Equivariant Matlis and the local duality

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Abstract

Generalizing the known results on graded rings and modules, we formulate and prove the equivariant version of the local duality on schemes with a group action. We also prove an equivariant analogue of Matlis duality.

1. Introduction

This paper is a continuation of [9], and study equivariant local cohomology. In this paper, utilizing an equivariant dualizing complex, we define the G-sheaf of matlis, an equivariant analogue of the injective hull of the residue field of a local ring. Using this, we formulate and prove Matlis and the local duality under equivariant settings.

Let R be a Gorenstein local ring, $T = R[x_1, \ldots, x_s]$ be the graded polynomial ring with $r_i := \deg x_i$ positive, I a homogeneous ideal of height h, and $A := T/I$. Assume that A is Cohen–Macaulay of dimension d. Set $\omega_T := T(-r)$, where $r = \sum_i r_i$, and $(-r)$ denotes the shift of degree. Set $\omega_A := \operatorname{Ext}^h_T(A, \omega_T)$. For a graded A-module M, set $M^\vee := \bigoplus_{i\in\mathbb{Z}} M^\dagger_{-i}$, where $(?)^\dagger = \operatorname{Hom}_R(?, E_R)$, where E_R is the injective hull of the residue field of R. Note that $M^\vee$ is a graded A-module again. Note also that ${}^*\operatorname{Hom}_A(M, A^\vee) \cong M^\vee$ (see for the notation ${}^*\operatorname{Hom}$, [1, page 33]).

For a finite graded A-module M, we have an isomorphism of *graded* A-modules

$$H^i_{\mathfrak{M}}(M) \cong \operatorname{Ext}^{d-i}_A(M, \omega_A)^\vee,$$

cf. [1, Theorem 3.6.19], see also Corollary 5.5.

The main purpose of this paper is to generalize this graded version of local duality to more general equivariant local duality. Note that a graded module over a $\mathbb{Z}$-graded ring is nothing but an equivariant module under the action of $\mathbb{G}_m = GL_1$, see [6, (II.1.2)]. On the way, we prove some basic properties on equivariant local cohomology.

In this introduction, let S be a noetherian scheme, G a flat S-group scheme of finite type, and X a noetherian G-scheme. In order to establish an analogy of the local duality on X, we need to define an equivariant analogue of a local ring or a local scheme. This is done in [9], and it is a G-local G-scheme. So let X be a G-local G-scheme. That is to say, X has a unique minimal nonempty G-stable closed G-subscheme, say Y. Next, we need to have an equivariant analogue of local cohomology. This is the main subject of [9]. Finally, we need to have an analogy of the Matlis duality. In other words, we need to have an analogue of the injective hull of the residue field of a local ring. The authors do not know how to define it quite generally. However, if X has a G-equivariant dualizing complex (see for the definition, [7, chapter 31]) $\mathbb{I}_X$, then we can define it as the unique nonzero cohomology group of $R\underline{\Gamma}_Y(\mathbb{I}_X)$. We call this sheaf the G-sheaf of Matlis. Thus we can formulate the equivariant local duality. The proof depends on the isomorphism $\mathfrak{H}$, see below.

Many ideas used in this paper have already appeared in the theory of graded rings [3], [4], [1], [10]. If H is a finitely generated abelian group, then letting $G = \operatorname{Spec} \mathbb{Z}H$, where $\mathbb{Z}H$ is the group algebra of H over $\mathbb{Z}$, an H-graded algebra is nothing but a G-algebra, and for a G-algebra A, a graded A-module is nothing but a (G, A)-module. However, we need to point out that for a general G and a G-local G-algebra $(A, \mathfrak{M})$ with the G-dualizing complex $\mathbb{I}$, the global section of the G-sheaf of Matlis E_A is not necessarily injective as a (G, A)-module, see Example 5.7. In particular, E_A is not the injective hull of $A/\mathfrak{M}$ in the category of (G, A)-modules.

Using the G-sheaf of Matlis, we can prove a weak version of the Matlis duality, too. It is a duality from the category of coherent $(G, \mathcal{O}_X)$-modules of finite length to itself, see Theorem 4.17. Note that a better Matlis duality exists over a complete local ring. It is a duality from the category of noetherian modules to the category of artinian modules ([1, Theorem 3.2.13]). The authors do not know a good analogue of a complete local ring, and thus cannot give an equivariant Matlis duality between noetherian quasi-coherent $(G, \mathcal{O}_X)$-modules and artinian modules in general. However, there is an example of graded case of that kind of duality, see Remark 5.6.

Section 2 is preliminaries. We give some basic properties of the duality map in a closed category. We also give some sufficient conditions to guarantee that injective objects in the category $\mathrm{Qch}(G, X)$ is acyclic with respect to some cohomological functors. We also prove a generalization of the flat base change ([9, Theorem 6.10]), see Lemma 2.14. We also describe the local cohomology over a diagram of schemes using the inductive limit of $\underline{\mathrm{Ext}}$ groups, as in the single-scheme case.

Section 3 treats the map $\mathfrak{H}$. For a small category I, an I^{op}-diagram of schemes X, an open subdiagram of schemes U of X, and an open subdiagram of schemes V of U, there is a natural map

$$\mathfrak{H} : \underline{\Gamma}_{U,V} \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \underline{\Gamma}_{U,V} \mathcal{N})$$

for $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(X)$. There is an obvious derived version of it, and $\mathfrak{H}$ is often an isomorphism (see Lemma 3.16 and Theorem 3.26). This is the key to the proof of the equivariant version of the local duality. In order to establish the existence and some basic properties of $\mathfrak{H}$, we need to prove various commutativity of diagrams. To do this, we utilize the basics on closed categories as in [7, chapter 1].

In section 4, we formulate and prove the equivariant analogues of Matlis and the local duality. We start with Matijevic–Roberts type theorem for G-local G-schemes, and prove an equivariant version of Nakayama's lemma, which is well-known for affine case.

In section 5, we give an example of the graded case. Note that in some cases, Matlis duality can be in more general form than the version described in section 4, see Remark 5.6.

2. Preliminaries

(2.1) We use the notation and terminology of [7], [9], and [8] freely.

(2.2) Let X be a symmetric monoidal closed category (see [11, (3.5.1)]), and $b, d \in X$. Then we denote the composite map

$$b \xrightarrow{\mathrm{tr}} [[b, d], b \otimes [b, d]] \xrightarrow{\gamma} [[b, d], [b, d] \otimes b] \xrightarrow{\mathrm{ev}} [[b, d], d]$$

by $\mathfrak{D}$, and we call it the *duality map*, where tr, γ, and ev denote the trace map [7, (1.30)], the twisting (symmetry) isomorphism [7, (1.28)], and the evaluation map [7, (1.30)], respectively.

2.3 Lemma. *$\mathfrak{D}$ is natural on b. Namely, for a morphism $\phi : b \to b'$, the diagram*

$$\begin{tikzcd}
b \arrow[r, "\phi"] \arrow[d, "\mathfrak{D}"] & b' \arrow[d, "\mathfrak{D}"] \\
{[[b,d],d]} \arrow[r, "\phi"] & {[[b',d],d]}
\end{tikzcd}$$

is commutative.

Proof. Consider the diagram

$$\begin{tikzcd}
& b \arrow[r, "\mathrm{tr}"] \arrow[dl, "\phi"'] \arrow[d, "\mathrm{tr}"] \arrow[dr, phantom, "\text{(a)}"] & {[[b,d], b \otimes [b,d]]} \arrow[d, "\phi"] \arrow[dd, bend left=60, "\mathrm{ev}\,\gamma"] \arrow[dd, phantom, shift left=8ex, "\text{(b)}"] \\
b' \arrow[d, "\mathrm{tr}"] \arrow[r, phantom, "\text{(c)}"] & {[[b',d], b \otimes [b',d]]} \arrow[r, "\phi"] \arrow[dl, "\phi"] \arrow[d, phantom, "\text{(d)}"] & {[[b',d], b \otimes [b,d]]} \arrow[dl, "\mathrm{ev}\,\gamma"] \\
{[[b',d], b' \otimes [b',d]]} \arrow[r, "\mathrm{ev}\,\gamma"] & {[[b',d],d]} & {[[b,d],d]} \arrow[l, "\phi"']
\end{tikzcd} .$$

(a) and (d) are commutative by [7, Lemma 1.32]. The commutativity of (b) and (c) are trivial. $\square$

2.4 Lemma. *For a morphism $\psi : d \to d'$, the diagram*

$$\begin{tikzcd}
b \arrow[r, "\mathfrak{D}"] \arrow[d, "\mathfrak{D}"] & {[[b,d],d]} \arrow[d, "\psi"] \\
{[[b,d'],d']} \arrow[r, "\psi"] & {[[b,d],d']}
\end{tikzcd}$$

is commutative.

Proof. Consider the diagram

$$\begin{tikzcd}
b \arrow[r, "\mathrm{tr}"] \arrow[d, "\mathrm{tr}"] \arrow[dr, phantom, "\text{(a)}"] & {[[b,d], b \otimes [b,d]]} \arrow[r, "\mathrm{ev}\,\gamma"] \arrow[d, "\psi"] \arrow[dr, phantom, "\text{(b)}"] & {[[b,d],d]} \arrow[dd, "\psi"] \\
{[[b,d'], b \otimes [b,d']]} \arrow[r, "\psi"] \arrow[d, "\mathrm{ev}\,\gamma"] \arrow[dr, phantom, "\text{(c)}"] & {[[b,d], b \otimes [b,d']]} \arrow[dr, "\mathrm{ev}\,\gamma"] & \\
{[[b,d'],d']} \arrow[rr, "\psi"] & & {[[b,d],d']}
\end{tikzcd} .$$

(a) is commutative by [7, Lemma 1.32]. (b) and (c) are obviously commutative. Hence the whole diagram is commutative. $\square$

2.5 Lemma. *Let $f : X \to Y$ be a symmetric monoidal functor* [11, (3.4.2)] *between symmetric monoidal closed categories. For $b, d \in X$, the diagram*

$$\begin{tikzcd}
fb \arrow[r, "\mathfrak{D}"] \arrow[d, "\mathfrak{D}"] & {f[[b,d],d]} \arrow[d, "H"] \\
{[[fb,fd],fd]} \arrow[r, "H"] & {[f[b,d],fd]}
\end{tikzcd}$$

is commutative.

Proof. Consider the diagram

$$\begin{tikzcd}
{[[fb,fd], fb\otimes[fb,fd]]} \arrow[d, "H"] \arrow[ddd, bend right=60, "\mathrm{ev}\,\gamma"'] \arrow[dr, phantom, "\text{(a)}"] & fb \arrow[l, "\mathrm{tr}"'] \arrow[d, "\mathrm{tr}"] \arrow[rdd, bend left, "\mathrm{tr}"] & \\
{[f[b,d], fb\otimes[fb,fd]]} \arrow[d, "\mathrm{ev}\,\gamma"] \arrow[d, phantom, "\text{(b)}"', shift right=8ex] \arrow[dr, phantom, "\text{(d)}"] & {[f[b,d], fb\otimes f[b,d]]} \arrow[l, "H"'] \arrow[d, "m"] \arrow[r, phantom, "\text{(c)}"] & {} \\
{[f[b,d], fd]} & {[f[b,d], f(b\otimes[b,d])]} \arrow[l, "\mathrm{ev}\,\gamma"'] \arrow[dr, phantom, "\text{(e)}"] & {f[[b,d], b\otimes[b,d]]} \arrow[l, "H"'] \arrow[dl, bend left, "\mathrm{ev}\,\gamma"] \\
{[[fb,fd],fd]} \arrow[u, "H"] & {f[[b,d],d]} \arrow[ul, "H"'] &
\end{tikzcd}.$$

(a) is commutative by [7, (1.32)]. The commutativity of (b) is trivial. (c) is [7, (1.37)] and is commutative. (d) is [7, (1.36)] and is commutative. (e) is commutative by the naturality of H. $\square$

(2.6) A symmetric monoidal functor $f : X \to Y$ is said to be *Lipman* if it has a left adjoint $g : Y \to X$ such that the natural maps $\Delta : g(b\otimes d) \to gb\otimes gd$ and $C : g\mathcal{O}_Y \to \mathcal{O}_X$ are isomorphisms, see [7, (1.48)]. We also say that (f, g) is a Lipman adjoint pair in this case.

By Lemma 2.5, we have:

2.7 Lemma. *Let $f : X \to Y$ and $g : Y \to X$ be a Lipman adjoint pair where X and Y are closed. Then the diagram*

$$\begin{tikzcd}
gb' \arrow[r, "\mathfrak{D}"] \arrow[d, "\mathfrak{D}"] & {g[[b',d'],d']} \arrow[d, "P"] \\
{[[gb',gd'],gd']} \arrow[r, "P"] & {[g[b',d'],gd']}
\end{tikzcd}$$

is commutative.

(2.8) Let $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be a ringed category. That is, $\mathbb{X}$ is a small category, and $\mathcal{O}_{\mathbb{X}}$ is a presheaf of commutative rings on $\mathbb{X}$. Then for $\mathcal{M}, \mathcal{N} \in \mathrm{PM}(\mathbb{X})$, the map

$$\mathfrak{D} : \mathcal{M} \to \underline{\mathrm{Hom}}_{\mathrm{PM}(\mathbb{X})}(\underline{\mathrm{Hom}}_{\mathrm{PM}(\mathbb{X})}(\mathcal{M}, \mathcal{N}), \mathcal{N})$$

is described as follows. At $x \in \mathbb{X}$,

$$\begin{aligned} \mathfrak{D} : \Gamma(x, \mathcal{M}) \to \Gamma(x, & \underline{\mathrm{Hom}}_{\mathrm{PM}(\mathbb{X})}(\underline{\mathrm{Hom}}_{\mathrm{PM}(\mathbb{X})}(\mathcal{M}, \mathcal{N}), \mathcal{N})) \\ & = \mathrm{Hom}_{\mathrm{PM}(\mathbb{X})/x}(\underline{\mathrm{Hom}}_{\mathrm{PM}(\mathbb{X})}(\mathcal{M}, \mathcal{N})|_x, \mathcal{N}|_x)) \end{aligned}$$

is given as follows. For $a \in \Gamma(x, \mathcal{M})$, $\mathfrak{D}(a) : \underline{\mathrm{Hom}}_{\mathrm{PM}(\mathbb{X})}(\mathcal{M}, \mathcal{N})|_x \to \mathcal{N}|_x$ is the map such that for $\phi : y \to x$, $\mathfrak{D}(a)_\phi : \mathrm{Hom}_{\mathrm{PM}(\mathbb{X}/y)}(\mathcal{M}|_y, \mathcal{N}|_y) \to \Gamma(y, \mathcal{N})$ is given by $\mathfrak{D}(a)_\phi(h) = h(a)$. This is proved easily using [7, (2.42)] and [7, (2.41)].

(2.9) Let $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be a ringed site, and $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\mathbb{X})$. Then the map

$$\mathfrak{D} : \mathcal{M} \to \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathbb{X}}}(\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathbb{X}}}(\mathcal{M}, \mathcal{N}), \mathcal{N})$$

is exactly the same map as the one described in (2.8). This follows from [7, (2.49)], Lemma 2.5, and (2.8).

(2.10) In the rest of this paper, S denotes a scheme, and G an S-group scheme. We write diagrams of schemes as X, Y, $Z, \ldots$ (not as $X_\bullet$, $Y_\bullet$, $Z_\bullet, \ldots$). Similarly, morphisms of diagrams of schemes are expressed as f, g, $h, \ldots$, not as $f_\bullet$, $g_\bullet$, $h_\bullet, \ldots$ This is a convention in [9].

2.11 Lemma. *Let I be a small category, and $f : X \to Y$ be a concentrated (i.e., quasi-compact quasi-separated) morphism of I^{op}-diagrams of schemes. Let (C_α) be a pseudo-filtered inductive system of complexes of $\mathcal{O}_X$-modules such that for each $j \in I$, one of the following holds:*

(a) *There exists some $n_j \in \mathbb{Z}$ such that for any α, $\tau^{\leq n_j - 1}(C_\alpha)_j$ is exact (see for the definition of $\tau^{\leq n_j - 1}$, see [7, (3.24)]);*

(b) *Each point of X_j has a noetherian open neighborhood of finite Krull dimension.*

(c) *For any α, $C_{\alpha,j}$ has quasi-coherent cohomology groups.*

Set $C = \varinjlim C_\alpha$. *Then the canonical map*

$$\varinjlim R^i f_* C_\alpha \to R^i f_* C \tag{1}$$

is an isomorphism for $i \in \mathbb{Z}$. *If, moreover, each* C_α *is* f_**-acyclic, then* C *is* f_**-acyclic.*

Proof. In view of [7, Example 8.23, **2**], it is easy to see that it suffices to show that

$$\varinjlim R^i (f_j)_* C_{\alpha,j} \to R^i (f_j)_* C_j$$

is an isomorphism for each j, to prove that (1) is an isomorphism. This is (3.9.3.1) and (3.9.3.2) of [11].

To prove the last assertion, it suffices to show that each C_j is $(f_j)_*$-acyclic. This is [11, (3.9.3.4)]. □

2.12 Corollary. *Let* $f : X \to Y$ *be as in* Lemma 2.11. *Let* C *be a complex of* $\mathcal{O}_X$*-modules such that each term of* C *is locally quasi-coherent and* f_**-acyclic. Then* C *is* f_**-acyclic.*

Proof. Similar to [11, (3.9.3.5)]. □

2.13 Lemma. *Let* X *and* Y *be* S*-groupoid (see for the definition,* [7, (12.1)]) *and* $f : X \to Y$ *a morphism (in the category* $\mathcal{P}(\Delta_M, \underline{\mathrm{Sch}}/S)$*, see for the notation,* [7, Glossary]). *Assume that* f *is cartesian,* Y *has affine arrows, and assume one of the following:*

(a) X_0 *is noetherian;*

(b) Y_0 *and* f_0 *are quasi-compact separated.*

Then

(i) f *is concentrated and* X_0 *is concentrated.*

(ii) *A* K*-injective complex* $\mathbb{I}$ *in* $K(\mathrm{Qch}(X))$ *is* f_**-acyclic.*

(iii) *The canonical maps*

$$F_Y \circ Rf_*^{\mathrm{Qch}} \cong R(F_Y \circ f_*^{\mathrm{Qch}}) \cong R(f_* \circ F_X) \to Rf_* \circ F_X$$

are all isomorphisms, where $F_Y : D(\mathrm{Qch}(Y)) \to D(Y)$ *and* $F_X : D(\mathrm{Qch}(X)) \to D(X)$ *are triangulated functors induced by inclusions, and* $f_*^{\mathrm{Qch}} : \mathrm{Qch}(X) \to \mathrm{Qch}(Y)$ *is the restriction of* $f_* : \mathrm{Mod}(X) \to \mathrm{Mod}(Y)$*, see* [7, Lemma 7.14].

Proof. **(i)** In either case, f_0 is concentrated. Since f is cartesian, each f_i ($i = 0, 1, 2$) is obtained as a base change of f_0, and hence is concentrated. It is easy to see that X_0 is concentrated in either case.

(ii) As f is concentrated cartesian, f_*^{Qch} is well-defined [7, Lemma 7.14]. Since X_0 is concentrated and X has affine arrows, $\mathrm{Qch}(X)$ is Grothendieck by [7, Lemma 12.8]. So $\mathbb{I}$ has a strictly injective resolution (that is, a K-injective resolution each of whose term is injective) $\mathbb{J}$ [2, Proposition 3.2]. As the mapping cone of $\mathbb{I} \to \mathbb{J}$ is null-homotopic, replacing $\mathbb{I}$ by $\mathbb{J}$, we may assume that $\mathbb{I}$ is strictly injective. By Corollary 2.12, it suffices to show that each term of $\mathbb{I}$ is f_*-acyclic. So we may assume that $\mathbb{I}$ is a single injective object of $\mathrm{Qch}(X)$. Let $\mathbb{I}_0 \to K$ be a monomorphism with K an injective object of $\mathrm{Qch}(X_0)$. This is possible, since $\mathrm{Qch}(X_0)$ is Grothendieck [7, Corollary 11.7]. Note that the restriction $(?)_0 : \mathrm{Qch}(X) \to \mathrm{Qch}(X_0)$ has the right adjoint $(d_0)_*^{\mathrm{Qch}} \circ \mathbb{A}$, see [7, Lemma 12.11]. As $(?)_0$ is faithful exact, the composite

$$\mathbb{I} \to (d_0)_*^{\mathrm{Qch}} \mathbb{A} \mathbb{I}_0 \to (d_0)_*^{\mathrm{Qch}} \mathbb{A} K$$

is a monomorphism into an injective object. This must split, and hence we may further assume that $\mathbb{I} = (d_0)_*^{\mathrm{Qch}} \mathbb{A} K$.

By restriction, it suffices to show that $R^j(f_i)_*\mathbb{I}_i = 0$ for $i = 0, 1, 2$ and $j > 0$. Since $(?)_i\mathbb{A} \cong r_0(i+1)^*$ (see for the notation, [7, (9.1)]) and (d_0) is affine,

$$\begin{aligned} R^j(f_i)_*\mathbb{I}_i &\cong R^j(f_i)_* d_0(i+1)_* r_0(i+1)^* K \cong R^j(f_i \circ d_0(i+1))_* r_0(i+1)^* K \\ &= R^j(d_0(i+1) \circ f_{i+1})_* r_0(i+1)^* K \cong d_0(i+1)_* R^j(f_{i+1})_* r_0(i+1)^* K \\ &\cong d_0(i+1)_* r_0(i+1)^* R^j(f_0)_* K = 0 \end{aligned}$$

for $j > 0$ by [7, Lemma 14.6, **1**] and its proof. This is what we wanted to prove.

(iii) Follows immediately from **(ii)**. □

The following is a generalization of [9, Theorem 6.10].

2.14 Lemma. *Let I be a small category, $h : X' \to X$ a flat morphism of I^{op}-diagrams of schemes. Let $f : U \hookrightarrow X$ be an open subdiagram of schemes, and $g : V \hookrightarrow U$ be an open subdiagram of schemes. Assume that f and g are quasi-compact. Let $f' : U' \hookrightarrow X'$ and $g' : V' \hookrightarrow U'$ be the base change of f and g, respectively. Then $\bar{\delta} : h^* R\underline{\Gamma}_{U,V} \to R\underline{\Gamma}_{U',V'} h^*$ in* [9, (6.1)] *is an isomorphism between functors from $D_{\mathrm{Lqc}}(X)$ to $D_{\mathrm{Lqc}}(X')$.*

Proof. As in the proof of [9, Corollary 6.3], we may assume that the problem is on single schemes. Consider the map of triangles

$$\begin{CD} h^* R\,\underline{\Gamma}_{U,V} @>>> h^* R f_* f^* @>>> h^* R f_* R g_* g^* f^* @>>> h^* R\,\underline{\Gamma}_{U,V}[1] \\ @V{\bar{\delta}}VV @V{d\theta}VV @V{dd\theta\theta}VV @VVV \\ R\,\underline{\Gamma}_{U',V'}\,h^* @>>> R f'_*(f')^* h^* @>>> R f'_* R g'_*(g')^*(f')^* h^* @>>> R\,\underline{\Gamma}_{U',V'}\,h^*[1] \end{CD} .$$

By [11, Proposition 3.9.5], the vertical arrows $d\theta$ and $dd\theta\theta$ are isomorphisms. Hence, $\bar{\delta}$ is also an isomorphism. □

(2.15) Let I be a small category, X an I^{op}-diagram of schemes, and Y a cartesian closed subdiagram of schemes of X defined by the quasi-coherent ideal sheaf $\mathcal{I}$ of $\mathcal{O}_X$. Assume that X is locally noetherian with flat arrows. Then, the canonical map

$$\Phi_Y : \varinjlim \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M}) \to \underline{\Gamma}_Y\,\mathcal{M}$$

is an isomorphism for $\mathcal{M} \in \mathrm{Lqc}(X)$, see [9, (3.21)]. By the way-out lemma [5, Proposition I.7.1], we have

2.16 Lemma. *Let the notation be as in* (2.15)*. Then for* $\mathbb{F} \in D^+_{\mathrm{Lqc}}(X)$*,*

$$\Phi_Y : R(\varinjlim \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, ?))(\mathbb{F}) \to R\,\underline{\Gamma}_Y\,\mathbb{F}$$

is an isomorphism. In particular, Φ_Y *induces an isomorphism*

$$\varinjlim \underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathbb{F}) \cong \underline{H}^i_Y(\mathbb{F}).$$

□

2.17 Lemma. *Let* X *be an* S*-groupoid with affine arrows. Let* U *be a cartesian open subdiagram of* X*, and* V *a cartesian open subdiagram of* Y*. Assume that* X_0 *is noetherian. If* $\mathbb{I}$ *is a* K*-injective complex in* $K(\mathrm{Qch}(X))$*, then* $\mathbb{I}$ *is* $\underline{\Gamma}_{U,V}$*-acyclic.*

Proof. Using [9, Corollary 6.7], it suffices to show that for an injective object K of $\mathrm{Qch}(X_0)$, $(d_0)_*^{\mathrm{Qch}}\mathbb{A}K$ is $\underline{\Gamma}_{U,V}$-acyclic, as in the proof of Lemma 2.13. Applying restrictions, it suffices to show that $\underline{H}^j_{U_i,V_i}(d_0(i+1)_* r_0(i+1)^* K) = 0$ for $j > 0$ and $i = 0, 1, 2$. By the independence [9, Corollary 4.17] and the flat base change Lemma 2.14, this sheaf is $d_0(i+1)_* r_0(i+1)^*\,\underline{H}^j_{U_0,V_0}\,K$. Since K is also injective in $\mathrm{Mod}(X)$ [5, Theorem II.7.18], it is a flabby sheaf, and $\underline{H}^j_{U_0,V_0}\,K = 0$. □

(2.18) A G-scheme X (i.e., an S-scheme with a left G-action) is said to be *standard* if X is noetherian, and the second projection $p_2 : G \times X \to X$ is flat of finite type.

Let X be a standard G-scheme. We denote the category of quasi-coherent (resp. coherent) $(G, \mathcal{O}_X)$-modules by $\mathrm{Qch}(G, X)$ (resp. $\mathrm{Coh}(G, X)$). Note that the sheaf theory discussed in [7, chapters 29–31] and [9], where we assume that G is flat of finite type over S, still works under our weaker assumption (p_2 is flat of finite type). In particular, $\mathrm{Qch}(G, X)$ is a locally noetherian category, and $\mathcal{M} \in \mathrm{Qch}(G, X)$ is a noetherian object if and only if $\mathcal{M} \in \mathrm{Coh}(G, X)$, see [7, Lemma 12.8].

(2.19) We say that a standard G-scheme X is *G-artinian* if there is no incidence relation between G-prime G-ideals (see for the definition, [8, (4.12)]) of X.

2.20 Lemma. *If X is G-artinian, then X is a disjoint union of finitely many G-artinian G-local G-schemes.*

Proof. Clearly, the set of all G-prime G-ideals $\mathrm{Spec}_G(X)$ agrees with the finite set $\mathrm{Min}_G(\mathcal{O}_X)$, the set of minimal G-primes of 0. Thus there are only finitely many G-prime G-ideals. For $\mathcal{P}, \mathcal{Q} \in \mathrm{Spec}_G(X)$ with $\mathcal{P} \neq \mathcal{Q}$, $\mathrm{Ass}_G(\mathcal{O}_X/(\mathcal{P} + \mathcal{Q})) = \emptyset$, since there is no G-prime G-ideal containing both $\mathcal{P}$ and $\mathcal{Q}$. Thus $\mathcal{P} + \mathcal{Q} = \mathcal{O}_X$. This shows that $X = \coprod_{\mathcal{P} \in \mathrm{Spec}_G(X)} V(\mathcal{P})$. As each $V(\mathcal{P})$ is clearly G-artinian G-local, we are done. $\square$

3. The map $\mathfrak{H}$

(3.1) Let $f : X \to Y$ be a symmetric monoidal functor between symmetric monoidal closed categories, and $g : Y \to X$ its right adjoint. For $b \in Y$ and $d \in X$, we denote the composite map

$$f[gb, d] \xrightarrow{H} [fgb, fd] \xrightarrow{u} [b, fd]$$

by ϑ.

3.2 Lemma. *Let $((?)^*, (?)_*)$ be an adjoint pair where $(?)_*$ is a covariant monoidal almost pseudofunctor on a category $\mathcal{S}$ and X_* is a symmetric monoidal closed category for $X \in \mathcal{S}$. Then for morphisms $f : X \to Y$*

and $g : Y \to Z$ *of* $\mathcal{S}$ *and* $b, d \in Z^*$, *the diagram*

(2)
$$\begin{tikzcd}
(gf)^*(b \otimes d) \arrow[rr, "\Delta"] \arrow[d, "d^{-1}"] & & (gf)^*b \otimes (gf)^*d \arrow[d, "d^{-1} \otimes d^{-1}"] \\
f^*g^*(b \otimes d) \arrow[r, "\Delta"] & f^*(g^*b \otimes g^*d) \arrow[r, "\Delta"] & f^*g^*b \otimes f^*g^*d
\end{tikzcd}$$

is commutative.

Proof. Consider the diagram

$$
\begin{tikzcd}
(gf)^*(b \otimes d) \arrow[r, "u\otimes u"] \arrow[d, "d^{-1}"] & (gf)^*((gf)_*(gf)^*b \otimes (gf)_*(gf)^*d) \arrow[r, "m"] \arrow[d, "d^{-1}"] & (gf)^*(gf)_*((gf)^*b \otimes (gf)^*d) \arrow[r, "\varepsilon"] \arrow[ddd, "d^{-1}c"] & (gf)^*b \otimes (gf)^*d \, . \arrow[dddd, "d^{-1}"] \\
f^*g^*(b \otimes d) \arrow[r, "u\otimes u"] \arrow[d, "u\otimes u"] \arrow[dr, phantom, "\text{(a)}"] & f^*g^*((gf)_*(gf)^*b \otimes (gf)_*(gf)^*d) \arrow[d, "d^{-1}c"] \arrow[dr, phantom, "\text{(b)}"] & \arrow[dr, phantom, "\text{(c)}"] & \\
f^*g^*(g_*g^*b \otimes g_*g^*d) \arrow[r, "u\otimes u"] \arrow[d, "m"] & f^*g^*(g_*f_*f^*g^*b \otimes g_*f_*f^*g^*d) \arrow[d, "m"] & & \\
f^*g^*g_*(g^*b \otimes g^*d) \arrow[r, "u\otimes u"] \arrow[d, "\varepsilon"] & f^*g^*g_*(f_*f^*g^*b \otimes f_*f^*g^*d) \arrow[r, "m"] \arrow[d, "\varepsilon"] & f^*g^*g_*f_*(f^*g^*b \otimes f^*g^*d) \arrow[d, "\varepsilon"] & \\
f^*(g^*b \otimes g^*d) \arrow[r, "u\otimes u"] & f^*(f_*f^*g^*b \otimes f_*f^*g^*d) \arrow[r, "m"] & f^*f_*(f^*g^*b \otimes f^*g^*d) \arrow[r, "\varepsilon"] & f^*g^*b \otimes f^*g^*d
\end{tikzcd}
$$

(a) is commutative by [7, Lemma 1.13]. The commutativity of (b) is one of our assumptions, see [11, (3.6.7.2)]. (c) is commutative by [7, Lemma 1.14]. Commutativity of the other squares is trivial. Thus the whole diagram is commutative, and we are done. □

3.3 Lemma. *Let $f : X \to Y$ be a symmetric monoidal functor between symmetric monoidal categories, and $g : Y \to X$ its adjoint. For $b \in X$ and $d \in Y$, the diagram*

$$\begin{tikzcd}
g(fb \otimes d) \arrow[rr, "\Delta"] \arrow[d, "u"] & & gfb \otimes gd \arrow[d, "\varepsilon"] \\
g(fb \otimes fgd) \arrow[r, "m"] & gf(b \otimes gd) \arrow[r, "\varepsilon"] & b \otimes gd
\end{tikzcd}$$

is commutative.

Proof. Follows from the commutativity of the diagram

$$\begin{tikzcd}
g(fb \otimes d) \arrow[r, "u\otimes u"] \arrow[d, "u"] & g(fgfb \otimes fgd) \arrow[r, "m"] \arrow[d, "\varepsilon"] & gf(gfb \otimes gd) \arrow[r, "\varepsilon"] \arrow[d, "\varepsilon"] & gfb \otimes gd \arrow[d, "\varepsilon"] \\
g(fb \otimes fgd) \arrow[ur, "u"] \arrow[r, "\mathrm{id}"] & g(fb \otimes fgd) \arrow[r, "m"] & gf(b \otimes gd) \arrow[r, "\varepsilon"] & b \otimes gd
\end{tikzcd}$$

□

3.4 Lemma. *Viewed as a functor on ?, $\vartheta : f[gb, ?] \to [b, ?]f$ is right conjugate to $\Delta : g(? \otimes b) \to g? \otimes gb$. In particular, if (f, g) is a Lipman symmetric monoidal adjoint pair, then ϑ is an isomorphism.*

Proof. Follows from the commutativity of the diagram

$$
\begin{tikzcd}
f[gb,d] \arrow[r, "\mathrm{tr}"] \arrow[d, "\mathrm{tr}"] \arrow[dr, phantom, "\text{(a)}"] & {[b, f[gb,d] \otimes b]} \arrow[r, "u"] \arrow[d, "u"] & {[b, fg(f[gb,d] \otimes b)]} \arrow[r, "\Delta"] \arrow[d, "u"] \arrow[dr, phantom, "\text{(b)}"] & {[b, f(gf[gb,d] \otimes gb)]}\ , \arrow[dd, "\varepsilon"] \\
{[fgb, f[gb,d] \otimes fgb]} \arrow[r, "u"] \arrow[d, "m"] & {[b, f[gb,d] \otimes fgb]} \arrow[r, "u"] \arrow[d, "m"] & {[b, fg(f[gb,d] \otimes fgb)]} \arrow[d, "m"] & \\
{[fgb, f([gb,d] \otimes gb)]} \arrow[r, "u"] \arrow[d, "\mathrm{ev}"] & {[b, f([gb,d] \otimes gb)]} \arrow[r, "u"] \arrow[d, "\mathrm{ev}"] & {[b, fgf([gb,d] \otimes gb)]} \arrow[r, "\varepsilon"] \arrow[d, "\mathrm{ev}"] & {[b, f([gb,d] \otimes gb)]} \arrow[d, "\mathrm{ev}"] \\
{[fgb, fd]} \arrow[r, "u"] & {[b, fd]} \arrow[r, "u"] \arrow[rr, bend right, "\mathrm{id}"'] & {[b, fgfd]} \arrow[r, "\varepsilon"] & {[b, fd]}
\end{tikzcd}
$$

where the commutativity of (a) and (b) follows from [7, (1.32)] and Lemma 3.3, respectively. $\square$

Consider that the diagram (2) is that of functors on b (consider that d is fixed), and then take a conjugate diagram, we immediately have:

3.5 Lemma. *Let $\mathcal{S}$, $((?)^*, (?)_*)$, f, and g be as in* Lemma 3.2. *Then for $d \in Z_*$ and $e \in X_*$, the diagram*

$$\begin{tikzcd}
{[d, (gf)_*e]} & & (gf)_*[(gf)^*d, e] \arrow[ll, "\vartheta"'] \\
{[d, g_*f_*e]} \arrow[u, "c^{-1}"] & g_*[g^*d, f_*e] \arrow[l, "\vartheta"'] & g_*f_*[f^*g^*d, e] \arrow[l, "\vartheta"'] \arrow[u, "c^{-1}d^{-1}"']
\end{tikzcd}$$

is commutative. $\square$

3.6 Lemma. *Let $\mathcal{S}$ and $((?)^*, (?)_*)$ be as in* Lemma 3.2. *Let*

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

be a commutative diagram in $\mathcal{S}$. Then for $b \in X_*$ and $d \in X'_*$, the diagram

$$\begin{tikzcd}
f_*g'_*[(g')^*b, d] \arrow[r, "\vartheta"] \arrow[d, "c"] & f_*[b, g'_*d] \arrow[r, "H"] & {[f_*b, f_*g'_*d]} \arrow[dr, "c"] & \\
g_*f'_*[(g')^*b, d] \arrow[r, "H"] & g_*[f'_*(g')^*b, f'_*d] \arrow[r, "\theta"] & g_*[g^*f_*b, f'_*d] \arrow[r, "\vartheta"] & {[f_*b, g_*f'_*d]}
\end{tikzcd}$$

is commutative.

Proof. Consider the diagram

$$\begin{tikzcd}
f_*g'_*[(g')^*b, d] \arrow[r, "H"] \arrow[d, "c"] & f_*[g'_*(g')^*b, g'_*d] \arrow[r, "u"] \arrow[d, "H"] \arrow[dr, phantom, "\text{(b)}"] & f_*[b, g'_*d] \arrow[d, "H"] \\
g_*f'_*[(g')^*b, d] \arrow[d, "H"] \arrow[r, phantom, "\text{(a)}"] & {[f_*g'_*(g')^*b, f_*g'_*d]} \arrow[r, "u"] \arrow[d, "c"] & {[f_*b, f_*g'_*d]} \arrow[dd, "c"] \\
g_*[f'_*(g')^*b, f'_*d] \arrow[r, "H"] \arrow[d, "\theta"] \arrow[dr, phantom, "\text{(c)}"] & {[g_*f'_*(g')^*b, g_*f'_*d]} \arrow[d, "\theta"] \arrow[r, phantom, "\text{(d)}"] & \\
g_*[g^*f_*b, f'_*d] \arrow[r, "H"] & {[g_*g^*f_*b, g_*f'_*d]} \arrow[r, "u"] & {[f_*b, g_*f'_*d]}
\end{tikzcd}$$

.

(a) is commutative by [7, Lemma 1.39]. The commutativity of (b) and (c) is trivial. (d) is commutative by [7, Lemma 1.24]. $\square$

(3.7) Let $f : X \to Y$ be a Lipman symmetric monoidal functor between closed categories, and $g : Y \to X$ its adjoint. We denote the composite

$$fg[b,d] \xrightarrow{P} f[gb,gd] \xrightarrow{\vartheta} [b,fgd]$$

by $\mathfrak{G}$.

3.8 Lemma. *Let $\mathcal{S}$, $((?)^*,(?)_*)$ and*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be as in Lemma 3.6. *Then for $b,d \in X_*$, the diagram*

$$\begin{tikzcd} g_*g^*f_*[b,d] \arrow[r, "H"] \arrow[d, "c\theta"] & g_*g^*[f_*b,f_*d] \arrow[r, "\mathfrak{G}"] & {[f_*b, g_*g^*f_*d]} \arrow[d, "c\theta"] \\ f_*g'_*(g')^*[b,d] \arrow[r, "\mathfrak{G}"] & f_*[b, g'_*(g')^*d] \arrow[r, "H"] & {[f_*b, f_*g'_*(g')^*d]} \end{tikzcd}$$

is commutative.

Proof. Left to the reader. Use [7, (1.24)], [7, (1.39)], and [7, (1.59)]. $\square$

3.9 Lemma. *Let $\mathcal{S}$, $((?)^*,(?)_*)$ and*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be as in Lemma 3.6. *Assume that $((?)^*,(?)_*)$ is Lipman. Then for $b,d \in Y_*$, the diagram*

$$\begin{tikzcd} g^*f_*f^*[b,d] \arrow[r, "\mathfrak{G}"] \arrow[d, "d\theta"] & g^*[b, f_*f^*d] \arrow[r, "P"] & {[g^*b, g^*f_*f^*d]} \arrow[d, "d\theta"] \\ f'_*(f')^*g^*[b,d] \arrow[r, "P"] & f'_*(f')^*[g^*b, g^*d] \arrow[r, "\mathfrak{G}"] & {[g^*b, f'_*(f')^*g^*d]} \end{tikzcd}$$

is commutative.

Proof. Left to the reader. Use [7, (1.26)], [7, (1.54)], and [7, (1.59)]. □

3.10 Lemma. *Let I be a small category, and $f : X \to Y$ a morphism of I^{op}-diagrams of schemes. Then for $\mathcal{M} \in \mathrm{Mod}(Y)$ and $\mathcal{N} \in \mathrm{Mod}(X)$, the composite*

$$\vartheta : f_* \underline{\mathrm{Hom}}_{\mathcal{O}_X}(f^*\mathcal{M}, \mathcal{N}) \xrightarrow{H} \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(f_*f^*\mathcal{M}, f_*\mathcal{N}) \xrightarrow{u} R\,\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{M}, f_*\mathcal{N})$$

is an isomorphism.

Proof. This is an immediate consequence of Lemma 3.4. □

3.11 Lemma. *Let I be a small category, and $f : U \to X$ be an open immersion of I^{op}-diagrams of schemes. Let $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(X)$. If either*

(i) *$\mathcal{M}$ is equivariant; or*

(ii) *f is cartesian,*

Then the canonical map

$$P : f^* \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \to \underline{\mathrm{Hom}}_{\mathcal{O}_U}(f^*\mathcal{M}, f^*\mathcal{N})$$

is an isomorphism of presheaves. In particular, it is an isomorphism of sheaves.

Proof. **(ii)** Taking the section at (i, V), where $i \in I$ and $V \in \mathrm{Zar}(U_i)$, it suffices to show that the map induced by the restriction

$$(3) \qquad \mathrm{Hom}_{\mathrm{Zar}(X)/(i,V)}(\mathcal{M}|_{(i,V)}, \mathcal{N}|_{(i,V)}) \to \mathrm{Hom}_{\mathrm{Zar}(U)/(i,V)}(\mathcal{M}|_{(i,V)}, \mathcal{N}|_{(i,V)})$$

is an isomorphism, see the description of P in [9, (2.8)]. But as U is cartesian, $\mathrm{Zar}(U)/(i, V) \hookrightarrow \mathrm{Zar}(X)/(i, V)$ is an equivalence. Indeed, if $(j, W) \to (i, V)$ is a morphism in $\mathrm{Zar}(X)$, it must be a morphism in $\mathrm{Zar}(U)$. Thus (3) is an isomorphism, and we are done.

(i) Similarly to the proof of [9, (2.13)], the problem is reduced to the case of single schemes. Then the assertion follows from **(ii)** immediately. □

3.12 Lemma. *Let $((?)^*, (?)_*)$ be a Lipman monoidal adjoint pair on a category $\mathcal{S}$ where X_* is closed for every $X \in \mathcal{S}$. For morphisms $g : X \to Y$ and $f : Y \to Z$ of $\mathcal{S}$ and $a, b \in Z^*$, the composite*

$$f_*f^*[a, b] \xrightarrow{\mathfrak{G}} [a, f_*f^*b] \xrightarrow{u} [a, f_*g_*g^*f^*b]$$

agrees with the composite

$$f_*f^*[a,b] \xrightarrow{u} f_*g_*g^*f^*[a,b] \xrightarrow{dc^{-1}} (fg)_*(fg)^*[a,b] \xrightarrow{\mathfrak{G}} [a,(fg)_*(fg)^*b] \xrightarrow{d^{-1}c} [a, f_*g_*g^*f^*b].$$

Proof. Left to the reader. Use [7, (1.39), (1.54), (1.56)]. □

3.13 Corollary. *Let* $((?)^*, (?)_*)$ *and* $g : X \to Y$ *be as in* Lemma 3.12. *Then the composite*

$$[a,b] \xrightarrow{u} g_*g^*[a,b] \xrightarrow{\mathfrak{G}} [a, g_*g^*b]$$

is u.

Proof. Let $f = \mathrm{id}$ in Lemma 3.12. □

3.14 Lemma. *Let* I *be a small category,* X *an* I^{op}*-diagram of schemes,* $f : U \hookrightarrow X$ *an open subdiagram. Let* $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(X)$*, and consider the map*

$$\mathfrak{G} : f_*f^* \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, f_*f^*\mathcal{N}).$$

If f *is cartesian or* $\mathcal{M}$ *is equivariant, then* $\mathfrak{G}$ *is an isomorphism.*

Proof. Note that $\mathfrak{G}$ is the composite

$$f_*f^* \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) \xrightarrow{P} f_* \underline{\mathrm{Hom}}_{\mathcal{O}_U}(f^*\mathcal{M}, f^*\mathcal{N}) \xrightarrow{\vartheta} \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, f_*f^*\mathcal{N}).$$

P is an isomorphism by Lemma 3.11. ϑ is an isomorphism by Lemma 3.10. So $\mathfrak{G}$ is an isomorphism. □

(3.15) Let I be a small category, X an I^{op}-diagram of schemes, $f : U \hookrightarrow X$ an open subdiagram, and $g : V \hookrightarrow U$ an open subdiagram. Then for $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(X)$, the diagram

(4)
$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\underline{\Gamma}_{U,V} \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) \arrow[d, "\iota"] & \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \underline{\Gamma}_{U,V}\mathcal{N}) \arrow[d, "\iota"] \\
f_*f^* \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) \arrow[r, "\mathfrak{G}"] \arrow[d, "u"] & \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, f_*f^*\mathcal{N}) \arrow[d, "u"] \\
f_*g_*g^*f^* \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) \arrow[r, "d^{-1}c\mathfrak{G}dc^{-1}"] & \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, f_*g_*g^*f^*\mathcal{N})
\end{tikzcd}$$

is commutative with exact columns by Lemma 3.12. So there is a unique natural map

$$\mathfrak{H} : \underline{\Gamma}_{U,V}\, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \underline{\Gamma}_{U,V}\, \mathcal{N}) \tag{5}$$

such that $\iota\mathfrak{H} = \mathfrak{G}\iota$.

3.16 Lemma. *Let the notation be as in* (3.15)*. If both f and g are cartesian, or $\mathcal{M}$ is equivariant, then $\mathfrak{H}$ in* (5) *is an isomorphism.*

Proof. Follows from Lemma 3.14 and the five lemma applied to the diagram (4). $\square$

(3.17) Let I be a small category, and $f : U \to X$ be an open immersion of I^{op}-diagram of schemes. Then $\Gamma((i, V), f^*_\heartsuit \mathcal{M}) = \Gamma((i, V), \mathcal{M})$ for $\mathcal{M} \in \heartsuit(X)$ almost by definition, where $\heartsuit = \mathrm{PM}$ or Mod. Thus if $j : \mathrm{Zar}\, U \hookrightarrow \mathrm{Zar}\, X$ is the inclusion, then $f^*_\heartsuit = j^\#_\heartsuit$. Thus $f^*_\heartsuit$ has a left adjoint $j^\heartsuit_\#$, as well as the right adjoint f_*. Hence $f^*_\heartsuit$ preserves arbitrary limits as well as arbitrary colimits. We denote $j^\heartsuit_\#$ by $f_!$ or $f^\heartsuit_!$ by an obvious reason.

Note that $\Gamma((i, V), f^{\mathrm{PM}}_!(\mathcal{M}))$ is $\Gamma((i, V), \mathcal{M})$ if $V \subset U_i$, and zero if $V \not\subset U_i$. In particular, $f^{\mathrm{PM}}_!$ is exact.

Note also that we have a commutative diagram

$$\begin{tikzcd} \mathrm{Zar}(U_i) \arrow[r, "Q(U,i)"] \arrow[d, "j"] & \mathrm{Zar}(U) \arrow[d, "j"] \\ \mathrm{Zar}(X_i) \arrow[r, "Q(X,i)"] & \mathrm{Zar}(X) \end{tikzcd},$$

where $Q(X, i)$ and $Q(U, i)$ are obvious inclusions, see [7, (4.5)]. By [7, (2.57)], Lipman's theta [7, (1.21)] $\theta : j^{\mathrm{PM}}_\# Q(U, i)^\# \to Q(X, i)^\# j^{\mathrm{PM}}_\#$, namely, $\theta : (f_i)^{\mathrm{PM}}_!(?)_i \to (?)_i f^{\mathrm{PM}}_!$ at (i, V) is the identity of $\Gamma((i, V), \mathcal{M})$ if $V \subset U_i$, and zero otherwise. In particular, θ is an isomorphism.

Note that $f^{\mathrm{Mod}}_! = j^{\mathrm{Mod}}_\# = a j^{\mathrm{PM}}_\# q = a f^{\mathrm{PM}}_! q$. By [7, (2.59)], $\theta : (?)_i f^{\mathrm{Mod}}_! \to (f_i)^{\mathrm{Mod}}_!(?)_i$ is an isomorphism. It is well-known that $(f_i)^{\mathrm{Mod}}_!$ is exact, and hence $f^{\mathrm{Mod}}_!$ is exact.

Since $f^*_\heartsuit$ has an exact left adjoint $f^\heartsuit_!$, $f^*_\heartsuit$ preserves injectives and K-injectives for $\heartsuit = \mathrm{PM}, \mathrm{Mod}$.

3.18 Lemma. *Let the notation be as in* (3.15)*. Then f^*, $(fg)^*$, and $\underline{\Gamma}_{U,V}$ preserves arbitrary limits.*

Proof. By the discussion in (3.17), f^* and $(fg)^*$ preserves limits.

Now let $(\mathcal{M}_\lambda)$ be a system in $\mathrm{Mod}(X)$. Then

$$\begin{tikzcd}
0 \arrow[r] & \underline{\Gamma}_{U,V} \varprojlim \mathcal{M}_\lambda \arrow[r, "\iota"] \arrow[d] & f_* f^* \varprojlim \mathcal{M}_\lambda \arrow[r, "u"] \arrow[d, "\cong"] & f_* g_* g^* f^* \varprojlim \mathcal{M}_\lambda \arrow[d, "\cong"] \\
0 \arrow[r] & \varprojlim \underline{\Gamma}_{U,V} \mathcal{M}_\lambda \arrow[r, "\iota"] & \varprojlim f_* f^* \mathcal{M}_\lambda \arrow[r, "u"] & \varprojlim f_* g_* g^* f^* \mathcal{M}_\lambda
\end{tikzcd}$$

is a commutative diagram with exact rows. By the five lemma, $\underline{\Gamma}_{U,V}$ preserves limits. $\square$

(3.19) Let the notation be as in (3.15). For a complex $\mathbb{F}$ in $\mathrm{Mod}(X)$, a natural map

$$\mathfrak{H} : \underline{\Gamma}_{U,V} \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?) \underline{\Gamma}_{U,V}$$

between functors on the category of complexes in $\mathrm{Mod}(X)$ is defined. By Lemma 3.18 and Lemma 3.16, it is an isomorphism if f and g are cartesian, or $\mathbb{F}$ is a complex of equivariant sheaves. Similarly,

$$\mathfrak{G} : f_* f^* \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?) f_* f^*$$

and

$$d^{-1} c \mathfrak{G} d c^{-1} : f_* g_* g^* f^* \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?) f_* g_* g^* f^*$$

are induced.

3.20 Lemma. *Let $(\mathbb{X}, \mathcal{O}_\mathbb{X})$ be a ringed site, $\mathbb{F}$ a complex of $\mathcal{O}_\mathbb{X}$-modules, and $\mathbb{G}$ a K-injective complex of $\mathcal{O}_\mathbb{X}$-modules. Then $\underline{\mathrm{Hom}}_{\mathcal{O}_\mathbb{X}}(\mathbb{F}, \mathbb{G})$ is weakly K-injective.*

Proof. Let $\mathbb{H}$ be any exact K-flat complex. Then

$$\mathrm{Hom}_{\mathcal{O}_\mathbb{X}}(\mathbb{H}, \underline{\mathrm{Hom}}_{\mathcal{O}_\mathbb{X}}(\mathbb{F}, \mathbb{G})) \cong \mathrm{Hom}_{\mathcal{O}_\mathbb{X}}(\mathbb{H} \otimes \mathbb{F}, \mathbb{G})$$

is exact, since $\mathbb{H} \otimes \mathbb{F}$ is exact [7, Lemma 3.21, **2**] and $\mathbb{G}$ is K-injective. By [7, Lemma 3.25, **5**], $\underline{\mathrm{Hom}}_{\mathcal{O}_\mathbb{X}}(\mathbb{F}, \mathbb{G})$ is weakly K-injective. $\square$

3.21 Lemma. *The canonical maps*

$$\zeta : R(\underline{\Gamma}_{U,V} \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?)) \to R \underline{\Gamma}_{U,V} R \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?),$$

$$\zeta : R(f_* f^* \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?)) \to R f_* f^* R \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?),$$

and

$$\zeta : R(f_*g_*g^*f^* \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?)) \to Rf_*Rg_*g^*f^*R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?)$$

are isomorphisms.

Proof. For a K-injective complex $\mathbb{G}$, $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, \mathbb{G})$ is weakly K-injective. So it is K-flabby, and $\underline{\Gamma}_{U,V}$-acyclic [9, (4.3)]. In particular, $f^* \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, \mathbb{G})$ and $g^*f^* \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, \mathbb{G})$ are K-limp by [9, (4.6)], and the assertion follows. $\square$

(3.22) By the lemma, the composite

$$\begin{aligned} \mathfrak{H} : R\,\underline{\Gamma}_{U,V}\, R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?) &\xrightarrow{\zeta^{-1}} R(\underline{\Gamma}_{U,V}\, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?)) \\ &\xrightarrow{R\mathfrak{H}} R(\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?)\, \underline{\Gamma}_{U,V}) \xrightarrow{\zeta} R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?)R\,\underline{\Gamma}_{U,V} \end{aligned}$$

is defined. Similarly,

$$\mathfrak{G} : Rf_*f^*R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?) \to R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?)Rf_*f^*$$

and

$$d^{-1}c\mathfrak{G}dc^{-1} : Rf_*Rg_*g^*f^*R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?) \to R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?)Rf_*Rg_*g^*f^*$$

are induced. Note that

(6)
$$\begin{tikzcd} R\,\underline{\Gamma}_{U,V}\, R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?) \arrow[r, "\mathfrak{H}"] \arrow[d, "\iota"] & R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?)R\,\underline{\Gamma}_{U,V} \arrow[d, "\iota"] \\ Rf_*f^*R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?) \arrow[r, "\mathfrak{G}"] \arrow[d, "u"] & R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?)Rf_*f^* \arrow[d, "u"] \\ Rf_*Rg_*g^*f^*R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?) \arrow[r, "d^{-1}c\mathfrak{G}dc^{-1}"] \arrow[d] & R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?)Rf_*Rg_*g^*f^* \arrow[d] \\ R\,\underline{\Gamma}_{U,V}\, R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?)[1] \arrow[r, "\mathfrak{H}[1]"] & R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?)R\,\underline{\Gamma}_{U,V}[1] \end{tikzcd}$$

is a commutative diagram with columns being triangles.

3.23 Lemma. *Let I be a small category, and $f : X \to Y$ a morphism of I^{op}-diagrams of schemes. Then the composite*

$$\begin{aligned} \vartheta : Rf_*R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(Lf^*\mathbb{F}, \mathbb{G}) &\xrightarrow{H} R\,\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(Rf_*Lf^*\mathbb{F}, Rf_*\mathbb{G}) \\ &\xrightarrow{u} R\,\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathbb{F}, Rf_*\mathbb{G}) \end{aligned}$$

is an isomorphism between functors on $D(Y)^{\mathrm{op}} \times D(X)$.

Proof. This is an immediate consequence of [7, (1.49)] and [7, (8.23), **5**]. □

3.24 Corollary. *Let $f : U \to X$ be a cartesian open immersion. Then*

$$P : f^* R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, \mathbb{G}) \to R\,\underline{\mathrm{Hom}}_{\mathcal{O}_U}(f^*\mathbb{F}, f^*\mathbb{G})$$

is an isomorphism for any $\mathbb{F}, \mathbb{G} \in D(X)$.

Proof. If $\mathbb{G}$ is a K-injective complex in $K(X)$, then so is $f^*\mathbb{G}$ by (3.17). So it suffices to show that

$$f^* \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, \mathbb{G}) \to \underline{\mathrm{Hom}}_{\mathcal{O}_U}(f^*\mathbb{F}, f^*\mathbb{G})$$

is an isomorphism of complexes, if $\mathbb{F}$ and $\mathbb{G}$ are complexes in $\mathrm{Mod}(X)$. This follows from Lemma 3.11 and the fact that f^* preserves direct product. □

3.25 Lemma. *Let I be a small category, and $f : X \to Y$ a morphism of I^{op}-diagrams of schemes. Let $\mathbb{F}$ and $\mathbb{G}$ be objects in $D(Y)$. Assume that one of the following holds:*

(i) *f is locally an open immersion, $\mathbb{F} \in D_{\mathrm{EM}}(Y)$, and one of the following holds:*

  (a) $\mathbb{G} \in D^+(Y)$;

  (b) $\mathbb{F} \in D^+_{\mathrm{EM}}(Y)$;

  (c) $\mathbb{G} \in D_{\mathrm{Lqc}}(Y)$.

(ii) *f is flat, Y is locally noetherian, $\mathbb{G} \in D^+(Y)$, and $\mathbb{F} \in D^-_{\mathrm{Coh}}(Y)$.*

(iii) *f is flat, Y is locally noetherian, $\mathbb{F} \in D_{\mathrm{Coh}}(Y)$, and both $\mathbb{G}$ and $f^*\mathbb{G}$ have finite injective dimension.*

Then the canonical map

$$P : f^* R\,\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathbb{F}, \mathbb{G}) \to R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(f^*\mathbb{F}, f^*\mathbb{G})$$

is an isomorphism.

Proof. Similarly to [7, Lemma 1.59], using [7, Lemma 1.56], it is easy to prove that the diagram

$$\begin{tikzcd}
(?)_i f^* R\,\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathbb{F},\mathbb{G}) \arrow[r, "P"] \arrow[d, "\theta^{-1}"] & (?)_i R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(f^*\mathbb{F}, f^*\mathbb{G}) \arrow[d, "H"] \\
f_i^*(?)_i R\,\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathbb{F},\mathbb{G}) \arrow[d, "H"] & R\,\underline{\mathrm{Hom}}_{\mathcal{O}_{X_i}}((?)_i f^*\mathbb{F}, (?)_i f^*\mathbb{G}) \arrow[d, "{[\theta,\theta^{-1}]}"] \\
f_i^* R\,\underline{\mathrm{Hom}}_{\mathcal{O}_{Y_i}}(\mathbb{F}_i,\mathbb{G}_i) \arrow[r, "P"] & R\,\underline{\mathrm{Hom}}_{\mathcal{O}_{X_i}}(f_i^*\mathbb{F}_i, f_i^*\mathbb{G}_i)
\end{tikzcd}$$

is commutative for $i \in I$. Note that the vertical morphisms are isomorphisms by [7, (13.9)] and [7, (6.25)]. So in order to prove that the top P is an isomorphism for each $i \in I$, it suffices to prove the bottom P is an isomorphism. So we may assume that the problem is on single schemes.

(i) We may assume that f is an open immersion. Then this is a special case of Corollary 3.24.

(ii) This is [5, (5.8)].

(iii) This follows from **(ii)** and the way-out lemma ([5, Proposition I.7.1, (iii)]). □

3.26 Theorem. *Let I be a small category, X an I^{op}-diagram of schemes, $f : U \hookrightarrow X$ an open subdiagram, and $g : V \hookrightarrow U$ an open subdiagram. Let $\mathbb{F}$ and $\mathbb{G}$ be in $D(X)$. If one of the following holds, then*

$$\mathfrak{H} : R\,\underline{\Gamma}_{U,V}\, R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?) \to R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, ?) R\,\underline{\Gamma}_{U,V}$$

is an isomorphism:

(i) *f and g are cartesian;*

(ii) *$\mathbb{F} \in D_{\mathrm{EM}}(X)$, and one of the following hold:* (a) *$\mathbb{G} \in D^+(X)$;* (b) *$\mathbb{F} \in D^+_{\mathrm{EM}}(X)$;* (c) *$\mathbb{G} \in D_{\mathrm{Lqc}}(X)$.*

Proof. By Lemma 3.23 and Lemma 3.25, the two maps ϑP and $d^{-1}c\vartheta P dc^{-1}$ in (6) are isomorphisms. As the columns of (6) are triangles, the third horizontal map $\mathfrak{H}$ is also an isomorphism. □

4. Matlis duality and the local duality

Let S be a scheme, G an S-group scheme, (X, Y) a standard G-local G-scheme. That is, X is a standard G-local G-scheme, and Y is its unique minimal closed G-subscheme. We denote the inclusion $Y \hookrightarrow X$ by j.

We denote the defining ideal sheaf of Y by $\mathcal{I}$. Thus $\mathcal{I}$ is the unique G-maximal G-ideal of $\mathcal{O}_X$. We fix the generic point of an irreducible component of Y and denote it by η.

4.1 Lemma. *Let $\mathcal{C}$ be a class of noetherian local rings. Assume that if $A \in \mathcal{C}$ and B is essentially of finite type over A, then $B \in \mathcal{C}$. Let $\mathbb{P}(A, M)$ be a property of a pair (A, M) of a finitely generated module M over a noetherian local ring A such that $A \in \mathcal{C}$. Assume that*

(i) *If $A \in \mathcal{C}$, $\mathbb{P}(A, M)$ holds, and $P \in \operatorname{Spec} A$, then $\mathbb{P}(A_P, M_P)$ holds.*

(ii) *If $A \in \mathcal{C}$, M a finite A-module, and $A \to B$ is a flat local homomorphism essentially of finite type with local complete intersection fibers (resp. geometrically regular fibers), then $\mathbb{P}(A, M)$ holds if and only if $\mathbb{P}(B, B \otimes_A M)$ holds.*

Assume that the all local rings of X belong to $\mathcal{C}$. For $\mathcal{M} \in \operatorname{Coh}(G, X)$, if $\mathbb{P}(\mathcal{O}_{X,\eta}, \mathcal{M}_\eta)$ holds (resp. $\mathbb{P}(\mathcal{O}_{X,\eta}, \mathcal{M}_\eta)$ holds and either the second projection $p_2 : G \times X \to X$ is smooth or $S = \operatorname{Spec} k$ with k a perfect field and G is of finite type over S), then $\mathbb{P}(\mathcal{O}_{X,x}, \mathcal{M}_x)$ holds for any $x \in X$.

Proof. Let Z be the unique integral closed subscheme of X whose generic point is x. Let Z^* be the unique minimal closed G-subscheme of X containing Z, see [8]. As $\eta \in Y \subset Z^*$, there exists some irreducible component Z_0 of Z such that $\eta \in Z_0$. Let ζ be the generic point of Z_0. Since $\mathbb{P}(\mathcal{O}_{X,\eta}, \mathcal{M}_\eta)$ holds and ζ is a generalization of η, $\mathbb{P}(\mathcal{O}_{X,\zeta}, \mathcal{M}_\zeta)$ holds. Then by [8, Corollary 7.6], $\mathbb{P}(\mathcal{O}_{X,x}, \mathcal{M}_x)$ holds. $\square$

4.2 Corollary. *Let m, n, and g be non-negative integers or ∞. Then*

(i) *Let $\mathcal{M} \in \operatorname{Coh}(G, X)$, and assume that $\mathcal{M}_\eta$ is maximal Cohen–Macaulay (resp. of finite injective dimension, projective dimension m, $\dim - \operatorname{depth} = n$, torsionless, reflexive, G-dimension g, zero) as an $\mathcal{O}_{X,\eta}$-module. Then $\mathcal{M}_x$ is so as an $\mathcal{O}_{X,x}$-module for any $x \in X$.*

(ii) *If $\mathcal{O}_{X,\eta}$ is a complete intersection, then X is locally a complete intersection.*

(iii) *Assume that $p_2 : G \times X \to X$ is smooth or $S = \operatorname{Spec} k$ with k a perfect field and G is of finite type over S. If $\mathcal{O}_{X,\eta}$ is regular, then X is regular.*

(iv) *In addition to the assumption of* **(iii)**, *assume further that X is a locally excellent $\mathbb{F}_p$-scheme, where p is a prime number. If $\mathcal{O}_{X,\eta}$ is F-regular (resp. F-rational), then the all local rings of X is F-regular (resp. F-rational).*

Proof. **(i)** Let $\mathcal{C}$ be the class of all noetherian local rings, and $\mathbb{P}(A, M)$ be "M is a maximal Cohen–Macaulay A-module." We can apply Lemma 4.1. Similarly for other properties.

(ii) Let $\mathcal{C}$ be the class of all noetherian local rings, and $\mathbb{P}(A, M)$ be "A is a complete intersection." Then as $\mathbb{P}(\mathcal{O}_{X,\eta}, 0)$ holds, $\mathbb{P}(\mathcal{O}_{X,x}, 0)$ holds for any $x \in X$.

(iii) Let $\mathcal{C}$ be the class of all noetherian local rings, and $\mathbb{P}(A, M)$ be "A is regular."

(iv) Let $\mathcal{C}$ be the class of all excellent noetherian local rings of characteristic p, and $\mathbb{P}(A, M)$ be "A is F-regular" (resp. "A is F-rational"). $\square$

4.3 Corollary. *The stalk functor $(?)_\eta : \operatorname{Qch}(G, X) \to \operatorname{Mod}(\mathcal{O}_{X,\eta})$ is faithfully exact.*

Proof. The exactness is well-known. Let $\mathcal{M} \in \operatorname{Qch}(G, X)$ and assume that $\mathcal{M} \neq 0$. Then as $\operatorname{Qch}(G, X)$ is locally noetherian and its noetherian object is nothing but a coherent $(G, \mathcal{O}_X)$-module, $\mathcal{M}$ contains a nonzero coherent $(G, \mathcal{O}_X)$-submodule $\mathcal{N}$. Then by Corollary 4.2, $\mathcal{M}_\eta \supset \mathcal{N}_\eta \neq 0$. This shows that $(?)_\eta$ is faithfully exact. $\square$

4.4 Remark. Formally, $(?)_\eta$ is a functor from $\operatorname{Qch}(G, X) = \operatorname{Qch}(B_G^M(X))$, or more generally, from $\operatorname{Mod}(G, X) = \operatorname{Mod}(B_G^M(X))$ to $\operatorname{Mod}(\mathcal{O}_{X,\eta})$, and is the composite

$$\operatorname{Mod}(B_G^M(X)) \xrightarrow{(?)_0} \operatorname{Mod}(B_G^M(X)_0) = \operatorname{Mod}(X_0) \xrightarrow{h^*} \operatorname{Mod}(\operatorname{Spec} \mathcal{O}_{X,\eta}),$$

where $h : \operatorname{Spec} \mathcal{O}_{X,\eta} \to X_0$ is the inclusion. Thus $(?)_\eta$ is sometimes written as $(?)_\eta(?)_0$, where $(?)_\eta$ means h^*.

4.5 Corollary (G-NAK). *Let $\mathcal{M} \in \operatorname{Coh}(G, X)$. If $j^*\mathcal{M} = 0$, then $\mathcal{M} = 0$, where $j : Y \hookrightarrow X$ is the inclusion.*

Proof. Since $j^*\mathcal{M} = 0$, $\mathcal{M}/\mathcal{I}\mathcal{M} = 0$. So $\mathcal{M}_\eta/\mathcal{I}_\eta\mathcal{M}_\eta = 0$. By Nakayama's lemma, $\mathcal{M}_\eta = 0$. By Corollary 4.3, $\mathcal{M} = 0$. $\square$

4.6 Proposition. *A standard G-artinian G-scheme is Cohen–Macaulay.*

Proof. By Lemma 2.20, we may assume that the G-scheme is G-local. So let X be a G-artinian G-local standard G-scheme. Let Y, η, and $\mathcal{I}$ be as above.

Then $\sqrt[G]{0} = \mathcal{I}$, since $\mathcal{I}$ is the only G-prime ideal (for the definition and basic properties of $\sqrt[G]{?}$, see [8, section 4]). So $Y = X$, set theoretically. Thus η is the generic point of an irreducible component of X. So $\mathcal{O}_{X,\eta}$ is an artinian ring, and hence is Cohen–Macaulay. By Corollary 4.2, X is Cohen–Macaulay. $\square$

4.7 Corollary. *Y is Cohen–Macaulay.*

Proof. Since Y is G-artinian G-local standard, the corollary follows immediately from Proposition 4.6. $\square$

(4.8) From now on, we assume that X has a G-dualizing complex $\mathbb{I}_X$ (see [7, (31.15)]). For a G-morphism $f : X' \to X$ which is separated of finite type, we denote $f^!\mathbb{I}_X$ by $\mathbb{I}_{X'}$, where $f^!$ is the twisted inverse functor $B_M^G(f)^!$ (see [7, chapter 29]). Note that $\mathbb{I}_{X'}$ is a G-dualizing complex of X' [7, Lemma 31.11]. By [7, Lemma 31.6], $\mathbb{I}_{X'}$, viewed as a complex of $\mathcal{O}_{X'}$-modules, is a dualizing complex of X'.

Since $\mathcal{O}_{Y,\eta}$ is Cohen–Macaulay, there is only one i such that $H^i(\mathbb{I}_Y)_\eta \neq 0$. This is equivalent to say that $H^i(\mathbb{I}_Y) \neq 0$. If this i is 0, then we say that $\mathbb{I}_X$ is G-normalized. If X has a G-dualizing complex, then by shifting, X has a G-normalized G-dualizing complex.

From now on, we always assume that $\mathbb{I}_X$ is G-normalized.

4.9 Lemma. *$\mathbb{I}_{X,\eta}$ is a normalized dualizing complex of the local ring $\mathcal{O}_{X,\eta}$. In particular, $H^0_{\mathfrak{m}_\eta}(\mathbb{I}_{X,\eta})$ is the injective hull of the residue field $\kappa(\eta)$ of $\mathcal{O}_{X,\eta}$, where $\mathfrak{m}_\eta$ is the maximal ideal of $\mathcal{O}_{X,\eta}$.*

Proof. Since $\mathbb{I}_X$ is a dualizing complex, $\mathbb{I}_{X,\eta}$ is also a dualizing complex of $\mathcal{O}_{X,\eta}$. We prove that $\mathbb{I}_{X,\eta}$ is normalized. Let $\mathbb{D}$ be a normalized dualizing complex of $\mathcal{O}_{X,\eta}$, and set $\mathbb{I}_{X,\eta} \cong \mathbb{D}[r]$. We want to prove that $r = 0$.

Consider the commutative diagram

$$\begin{tikzcd} X & \operatorname{Spec} \mathcal{O}_{X,\eta} \arrow[l, "p"'] \\ Y \arrow[u, "j"] & \operatorname{Spec} \mathcal{O}_{Y,\eta} \arrow[l, "q"'] \arrow[u, "j'"'] \end{tikzcd} .$$

By the commutativity with restrictions [7, Proposition 18.14],

$$H^0(\mathbb{I}_Y)_\eta \cong H^0(q^* j^! \mathbb{I}_X) \cong H^0((j')^! \mathbb{I}_{X,\eta}) \cong \mathrm{Ext}^r_{\mathcal{O}_{X,\eta}}(\mathcal{O}_{Y,\eta}, \mathbb{D}) \neq 0.$$

The Matlis dual of the last module is $H^{-r}_{\mathfrak{m}_\eta}(\mathcal{O}_{Y,\eta})$, by the local duality [5, (V.6.3)]. Since $\mathcal{O}_{Y,\eta}$ is an $\mathcal{O}_{X,\eta}$-module of finite length, $H^{-r}_{\mathfrak{m}_\eta}(\mathcal{O}_{Y,\eta}) \neq 0$ implies $r = 0$. $\square$

4.10 Lemma. *$\underline{H}^i_Y(\mathbb{I}_X) = 0$ for $i \neq 0$, and $\underline{H}^0_Y(\mathbb{I}_X)_\eta$ is the injective hull of the residue field $\kappa(\eta)$ of the local ring $\mathcal{O}_{X,\eta}$.*

Proof. By [9, Theorem 6.10],

$$(\underline{H}^i_Y(\mathbb{I}_X))_\eta \cong H^i((?)_\eta R\,\underline{\Gamma}_Y\,\mathbb{I}_X) \cong H^i(R\,\underline{\Gamma}_{\mathcal{I}_\eta}(?)_\eta \mathbb{I}_X) \cong H^i_{\mathfrak{m}_\eta}(\mathbb{I}_{X,\eta}).$$

Since $\mathbb{I}_{X,\eta}$ is a normalized dualizing complex of $\mathcal{O}_{X,\eta}$, the last module is zero if $i \neq 0$ and is the injective hull of the residue field $\kappa(\eta)$ of the local ring $\mathcal{O}_{X,\eta}$ if $i = 0$. As $(?)_\eta$ is faithfully exact, we are done. $\square$

(4.11) We set $\mathcal{E} := \underline{H}^0_Y(\mathbb{I}_X)$, and call it the *G-sheaf of Matlis*. Note that the definition of $\mathcal{E}$ depends on the choice of $\mathbb{I}_X$. Note also that $\mathcal{E}_\eta$ is the injective hull of the residue field of $\mathcal{O}_{X,\eta}$.

4.12 Lemma. *$\mathcal{E}$ is of finite injective dimension as an object of* $\mathrm{Mod}(G, X)$.

Proof. We may assume that $\mathbb{I}_X$ is a bounded complex of injective objects. By Lemma 4.10, $\mathcal{E}$ is isomorphic to $\underline{\Gamma}_Y(\mathbb{I}_X)$ in $D(X)$. On the other hand, $\underline{\Gamma}_Y(\mathbb{I}_X)$ is quasi-isomorphic to $\mathbb{J} = \mathrm{Cone}(\mathbb{I}_X \to f_* f^* \mathbb{I}_X)[-1]$, where $f : X \setminus Y \to X$ is the inclusion. As $f_* f^*$ has an exact left adjoint $f_! f^*$ (see (3.17)), $\mathbb{J}$ is a bounded injective resolution of $\mathcal{E}$. $\square$

4.13 Lemma. *$\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{M}, \mathcal{E}) = 0$ for $i > 0$ and $\mathcal{M} \in \mathrm{Coh}(G, X)$.*

Proof. $\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{M}, \mathcal{E})_\eta \cong \mathrm{Ext}^i_{\mathcal{O}_{X,\eta}}(\mathcal{M}_\eta, \mathcal{E}_\eta)$. As $\mathcal{E}_\eta$ is injective, we are done. $\square$

4.14 Corollary. $\mathbb{D} := \underline{\mathrm{Hom}}_{\mathcal{O}_X}(?, \mathcal{E})$ *is an exact functor on* $\mathrm{Coh}(G, X)$.

4.15 Lemma. *For $\mathcal{M} \in \mathrm{Qch}(G, X)$, the following are equivalent:*

(i) *$\mathcal{M}$ is of finite length;*

(ii) *$\mathcal{M} \in \mathrm{Coh}(G, X)$, and $\mathcal{I}^n \mathcal{M} = 0$ for some n.*

(iii) $\mathcal{M}_\eta$ *is an* $\mathcal{O}_{X,\eta}$*-module of finite length;*

Proof. **(i)⇒(ii)** As $\mathcal{M}$ is of finite length, it is a noetherian object. Hence it is coherent by [7, Lemma 12.8]. As $\mathcal{M}$ is also an artinian object, $\mathcal{I}^n\mathcal{M} = \mathcal{I}^{n+1}\mathcal{M}$ for sufficiently large n. By Corollary 4.5, $\mathcal{I}^n\mathcal{M} = 0$.

(ii)⇒(iii) As $\mathcal{M}$ is coherent, $\mathcal{M}_\eta$ is a finitely generated $\mathcal{O}_{X,\eta}$-module. Since $\mathcal{I}_\eta^n\mathcal{M}_\eta = 0$, the support of $\mathcal{M}_\eta$ is one point, and hence $\mathcal{M}_\eta$ is a module of finite length.

(iii)⇒(i) This is because $(?)_\eta$ is faithfully exact. □

(4.16) We denote by $\mathcal{F}$ the full subcategory of those objects $\mathcal{M} \in \mathrm{Qch}(G, X)$ such that the equivalent conditions in the lemma are satisfied.

4.17 Theorem (Matlis duality). *Set* $\mathbb{D} := \underline{\mathrm{Hom}}_{\mathcal{O}_X}(?, \mathcal{E})$*. Then*

(i) $\mathbb{D}$ *is an exact functor from* $\mathcal{F}$ *to itself.*

(ii) $\mathbb{D}^2 \cong \mathrm{Id}$ *as functors on* $\mathcal{F}$*. In particular,* $\mathbb{D} : \mathcal{F} \to \mathcal{F}$ *is an anti-equivalence.*

Proof. **(i)** If $\mathcal{M} \in \mathcal{F}$, then $\mathbb{D}(\mathcal{M}) = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{E})$ is in $\mathrm{Qch}(G, X)$, and $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{E})_\eta = \mathrm{Hom}_{\mathcal{O}_{X,\eta}}(\mathcal{M}_\eta, \mathcal{E}_\eta)$ is of finite length, because this module is the Matlis dual of the module $\mathcal{M}_\eta$, which is of finite length. So the condition **(iii)** in Lemma 4.15 is satisfied, and hence $\mathbb{D}(\mathcal{M}) \in \mathcal{F}$. The exactness of $\mathbb{D}$ is already checked.

(ii) Let $\mathfrak{D} : \mathcal{M} \to \mathbb{D}\mathbb{D}\mathcal{M} = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{E}), \mathcal{E})$ be the canonical map, see (2.2). Note that by Lemma 2.5 and Lemma 2.7, applying $(?)_\eta$ to this map, we get the duality map $\mathfrak{D} : \mathcal{M}_\eta \to \mathrm{Hom}_{\mathcal{O}_{X,\eta}}(\mathrm{Hom}_{\mathcal{O}_{X,\eta}}(\mathcal{M}_\eta, \mathcal{E}_\eta), \mathcal{E}_\eta)$, which is an isomorphism, since $\mathcal{E}_\eta$ is the injective hull of the residue field $\kappa(\eta)$. Since $(?)_\eta$ is faithful, $\mathfrak{D} : \mathcal{M} \to \mathbb{D}\mathbb{D}\mathcal{M}$ is an isomorphism. □

4.18 Theorem (Local duality). *For* $\mathbb{F} \in D_{\mathrm{Coh}}(G, X)$*, the composite*

$$\mathfrak{d} : R\,\underline{\Gamma}_Y\,\mathbb{F} \xrightarrow{\mathfrak{D}} R\,\underline{\Gamma}_Y\,R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, \mathbb{I}_X), \mathbb{I}_X) \xrightarrow{\mathfrak{H}}$$
$$R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, \mathbb{I}_X), R\,\underline{\Gamma}_Y\,\mathbb{I}_X) \cong R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, \mathbb{I}_X), \mathcal{E})$$

is an isomorphism. It induces an isomorphism

$$\underline{H}^i_Y(\mathbb{F}) \cong \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\underline{\mathrm{Ext}}^{-i}_{\mathcal{O}_X}(\mathbb{F}, \mathbb{I}_X), \mathcal{E})$$

for each $i \in \mathbb{Z}$*.*

Proof. $\mathfrak{D}$ in the composition is an isomorphism by [7, (31.9)]. $\mathfrak{H}$ is an isomorphism by Theorem 3.26, **(i)**. Thus $\mathfrak{d}$ is an isomorphism.

To prove the second assertion, it suffices to show that

$$\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathbb{G}, \mathcal{E}) \cong \underline{\mathrm{Hom}}_{\mathcal{O}_X}(H^{-i}(\mathbb{G}), \mathcal{E}),$$

where $\mathbb{G} = R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, \mathbb{I}_X)$. Note that $\mathbb{G} \in D_{\mathrm{Coh}}(X)$ by [7, (31.9)]. Let $\mathbb{J}$ be a bounded injective resolution of $\mathcal{E}$ (it does exist, see Lemma 4.12). Consider the spectral sequence

$$E_2^{p,q} = H^p(\underline{\mathrm{Hom}}_{\mathcal{O}_X}(H^{-q}(\mathbb{G}), \mathbb{J})) \Rightarrow \underline{\mathrm{Ext}}^{p+q}_{\mathcal{O}_X}(\mathbb{G}, \mathcal{E}).$$

By Lemma 4.13, $E_2^{p,q} = 0$ for $p \neq 0$, and the spectral sequence collapses, and we get the desired assertion. □

4.19 Lemma. *Let $\mathbb{F} \in D_{\mathrm{Coh}}(X)$. Then the diagram*

$$\begin{tikzcd}
(?)_\eta(?)_0 R\,\underline{\Gamma}_Y(\mathbb{F}) \arrow[r, "\mathfrak{d}"] \arrow[dd, "\bar{\delta}\hat{\gamma}^{-1}"] & (?)_\eta(?)_0 R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, \mathbb{I}_X), \mathcal{E}) \arrow[d, "PH"] \\
& R\,\mathrm{Hom}_{\mathcal{O}_{X,\eta}}((?)_\eta(?)_0 R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{F}, \mathbb{I}_X), \mathcal{E}_\eta) \arrow[d, "P^{-1}H^{-1}"] \\
R\Gamma_{\mathfrak{m}_\eta}(\mathbb{F}_\eta) \arrow[r, "\mathfrak{d}"] & R\,\mathrm{Hom}_{\mathcal{O}_{X,\eta}}(R\,\mathrm{Hom}_{\mathcal{O}_{X,\eta}}(\mathbb{F}_\eta, \mathbb{I}_{X,\eta}), \mathcal{E}_\eta)
\end{tikzcd}$$

is commutative (*see for the definition of $\hat{\gamma}$ and $\bar{\delta}$, see* [9, section 4] *and* [9, (6.1)], *respectively*).

Proof. Note that H^{-1} in the diagram exists by [7, (13.9)]. The P^{-1} exists by Lemma 3.25, **(iii)**. The commutativity of the diagram follows from Lemma 2.5 and Lemma 2.7 immediately. □

5. An example of graded rings

(5.1) Let $(R, \mathfrak{m})$ be a noetherian local ring with a normalized dualizing complex $\mathbb{I}_R$. Set $S = \operatorname{Spec} R$. Let H be a flat R-group scheme of finite type, and $G = \mathbb{G}_m \times H$. Let A be a G-algebra which is of finite type over R. So A is $\mathbb{Z}$-graded and each homogeneous component is an H-submodule of A. Assume that $A = \bigoplus_{i \geq 0} A_i$ is $\mathbb{N}$-graded and $A_0 = R$. Let $\pi : X \to S$ be the canonical map, where $X := \operatorname{Spec} A$. Set $\mathbb{I}_X := \pi^! \mathbb{I}_R$.

5.2 Lemma. *Under the notation as above, X is G-local, and $\mathbb{I}_X$ is G-normalized.*

Proof. Let I be a proper G-ideal of A. Then I is a homogeneous ideal, and is contained in the unique graded maximal ideal $\mathfrak{M} := \mathfrak{m} + A_+$, where $A_+ = \bigoplus_{i>0} A_i$. Clearly, $\mathfrak{M}$ is a G-ideal, and hence is the unique G-maximal G-ideal. So X is G-local.

Let $\varphi : S \to X$ be the closed immersion induced by $A \to A/A_+ = R$. Let $\psi : Y \to S$ be the closed immersion induced by $R \to R/\mathfrak{m} \cong A/\mathfrak{M}$, where $Y = \operatorname{Spec} A/\mathfrak{M}$. Then since $\pi\varphi = \mathrm{id}_S$,

$$\mathbb{I}_Y = (\varphi\psi)^!(\mathbb{I}_X) = \psi^!\varphi^!\pi^!\mathbb{I}_R = \psi^!\mathbb{I}_R.$$

So $H^i(\mathbb{I}_Y) \cong \operatorname{Ext}^i_R(R/\mathfrak{m}, \mathbb{I}_R)$, whose Matlis dual is $H^i_{\mathfrak{m}}(R/\mathfrak{m})$. This is nonzero if and only if $i = 0$. Thus $\mathbb{I}_X$ is G-normalized. $\square$

(5.3) For a finite R-module V, set $V^\dagger := \operatorname{Hom}_R(V, E_R)$, where E_R is the injective hull of the residue field $R/\mathfrak{m}$ of R. For an A-finite G-module M, set $M^\vee = \varinjlim \operatorname{Hom}_R(M/\mathfrak{M}^n M, E_R)$. As each $M/\mathfrak{M}^n M$ is an R-finite (G, A)-module, each $\operatorname{Hom}_R(M/\mathfrak{M}^n M, E_R)$ is a (G, A)-module, and hence $M^\vee$ is also a (G, A)-module. It is easy to see that $M^\vee \cong \operatorname{Hom}_A(M, A^\vee)$. Note that the degree i component of $M^\vee$ is $M^\dagger_{-i}$. That is, $M^\vee = \bigoplus_{i\in\mathbb{Z}} M^\dagger_{-i}$.

5.4 Lemma. *$A^\vee$ is isomorphic to $E_A := \Gamma(X, \mathcal{E})$ as a (G, A)-module.*

Proof. We may assume that $\mathbb{I}_R$ is the normalized fundamental dualizing complex. We have

$$\begin{aligned}(7)\quad \mathcal{E} = \underline{H}^0_Y(\mathbb{I}_X) &= \varinjlim \underline{\operatorname{Ext}}^0_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathbb{I}_X)\\ &= \varinjlim H^0((\psi_n)_*\psi_n^!\pi^!\mathbb{I}_S) = \varinjlim \operatorname{Ext}^0_R(A/\mathfrak{M}^n, \mathbb{I}_R)^\sim,\end{aligned}$$

where $\psi_n : \operatorname{Spec} A/\mathfrak{M}^n \to \operatorname{Spec} A$ is the canonical closed immersion, and $(?)^\sim$ denotes the quasi-coherent sheaf associated to a module. On the other hand, $A/\mathfrak{M}^n$ has finite length as an R-module, so

$$\begin{aligned}\operatorname{Ext}^0_R(A/\mathfrak{M}^n, \mathbb{I}_R) &\cong H^0(\operatorname{Hom}_R(A/\mathfrak{M}^n, \mathbb{I}_R)) \cong H^0(\operatorname{Hom}_R(A/\mathfrak{M}^n, \Gamma_{\mathfrak{m}}\mathbb{I}_R))\\ &\cong H^0(\operatorname{Hom}_R(A/\mathfrak{M}^n, E_R)) = \operatorname{Hom}_R(A/\mathfrak{M}^n, E_R).\end{aligned}$$

We prove that the map $\operatorname{Hom}_R(A/\mathfrak{M}^m, E_R) \to \operatorname{Hom}_R(A/\mathfrak{M}^n, E_R)$ in the inductive system is induced by the projection $A/\mathfrak{M}^n \to A/\mathfrak{M}^m$ for $n \geq m$.

Note that $\underline{\mathrm{Ext}}^0_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^m, \mathbb{I}_X) \to \underline{\mathrm{Ext}}^0_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathbb{I}_X)$ in (7) is induced by the projection. So by the description of the twisted inverse for finite morphisms [7, (27.7)], the map $(\psi_m)_*\psi_m^! \to (\psi_n)_*\psi_n^!$ is induced by the counit map. That is, the map is the composite

$$(\psi_m)_*\psi_m^! \cong (\psi_n)_*(\psi_{n,m})_*\psi_{n,m}^!\psi_n^! \xrightarrow{\varepsilon} (\psi_n)_*\psi_n^!,$$

where $\psi_{n,m} : \operatorname{Spec} A/\mathfrak{M}^m \to \operatorname{Spec} A/\mathfrak{M}^n$ is the map induced by the projection. So again by [7, (27.7)], the map $\mathrm{Ext}^0_R(A/\mathfrak{M}^m, \mathbb{I}_R) \to \mathrm{Ext}^0_R(A/\mathfrak{M}^n, \mathbb{I}_R)$ in (7) is also induced by the projection, and we are done.

Hence

$$E_A = \varinjlim \mathrm{Hom}_R(A/\mathfrak{M}^n, E_R) = A^\vee. \quad \square$$

5.5 Corollary. *Assume that A is Cohen–Macaulay and $\dim A = d$. Set $\Gamma(X, H^{-d}(\mathbb{I}_X))$ to be ω_A. For a A-finite (G,A)-module M, the canonical map*

$$\mathfrak{d} : H^i_{\mathfrak{M}}(M) \to \mathrm{Ext}^{d-i}_A(M, \omega_A)^\vee$$

is an isomorphism of (G,A)-modules. That is, this isomorphism preserves grading and H-action.

5.6 Remark. Assume that $R = k$ is a field. Let $\mathcal{G}$ be the full subcategory of (G,A)-modules consisting of M such that M_i is finite dimensional for every i. Then we define $M^\vee = \bigoplus_{i\in\mathbb{Z}} M^\dagger_{-i}$ for $M \in \mathcal{G}$, where $M^\dagger_{-i} = \mathrm{Hom}_k(M_{-i}, k)$. We have an isomorphism $\Phi : {}^*\mathrm{Hom}_A(M, A^\vee) \to M^\vee$. See for the notation ${}^*\mathrm{Hom}_A$, [1]. Note that

$$\begin{aligned}\Phi_n : {}^*\mathrm{Hom}_A(M, A^\vee)_n = {}^*\mathrm{Hom}_A(M(-n), A^\vee)_0 \\ \to \mathrm{Hom}_k(M_{-n}, A^\vee_0) = \mathrm{Hom}_k(M_{-n}, k)\end{aligned}$$

is given by the restriction. It is easy to see that $(?)^\vee$ is an anti-equivalence from $\mathcal{G}$ to itself. This also gives an anti-equivalence between the category of noetherian (G,A)-modules to that of artinian (G,A)-modules. This is not contained in Theorem 4.17, which treats only objects of finite length.

5.7 Example. Let k be an algebraically closed field of characteristic two, and we set $R = k$ and $S = \operatorname{Spec} R$. Let $V = k^2$, and $H = GL(V)$. Let $A = \operatorname{Sym} V$, and $X = V^* = \operatorname{Spec} A$. Then A_2^* is not injective as a G-module. So $E_A = \bigoplus_{i\geq 0} A_i^*$ is not injective as a G-module either. So E_A is not injective as a (G,A)-module either by [6, Corollary II.1.1.9]. In particular, E_A is not the injective hull of $A/\mathfrak{M}$ as a (G,A)-module.

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