$, $\kappa(P) \otimes_A B$ is module finite over $\kappa(P)$. A quasi-finite homomorphism (that is, finite-type homomorphism with zero-dimensional fibers) is almost quasi-finite. A localization $A \to A_S$ is almost quasi-finite. A composite of almost quasi-finite homomorphisms is almost quasi-finite. A base change of an almost quasi-finite homomorphism is almost quasi-finite.

2.19 Theorem. *Let $f : A \to B$ be an almost quasi-finite homomorphism between noetherian rings of characteristic p. Then the following are equivalent.*

1 *f is regular.*

2 *f is F-pure.*

3 *f is Dumitrescu.*

Proof. Note that **1**⇒**2**⇒**3** is trivial. So it suffices to prove **3**⇒**1**. Note that each fiber $\kappa(P) \otimes_A B$ is Dumitrescu over $\kappa(P)$ by Proposition 2.8, **7**. So it is geometrically reduced by Dumitrescu's theorem [5]. As we assume that $\kappa(P) \otimes_A B$ is finite over $\kappa(P)$, we have that $\kappa(P) \otimes_A B$ is étale over $\kappa(P)$. So in order to prove **1**, it suffices to prove that f is flat.

Thanks to Proposition 2.8, **9**, we may assume that $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ is a local homomorphism between local rings. By the local criterion of flatness ((5)⇒(1) of [25, Theorem 22.3]) and Proposition 2.8, **7**, we may assume that A is artinian. Then B is module finite over A. It is easy to see that if $l_A(B) \geq l_A(A) l_A(B/\mathfrak{m}B)$, then f is flat by (4)⇒(1) of [25, Theorem 22.3], where l_A denotes the length as an A-module.

Take $e > 0$ sufficiently large so that $\mathfrak{m}^{(e)} A = 0$, that is, for any $x \in \mathfrak{m}$, $x^{p^e} = 0$. Then $\Phi_e(A, B)$ is identified with

$$(B/\mathfrak{m}B)^{(e)} \otimes_{(A/\mathfrak{m})^{(e)}} A \to B.$$

This map is an injective A-linear map with $l_A((B/\mathfrak{m}B)^{(e)} \otimes_{(A/\mathfrak{m})^{(e)}} A) = l_A(A) l_A(B/\mathfrak{m}B)$. So $l_A(B) \geq l_A(A) l_A(B/\mathfrak{m}B)$, as desired. □

2.20 Remark. Theorem 2.19 for the case that $f : A \to B$ is a finite homomorphism between integral domains (the crucial case) is due to K.-i. Watanabe.

2.21 Proposition. *Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local homomorphism between noetherian local rings of characteristic p, and $t \in \mathfrak{m}$. Assume that A is normally flat along tA, A/tA is reduced, $A/tA \to B/tB$ is flat, and f is Dumitrescu. Then f is flat.*

Proof. It suffices to show that $A/t^nA \to B/t^nB$ is flat for all $n \geq 1$. So we may assume that $t^n = 0$ for some $n \geq 1$. We prove the assertion by induction on n. If $n = 1$, then the assertion is assumed by the assumption of the proposition.

So we consider the case that $n \geq 2$. It suffices to show that the canonical map

$$\gamma_i : B/tB \otimes_{A/tA} (t^iA/t^{i+1}A) \to t^iB/t^{i+1}B \qquad \bar{b} \otimes \overline{t^i a} \mapsto \overline{t^i ab}$$

is injective for $i = 1, \ldots, n-1$. This is true for $i = 1, \ldots, n-2$, as $A/t^{n-1}A \to B/t^{n-1}B$ is flat by induction assumption. So it suffices to show that $\gamma_{n-1} : B/tB \otimes_{A/tA} t^{n-1}A \to t^{n-1}B$ is injective. Let $x \in B$ be an element such that $\bar{x} \otimes t^{n-1} \in \operatorname{Ker} \gamma_{n-1}$, or equivalently, $xt^{n-1} = 0$. Take $e \gg 0$ such that $p^e > n$. Obviously, we have $x^{p^e} t^{n-1} = 0$ in B. As

$$\Phi_e(A, B) : (B/tB)^{(e)} \otimes_{(A/tA)^{(e)}} A \cong B^{(e)} \otimes_{A^{(e)}} A \to B$$

is injective, $\bar{x}^{(e)} \otimes t^{n-1} = 0$ in $(B/tB)^{(e)} \otimes_{(A/tA)^{(e)}} A$. As $(B/tB)^{(e)}$ is $(A/tA)^{(e)}$-flat, $\bar{x}^{(e)} \otimes t^{n-1} = 0$ in $(B/tB)^{(e)} \otimes_{(A/tA)^{(e)}} t^{n-1}A$. Since $t^{n-1}A$ is a rank-one free A/tA-module with t^{n-1} its basis, $\bar{x}^{(e)} \otimes 1 \in (B/tB)^{(e)} \otimes_{(A/tA)^{(e)}} A/tA$ is zero. As A/tA is reduced, $(A/tA)^{(e)} \to A/tA$ is injective. So $(B/tB)^{(e)} \to (B/tB)^{(e)} \otimes_{(A/tA)^{(e)}} A/tA$ is injective. Hence $\bar{x}^{(e)} = 0$ in $(B/tB)^{(e)}$. So $\bar{x} = 0$ in B/tB. This shows that $\bar{x} \otimes t^{n-1} = 0$ in $B/tB \otimes_{A/tA} t^{n-1}A$, and thus γ_{n-1} is injective, as desired. $\square$

2.22 Corollary. *Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a Dumitrescu local homomorphism between noetherian local rings of characteristic p. If $t \in \mathfrak{m}$ is a nonzerodivisor, A/tA is reduced, and B/tB is A/tA-flat, then B is A-flat.* $\square$

2.23 Corollary. *Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a Dumitrescu local homomorphism between noetherian local rings of characteristic p. If A is regular, then f is flat.*

Proof. We prove this by induction on $\dim A$. If $\dim A = 0$, then A is a field, and f is flat.

Next consider the case $\dim A > 0$. Then take $t \in \mathfrak{m} \setminus \mathfrak{m}^2$. Then t is a nonzerodivisor, A/tA is regular, and $A/tA \to B/tB$ is flat by induction assumption. By Corollary 2.22, f is flat. □

3. Strong F-regularity

(3.1) For a ring R, we define $R^\circ := R \setminus \bigcup_{P \in \operatorname{Min} R} P$, where $\operatorname{Min} R$ denotes the set of minimal primes of R. Let R be a ring of characteristic p. Let M be an R-module, and N its submodule. We define

$$\operatorname{Cl}_R(N, M) = N^*_M := \{x \in M \mid \exists c \in R^\circ\ \exists e_0 \geq 1 \\ \forall e \geq e_0\ x \otimes c^{(-e)} \in M/N \otimes_R R^{(-e)} \text{ is zero}\},$$

and call it the tight closure of N in M. Note that $\operatorname{Cl}_R(N, M)$ is an R-submodule of M containing N [19, section 8].

3.2 Lemma. *Let R be a noetherian commutative ring of characteristic p, and S a multiplicatively closed subset of R. Let M be an R_S-module, and N its R_S-submodule. Then $\operatorname{Cl}_R(N, M) = \operatorname{Cl}_{R_S}(N, M)$.*

Proof. Note that

$$x \in \operatorname{Cl}_R(N, M) \iff \exists c \in R^\circ\ \exists q'\ \forall q \geq q'\ x \otimes c^{(-e)} \in \\ \operatorname{Ker}(M \otimes_R R^{(-e)} \to M/N \otimes_R R^{(-e)}) =: N^{[q]}$$

and that

$$x \in \operatorname{Cl}_{R_S}(N, M) \iff \exists c \in R_S^\circ\ \exists q'\ \forall q \geq q'\ x \otimes c^{(-e)} \in \\ \operatorname{Ker}(M \otimes_{R_S} R_S^{(-e)} \to M/N \otimes_{R_S} R_S^{(-e)}) = N^{[q]},$$

where $q = p^e$ and q' denote some power of p. If $c \in R^\circ$, then $c/1 \in R_S^\circ$. So $\operatorname{Cl}_R(N, M) \subset \operatorname{Cl}_{R_S}(N, M)$.

Let $c \in R$, $c/1 \in R_S^\circ$, and assume that there exists some q' such that for all $q \geq q'$, $x \otimes c^{(-e)} \in N^{[q]}$. Take $\delta \in R$ such that for any $P \in \operatorname{Min} R$, $\delta \in P$ if and only if $c \notin P$. Then $\delta/1$ is nilpotent in R_S. Replacing δ by its some power, we may assume that $\delta/1 = 0$ in R_S. Then $x \otimes (c+\delta)^{(-e)} \in N^{[q]}$ for $q \geq q'$ and $c+\delta \in R^\circ$. Hence $x \in \operatorname{Cl}_R(N, M)$. This shows $\operatorname{Cl}_R(N, M) \supset \operatorname{Cl}_{R_S}(N, M)$. □

3.3 Definition (Hochster [17, p. 166]). We say that a noetherian ring R of characteristic p is *strongly F-regular* if $\mathrm{Cl}_R(N, M) = N$ for any R-module M and any submodule N of M.

3.4 Definition (cf. [20, (5.3)]). We say that a noetherian ring R of characteristic p is *very strongly F-regular* if for any $c \in R^\circ$ there exists some $e > 0$ such that the map $c^{(-e)}F^e : R \to R^{(-e)}$ ($x \mapsto (cx^{p^e})^{(-e)}$) is R-pure.

3.5 Lemma. *Let $(R, \mathfrak{m})$ be a local ring, and E be the injective hull of the residue field. Let S be an R-module, and $h : R \to S$ be an R-linear map. If $1_E \otimes h : E \to E \otimes_R S$ is injective, then h is R-pure.*

Proof. Exactly the same proof as [21, (6.11)] works. □

3.6 Lemma. *Let R be a noetherian ring of characteristic p. Then the following are equivalent.*

1 *R is strongly F-regular.*

2 *For any multiplicatively closed subset S of R, R_S is strongly F-regular.*

3 *$R_\mathfrak{m}$ is strongly F-regular for any $\mathfrak{m} \in \mathrm{Max}(R)$.*

4 *For any $\mathfrak{m} \in \mathrm{Max}(R)$, $\mathrm{Cl}_R(0, E_R(R/\mathfrak{m})) = 0$.*

5 *For any $\mathfrak{m} \in \mathrm{Max}(R)$, $\mathrm{Cl}_{R_\mathfrak{m}}(0, E_{R_\mathfrak{m}}(R/\mathfrak{m})) = 0$.*

6 *$R_\mathfrak{m}$ is very strongly F-regular for any $\mathfrak{m} \in \mathrm{Max}(R)$.*

Proof. **1⇒4**, **2⇒3**, and **3⇒5** are obvious. **1⇒2** and **4⇒5** follow from Lemma 3.2.

We prove **5⇒6**. We may assume that $(R, \mathfrak{m})$ is local. Let $E = E_R(R/\mathfrak{m})$ be the injective hull of the residue field. The kernel of the map

$$1_E \otimes F_R : E \to E \otimes_R R^{(-1)} \qquad (x \mapsto x \otimes 1)$$

is contained in $\mathrm{Cl}_R(0, E) = 0$. Thus $F_R : R \to R^{(-1)}$ is pure by [21, (6.11)]. In other words, R is F-pure.

Now let $c \in R^\circ$, and set $K_e := \mathrm{Ker}(1_E \otimes c^{(-e)}F^e : E \to E \otimes_R R^{(-e)})$, where $1_E \otimes c^{(-e)}F^e$ sends x to $x \otimes c^{(-e)}$. For $e' > e$, we have a commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{1_E \otimes c^{(-e)} F^e} & E \otimes_R R^{(-e)} \\
\downarrow{\scriptstyle 1_E \otimes c^{(-e')} F^{e'}} & & \downarrow{\scriptstyle 1 \otimes F^{e'-e}} \\
E \otimes_R R^{(-e')} & \xrightarrow{1_E \otimes (c^{p^{e'-e}-1})^{(-e')}} & E \otimes_R R^{(-e')}
\end{array} . \tag{1}
$$

As $1 \otimes F^{e'-e}$ is injective, we have

$$E \supset K_1 \supset K_2 \supset \cdots .$$

As E is an artinian module, there exists some $e \gg 0$ such that $K_e = \bigcap_{e'} K_{e'} \subset \mathrm{Cl}_R(0, E) = 0$. By Lemma 3.5, $c^{(-e)} F^e : R \to R^{(-e)}$ is pure, as desired.

6⇒3 We may assume that $(R, \mathfrak{m})$ is local. Considering the case that $c = 1$, $F^e : R \to R^{(-e)}$ is pure for some e. Hence R is F-pure. Now let $c \in R^\circ$. Then there exists some $e > 0$ such that $c^{(-e)} F^e$ is R-pure. Considering the commutative diagram (1), we have that $c^{(-e')} F^{e'}$ is R-pure for $e' \geq e$.

Now let M be an R-module, N its submodule. Then $1_{M/N} \otimes c^{(-e')} F^{e'} : M/N \to M/N \otimes_R R^{(-e')}$ is injective by the purity for $e' \geq e$. This shows that $\mathrm{Cl}_R(N, M) = N$.

3⇒1 Let M be an R-module and N its submodule. Let $\mathfrak{m} \in \mathrm{Max}(R)$. Then

$$N_\mathfrak{m} \subset \mathrm{Cl}_R(N, M)_\mathfrak{m} \subset \mathrm{Cl}_{R_\mathfrak{m}}(N_\mathfrak{m}, M_\mathfrak{m}) = N_\mathfrak{m}.$$

Hence $\mathrm{Cl}_R(N, M)_\mathfrak{m} = N_\mathfrak{m}$ for any $\mathfrak{m} \in \mathrm{Max}(R)$. This shows that $\mathrm{Cl}_R(N, M) = N$. □

3.7 Corollary. *A strongly F-regular noetherian ring of characteristic p is F-regular. In particular, it is normal and F-pure.*

Proof. By the definition of strong F-regularity, a strongly F-regular implies weakly F-regular. By **1⇒2** of the lemma, it is also F-regular.

The normality assertion is a consequence of [20, (4.2)].

For the F-purity assertion, it suffices to point out that a weakly F-regular noetherian ring R of characteristic p is F-pure [9, Remark 1.6]. Almost by the definition of weak F-regularity, for any ideal I of R, $1 \otimes F : R/I \otimes_R R \to R/I \otimes_R R^{(-1)}$ is injective. Thus $F_R : R \to R^{(-1)}$ is cyclically pure. But as R is normal, it is approximately Gorenstein, and thus F_R is pure [16]. □

3.8 Lemma. *If R is very strongly F-regular noetherian ring of characteristic p, then R_S is very strongly F-regular for any multiplicatively closed subset S of R. In particular, a very strongly F-regular noetherian ring of characteristic p is strongly F-regular.*

Proof. Let $c/s \in (R_S)^\circ$, where $c \in R$ and $s \in S$. Take $\delta \in R$ such that $\delta \in P$ if and only if $c \notin P$ for $P \in \operatorname{Min} R$. Replacing δ by its power, we may assume that $\delta/1 = 0$ in R_S. Set $d = c + \delta$. Then $d^{(-e)}F^e : R \to R^{(-e)}$ is R-pure for some $e \geq 0$. So $d^{(-e)}F^e : R_S \to R_S^{(-e)}$ is R_S-pure. It follows that $(c/s)^{(-e)}F^e : R_S \to R_S^{(-e)}$ is also R_S-pure, since $c/s = d/s$. The last assertion follows from Lemma 3.6. □

3.9 Lemma. *Let R be an F-finite noetherian ring of characteristic p. If R is strongly F-regular, then it is very strongly F-regular.*

Proof. $R_\mathfrak{m}$ is very strongly F-regular for $\mathfrak{m} \in \operatorname{Max} R$ by Lemma 3.6. Then by [22, (5.2)], $R_\mathfrak{m}$ is "strongly F-regular" in the sense of [18] for each $\mathfrak{m}$. By [18, (3.1)], R is "strongly F-regular" in the sense of [18]. So R is very strongly F-regular. □

3.10 Lemma. *Let $R = R_1 \times R_2$ be a noetherian ring of characteristic p. Then R is very strongly F-regular if and only if both R_1 and R_2 are.*

Proof. If R is very strongly F-regular, then R_1 and R_2 are very strongly F-regular, as R_1 and R_2 are localizations of R.

Conversely, let R_1 and R_2 be very strongly F-regular. Let e_1 and e_2 be respectively the idempotents corresponding to R_1 and R_2. Let $c \in R^\circ$. Then $ce_i \in R_i^\circ$, and hence there exist some r_i ($i = 1, 2$) such that $(ce_i)^{(-r_i)}F^{r_i} : R_i \to R_i^{(-r_i)}$ is R_i-pure for each i. As each R_i is F-pure, it is easy to see that letting $r = \max(r_1, r_2)$, $(ce_i)^{(-r)}F^r : R_i \to R_i^{(-r)}$ is R_i-pure for $i = 1, 2$. Then $c^{(-r)}F^r : R \to R^{(-r)}$ is R-pure. □

3.11 Lemma. *Let $R \to S$ be a ring homomorphism. Let $S = S_1 \times \cdots \times S_r$ be a finite direct product of integrally closed domains. Assume that for any nonzerodivisor a of R, $aS \cap R = aR$. Then R is integrally closed in the total quotient ring $Q(R)$.*

Proof. Let $\alpha = b/a$ be an element of $Q(R)$, where $a, b \in R$ with a a nonzero divisor of R. Assume that α is integral over R.

Let $1 \leq i \leq r$. Consider the case that a is nonzero in S_i. Then b/a makes sense in the field of fractions $Q(S_i)$ of S_i, and it is integral over S_i. As S_i is integrally closed, $b/a \in S_i$. Hence $b \in aS_i$.

Next consider the case that a is zero in S_i. Since b/a is integral over R and a is a nonzerodivisor, there exists some $n \geq 1$ such that $b^n \in aR$. This shows that $b^n = 0$ in S_i. As S_i is a domain, $b = 0$ in S_i. This shows that $b \in aS_i$.

As $b \in aS_i$ for any i, $b \in aS \cap R = aR$. Hence $\alpha = b/a \in R$, and R is integrally closed in $Q(R)$. □

3.12 Corollary. *Let S be a noetherian normal ring, and R its cyclically pure subring. Then R is a noetherian normal ring, and hence R is a pure subring of S.*

Proof. If $I_1 \subset I_2 \subset \cdots$ is an ascending chain of ideals in R, then $I_1S \subset I_2S \subset \cdots$ is that of S, and hence $I_NS = I_{N+1}S = \cdots$ for some N. Hence $I_NS \cap R = I_{N+1}S \cap R = \cdots$. By cyclic purity, $I_N = I_{N+1} = \cdots$, and hence R is noetherian.

As R is a subring of S, R is reduced. By Lemma 3.11, R is integrally closed in $Q(R)$. So R is a noetherian normal ring.

Hence R is approximately Gorenstein [16], and hence R is a pure subring of S. □

3.13 Corollary. *Let S be a noetherian ring of characteristic p, and R is cyclically pure subring of S. If S is weakly F-regular (resp. F-regular), then so is R.*

Proof. In any case, S is normal. So R is normal and pure in S. Now the assertion follows from [14, (9.11)]. □

3.14 Lemma. *Let $R \to S$ be a cyclically pure ring homomorphism between noetherian rings of characteristic p. If S is very strongly F-regular, then so is R.*

Proof. Note that S is F-regular. So R is normal and is a pure subring of S by Corollary 3.12. By Lemma 3.10, we may assume that R is a normal domain. We can write $S = S_1 \times \cdots \times S_r$, where S_i is a very strongly F-regular domain for each i. Assume that the induced map $f_i : R \to S_i$ is not injective. Let $I_i = \operatorname{Ker} f_i$. For $\mathfrak{m} \in \operatorname{Max} R$, $(I_i)_{\mathfrak{m}} \neq 0$. So $R_{\mathfrak{m}} \to (S_i)_{\mathfrak{n}}$ cannot be injective for any $\mathfrak{n} \in \operatorname{Max} S_i$. By [14, (9.10)], $R_{\mathfrak{m}} \to S'_{\mathfrak{n}}$ is pure for some maximal ideal

$\mathfrak{n}$ of $S' := S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_r$. This shows that $R \to S'$ is still pure. Removing redundant S_i, we may assume that f_i is injective for each i.

Now let $c \in R^\circ$. Then by our additional assumption, $c \in S^\circ$. So $c^{(-e)}F_S^e$ is S-pure for some $e \geq 1$. As the diagram

$$\begin{array}{ccc} S & \xrightarrow{c^{(-e)}F_S^e} & S^{(-e)} \\ \uparrow & & \uparrow \\ R & \xrightarrow{c^{(-e)}F_R^e} & R^{(-e)} \end{array}$$

is commutative, we have that $c^{(-e)}F_R^e$ is R-pure. Hence R is very strongly F-regular. □

3.15 Lemma. *Let $R \to R'$ be a homomorphism between noetherian rings of characteristic p. Assume that the induced map $\operatorname{Spec} R' \to \operatorname{Spec} R$ is an open immersion. If R is very strongly F-regular, then R' is very strongly F-regular.*

Proof. $\operatorname{Spec} R'$ has a finite affine open covering $\operatorname{Spec} R' = \bigcup U_i$ such that $U_i = \operatorname{Spec} R[1/f_i]$ for some $f_i \in R$. Then $R[1/f_i]$ is very strongly F-regular by Lemma 3.8. So $R'' := \prod_i R[1/f_i]$ is also very strongly F-regular by Lemma 3.10. As R'' is faithfully flat over R', R' is also very strongly F-regular by Lemma 3.14. □

3.16 Corollary. *Let R be a noetherian ring of characteristic p, and $\operatorname{Spec} R = \bigcup_i \operatorname{Spec} R_i$ an affine open covering. Then R is very strongly F-regular if and only if R_i is very strongly F-regular for each i.*

Proof. Note that each R_i is noetherian. If R is very strongly F-regular, then each R_i is very strongly F-regular by Lemma 3.15. Conversely, assume that each R_i is very strongly F-regular for each i. Then we can take a finite open subcovering $\operatorname{Spec} R = \bigcup_{j=1}^r \operatorname{Spec} R_{i_j}$. As $R' = \prod_{j=1}^r R_{i_j}$ is very strongly F-regular by Lemma 3.10 and R' is faithfully flat over R, R is also very strongly F-regular by Lemma 3.14. □

3.17 Lemma. *Let $R \to S$ be a cyclically pure ring homomorphism between noetherian rings of characteristic p. If S is strongly F-regular, then so is R.*

Proof. Note that R is pure in S. Let $\mathfrak{m} \in \operatorname{Max} R$. Then there exists some $\mathfrak{n} \in \operatorname{Max} S$ such that $\mathfrak{n}$ lies on $\mathfrak{m}$ and $R_\mathfrak{m} \to S_\mathfrak{n}$ is pure, see [14, (9.10)]. As $S_\mathfrak{n}$ is very strongly F-regular by Lemma 3.6, $R_\mathfrak{m}$ is very strongly F-regular by Lemma 3.14. So R is strongly F-regular by Lemma 3.6 again. □

(3.18) Let $(R, \mathfrak{m}, K)$ be a complete noetherian local ring with a coefficient field $K \subset R$ of characteristic p. We fix a p-base Λ of K. A subset Γ of Λ is said to be cofinite if $\Lambda \setminus \Gamma$ is a finite set. For a cofinite subset Γ of Λ and $e \geq 0$, we denote K_e^Γ the extension field of K generated by the all p^eth roots of elements in Γ. We define R_e^Γ to be the completion of the noetherian local ring $K_e^\Gamma \otimes_K R$. We denote the inductive limit $\varinjlim_e R_e^\Gamma$ by R^Γ. For an R-algebra A essentially of finite type, we define $A^\Gamma := A \otimes_R R^\Gamma$.

3.19 Lemma. *The canonical map $A \to A^\Gamma$ is a faithfully flat homomorphism with complete intersection fibers.*

Proof. We may assume that A is a field. Note that $R \to K_e^\Gamma \otimes_K R$ is a faithfully flat map with complete intersection fibers (to verify this, we may assume that $R = K[[x_1, \ldots, x_n]]$). As R is complete, $K_e^\Gamma \otimes_K R$ is a homomorphic image of a regular local ring, and hence the completion $K_e^\Gamma \otimes_K R \to R_e^\Gamma$ has complete intersection fibers.

Thus $A^\Gamma = A \otimes_R \varinjlim R_e^\Gamma \cong \varinjlim A \otimes_R R_e^\Gamma$ is a noetherian inductive limit of artinian local rings and faithfully flat purely inseparable complete intersection homomorphisms. As A^Γ is noetherian, and an element of A^Γ is either a unit or nilpotent, A^Γ is artinian local. The maximal ideal $\mathfrak{m}^\Gamma$ of A^Γ is generated by the maximal ideal $\mathfrak{m}_e^\Gamma$ of the artinian local ring $A_e^\Gamma := A \otimes_R R_e^\Gamma$ for sufficiently large e. Then A_e^Γ is a complete intersection, and the fiber of $A_e^\Gamma \to A^\Gamma$ is a field. So A^Γ is a complete intersection, as desired. $\square$

3.20 Lemma. *Set $K^\Gamma = \bigcup_e K_e^\Gamma$. Then $\bigcap_\Gamma K^\Gamma = K$.*

Proof. Note that K^Γ has a basis

$$B^\Gamma := \{\xi_1^{\lambda_1} \cdots \xi_n^{\lambda_n} \mid n \geq 0,\ \xi_1, \ldots, \xi_n \text{ are distinct elements in } \Lambda,\\ 0 < \lambda_i < 1,\ \lambda_i \in \mathbb{Z}[1/p]\}.$$

Thus a linear combination of elements of B^Λ lies in K^Γ if and only if any basis element with a nonzero coefficient lies in B^Γ. Thus $\bigcap_\Gamma K^\Gamma$ has a basis $\{1\}$, and hence $\bigcap_\Gamma K^\Gamma = K$. $\square$

3.21 Lemma. *Let $K \subset L$ be a field extension, and $\{K_\lambda\}$ a family of intermediate fields. Assume that $\bigcap_\lambda K_\lambda = K$. Let V be a K-vector space. Then $\bigcap_\lambda (K_\lambda \otimes_K V) = V$.*

Proof. Let A be a basis of V over K, and $\sum_{\alpha \in A} c_\alpha \alpha$ be an element of $L \otimes_K V$ with $c_\alpha \in L$. It lies in $K_\lambda \otimes_K V$ if and only if for any α, $c_\alpha \in K_\lambda$. So it lies in $\bigcap_\lambda (K_\lambda \otimes_K V)$ if and only if for any α, $c_\alpha \in K$. Thus $\bigcap_\lambda (K_\lambda \otimes_K V) = V$. $\square$

3.22 Lemma. *Let $(R, \mathfrak{m})$ be a noetherian complete local ring of characteristic p with a coefficient field K. Let Λ be a p-base of K. Then $\bigcap_\Gamma R^\Gamma = R$, where the intersection is taken over the all cofinite subsets Γ of Λ.*

Proof. First consider the case that R is artinian. Then $R_e^\Gamma = K_e^\Gamma \otimes_K R$ (completion is unnecessary, since $K_e^\Gamma \otimes_K R$ is already complete). So $R^\Gamma = K^\Gamma \otimes_K R$. So $\bigcap_\Gamma R^\Gamma = \bigcap_\Gamma (K^\Gamma \otimes_K R) = R$ by Lemma 3.20 and Lemma 3.21.

Next consider the general case. Let a be an element of $\bigcap_\Gamma R^\Gamma$. Then a modulo $\mathfrak{m}^n R^\Lambda$ lies in $\bigcap_\Gamma (R/\mathfrak{m}^n)^\Gamma = R/\mathfrak{m}^n$. So a lies in $\varprojlim R/\mathfrak{m}^n = R$. $\square$

3.23 Lemma. *Let R be as above, and $\pi^\Gamma : \operatorname{Spec} A^\Gamma \to \operatorname{Spec} A$ be the canonical homeomorphism. Then there exists some cofinite subset $\Gamma_0 \subset \Lambda$ such that for every cofinite subset $\Gamma \subset \Gamma_0$, $\operatorname{Reg}(A^\Gamma) = (\pi^\Gamma)^{-1}(\operatorname{Reg}(A))$. Similarly for complete intersection, Gorenstein, Cohen–Macaulay, (S_i), (R_i), normal, and reduced loci.*

Proof. We prove the assertion for the regular property. Set Z^Γ to be $\pi^\Gamma(\operatorname{Sing}(A^\Gamma))$. By Kunz's theorem [24], Z_Γ is closed. The set $\{Z^\Gamma \mid \Gamma : \text{ cofinite in } \Lambda\}$ has a minimal element Z^{Γ_0}. For any $\Gamma \subset \Gamma_0$, $\operatorname{Sing} R \subset Z^\Gamma = Z^{\Gamma_0}$. So it suffices to show that $Z^{\Gamma_0} \subset \operatorname{Sing} R$. Assume the contrary, and let $P \in Z^{\Gamma_0} \setminus \operatorname{Sing} R$. By [20, (6.13)], there exists some $\Gamma \subset \Gamma_0$ such that PA^Γ is a prime and $\kappa(P) \otimes_A A^\Gamma$ is a field. As A_P is regular, $A^\Gamma_{PA^\Gamma}$ is regular. So $P \notin Z^\Gamma = Z^{\Gamma_0}$. A contradiction.

The assertion for (R_i) follows easily from this.

The assertions for complete intersection, Gorenstein, Cohen–Macaulay, and (S_i) properties are true for $\Gamma = \Lambda$, because π^Λ is a faithfully flat homomorphism with complete intersection fibers by Lemma 3.19.

The normality is $(R_1) + (S_2)$, and reduced property is $(R_0) + (S_1)$. They are proved easily from the assertions for (R_i) and (S_i). $\square$

3.24 Lemma. *Let R be as above. Let A be a field which is essentially of finite type over R. Then $\bigcap_\Gamma A^\Gamma = A$.*

Proof. Let P be the kernel of $R \to A$. As A is a field, P is a prime ideal. Replacing R by R/P, we may assume that R is a domain and $R \to A$ is injective. In view of Lemma 3.21, replacing A by the field of fractions $Q(R)$ of R, we may assume that $A = Q(R)$.

Let B be the normalization of R. Take a cofinite subset $\Gamma_0 \subset \Lambda$ such that R^Γ is a domain and B^Γ is a normal domain for $\Gamma \subset \Gamma_0$. This is possible by [20, (6.13)] and Lemma 3.23. As $Q(R) \to Q(R) \otimes_R B$ is an isomorphism, $Q(R^\Gamma) = Q(R) \otimes_R R^\Gamma = Q(R) \otimes_R B^\Gamma$. Hence $R^\Gamma \to B^\Gamma$ is the normalization. In particular, $Q(R^\Gamma) = Q(B^\Gamma) = Q(R)^\Gamma$.

Note that for $\Gamma \subset \Gamma_0$, $B^{\Gamma_0} \cap Q(R^\Gamma)$ is purely inseparable, hence is integral, over B^Γ. Hence $B^{\Gamma_0} \cap Q(B^\Gamma) = B^\Gamma$.

Let $d \in \bigcap_\Gamma B^\Gamma$. If $c \neq 0$ is an element of the conductor $R : B$, then $c \in R^\Gamma : B^\Gamma$. So $cd \in \bigcap_\Gamma R^\Gamma = R$ by Lemma 3.22. So $d \in Q(R)$. As d is integral over B and B is normal, $d \in B$. This shows that $\bigcap_\Gamma B^\Gamma = B$.

Now let $\alpha \in \bigcap_\Gamma Q(R)^\Gamma$. Then there exists some $a \in R^\circ$ and $b \in R^{\Gamma_0}$ such that $\alpha = b/a$. As $\alpha \in \bigcap_\Gamma Q(R)^\Gamma$ and $a \in R^\circ$,

$$b = a\alpha \in \bigcap_\Gamma (Q(R)^\Gamma \cap R^{\Gamma_0}) \subset \bigcap_\Gamma B^\Gamma = B.$$

Hence $\alpha = b/a \in Q(R)$. So $\bigcap_\Gamma Q(R)^\Gamma = Q(R)$. This is what we wanted to prove. □

3.25 Lemma. *Let $(R, \mathfrak{m})$ be a noetherian local ring, and M an R-module. If* $\operatorname{Supp} M = \{\mathfrak{m}\}$*,* $\operatorname{Ext}^1_R(R/\mathfrak{m}, M) = 0$*, and* $\operatorname{Hom}_R(R/\mathfrak{m}, M) \cong R/\mathfrak{m}$*, then M is isomorphic to the injective hull of $R/\mathfrak{m}$.*

Proof. As $\operatorname{Supp} M = \{\mathfrak{m}\}$, M is an essential extension of $\operatorname{Hom}_R(R/\mathfrak{m}, M) \cong R/\mathfrak{m}$. So there is an exact sequence of the form

$$0 \to M \to E \to W \to 0,$$

where E is the injective hull of $R/\mathfrak{m}$. As $\operatorname{Ext}^1_R(R/\mathfrak{m}, M) = 0$, we have that

$$0 \to \operatorname{Hom}_R(R/\mathfrak{m}, M) \to \operatorname{Hom}_R(R/\mathfrak{m}, E) \to \operatorname{Hom}_R(R/\mathfrak{m}, W) \to 0$$

is exact. So $\operatorname{Hom}_R(R/\mathfrak{m}, W) = 0$. As $\operatorname{Supp} W \subset \{\mathfrak{m}\}$, $W = 0$. This shows that $M \cong E$. □

3.26 Lemma. *Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism between noetherian local rings of characteristic p. Assume that φ is flat, and $S/\mathfrak{m}S$ is Gorenstein of dimension zero. Then*

1 *$E_R \otimes_R S \cong E_S$, where E_R and E_S respectively denote the injective hulls of the residue field of R and S.*

2 *Assume further that* $\mathfrak{n} = \mathfrak{m}S$*. If* $c \in R$*,* $e \geq 0$*, and* $c^{(-e)}F_R^e : R \to R^{(-e)}$ *is* R*-pure, then* $c^{(-e)}F_S^e : S \to S^{(-e)}$ *is* S*-pure.*

Proof. **1** It is easy to see that $\operatorname{Supp} E_R \otimes_R S = \{\mathfrak{n}\}$.

There is a spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_{S/\mathfrak{m}S}(S/\mathfrak{n}, \operatorname{Ext}^q_S(S/\mathfrak{m}S, E_R \otimes_R S)) \Rightarrow \operatorname{Ext}^{p+q}_S(S/\mathfrak{n}, E_R \otimes_R S).$$

Note that

$$\operatorname{Ext}^q_S(S/\mathfrak{m}S, E_R \otimes_R S) \cong \operatorname{Ext}^q_R(R/\mathfrak{m}, E_R) \otimes_R S \cong \begin{cases} S/\mathfrak{m}S & (q = 0) \\ 0 & (q \neq 0) \end{cases}.$$

So $E_2^{p,q} = 0$ for $q \neq 0$. As $S/\mathfrak{m}S$ is the injective hull of the residue field of $S/\mathfrak{m}S$, $E_2^{p,0} = 0$ for $p > 0$, and $E_2^{0,0} \cong S/\mathfrak{n}$. By Lemma 3.25, $E_R \otimes_R S \cong E_S$.

2 Let ξ be a generator of the socle of E_R. Then $\xi \otimes 1 \in E_R \otimes_R S$ generates a submodule isomorphic to $R/\mathfrak{m} \otimes_R S \cong S/\mathfrak{n}$. Thus $\xi \otimes 1$ is a generator of the socle of E_S. Consider the commutative diagram

$$\begin{array}{ccc} E_R \otimes_R S & \xrightarrow{1 \otimes c^{(-e)}F_S^e} & (E_R \otimes_R S) \otimes_S S^{(-e)} = E_R \otimes_R S^{(-e)} \\ \uparrow \varphi & & \uparrow 1 \otimes \varphi^{(-e)} \\ E_R & \xrightarrow{1 \otimes c^{(-e)}F_R^e} & E_R \otimes_R R^{(-e)} \end{array}.$$

Then $\xi \in E_R$ goes to a nonzero element in $E_R \otimes_R S^{(-e)}$, since $c^{(-e)}F_R^e$ is R-pure, and $\varphi^{(-e)}$ is faithfully flat. Thus the socle element $\xi \otimes 1 \in E_R \otimes_R S$ goes to a nonzero element by $1 \otimes c^{(-e)}F_S^e$. This shows that $c^{(-e)}F_S^e : S \to S^{(-e)}$ is S-pure. □

3.27 Lemma. *Let* R *be an excellent local ring of characteristic* p*, and* A *an* R*-algebra essentially of finite type. Let* $c \in A$ *such that* $A[1/c]$ *is regular. If* $c^{(-e)}F^e : A \to A^{(-e)}$ *is* A*-pure for some* $e \geq 0$*, then* A *is very strongly* F*-regular.*

Proof. Let $\hat{R}$ be the completion of R, and $\hat{A} := \hat{R} \otimes_R A$. As R is excellent, $\hat{A}[1/c]$ is regular. Note that $c^{(-e)}F^e_{\hat{A}} : \hat{A} \to \hat{A}^{(-e)}$ is the composite

$$\hat{A} \xrightarrow{1 \otimes c^{(-e)}F_A^e} \hat{A} \otimes_A A^{(-e)} \xrightarrow{\Psi_e(A, \hat{A})} \hat{A}^{(-e)}.$$

As $A \to \hat{A}$ is regular, this is $\hat{A}$-pure. Replacing R by its completion $\hat{R}$ and A by $\hat{A}$, we may assume that R is complete local by Lemma 3.14.

Take a coefficient field K of R, fix a p-base Λ of K, and take a cofinite subset Γ_0 of Λ such that $A^\Gamma[1/c]$ is regular for any cofinite subset $\Gamma \subset \Gamma_0$.

For a cofinite subset Γ of Λ, let $\pi^\Gamma : \operatorname{Spec} A^\Gamma \to \operatorname{Spec} A$ be the canonical map. Let W^Γ be the closed subset of $\operatorname{Spec} A^\Gamma$ consisting of prime ideals P such that $c^{(-e)}F^e_{A^\Gamma_P} : A^\Gamma_P \to (A^\Gamma_P)^{(-e)}$ is not A^Γ_P-pure (it is closed, since A^Γ is F-finite by [20, (6.6), (6.8)]). Let $Z^\Gamma = \pi^\Gamma(W^\Gamma)$. It is easy to see that if $\Gamma' \subset \Gamma$, then $Z^{\Gamma'} \subset Z^\Gamma$. Let $\Gamma_1 \subset \Gamma_0$ be a cofinite subset such that Z^{Γ_1} is minimal. We show that Z^{Γ_1} is empty. Assume the contrary. Then there is a prime ideal $P \in Z^{\Gamma_1}$. Take $\Gamma_2 \subset \Gamma_1$ such that PA^{Γ_2} is a prime ideal. This can be done by [20, (6.13)]. As $c^{(-e)}F^e_{A_P} : A_P \to A_P^{(-e)}$ is A_P pure, $c^{(-e)}F^e_{A_P^{\Gamma_2}} : A_P^{\Gamma_2} \to (A_P^{\Gamma_2})^{(-e)}$ is $A_P^{\Gamma_2}$-pure by Lemma 3.26. On the other hand, as $P \in Z^{\Gamma_1} = Z^{\Gamma_2}$, $PA^{\Gamma_2} \in W^{\Gamma_2}$. A contradiction. So $c^{(-e)}F^e_{A^{\Gamma_1}} : A^{\Gamma_1} \to (A^{\Gamma_1})^{(-e)}$ is A^{Γ_1}-pure.

As A^{Γ_1} is F-finite [20, (6.6), (6.8)], $A^{\Gamma_1}[1/c]$ is regular, and $c^{(-e)}F^e : A^{\Gamma_1} \to (A^{\Gamma_1})^{(-e)}$ is A^{Γ_1}-pure (in particular, c is a nonzerodivisor of A^{Γ_1}), A^{Γ_1} is very strongly F-regular by [18, Theorem 3.3]. Since $A \to A^{\Gamma_1}$ is faithfully flat, A is very strongly F-regular. □

3.28 Lemma. *Let $A \to B$ be a regular homomorphism of noetherian rings of characteristic p. Assume that A is very strongly F-regular* (*resp. strongly F-regular*)*, and is excellent. Assume also that B is essentially of finite type over an excellent local ring* (*resp. locally excellent*)*. Then B is very strongly F-regular* (*resp. strongly F-regular*)*.*

Proof. First consider the case that A is very strongly F-regular. Take $c \in A^\circ$ such that $A[1/c]$ is regular. This is possible, since A is normal and excellent. Then we can take $e > 0$ such that $c^{(-e)}F^e : A \to A^{(-e)}$ is A-pure. Then plainly, $1_B \otimes c^{(-e)}F^e : B \to B \otimes_A A^{(-e)}$ is B-pure. As $\Psi_e : B \otimes_A A^{(-e)} \to B^{(-e)}$ is faithfully flat by Radu [27] and André [2], $c^{(-e)}F^e_B : B \to B^{(-e)}$ is B-pure. As $B[1/c]$ is regular, B is very strongly F-regular by Lemma 3.27.

Next consider the case that A is strongly F-regular. Take $\mathfrak{n} \in \operatorname{Max} B$, and set $\mathfrak{m} := A \cap \mathfrak{n}$. Then $A_\mathfrak{m}$ is very strongly F-regular by Lemma 3.6 and Lemma 3.8. By the first paragraph, $B_\mathfrak{n}$ is very strongly F-regular. By Lemma 3.6, B is strongly F-regular. □

For a noetherian ring A of characteristic p, set

$$\begin{aligned} \mathrm{SFR}(A) &:= \{P \in \operatorname{Spec} A \mid A_P \text{ is strongly } F\text{-regular}\}, \\ \mathrm{NonSFR}(A) &:= \operatorname{Spec} A \setminus \mathrm{SFR}(A). \end{aligned}$$

3.29 Lemma. *Let A be an F-finite noetherian ring of characteristic p. Then* $\mathrm{SFR}(A)$ *is a Zariski open subset of* $\operatorname{Spec} A$.

Proof. As the reduced locus of A is open, we may assume that A is reduced. Take $c \in A^\circ$ such that $A[1/c]$ is regular. For each $e \geq 0$, let U_e be the complement of the support of the cokernel of the map

$$c^{(-e)}F^e : \mathrm{Hom}_A(A^{(-e)}, A) \to \mathrm{Hom}_A(A, A) = A \qquad (\varphi \mapsto \varphi c^{(-e)} F^e).$$

Then U_e is open, and $\mathrm{SFR}(A) = \bigcup_e U_e$ is also open. □

3.30 Lemma. *Let $(R, \mathfrak{m})$ be a complete local ring with the coefficient field K, and A an R-algebra essentially of finite type. Let Λ be a p-base of K. For a cofinite subset $\Gamma \subset \Lambda$, let $\pi^\Gamma : \operatorname{Spec} A^\Gamma \to \operatorname{Spec} A$ be the canonical map. Then there exists some cofinite subset Γ_0 of Λ such that $\pi^\Gamma(\mathrm{SFR}(A^\Gamma)) = \mathrm{SFR}(A)$ for any cofinite subset Γ of Γ_0. In particular,* $\mathrm{SFR}(A)$ *is a Zariski open subset of* $\operatorname{Spec} A$.

Proof. Let Z^Γ be the closed subset $\pi^\Gamma(\mathrm{NonSFR}(A^\Gamma))$ of $\operatorname{Spec} A$. Take Γ_0 such that Z^{Γ_0} is minimal. Then $Z^\Gamma = Z^{\Gamma_0} \supset \mathrm{NonSFR}(A)$ for any cofinite subset $\Gamma \subset \Gamma_0$. Assume that $P \in Z^{\Gamma_0} \setminus \mathrm{NonSFR}(A)$. By [20, (6.13)], we can take $\Gamma_1 \subset \Gamma_0$ such that PA^Γ is a prime ideal for any cofinite subset $\Gamma \subset \Gamma_1$. Take $c \in A_P^\circ$ such that $A_P[1/c]$ is regular. We can take $\Gamma \subset \Gamma_1$ such that $A_P^\Gamma[1/c] \cong A_P[1/c] \otimes_R R^\Gamma$ is regular by Lemma 3.23. As A_P is very strongly F-regular, there exists some $e > 0$ such that $c^{(-e)}F^e_{A_P} : A_P \to A_P^{(-e)}$ is A_P-pure. By Lemma 3.26, $c^{(-e)}F^e_{A_P^\Gamma} : A_P^\Gamma \to (A_P^\Gamma)^{(-e)}$ is A_P^Γ-pure. As $A_P^\Gamma[1/c]$ is regular, A_P^Γ is strongly F-regular. This contradicts the choice of P. So $\pi^\Gamma(\mathrm{SFR}(A^\Gamma)) = \mathrm{SFR}(A)$ for $\Gamma \subset \Gamma_0$, as desired.

Now the openness of $\mathrm{SFR}(A)$ follows from Lemma 3.29. □

3.31 Corollary. *Let A be as in* Lemma 3.30. *Assume that A is strongly F-regular. Then there exists some cofinite subset Γ_0 of Λ such that for any cofinite subset Γ of Γ_0, A^Γ is strongly F-regular.* □

3.32 Lemma. *Let $\varphi : X \to Y$ be a continuous map between topological spaces, and $Z \subset X$ a closed subset. Assume that X is a noetherian topological space, and each irreducible closed subset of X has a generic point. If $\varphi(Z)$ is closed under specialization, then $\varphi(Z) \subset Y$ is closed.*

Proof. By assumption, there is a finite set of points $C = \{z_1, \ldots, z_r\}$ of Z such that the closure of C is Z. As $\varphi(C) \subset \varphi(Z)$ and $\varphi(Z)$ is closed under specialization, we have $\overline{\varphi(C)} = \bigcup_i \overline{\{\varphi(z_i)\}} \subset \varphi(Z)$. As φ is continuous,

$$\varphi(Z) = \varphi(\bar{C}) \subset \overline{\varphi(C)} \subset \varphi(Z).$$

Hence $\varphi(Z) = \overline{\varphi(C)}$ is closed. □

3.33 Proposition. *Let R be an excellent local ring of characteristic p, and A an R-algebra essentially of finite type. Then* $\mathrm{SFR}(A)$ *is Zariski open in* $\operatorname{Spec} A$.

Let $\hat{R}$ be the completion of R, and set $\hat{A} := \hat{R} \otimes_R A$. Let $\rho : \operatorname{Spec} \hat{A} \to \operatorname{Spec} A$ be the map associated with the base change of the completion. Let Q be a prime ideal of $\hat{A}$. Then $\hat{A}_Q$ is strongly F-regular if and only if so is $A_{\rho(Q)}$ by Lemma 3.17 and Lemma 3.28. Thus $\rho^{-1}(\mathrm{NonSFR}(A)) = \mathrm{NonSFR}(\hat{A})$. Letting $X = \operatorname{Spec} \hat{A}$, $Y = \operatorname{Spec} A$, and $Z = \mathrm{NonSFR}(\hat{A})$, $\rho(Z) = \rho(\rho^{-1}(\mathrm{NonSFR}(A))) = \mathrm{NonSFR}(A)$ is closed by Lemma 3.32, since it is closed under specialization by Lemma 3.6. □

3.34 Proposition. *Let R be an excellent noetherian local ring of characteristic p, and A an R-algebra essentially of finite type. Let $c \in A$ such that $A[1/c]$ is strongly F-regular. If $c^{(-e)} F_A^e : A \to A^{(-e)}$ is A-pure for some $e \geq 0$, then A is very strongly F-regular.*

Proof. Let $\hat{R}$ be the completion of R, and set $\hat{A} := \hat{R} \otimes_R A$. Then $\hat{A}[1/c]$ is strongly F-regular by Lemma 3.28. Moreover, $c^{(-e)} F_{\hat{A}}^e : \hat{A} \to \hat{A}^{(-e)}$ is $\hat{A}$-pure as in the proof of Lemma 3.27. By Lemma 3.14, we may assume that R is complete local.

Now take a coefficient field K of R, and take a p-base Λ of K. Then by Corollary 3.31, there exists some cofinite subset Γ_0 of Λ such that for each cofinite subset Γ of Γ_0, $A^\Gamma[1/c] = A[1/c]^\Gamma$ is strongly F-regular. As in the proof of Lemma 3.27, there exists some $\Gamma_1 \subset \Gamma_0$ such that $c^{(-e)} F_{A^{\Gamma_1}}^e : A^{\Gamma_1} \to (A^{\Gamma_1})^{(-e)}$ is A^{Γ_1}-pure. As A^{Γ_1} is F-finite, A^{Γ_1} is very strongly F-regular by [18, (3.3)]. By Lemma 3.14, A is very strongly F-regular. □

3.35 Corollary. *Let R be an excellent noetherian local ring of characteristic p, and A an R-algebra essentially of finite type. Then A is very strongly F-regular if and only if it is strongly F-regular.*

Proof. By Lemma 3.8, a very strongly F-regular implies strongly F-regular.

Conversely, assume that A is strongly F-regular. Then letting $c = 1$, $A[1/c]$ is strongly F-regular, and $c^{(0)}F_A^0 : A \to A^{(0)}$ is A-pure, since it is the identity map. By Lemma 3.34, A is very strongly F-regular. □

3.36 Corollary. *Let A be an locally excellent noetherian ring of characteristic p. Let $c \in A$ such that $A[1/c]$ is strongly F-regular. If $c^{(-e)}F_A^e : A \to A^{(-e)}$ is A-pure for some $e \geq 0$, then A is strongly F-regular.*

Proof. We may assume that A is local. Then the assertion is obvious by Lemma 3.34. □

3.37 Theorem. *Let $\varphi : A \to B$ be a homomorphism of noetherian rings of characteristic p. Assume that A is a strongly F-regular domain. Assume that the generic fiber $Q(A) \otimes_A B$ is strongly F-regular, where $Q(A)$ is the field of fractions of A. If φ is F-pure and B is locally excellent, then B is strongly F-regular.*

Proof. Assume the contrary. Then there is a prime ideal P of B such that B_P is not strongly F-regular, but B_Q is strongly F-regular for any prime ideal $Q \subsetneq P$. Replacing B by B_P and A by $A_{P \cap A}$, we may assume that $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ are local and φ is local, and we may assume that $\mathrm{NonSFR}(B) = \{\mathfrak{n}\}$. By assumption, $A \neq Q(A)$. So there is a nonzero element $c \in \mathfrak{m}$. Then $B[1/c]$ is strongly F-regular, since $c \in \mathfrak{n}$. As A is a very strongly F-regular domain, there exists some $e \geq 0$ such that $c^{(-e)}F_A^e : A \to A^{(-e)}$ is A-pure. As $c^{(-e)}F_B^e : B \to B^{(-e)}$ is the composite

$$B = B \otimes_A A \xrightarrow{1_B \otimes c^{(-e)}F_A^e} B \otimes_A A^{(-e)} \xrightarrow{\Psi_e(A,B)} B^{(-e)}$$

and $\Psi_e(A, B)$ is pure, $c^{(-e)}F_B^e$ is B-pure. By Corollary 3.36, B is strongly F-regular. This is a contradiction. □

4. Cohen–Macaulay F-injective property

(4.1) We say that a noetherian local ring $(R, \mathfrak{m})$ of characteristic p is Cohen–Macaulay F-injective (CMFI for short) if R is Cohen–Macaulay, and the Frobenius map $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R) \otimes_R R^{(-e)} \cong H^d_{\mathfrak{m}^{(-e)}}(R^{(-e)})$ is injective for some (or equivalently, any) $e > 0$, where d is the dimension of R. Obviously, R is CMFI if and only if its completion is. A noetherian ring of characteristic p is said to be CMFI if its localization at any maximal ideal is CMFI.

(4.2) Let I be an ideal of a noetherian ring R of characteristic p. The Frobenius closure of I is defined to be

$$I^F := \{x \in R \mid x^q \in I^{[q]} \text{ for some } q = p^e\},$$

where $I^{[q]} = I^{(e)}R$. If $I = I^F$, then we say that I is Frobenius closed.

(4.3) Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of characteristic p with dimension d. Take a system of parameters $x_1, \ldots, x_d$ of R. We see that the local cohomology $H^d_{\mathfrak{m}}(R)$ is the dth cohomology group of the modified Čech complex, see [3, (3.5)]. It is identified with the inductive limit of the inductive system:

$$(2) \qquad R/(x_1, \ldots, x_d) \xrightarrow{x_1 \cdots x_d} R/(x_1^2, \ldots, x_d^2) \xrightarrow{x_1 \cdots x_d} R/(x_1^3, \ldots, x_d^3) \to \cdots ,$$

where $a \in R/(x_1^t, \ldots, x_d^t)$ corresponds to $a/(x_1 \cdots x_d)^t$ in $H^d_{\mathfrak{m}}(R)$. Note that the maps of the inductive system (2) are injective, because $x_1, \ldots, x_d$ is a regular sequence. In particular, $R/(x_1^t, \ldots, x_d^t)$ can be identified with a submodule of $H^d_{\mathfrak{m}}(R)$.

So if R is CMFI and $F : H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R) \otimes_R R^{(-e)}$ is injective, then $F : R/(x_1, \ldots, x_d) \to R/(x_1, \ldots, x_d) \otimes_R R^{(-e)}$ is injective. In other words, if $x \in R$ and $x^q \in (x_1^q, \ldots, x_d^q)$, then $x \in (x_1, \ldots, x_d)$.

On the other hand, in (2), the socle of $R/(x_1^t, \ldots, x_d^t)$ is mapped bijectively onto the socle of $R/(x_1^{t+1}, \ldots, x_d^{t+1})$, since the map is injective, and the dimensions of the socles are equal (they agree with the Cohen–Macaulay type of R). Hence $\operatorname{Soc} R/(x_1, \ldots, x_d) \to \operatorname{Soc} H^d_{\mathfrak{m}}(R)$ is isomorphic. This shows that if $F : R/(x_1, \ldots, x_d) \to R/(x_1, \ldots, x_d) \otimes_R R^{(-1)}$ is injective, then R is CMFI. Hence we have

4.4 Lemma. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of characteristic p. Then the following are equivalent.*

1 *R is CMFI.*

2 *Any parameter ideal of R is Frobenius closed, where a parameter ideal means an ideal generated by a system of parameters.*

3 *For one system of parameters* $x_1, \dots, x_d$ *of* R, $x \in R$, $x^p \in (x_1^p, \dots, x_d^p)$ *implies* $x \in (x_1, \dots, x_d)$.

□

This lemma is a simplified variation of [9, Proposition 2.2].

(4.5) If R is CMFI and $(x_1, \dots, x_d)$ is a system of parameters of R, and $0 \leq s \leq d$, then

$$\begin{aligned}(x_1, \dots, x_s)^F = (\bigcap_{i_{s+1},\dots,i_d \geq 0} (x_1, \dots, x_s, x_{s+1}^{i_{s+1}}, \dots, x_d^{i_d}))^F \subset \\ \bigcap_{i_{s+1},\dots,i_d \geq 0} (x_1, \dots, x_s, x_{s+1}^{i_{s+1}}, \dots, x_d^{i_d})^F = \\ \bigcap_{i_{s+1},\dots,i_d \geq 0} (x_1, \dots, x_s, x_{s+1}^{i_{s+1}}, \dots, x_d^{i_d}) = (x_1, \dots, x_s).\end{aligned}$$

Thus, an ideal generated by a regular sequence of R is Frobenius closed.

4.6 Lemma. *Let* $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ *be a flat local homomorphism between noetherian local rings of characteristic* p*. If* S *is CMFI, then so is* R.

Proof. As S is Cohen–Macaulay and f is flat local, R is Cohen–Macaulay. Let $x_1, \dots, x_d$ be a system of parameters of R, and set $I := (x_1, \dots, x_d)$. Then $I^F \subset (IS)^F \cap R \subset IS \cap R = I$, because IS is generated by a regular sequence, and S is CMFI. □

Let R be a noetherian ring of characteristic p. We define

$$\mathrm{CMFI}(R) = \{P \in \operatorname{Spec} R \mid R_P \text{ is CMFI}\}$$

and $\mathrm{NonCMFI}(R) := \operatorname{Spec} R \setminus \mathrm{CMFI}(R)$.

4.7 Lemma. *Let* R *be a Cohen–Macaulay ring, and* M *a finite* R*-module. Then the locus*

$$\mathrm{MCM}(M) := \{P \in \operatorname{Spec} R \mid M_P \textit{ is a maximal Cohen–Macaulay } R_P\textit{-module}\}$$

is a Zariski open subset of $\operatorname{Spec} R$.

Proof. Set $S = R \oplus M$ to be the idealization of M. S is a noetherian ring with the product $(r, m)(r', m') = (rr', rm' + mr')$. Note that for $P \in \operatorname{Spec} R$, M_P is maximal Cohen–Macaulay (MCM for short) if and only if S_P is an MCM R-module if and only if S_P is a Cohen–Macaulay ring. Note that $\pi : S \to S/M = R$ induces a homeomorphism $\pi^{-1} : \operatorname{Spec} R \to \operatorname{Spec} S$, and $S_P = S_{\pi^{-1}(P)}$. So it suffices to show that the Cohen–Macaulay locus of the ring S is open. This is well-known, as S is finitely generated over a Cohen–Macaulay ring, see [25, Exercise 24.2]. $\square$

4.8 Lemma. *Let R be an F-finite noetherian ring of characteristic p. Then* $\mathrm{CMFI}(R)$ *is Zariski open in* $\operatorname{Spec} R$.

Proof. Note that R is excellent [24]. So the Cohen–Macaulay locus of R is Zariski open by Nagata's criterion [25]. Hence we may assume that R is Cohen–Macaulay. Let M be the cokernel of the Frobenius map $F_R : R \to R^{(-1)}$. Then R_P is CMFI if and only if M_P is a maximal Cohen–Macaulay R_P-module [12]. So $\mathrm{CMFI}(R) = \mathrm{MCM}(M)$ is Zariski open by Lemma 4.7. $\square$

4.9 Lemma. *Let $(R, \mathfrak{m})$ be a complete noetherian local ring of characteristic p with a coefficient field K. Let Λ be a p-base of K. Let A be an R-algebra essentially of finite type. Then there exists some cofinite subset Γ_0 of Λ such that for any cofinite subset Γ of Γ_0, $\pi^\Gamma(\mathrm{CMFI}(A^\Gamma)) = \mathrm{CMFI}(A)$, where $\pi^\Gamma : \operatorname{Spec} A^\Gamma \to \operatorname{Spec} A$ is the canonical morphism.*

Proof. Set $Z^\Gamma := \pi^\Gamma(\mathrm{NonCMFI}(A^\Gamma))$. The set $\{Z^\Gamma \mid \Gamma \text{ is cofinite in } \Lambda\}$ is a non-empty set of closed subsets of the noetherian space $\operatorname{Spec} A$. Take Γ_0 so that Z^{Γ_0} is minimal. By Lemma 4.6, it is easy to see that $Z^\Gamma = Z^{\Gamma_0}$ for $\Gamma \subset \Gamma_0$.

Clearly, $\mathrm{CMFI}(A) \supset \pi^\Gamma(\mathrm{CMFI}(A^{\Gamma_0})) = \operatorname{Spec} A \setminus Z^{\Gamma_0}$ by Lemma 4.6. Hence $Z^{\Gamma_0} \supset \mathrm{NonCMFI}(A)$. So it suffices to show that $Z^{\Gamma_0} \subset \mathrm{NonCMFI}(A)$.

Assume the contrary, and take $P \in Z^{\Gamma_0} \cap \mathrm{CMFI}(A)$. We can take $\Gamma_1 \subset \Gamma_0$ such that PA^{Γ_1} is a prime ideal of A^{Γ_1}. Set $d := \operatorname{ht} P$, and take a parameter ideal $I = (x_1, \ldots, x_d)$ of A_P.

Set $k := \kappa(P)$. For $\Sigma \subset \Gamma_1$,

$$\operatorname{Soc}(A_P^\Sigma / I A_P^\Sigma) = \operatorname{Hom}_{A_P^\Sigma}(k \otimes_{A_P} A_P^\Sigma, A_P/I \otimes_{A_P} A_P^\Sigma) \cong \operatorname{Hom}_{A_P}(k, A_P/I) \otimes_k k^\Sigma = \operatorname{Soc}(A_P/I) \otimes_k k^\Sigma,$$

where $k^\Sigma = k \otimes_{A_P} A_P^\Sigma$. Thus $V_k := \operatorname{Soc}(A_P/I)$ gives a k-structure of the finite dimensional k^{Γ_1}-vector space $V := \operatorname{Soc}(A_P^{\Gamma_1}/IA_P^{\Gamma_1})$. For $\Sigma \subset \Gamma_1$, we set $V^\Sigma := V_k \otimes_k k^\Sigma \cong \operatorname{Soc}(A_P^\Sigma/IA_P^\Sigma)$.

Now consider $M := \operatorname{Ker}(A_P^{\Gamma_1}/IA_P^{\Gamma_1} \xrightarrow{F} (A_P^{\Gamma_1})^{(-1)}/(IA_P^{\Gamma_1})^{(-1)})$ and $E := M \cap V = \operatorname{Soc} M$. For each $\Sigma \subset \Gamma_1$, we set $E^\Sigma = E \cap V^\Sigma$. For $\Sigma' \subset \Sigma \subset \Gamma_1$, the canonical map $k^\Sigma \otimes_{k^{\Sigma'}} E^{\Sigma'} \to E^\Sigma$ is injective, and hence $\dim_{k^{\Sigma'}} E^{\Sigma'} \leq \dim_{k^\Sigma} E^\Sigma$. Take a cofinite subset $\Omega \subset \Gamma_1$ such that $\dim_{k^\Omega} E^\Omega$ is small as possible. Let κ be the smallest field of definition of E^Ω over k. Namely, κ is the smallest intermediate field $k \subset \kappa \subset k^\Omega$ such that $k^\Omega \otimes_\kappa (E^\Omega \cap (V_k \otimes_k \kappa)) \to E^\Omega$ is surjective, see [26, (3.10)]. Then by the choice of Ω, for any cofinite subset Ω' of Ω, $k^{\Omega'} \supset \kappa$. Hence $\kappa \subset \bigcap_{\Omega'} k^{\Omega'} = k$ by Lemma 3.24. Hence $\kappa = k$.

As the diagram

$$\begin{array}{ccc} A_P/I \otimes_{A_P} A_P^\Sigma & \xrightarrow{F} & A_P/I \otimes_{A_P} (A_P^\Sigma)^{(-1)} \\ \uparrow & & \uparrow \\ A_P/I & \xrightarrow{F} & A_P/I \otimes_{A_P} A_P^{(-1)} \end{array}$$

is commutative and the bottom F is injective by the assumption $P \in \operatorname{CMFI}(A)$, $M \cap A_P/I = 0$. In particular, $E_k := E^\Omega \cap V_k = 0$. As k is the field of definition of E^Ω, $E^\Omega = 0$. This shows that $F : A_P^\Omega/IA_P^\Omega \to (A_P^\Omega)^{-1}/I(A_P^\Omega)^{(-1)}$ is injective. As A_P^Ω is Cohen–Macaulay, A_P^Ω is CMFI by Lemma 4.4. This contradicts $P \in Z^{\Gamma_0} = Z^\Omega$. $\square$

4.10 Corollary. *Let $(R, \mathfrak{m})$ be a noetherian local ring of characteristic p, and A a finite R-algebra. If A is CMFI, then there exists some faithfully flat F-finite local R-algebra R' such that $A' = R' \otimes_R A$ is CMFI.*

Proof. Let $\hat{R}$ be the completion of R. Then A is a semilocal ring, and $\hat{R} \otimes_R A$ is the direct product of the completions of the local rings of A at the maximal ideals. Hence $\hat{R} \otimes_R A$ is CMFI. So replacing R by $\hat{R}$ and A by $\hat{R} \otimes_R A$, we may assume that R is complete.

Let K be a coefficient field of R, and take a p-base Λ of K. Then by Lemma 4.9, there exists some cofinite subset Γ such that $A' = A^\Gamma$ is CMFI. $\square$

4.11 Corollary. *Let $(R, \mathfrak{m})$ be a noetherian local CMFI ring of characteristic p. Then for any prime ideal P of R, R_P is CMFI.*

Proof. Let $(R', \mathfrak{m}')$ be an F-finite CMFI local R-algebra which is faithfully flat over R. There exists some $Q \in \operatorname{Spec} R'$ such that $Q \cap R = P$. By Lemma 4.8, the CMFI locus of R' is open. As $\mathfrak{m}' \in \mathrm{CMFI}(R')$, we have that $\mathrm{CMFI}(R') = \operatorname{Spec} R'$. Thus R'_Q is CMFI. By Lemma 4.6, R_P is also CMFI. □

4.12 Lemma. *A flat module over an artinian local ring is free.*

Proof. Let $(R, \mathfrak{m})$ be an artinian local ring, and F a flat R-module. Then by [11, (III.2.1.8)], there is a short exact sequence

$$0 \to P \to F \to G \to 0$$

of R-modules in which P is R-free, G is R-flat, and $G/\mathfrak{m}G = 0$. Then by [11, (I.2.1.6)], $G = 0$. □

4.13 Corollary. *Let $(R, \mathfrak{m})$ be an artinian local ring, and $f : P \to F$ an R-linear map between flat R-modules. If $\bar{f} : P/\mathfrak{m}P \to F/\mathfrak{m}F$ is injective, then f is R-pure and* Coker f *is R-free.*

Proof. Follows immediately by Lemma 4.12 and [11, (I.2.1.4)]. □

4.14 Corollary. *Let $(R, \mathfrak{m})$ be an artinian local ring, A an R-algebra, and M an R-flat A-module. Let $(x_1, \dots, x_n)$ be a sequence in A. If $(x_1, \dots, x_n)$ is a (weak) $M/\mathfrak{m}M$-sequence, then $(x_1, \dots, x_n)$ is a (weak) M-sequence, and $M/(x_1, \dots, x_n)M$ is R-flat.*

Proof. Set $M_0 = M$, $M_i = M/(x_1, \dots, x_i)M$, $\bar{M} = M/\mathfrak{m}M$, and $\bar{M}_i = \bar{M}/(x_1, \dots, x_i)\bar{M}$. We prove that M_i is R-flat and $(x_1, \dots, x_i)$ is a weak M-sequence by induction on i. If $i = 0$, then M is R-flat by assumption, and the empty sequence is an M-sequence of length zero. If $i > 0$, then M_{i-1} is R-flat and $(x_1, \dots, x_{i-1})$ is an M-sequence by induction assumption. As $x_i : \bar{M}_{i-1} \to \bar{M}_{i-1}$ is an injective R-linear map, $x_i : M_{i-1} \to M_{i-1}$ is injective, and $M_i = M_{i-1}/x_i M_{i-1}$ is R-flat. Clearly, if $\bar{M}_n \neq 0$, then $M_n \neq 0$, and $(x_1, \dots, x_n)$ is a regular sequence. □

4.15 Lemma. *Let $(R, \mathfrak{m})$ be an artinian local ring, A an R-algebra, and $f : M \to N$ an A-linear map between R-flat A-modules. Let $x_1, \dots, x_n$ be a sequence of elements in A, and assume that $x_1, \dots, x_n$ is both a weak $M/\mathfrak{m}M$-sequence and a weak $N/\mathfrak{m}N$-sequence. Assume that $\bar{f}_n : \bar{M}_n \to \bar{N}_n$ is injective, where $M_i := M/(x_1, \dots, x_i)M$, $N_i := N/(x_1, \dots, x_i)N$, $\bar{M}_i =$*

$R/\mathfrak{m} \otimes_R M_i$, and $\bar{N}_i = R/\mathfrak{m} \otimes_R N_i$. Then $f_n : M_n \to N_n$ is injective, and $\operatorname{Coker} f_n$ *is R-flat.*

Proof. By Corollary 4.14, M_n and N_n are R-flat. Since $\bar{f}_n$ is injective, the assertions follow immediately by Corollary 4.13. □

The following was first proved by Aberbach–Enescu [1], see [7].

4.16 Proposition. *Let $\varphi : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a flat local homomorphism of noetherian local rings of characteristic p. If A is CMFI and $B/\mathfrak{m}B$ is geometrically CMFI over $A/\mathfrak{m}$ (see* [12, Definition 5.3]*), then B is CMFI.*

Proof. Clearly, B is Cohen–Macaulay. Take a system of parameters $x_1, \ldots, x_n$ of A and a sequence $y_1, \ldots, y_m$ in $\mathfrak{n}$ whose image in $B/\mathfrak{m}B$ is a system of parameters. Set $I = (x_1, \ldots, x_n)A$, $J = (y_1, \ldots, y_m)B$, and $\mathfrak{a} = IB + J$. B/J is A-flat by [25, Corollary to (22.5)]. Since $F_A : A/I \to A/I \otimes_A A^{(-1)}$ is injective by the CMFI property of A, $F_A : B/J \otimes_A A/I \to B/J \otimes_A A/I \otimes_A A^{(-1)}$ is also injective. In other words, $F_A : B/\mathfrak{a} \to (B \otimes_A A^{(-1)})/\mathfrak{a}(B \otimes_A A^{(-1)})$ is injective.

Let $L \subset \kappa(\mathfrak{m})^{(-1)}$ be a finite extension field of $\kappa(\mathfrak{m})$. As $B/\mathfrak{m}B \otimes_{\kappa(\mathfrak{m})} L$ is CMFI,

$$F : B/(\mathfrak{m}B+J) \otimes_{B/\mathfrak{m}B} B/\mathfrak{m}B \otimes_{\kappa(\mathfrak{m})} L \to B/(\mathfrak{m}B+J) \otimes_{B/\mathfrak{m}B} (B/\mathfrak{m}B \otimes_{\kappa(\mathfrak{m})} L)^{(-1)}$$

is injective. Taking the inductive limit,

$$F : B/(\mathfrak{m}B + J) \otimes_{B/\mathfrak{m}B} B/\mathfrak{m}B \otimes_{\kappa(\mathfrak{m})} \kappa(\mathfrak{m})^{(-1)} \to B/(\mathfrak{m}B + J) \otimes_{B/\mathfrak{m}B} (B/\mathfrak{m}B \otimes_{\kappa(\mathfrak{m})} \kappa(\mathfrak{m})^{(-1)})^{(-1)}$$

is injective. By [12, Lemma 4.1], **7**,

$$1 \otimes \Psi_1(\kappa(\mathfrak{m}), B/\mathfrak{m}B) : B/(\mathfrak{m}B + J) \otimes_{B/\mathfrak{m}B} B/\mathfrak{m}B \otimes_{\kappa(\mathfrak{m})} \kappa(\mathfrak{m})^{-1} \to B/(\mathfrak{m}B + J) \otimes_{B/\mathfrak{m}B} (B/\mathfrak{m}B)^{-1}$$

is injective. As $B \otimes_A A^{(-1)}$ and $B^{(-1)}$ are $A^{(-1)}$-flat, $B/IB \otimes_{A/I} A^{(-1)}/IA^{(-1)}$ and $B^{(-1)}/IB^{(-1)}$ are flat modules over the artinian local ring $A^{(-1)}/IA^{(-1)}$. Clearly, $y_1, \ldots, y_m$ is a $B/\mathfrak{m}B \otimes_{\kappa(\mathfrak{m})} \kappa(\mathfrak{m})^{(-1)}$-sequence. Moreover,

$$A^{(-1)}/\mathfrak{m}A^{(-1)} \to B^{(-1)}/\mathfrak{m}B^{(-1)}$$

is a flat homomorphism with an m-dimensional Cohen–Macaulay closed fiber. As $A^{(-1)}/\mathfrak{m}A^{(-1)}$ is artinian, $B^{(-1)}/\mathfrak{m}B^{(-1)}$ is m-dimensional Cohen–Macaulay.

It is easy to see that $B^{(-1)}/(\mathfrak{m}B^{(-1)}+JB^{(-1)})$ is artinian, so $y_1,\ldots,y_m$ is a $B^{(-1)}/\mathfrak{m}B^{(-1)}$-sequence. So by Lemma 4.15,

$$1\otimes\Psi_1(A,B):(B\otimes_A A^{(-1)})/\mathfrak{a}(B\otimes_A A^{(-1)})\to B^{(-1)}/\mathfrak{a}B^{(-1)}$$

is injective.

Hence $1\otimes F_B: B/\mathfrak{a}\to B^{(-1)}/\mathfrak{a}B^{(-1)}$ is injective. By Corollary 4.4, B is CMFI. $\square$

4.17 Corollary. *Let $(B,\mathfrak{n})$ be a noetherian local ring of characteristic p, and $t\in\mathfrak{n}$ be a nonzerodivisor. If B/tB is CMFI, then B is CMFI.*

Proof. Let A be the localization $\mathbb{F}_p[T]_{(T)}$ of the polynomial ring $\mathbb{F}_p[T]$ at the maximal ideal (T). Then the canonical map $A\to B$ which maps T to t is flat, as t is a nonzerodivisor. Let L be a finite extension field of $\mathbb{F}_p$. Then L is a separable extension of $\mathbb{F}_p$, and hence $B/tB\otimes_{\mathbb{F}_p}L$ is étale over B/tB, and hence is CMFI by Proposition 4.16. Thus B/tB is geometrically CMFI over A/tA. On the other hand, A is regular, and hence is CMFI. By Proposition 4.16 again, B is CMFI. $\square$

4.18 Corollary. *Let R be an excellent local ring of characteristic p, and A an R-algebra essentially of finite type. Then* CMFI(A) *is a Zariski open subset of* Spec A.

Proof. Let $\hat{R}$ be the completion of R, and $\hat{A}:=\hat{R}\otimes_R A$. Note that CMFI$(\hat{A})$ is a Zariski open subset of Spec $\hat{A}$ by Lemma 4.9 and Lemma 4.8.

Let $\rho:\operatorname{Spec}\hat{A}\to\operatorname{Spec}A$ be the morphism associated with the base change of the completion. Then $\rho^{-1}(\mathrm{NonCMFI}(A))=\mathrm{NonCMFI}(\hat{A})$ by Proposition 4.16 and Lemma 4.6. By Corollary 4.11 and Lemma 3.32, $\mathrm{NonCMFI}(A)=\rho(\mathrm{NonCMFI}(\hat{A}))$ is closed, as desired. $\square$

5. Matijevic–Roberts type theorem

(5.1) Let S be a scheme, G an S-group scheme, and X a standard G-scheme [15, (2.18)] (that is, X is noetherian and the second projection $p_2: G\times X\to X$ is flat of finite type).

5.2 Theorem. *Let y be a point of X, and Y the integral closed subscheme of X whose generic point is y. Let η be the generic point of an irreducible component of Y^*, where Y^* is the smallest G-stable closed subscheme of X containing Y. Assume either that the second projection $p_2: G\times X\to X$ is* *smooth, or that* $S = \operatorname{Spec} k$ *with* k *a perfect field and* G *is of finite type over* S*. Assume that* $\mathcal{O}_{X,\eta}$ *is of characteristic* p*. Then* $\mathcal{O}_{X,y}$ *is of characteristic* p*. Moreover,*

1 *If* $\mathcal{O}_{X,\eta}$ *is* F*-pure, then* $\mathcal{O}_{X,y}$ *is* F*-pure.*

2 *If* $\mathcal{O}_{X,\eta}$ *is excellent and strongly* F*-regular, then* $\mathcal{O}_{X,y}$ *is strongly* F*-regular.*

3 *If* $\mathcal{O}_{X,\eta}$ *is CMFI, then* $\mathcal{O}_{X,y}$ *is CMFI.*

Proof. We set $\mathcal{C}$ and $\mathcal{D}$ to be the class of all noetherian local rings of characteristic p, and $\mathbb{P}(A, M)$ to be "always true" in [13, Corollary 7.6]. Then the conditions **(i)** and **(ii)** there are satisfied, and by [13, Corollary 7.6], $\mathcal{O}_{X,y}$ is of characteristic p.

1 We set $\mathcal{C}$ and $\mathcal{D}$ to be the class of all F-pure noetherian local rings of characteristic p, and $\mathbb{P}(A, M)$ to be "always true" in [13, Corollary 7.6]. Then **(i)** there (the smooth base change) is satisfied by Lemma 2.4, **4** and **6**. The condition **(ii)** (the flat descent) holds by Lemma 2.4, **5**. So the assertion follows by [13, Corollary 7.6].

2 Set $\mathcal{C}$ to be the class of excellent strongly F-regular noetherian local domains of characteristic p, and $\mathcal{D}$ to be the class of strongly F-regular noetherian local domains of characteristic p. Then **(i)** and **(ii)** of [13, Corollary 7.6] are satisfied by Lemma 3.28 and Lemma 3.17.

3 Set $\mathcal{C} = \mathcal{D}$ be the class of CMFI noetherian local rings of characteristic p. Then **(i)** and **(ii)** of [13, Corollary 7.6] are satisfied by Proposition 4.16 and Lemma 4.6. □

5.3 Corollary. *Let* p *be a prime number, and* A *a* $\mathbb{Z}^n$*-graded noetherian ring of characteristic* p*. Let* P *be a prime ideal of* A*, and* P^* *be the prime ideal of* A *generated by the homogeneous elements of* P*. If* A_{P^*} *is* F*-pure* (*resp. excellent strongly* F*-regular, CMFI*)*, then* A_P *is* F*-pure* (*resp. strongly* F*-regular, CMFI*)*.*

Proof. Let $S = \operatorname{Spec} \mathbb{Z}$, $G = \mathbb{G}_m^n$, and $X = \operatorname{Spec} A$. If $y = P$, then η in Theorem 5.2 is P^*. The assertion follows immediately by Theorem 5.2. □

5.4 Corollary. *Let* A *be a* $\mathbb{Z}^n$*-graded noetherian ring of characteristic* p*. If* $A_\mathfrak{m}$ *is* F*-pure* (*resp. excellent strongly* F*-regular, CMFI*) *for any maximal graded ideals* (*that is,* G*-maximal ideals for* $G = \mathbb{G}_m^n$ (*called* **maximal ideal in* [3]))*, then* A *is* F*-pure* (*resp. strongly* F*-regular, Cohen–Macaulay* F*-injective*)*.*

Proof. Similar to [13, Corollary 7.11]. □

5.5 Corollary. *Let $A = \bigoplus_{n\geq 0} A_n$ be an $\mathbb{N}$-graded noetherian ring of characteristic p. Let $t \in A_+ := \bigoplus_{n>0} A_n$ be a nonzerodivisor of A. If A/tA is CMFI, then A is CMFI.*

Proof. Similar to [13, Corollary 7.13]. □

5.6 Corollary. *Let A be a ring of characteristic p, and $(F_n)_{n\geq 0}$ a filtration of A. That is, $F_0 \subset F_1 \subset F_2 \subset \cdots \subset A$, $1 \in F_0$, $F_iF_j \subset F_{i+j}$, and $\bigcup_{n\geq 0} F_n = A$. Set $R = \bigoplus_{n\geq 0} F_n t^n \subset A[t]$, and $G = R/tR$. If G is noetherian and Cohen–Macaulay F-injective, then A is also noetherian and Cohen–Macaulay F-injective.*

Proof. Similar to [13, Corollary 7.14]. □

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