Classification of the linearly reductive finite subgroup schemes of SL_2

Mitsuyasu Hashimoto

Department of Mathematics, Okayama University
Okayama 700–8530, JAPAN
E-mail: mh@okayama-u.ac.jp

Dedicated to Professor Ngo Viet Trung
on the occasion of his sixtieth birthday

Abstract

We classify the linearly reductive finite subgroup schemes G of $SL_2 = SL(V)$ over an algebraically closed field k of positive characteristic, up to conjugation. As a corollary, we prove that such G is in one-to-one correspondence with an isomorphism class of two-dimensional F-rational Gorenstein complete local rings with the coefficient field k by the correspondence $G \mapsto ((\operatorname{Sym} V)^G)\widehat{\ }$.

1. Introduction

The classification of the finite subgroups of $SL_2(\mathbb{C})$ is well-known ([Dor, Section 26], [LW, (6.2)], see Theorem 3.2), and such a group corresponds to a Dynkin diagram of type A, D, or E. A two-dimensional singularity is Gorenstein and rational if and only if it is a quotient singularity by a finite subgroup of $SL_2(\mathbb{C})$, and such singularities (also called Kleinian singularities) are classified via these subgroups, see [Dur]. Indeed, a two-dimensional singularity

2010 *Mathematics Subject Classification.* Primary 14L15; Secondary 13A50. Key Words and Phrases. group scheme, Kleinian singularity, invariant theory,

is Gorenstein and rational if and only if it is a quotient singularity by a finite subgroup of SL_2.

It is known that the F-rationality is the characteristic p version of the rational singularity. More precisely, a finite-type algebra over a field of characteristic zero has rational singularities if and only if its modulo p reduction is F-rational for almost all prime numbers p [Sm], [Har]. The two-dimensional complete local F-rational Gorenstein rings over an algebraically closed field k of characteristic $p > 0$ is classified using Dynkin diagrams A, D, and E, based on Artin's classification of rational double points [Art], see [WY], [HL]. Then we might well ask whether such a ring is obtained as an invariant subring $k[[x, y]]^G$ with G a finite subgroup of $SL_2 = SL(V)$, where $V = kx \oplus ky$.

Before considering this question, we have to consider several things.

First, any finite subgroup of $SL_2(\mathbb{C})$ is small in the sense that it does not have a pseudo-reflection, where an element g of $GL(V)$ is called a pseudo-reflection if $\operatorname{rank}(g - 1_V) = 1$. This is important in studying the ring of invariants. If G is a small finite subgroup of $GL(V)$ ($V = \mathbb{C}^2$), then G can be recovered from $\hat{R} = \hat{S}^G$, where $\hat{S}$ is the completion of $S = \operatorname{Sym} V$, in the sense that the fundamental group of $\operatorname{Spec} \hat{R} \setminus \{0\}$ is G, where 0 is the unique closed point. Moreover, the category of maximal Cohen–Macaulay modules of $\hat{R}$ is canonically equivalent to the category of $\hat{S}$-finite $\hat{S}$-free $(G, \hat{S})$-modules [Yos, (10.9)]. However, this is not the case for $SL_2(k)$ with $\operatorname{char}(k) > 0$. Indeed, a finite subgroup of $SL_2(k)$ may have a transvection, where $g \in GL(V)$ is called a transvection if it is a pseudo-reflection and $g - 1_V$ is nilpotent. Even if G is a non-trivial subgroup of SL_2, $\hat{S}^G$ may be a formal power series ring again, see [KS, Proposition 4.6].

Next, even if G is a finite subgroup of $SL(V)$, the ring of invariants $R = (\operatorname{Sym} V)^G$ may not be F-regular. Indeed, Singh [Sin] proved that if G is the alternating group A_n acting canonically on $V = k^n$, then $R = (\operatorname{Sym} V)^G$ is strongly F-regular if and only if $p = \operatorname{char}(k)$ does not divide the order $(n!)/2$ of $G = A_n$. More generally, Yasuda [Yas] proved that if G is a small subgroup of $GL(V)$, then the ring of invariants $(\operatorname{Sym} V)^G$ is strongly F-regular if and only if $p = \operatorname{char}(k)$ does not divide the order of G.

So we want to classify the subgroups $G \subset SL_2$ with the order of G is not divisible by $p = \operatorname{char}(k)$. It is easy to see that such G must be small. The classification is known (see Theorem 3.2), and the result is the same as that over $\mathbb{C}$, except that small p which divides the order $|G|$ of G is not allowed. More precisely, for the type (A_n), p must not divide $n + 1$, for (D_n), p must not divide $4n - 8$, and we must have $p \geq 5$, $p \geq 5$, $p \geq 7$ for type (E_6), (E_7), and (E_8), respectively. However, the restriction on p for the classification of two-dimensional F-rational Gorenstein complete local rings is different [HL], and it is p arbitrary for (A_n), $p \geq 3$ for (D_n), and $p \geq 5$, $p \geq 5$, $p \geq 7$ for type (E_6), (E_7), and (E_8), respectively.

The purpose of this paper is to show the gap occuring on the type (A_n) and (D_n) comes from the non-reduced group schemes, as shown in Theorem 3.8. As a corollary, we show that all the two-dimensional F-rational Gorenstein complete local rings with the algebraically closed coefficient field appear as the ring of invariants under the action of a linearly reductive finite subgroup scheme of SL_2, see Corollary 3.10. This is already pointed out by Artin [Art] for the type (A_n), and is trivial for (E_6), (E_7), and (E_8) because of the order of the group and the restriction on p. What is new in this paper is the case (D_n). At this moment, the author does not know how to recover the group scheme G from $R = S^G$. So although the classification of R, the two-dimensional F-rational Gorenstein singularities are well-known, the classification of G seems to be nontrivial for the author. As a result, we can recover G from R in the sense that the correspondence from G to $\hat{R} = \hat{S}^G$ is one-to-one.

The key to the proof is Sweedler's theorem (Theorem 2.8) which states that a connected linearly reductive group scheme over a field of positive characteristic is abelian.

The author thanks Professor Kei-ichi Watanabe for valuable advice.

2. Preliminaries

(2.1) Let k be a field. For a k-scheme X, we denote the ring $H^0(X, \mathcal{O}_X)$ by $k[X]$.

We say that an affine algebraic k-group scheme G is linearly reductive if any G-module is semisimple.

Lemma 2.2. *Let*

$$1 \to N \to G \to H \to 1$$

be an exact sequence of affine algebraic k-group schemes. Then G is linearly reductive if and only if H and N are linearly reductive.

Proof. We prove the 'if' part. If M is a G-module, then the Lyndon-Hochschild-Serre spectral sequence [Jan, (I.6.6)]

$$E_2^{p,q} = H^p(H, H^q(N, M)) \Rightarrow H^{p+q}(G, M)$$

degenerates, and $E_2^{p,q} = 0$ for $(p,q) \neq (0,0)$ by assumption. Thus $H^n(G,M) = 0$ for $n > 0$, as required.

We prove the 'only if' part. First, given a short exact sequence of H-modules, it is also a short exact sequence of G-modules by restriction. By assumption, it G-splits, and hence it H-splits. Thus any short exact sequence of H-modules H-splits, and H is linearly reductive. Next, we prove that N is linearly reductive. Let M be a finite dimensional N-module. Then there is a spectral sequence

$$E_2^{p,q} = H^p(G, R^q \operatorname{ind}_N^G(M)) \Rightarrow H^{p+q}(N,M),$$

see [Jan, (I.4.5)]. As $G/N \cong H$ is affine, $R^q \operatorname{ind}_N^G(M) = 0$ $(q > 0)$ by [Jan, (I.5.13)]. As G is linearly reductive by assumption, $E_2^{p,q} = 0$ for $(p,q) \neq (0,0)$. Thus $H^n(N,M) = 0$ for $n > 0$, and thus N is linearly reductive. □

(2.3) Let $C = (C, \Delta, \varepsilon)$ be a k-coalgebra. An element $c \in C$ is said to be group-like if $c \neq 0$ and $\Delta(c) = c \otimes c$ [Swe]. If so, $\varepsilon(c) = 1$. The set of group-like elements of C is denoted by $\mathcal{X}(C)$. Note that $\mathcal{X}(C)$ is linearly independent.

Let H be a k-Hopf algebra. Then for $h \in \mathcal{X}(H)$, $\mathcal{S}(h) = h^{-1}$, where $\mathcal{S}$ is the antipode. Note that $\mathcal{X}(H)$ is a subgroup of the unit group $H^\times$. We denote $GL_1 = \operatorname{Spec} k[t, t^{-1}]$ with t group-like by $\mathbb{G}_m$, and its subgroup scheme $\operatorname{Spec} k[t]/(t^r - 1)$ by $\boldsymbol{\mu}_r$ for $r \geq 0$. Note that $\boldsymbol{\mu}_r$ represents the group of the rth roots of unity, but it is not a reduced scheme if $\operatorname{char}(k) = p$ divides r.

(2.4) In the rest of this paper, let k be algebraically closed. For an affine algebraic group scheme G over k, let $\mathcal{X}(G)$ denote the group of characters (one-dimensional representations) of G. Note that $\mathcal{X}(G)$ is canonically identified with $\mathcal{X}(k[G])$, see [Wat, (2.1)].

Lemma 2.5. *Let G be an affine algebraic k-group scheme. Then the following are equivalent.*

1 *G is abelian (that is, the product is commutative) and linearly reductive.*

2 *G is linearly reductive, and any simple G-module is one-dimensional.*

3 *G is diagonalizable. That is, a closed subgroup scheme of a torus $\mathbb{G}_m^n$.*

4 *The coordinate ring $k[G]$ is group-like as a coalgebra. That is, $k[G]$ is the group ring $k\Gamma$, where $\Gamma = \mathcal{X}(G)$.*

5 *G is a finite direct product of $\mathbb{G}_m$ and $\boldsymbol{\mu}_r$ with $r \geq 2$.*

Proof. **1⇒2** Follows easily from [Swe, (8.0.1)].

2⇒3 Take a finite dimensional faithful G-module V (this is possible [Wat, (3.4)]). Take a basis $v_1, \ldots, v_n$ of V such that each kv_i is a one-dimensional G-submodule of V. Then the embedding $G \to GL(V)$ factors through $GL(kv_1) \times \cdots \times GL(kv_n) \cong \mathbb{G}_m^n$.

3⇒4 Let $G \subset \mathbb{G}_m^n = T$. Then $k[T]$ is a Laurent polynomial ring $k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. As each Laurent monomial $t_1^{\lambda_1} \cdots t_n^{\lambda_n}$ is group-like, $k[T]$ is generated by its group-like elements. This property is obviously inherited by its quotient Hopf algebra $k[G]$, and we are done.

4⇒5 Apply the fundamental theorem of abelian groups on Γ.

5⇒4⇒2⇒1 is easy. □

(2.6) The category of finitely generated abelian groups and the category of diagonalizable k-group schemes are contravariantly equivalent with the equivalences $\Gamma \mapsto \operatorname{Spec}(k\Gamma)$ and $G \mapsto \mathcal{X}(G)$. For a diagonalizable k-group scheme G, a G-module is identified with an $\mathcal{X}(G)$-graded k-vector space. A G-algebra is nothing but a $\mathcal{X}(G)$-graded k-algebra.

(2.7) For a diagonalizable group scheme $G = \operatorname{Spec} k\Gamma$, the closed subgroup schemes H of G is in one-to-one correspondence with the quotient groups M of Γ with the correspondence $H \mapsto \mathcal{X}(H)$ and $M \mapsto \operatorname{Spec} kM$. In particular, the only closed subgroup schemes of $\mathbb{G}_m$ is $\boldsymbol{\mu}_r$ with $r \geq 0$, since the only quotient groups of $\mathbb{Z}$ are $\mathbb{Z}/r\mathbb{Z}$.

The following is due to Sweedler [Swe2].

Theorem 2.8. *Let G be a connected linearly reductive affine algebraic k-group scheme over an algebraically closed field of positive characteristic p. Then G is an abelian group (and hence is diagonalizable). So G is, up to isomorphisms, of the form*

$$\mathbb{G}_m^r \times \boldsymbol{\mu}_{p^{e_1}} \times \cdots \times \boldsymbol{\mu}_{p^{e_s}}$$

for some $r \geq 0$, $s \geq 0$, and $e_1 \geq \cdots e_s \geq 1$.

(2.9) Let G be an affine algebraic k-group scheme. Note that $\operatorname{Spec} k$, G_{red} and $G_{\mathrm{red}} \times G_{\mathrm{red}}$ are all reduced. Hence the unit map $e : \operatorname{Spec} k \to G$, the inverse $\iota : G_{\mathrm{red}} \to G$, and the product $\boldsymbol{\mu} : G_{\mathrm{red}} \times G_{\mathrm{red}} \to G$ all factor through $G_{\mathrm{red}} \hookrightarrow G$, and so G_{red} is a closed subgroup scheme of G. Thus G_{red} is k-smooth.

(2.10) We denote the identity component (the connected component containing the identity element) of G by G°. As $G_{\mathrm{red}} \hookrightarrow G$ is a homeomorphism and G_{red} is k-smooth, each connected component of G is irreducible, and is isomorphic to G°. As $\operatorname{Spec} k$, G°, and $G^\circ \times G^\circ$ are all irreducible, it is easy to see that the unit map, the inverse, the product from them all factor through $G^\circ \hookrightarrow G$, and hence G° is a closed open subgroup of G. If C is any irreducible component of G, then the image of the map $C \times G^\circ \to G$ given by $(g, n) \mapsto gng^{-1}$ is contained in G°. Thus G° is a normal subgroup scheme of G. That is, the map $G \times G^\circ \to G$ given by $(g, n) \mapsto gng^{-1}$ factors through G°.

(2.11) As the inclusion $G^\circ \cdot G_{\mathrm{red}} \hookrightarrow G$ is a surjective open immersion, we have that $G^\circ \cdot G_{\mathrm{red}} = G$. As G° is an open subscheme of G, $G^\circ \cap G_{\mathrm{red}} = G^\circ_{\mathrm{red}}$. So if G is finite, then G is a semidirect product $G = G^\circ \rtimes G_{\mathrm{red}}$.

3. The classification

(3.1) Throughout this section, let k be an algebraically closed field of characteristic $p > 0$.

The purpose of this section is to classify the linearly reductive finite subgroup schemes of SL_2 over k, up to conjugation. Our starting point is the reduced case, which is well-known. Unfortunately, the author does not know the proof of the theorem below exactly as stated, but the proof in [Dor, Section 26] also works for the case of positive characteristic. See also [LW, Chapter 6, Section 2].

Theorem 3.2. *Let k be an algebraically closed field of characteristic $p > 0$, and G a finite nontrivial subgroup of SL_2. Assume that the order $|G|$ of G is not divisible by p. Then G is conjugate to one of the following, where ζ_r denotes a primitive rth root of unity.*

(A_n) $(n \geq 1)$ *The cyclic group generated by*

$$\begin{pmatrix} \zeta_{n+1} & 0 \\ 0 & \zeta_{n+1}^{-1} \end{pmatrix}.$$

(D_n) $(n \geq 4)$ *The binary dihedral group generated by* (A_{2n-5}) *and*

$$\begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{pmatrix}.$$

(E_6) *The binary tetrahedral group generated by* (D_4) *and*

$$\frac{1}{\sqrt{2}}\begin{pmatrix}\zeta_8^7 & \zeta_8^7\\ \zeta_8^5 & \zeta_8\end{pmatrix}.$$

(E_7) *The binary octahedral group generated by* (E_6) *and* (A_7).

(E_8) *The binary icosahedral group generated by* (A_9),

$$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad \textit{and} \quad \frac{1}{\zeta_5^2-\zeta_5^3}\begin{pmatrix}\zeta_5+\zeta_5^{-1} & 1\\ 1 & -(\zeta_5+\zeta_5^{-1})\end{pmatrix}.$$

Conversely, if $g = n+1$ (*resp.* $4n-8$, 24, 48, *and* 120) *is not zero in* k, *then* (A_n) (*resp.* (D_n), (E_6), (E_7), *and* (E_8)) *above is defined, and is a linearly reductive finite subgroup of* SL_2 *of order* g.

(3.3) Let G be a linearly reductive finite subgroup scheme of $SL_2 = SL(V)$. As the sequence

$$1 \to G^\circ \to G \to G_{\mathrm{red}} \to 1$$

is exact, both G° and G_{red} are linearly reductive by Lemma 2.2.

(3.4) First, consider the case that G is abelian. Then the vector representation V is the direct sum of two one-dimensional G-modules, say V_1 and V_2, and hence we may assume that G is diagonalized. As $G \subset SL_2$, $V_2 \cong V_1^*$. Thus $G \to GL(V_1) = \mathbb{G}_m$ is also a closed immersion, and $G \cong \boldsymbol{\mu}_m$ is

$$\left\{\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix} \mid a \in \boldsymbol{\mu}_m\right\}.$$

(3.5) So assume that G is not abelian. If G° is trivial, then $G = G_{\mathrm{red}}$, and the classification for this case is done in Theorem 3.2. So assume further that G° is non-trivial.

G° is diagonalized as above, since G° is linearly reductive and connected (and hence is also abelian by Theorem 2.8). We have $G^\circ \cong \boldsymbol{\mu}_r$ with $r = p^e$ for some $e \geq 0$.

(3.6) We consider the case that G° is contained in the group of scalar matrices. In this case, $r = 2$ (so $p = 2$), as $G \subset SL_2$. Then by Maschke's theorem, the order of G_{red} is odd. According to the classification in Theorem 3.2, G_{red} must be of type (A_n) and is cyclic. This shows that G is abelian, and this is a contradiction.

(3.7) So G° is not contained in the group of scalar matrices. Note that if $a, b, c, d \in k$ with $ad - bc = 1$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \zeta_r & 0 \\ 0 & \zeta_r^{-1} \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for some $\lambda, \mu \in A = k[T, T^{-1}]/(T^r - 1)$, where ζ_r is the image of T in A, then (1) $\lambda = \zeta_r$, $\mu = \zeta_r^{-1}$ and $b = c = 0$, or (2) $\lambda = \zeta_r^{-1}$, $\mu = \zeta_r$ and $a = d = 0$. This is because $\zeta_r \neq \zeta_r^{-1}$. Then it is easy to see that the centralizer $C = Z_G(G^\circ)$ is contained in the subgroup of diagonal matrices in SL_2. As we assume that G is not abelian, $C \neq N_G(G^\circ) = G$. Clearly, C_{red} has index two in G_{red}. This shows that the order of G_{red} is divided by 2. By Maschke's theorem, $p \neq 2$. There exists some matrix

$$\begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}$$

in G_{red} for some $b \in k^\times$. After taking conjugate by

$$\begin{pmatrix} b^{-1/2}\zeta_8 & 0 \\ 0 & b^{1/2}\zeta_8^{-1} \end{pmatrix},$$

we obtain the group scheme of type (D_n) below (see Theorem 3.8) for appropriate n.

In conclusion, we have the following.

Theorem 3.8. *Let k be an algebraically closed field of arbitrary characteristic p (so p is a prime number, or ∞). Let G be a linearly reductive finite subgroup scheme of SL_2. Then, up to conjugation, G agrees with one of the following, where ζ_r denotes a primitive rth root of unity.*

(A_n) $(n \geq 1)$ *The group scheme $\boldsymbol{\mu}_{n+1}$ lying in SL_2 as*

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} . \mid a \in \boldsymbol{\mu}_{n+1} \right\}.$$

(D_n) $(n \geq 4)$ $p \geq 3$. *The subgroup scheme generated by (A_{2n-5}) and*

$$\begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{pmatrix}.$$

(E_6) $p \geq 5$. *The binary tetrahedral group generated by* (D_4) *and*

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \zeta_8^7 & \zeta_8^7 \\ \zeta_8^5 & \zeta_8 \end{pmatrix}.$$

(E_7) $p \geq 5$. *The binary octahedral group generated by* (E_6) *and* (A_7).

(E_8) $p \geq 7$. *The binary icosahedral group generated by* (A_9),

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \textit{and} \quad \frac{1}{\zeta_5^2 - \zeta_5^3}\begin{pmatrix} \zeta_5 + \zeta_5^{-1} & 1 \\ 1 & -(\zeta_5 + \zeta_5^{-1}) \end{pmatrix}.$$

Conversely, any of above is a linearly reductive finite subgroup scheme of SL_2, *and a different type gives a non-isomorphic group scheme.*

(3.9) For a finite k-group scheme G over k, we define $|G| := \dim_k k[G]$. Then in the theorem, $|G|$ is $n+1$ for (A_n), $4n-8$ for (D_n), and 24, 48, and 120 for (E_6), (E_7), and (E_8), respectively. This is independent of p, and hence is the same as the case for $p = \infty$.

Corollary 3.10. *Let* k *be an algebraically closed field of positive characteristic. Let* $\hat{R}$ *be a two-dimensional* F*-rational Gorenstein complete local ring with the coefficient field* k*. Then there is a linerly reductive finite subgroup scheme* G *of* $SL_2 = SL(V)$, *where* $V = k^2$, *such that the completion of* $(\operatorname{Sym} V)^G$ *with respect to the irrelevant maximal ideal is isomorphic to* $\hat{R}$. *Conversely, if* G *is such a group scheme, then the completion of* $(\operatorname{Sym} V)^G$ *is a two-dimensional* F*-rational Gorenstein complete local ring with the coefficient field* k.

Proof. This follows from the theorem and the list in [HL, Example 18].

Let u, v be the standard basis of $V = k^2$ and G be as in the list of the theorem. Let $S = k[u, v]$ and $R = S^G$.

The case that $G = (A_n)$. Then a G-algebra is nothing but a $\mathcal{X}(G) = \mathbb{Z}/(n+1)\mathbb{Z}$-graded k-algebra. S is a G-algebra with $\deg u = 1$ and $\deg v = -1$, and $R = S_0$, the degree 0 component with respect to this grading. Set $x = u^{n+1}$, $y = -v^{n+1}$ and $z = uv$. Then it is easy to see that $R = k[x, y, z]$. Obviously, it is a quotient of $R_1 = k[X, Y, Z]/(XY + Z^{n+1})$. As R_1 is a normal domain of dimension two, $R_1 = R$. So $\hat{R}$ is of type (A_n).

The case that $G = (D_n)$. Set $x = uv(u^{2n-4} - (-1)^{n-2}v^{2n-4})$, $y = -2^{2/(n-1)}u^2v^2$ and $z = 2^{-1/(n-1)}(u^{2n-4} + (-1)^{n-2}v^{2n-4})$. Then

$$k[x,y,z] \subset R = S^G = (S^{G'})^{G/G'} \subset k[u^{2n-4}, uv, v^{2n-4}] = S^{G'} \subset S,$$

where G' is the group scheme of type (A_{2n-5}). Note that $k[x,y,z]$ is a quotient of $R_1 = k[x,y,z] = k[X,Y,Z]/(X^2+YZ^2+Y^{n-1})$. As R_1 is a two-dimensional normal domain, $R_1 \to k[x,y,z]$ is an isomorphism, and hence $k[x,y,z]$ is normal. It is easy to see that $Q(S^{G'}) = k(x,y,z,uv)$ and $[k(x,y,z,uv) : k(x,y,z)] \leq 2$. As $|G/G'| = 2$, $R_1 = k[x,y,z] \to R$ is finite and birational. As R_1 is normal, $R_1 = R$. Thus $\hat{R}$ is of type (D_n).

The cases of constant groups $G = (E_6), (E_7), (E_8)$ are well-known [LW], and we omit the proof. □

Remark 3.11. Note that the converse in the corollary is also checked theoretically. As G is linearly reductive, $R = (\operatorname{Sym} V)^G$ is a direct summand subring of $S = \operatorname{Sym} V$, and hence is strongly F-regular. Thus its completion is also strongly F-regular, see for example, [Has2, (3.28)]. Gorenstein property of R is a consequence of [Has, (32.4)].

Nevertheless, at this moment, the author does not know a theoretical reason why G can be recovered from the isomorphism class of $\hat{R}$ (this is true, as can be seen from the result of the classification).

Remark 3.12. Let V and G be as above. Set $S := (\operatorname{Sym} V)^G$, and let $\hat{S}$ be its completion with respect to the irrelevant maximal ideal so that $\hat{S} \cong k[[x,y]]$. As G° is infinitesimal, $\hat{S}^{G^\circ} \to \hat{S}$ is purely inseparable. So $\operatorname{Spec} \hat{S}^{G^\circ} \setminus 0$ is simply connected. As $\operatorname{Spec} \hat{S}^{G^\circ} \setminus 0 \to \operatorname{Spec} \hat{S}^G \setminus 0$ is a Galois covering of the Galois group $G/G^\circ = G_{\mathrm{red}}$, the fundamental group of $\operatorname{Spec} \hat{S}^G \setminus 0$ is G_{red}, which is linearly reductive, as stated in [Art].

References

[Art] M. Artin, Coverings of rational double points in characteristic p, in *Complex Analysis and Algebraic Geometry, dedicated to K. Kodaira,* W. L. Baily and T. Shioda (eds.), Cambridge (1977), pp. 11–22.

[Dor] L. Dornhoff, *Group representation theory. Part A: Ordinary representation theory,* Dekker (1971).

[Dur] A. Durfee, Fifteen characterizations of rational double points and simple critical points, *L'Enseignement Math.* **25** (1979), 131–163.

[Har] N. Hara, A characterization of rational singularities in terms of injectivity of Frobenius maps, *Amer. J. Math.* **120** (1998), 981–996.

[Has] M. Hashimoto, Equivariant twisted inverses, *Foundations of Grothendieck Duality for Diagrams of Schemes* (J. Lipman, M. Hashimoto), Lecture Notes in Math. **1960**, Springer (2009), pp. 261–478.

[Has2] M. Hashimoto, F-pure homomorphisms, strong F-regularity, and F-injectivity, *Comm. Algebra* **38** (2010), 4569–4596.

[HL] C. Huneke and G. J. Leuschke, Two theorems about maximal Cohen–Macaulay modules, *Math. Ann.* (2002), 391–404.

[LW] G. J. Leuschke and R. Wiegand, *Cohen–Macaulay Representations,* AMS (2012).

[Jan] J. C. Jantzen, *Representations of algebraic groups,* Second edition, AMS (2003).

[KS] D. B. Karagueuzian and P. Symonds, The module structure of a group action on a polynomial ring: A finiteness theorem, *J. Amer. Math. Soc.* **20** (2007), 931–967.

[Sin] A. K. Singh, Failure of F-purity and F-regularity in certain rings of invariants, *Illinois J. Math.* **42** (1998), 441–448.

[Sm] K. E. Smith, F-rational rings have rational singularities, *Amer. J. Math.* **119** (1997), 159–180.

[Swe] M. Sweedler, *Hopf Algebras,* Benjamin (1969).

[Swe2] M. E. Sweedler, Connected fully reducible affine group schemes in positive characteristic are Abelian, *J. Math. Kyoto Univ.* **11** (1971), 51–70.

[Wat] W. C. Waterhouse, *Introduction to Affine Group Schemes,* Springer (1979).

[Yas] T. Yasuda, Pure subrings of regular local rings, endomorphism rings and Frobenius morphisms, *J. Algebra* **370** (2012), 15–31.

[WY] K.-i. Watanabe and K.-i. Yoshida, Hilbert–Kunz multiplicity and an inequality between multiplicity and colength, *J. Algebra* **230** (2000), 295–317.

[Yos] Y. Yoshino, *Cohen–Macaulay Modules over Cohen–Macaulay Rings,* Cambridge (1990).