Matijevic–Roberts type theorems for F-singularities

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1. Matijevic–Roberts type theorem

Consider the following statement.

1.1 Statement (Matijevic–Roberts type theorem (MRTT)). Let $\mathcal{C}$ be a class of noetherian local rings. Let R be a noetherian $\mathbb{Z}^n$-graded ring, and P its prime ideal. Let P^* be the prime ideal generated by the all homogeneous elements of P. If $R_{P^*} \in \mathcal{C}$, then $R_P \in \mathcal{C}$.

Clearly, the truth of the statement depends on the choice of $\mathcal{C}$. Nagata conjectured the Matijevic–Roberts type theorem for the case that $\mathcal{C}$ is the class of Cohen–Macaulay local rings, and $n = 1$. Nagata's conjecture was solved affirmatively by Hochster–Ratliff [25] and Matijevic–Roberts [29] independently.

After that, due to the contribution of Aoyama–Goto [1], Avramov–Achilles [2], Cavaliere–Niesi [6], Goto–Watanabe [14], and Matijevic [28], it was proved that the Matijevic–Roberts type theorem is true for the case that $\mathcal{C}$ is the class of Cohen–Macaulay, Gorenstein, complete intersection, and regular local rings, for arbitrary n.

After that, the result was generalized to an assertion for group actions in [17], and then M. Miyazaki and the author [20] proved the following.

1.2 Theorem. *Let S be a scheme, G a smooth S-group scheme of finite type, X a noetherian G-scheme, $y \in X$, $Y := \overline{\{y\}}$, Y^* the smallest G-stable closed subscheme of X containing Y. Let η be the generic point of an irreducible component of Y^*. Let $\mathcal{C}$ and $\mathcal{D}$ be classes of noetherian local rings. Assume:*

1. (Smooth base change) *If $A \to B$ is a regular (i.e., flat with geometrically regular fibers) local homomorphism essentially of finite type and $A \in \mathcal{C}$, then $B \in \mathcal{D}$.*

2. (Flat descent) *If $A \to B$ is a regular local homomorphism essentially of finite type and $B \in \mathcal{D}$, then $A \in \mathcal{D}$.*

If $\mathcal{O}_{X,\eta} \in \mathcal{C}$, then $\mathcal{O}_{X,y} \in \mathcal{D}$.

Considering the case that $S = \operatorname{Spec} \mathbb{Z}$, $G = \mathbb{G}_m^n$, and $X = \operatorname{Spec} R$ is affine, we immediately have the following.

1.3 Corollary. *Let R be a $\mathbb{Z}^n$-graded noetherian ring, and $P \in \operatorname{Spec} R$. Let $\mathcal{C}$ and $\mathcal{D}$ be classes of noetherian local rings which satisfy 1 and 2 in the theorem. If $R_{P^*} \in \mathcal{C}$, then $R_P \in \mathcal{D}$.*

When we let $\mathcal{C} = \mathcal{D}$ be the class of Cohen–Macaulay, Gorenstein, complete intersection, or regular local rings, then the conditions 1 and 2 in the theorem are well-known, and the classical Matijevic–Roberts type theorem for these properties follows from the corollary.

Although the theorem requires some generality on group actions, it is easy to give a proof of the corollary.

Proof of Corollary 1.3. Let $A = R[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ be the Laurent polynomial ring. Let $i : R \hookrightarrow A$ be the inclusion. Let $\varphi : R \to A$ be the ring homomorphism given by $\varphi(x) = xt^\lambda$ for $x \in R_\lambda$, where $t^\lambda = t_1^{\lambda_1} \cdots t_n^{\lambda_n}$ for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$. Let $\rho^+ : A \to A$ and $\rho^- : A \to A$ be the ring homomorphisms given by $\rho^+(xt^\mu) = xt^{\lambda+\mu}$ and $\rho^-(xt^\mu) = xt^{-\lambda+\mu}$ for $x \in R_\lambda$ and $\mu \in \mathbb{Z}^n$.

It is easy to check that

1. ρ^+ and ρ^- are inverse each other.

2. $\varphi = \rho^+ \circ i$.

3. i and φ are smooth.

Let $Q := i(P) \cdot A = P[t_1^{\pm}, \ldots, t_n^{\pm}]$. Letting xt^λ be of degree λ for $x \in R$, φ is degree-preserving. If $x \in R_\lambda$ is homogeneous, then $\varphi(x) = xt^\lambda \in Q$ if and only if $x \in P$. This shows that $\varphi^{-1}(Q) = P^*$. So φ induces $R_{P^*} \to A_Q$, which is regular local, essentially of finite type. By the smooth base change (1 of the theorem), $A_Q \in \mathcal{D}$. As i induces $R_P \to A_Q$, which is regular local, essentially of finite type, $R_P \in \mathcal{D}$, by the flat descent (2 of the theorem). □

The purpose of this survey paper is to introduce Matijevic–Roberts type theorems on F-singularities developed by the author and Miyazaki [20] and the author [19]. We treat (strong, weak) F-regularity, F-rationality, F-purity, and Cohen–Macaulay F-injectivity. As we have already seen, the smooth base change and the flat descent are important.

We also introduce the notion of F-purity of homomorphisms, and discuss some basic properties. In particular, we discuss the flatness of F-pure homomorphisms. It seems that strong F-regularity without F-finite assumption was not studied so much before. We prove the F-pure base change theorem of strong F-regularity.

Finally, we mention the openness of loci of F-singularities. Vélez [36] proved the openness of F-rationality under mild hypothesis, using Γ-construction. We apply the same technique to the strong F-regularity and Cohen–Macaulay F-injectivity. Hoshi proved the openness of the F-pure locus using the same technique.

In section 2, we introduce (weak, strong, very strong) F-regularity and F-rationality. In section 3, we discuss the smooth base change and the flat descent for (weak) F-regularity and F-rationality. In section 4, we treat strong F-regularity, F-purity, and Cohen–Macaulay F-injectivity. F-purity of homomorphisms is introduced in this section.

2. Some F-singularities

From now on, p denotes a prime number, and R denotes a noetherian ring of characteristic p. We set $R^\circ := R \setminus \bigcup_{P \in \operatorname{Min} R} P$.

For $e \geq 0$, let ${}^eR$ denote the R-algebra R with the structure map $F_R^e : R \to {}^eR$, where F_R^e is the eth power of the Frobenius map. For $c \in R$, c viewed as an element of ${}^eR = R$ is denoted by ${}^ec$. So for example, $F_R^e(c) = {}^ec^{p^e}$.

2.1 Definition. For an R-module M and its submodule N, define

$$N_M^* = \operatorname{Cl}_R(N, M) := \{x \in M \mid \exists c \in R^\circ \; \exists e_0 \geq 1 \; \forall e \geq e_0 \quad x \otimes {}^ec \in M/N \otimes_R {}^eR \text{ is zero}\}.$$

We call N_M^* the *tight closure* of N in M. For an ideal I of R, I_R^* is simply denoted by I^*, and called the tight closure of I.

It is easy to see that N_M^* is an R-submodule of M containing N. If $N = N_M^*$, then we say that N is *tightly closed* in M. For an ideal I, we say that I is tightly closed if $I = I^*$.

2.2 Definition. Let A be a commutative ring, and $\varphi : M \to N$ an A-linear map of A-modules. We say that φ is *pure* (or A-pure) if for any A-module W, $1_W \otimes \varphi : W \otimes_A M \to W \otimes_A N$ is injective. A submodule $N \subset M$ is said to be pure if the inclusion map $N \hookrightarrow M$ is A-pure.

2.3 Definition. A ring homomorphism $f : A \to B$ is said to be *pure* if f is pure as an A-linear map. A subring $A \subset B$ is said to be pure if the inclusion map $A \hookrightarrow B$ is pure.

2.4 Definition. We say that R is

1. (cf. [24]) *very strongly F-regular* if for any $c \in R^\circ$, there exists some $e \geq 1$ such that ${}^e cF^e : R \to {}^e R \quad (x \mapsto {}^e(cx^{p^e}))$ is R-pure.
2. (Hochster [21]) *strongly F-regular* if for any R-module M and its submodule N, $N^*_M = N$.
3. (Hochster–Huneke [23]) *F-regular* if R_P is weakly F-regular (see below) for any $P \in \operatorname{Spec} R$.
4. (Hochster–Huneke [23]) *weakly F-regular* if $I^* = I$ for any ideal I of R.
5. (Fedder–Watanabe [13]) *F-rational* if for any ideal I of R such that I is generated by $\operatorname{ht} I$ elements, $I = I^*$.

In [24], very strong F-regularity is simply called "strong F-regularity." As I do not know if the definitions 1 and 2 agree, I give different names to them.

2.5 Definition. We say that R is *F-finite* if ${}^1R$ is a finite R-module.

2.6 Lemma. *The following hold.*

- ([24], [19]) *Very strongly F-regular implies strongly F-regular. Strongly F-regular implies F-regular. F-regular implies weakly F-regular. Weakly F-regular implies F-rational. F-rational implies normal.*
- (*Vélez* [36]) *Excellent F-rational implies Cohen–Macaulay.*
- *F-rational Gorenstein implies strongly F-regular.*
- (*Lyubeznik–Smith* [27]) *For a positively graded finitely generated algebra over a field, weakly F-regular implies strongly F-regular.*

- ([22], [19]) *For a local ring, F-finite ring, and an essentially finite-type algebra over an excellent local ring, strongly F-regular implies very strongly F-regular.*

Some F-singularities are known to be deeply related to singularities in characteristic zero, using the modulo p reduction.

2.7 Definition. Let k be a field of characteristic zero, and X a k-scheme of finite type. We say that X has *rational singularities* if X is normal, and for any (or equivalently, some) resolution of singularities $\pi : Y \to X$, $R^i\pi_*\mathcal{O}_Y = 0$ for $i > 0$.

2.8 Definition. Let X be a $\mathbb{Q}$-Gorenstein normal variety over a field of characteristic zero. Let $\pi : Y \to X$ be a resolution such that the exceptional set is a simple normal crossing divisor with the irreducible components $E_1, \ldots, E_r$. Then we can write

$$K_Y = \pi^* K_X + \sum_i a_i E_i.$$

We say that X is *log terminal* (resp. *log canonical*) if $a_i > -1$ (resp. $a_i \geq -1$) for every i.

2.9 Definition. Let k be a field of characteristic zero, and A a k-algebra of finite type. Let $\mathbb{P}$ be a property of finitely generated algebras over finite fields. We say that A has open (resp. dense) $\mathbb{P}$ type if there exists some finitely generated $\mathbb{Z}$-subalgebra B of k and a finitely generated B-algebra A_B such that $k \otimes_B A_B \cong A$, and there exists some dense open subset U (resp. dense set D of closed points) of $\operatorname{Spec} B$ such that for any closed point x of U (resp. any point x of D), $\kappa(x) \otimes_B A_B$ satisfies $\mathbb{P}$.

2.10 Theorem (Smith [34], Hara [15], Mehta–Srinivas [30]). *Let k be a field of characteristic zero, and A a k-algebra of finite type. Then* $\operatorname{Spec} A$ *has rational singularities if and only if A has open F-rational type.*

2.11 Theorem (K.-i. Watanabe–Hara [16], [15], Smith [35]). *Let k be a field of characteristic zero, and A a k-algebra of finite type. Assume that A is a $\mathbb{Q}$-Gorenstein normal domain. Namely, A is normal, and the canonical divisor of* $\operatorname{Spec} A$ *is $\mathbb{Q}$-Cartier. Then A has log-terminal singularities if and only if A is of open F-regular type.*

Thus F-rationality and $\mathbb{Q}$-Gorenstein F-regular properties are related to rational and log-terminal singularities, respectively.

3. Smooth base change and flat descent for F-singularities

The smooth base change for the (weak) F-regularity was proved by Hochster and Huneke [24].

3.1 Theorem. *Let $R \to S$ be a regular local homomorphism between noetherian local rings of characteristic p. If R is weakly F-regular and S is excellent, then S is weakly F-regular.*

3.2 Corollary. *Let $R \to S$ be a regular homomorphism between noetherian rings of characteristic p. If R is F-regular and S is locally excellent, then S is F-regular.*

Next we consider the F-rationality. As for the characteristic zero counterpart, the rational singularity, the following is known.

3.3 Theorem (Elkik [9]). *Let k be a field of characteristic zero, and $f : X \to Y$ a flat k-morphism between k-schemes of finite type. If Y has rational singularities and f has fibers with rational singularities, then X has rational singularities.*

The smooth base change for F-rationality was proved by Vélez [36].

3.4 Theorem (Vélez). *Let $R \to S$ be a regular homomorphism between locally excellent noetherian rings of characteristic p. If R is F-rational, then S is F-rational.*

After that, Aberbach and Enescu [3] proved the following (see also the weaker results in [10] and [18]).

3.5 Theorem. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism between noetherian local rings of characteristic p. Assume*

1. *R is Cohen–Macaulay F-rational.*
2. *S is excellent.*
3. *For any minimal prime P of R, S_P is F-rational.*
4. *$S/\mathfrak{m}S$ is Cohen–Macaulay and geometrically F-injective over $R/\mathfrak{m}$.*

Then S is F-rational.

Next we consider the flat descent. The following is proved in [19].

3.6 Lemma. *Let $\varphi : A \to B$ be a homomorphism of rings. Assume that φ is cyclically pure. That is, $IS \cap R = I$ for any ideal I of R. If B is noetherian, of characteristic p, and is weakly F-regular (resp. F-regular, strongly F-regular, very strongly F-regular, normal), then so is A.*

Note that faithfully flat implies pure, and pure implies cyclically pure. Thus the flat descent is true for weak F-regularity, F-regularity, strong F-regularity, and very strong F-regularity.

The flat descent for (Cohen–Macaulay) F-rationality is also true.

3.7 Lemma. *Let $A \to B$ be a faithfully flat homomorphism of rings. If B is noetherian, characteristic p, Cohen–Macaulay and F-rational, then so is A.*

Because of [24, (4.2)], we may assume that $A \to B$ is a local homomorphism of local rings in order to prove the lemma. Let I be an ideal of A generated by a regular sequence. Then IB is so, and hence $(IB)^* = IB$ by the F-rationality. By the faithful flatness, it is easy to see that $I^* \subset I^*B \cap A \subset (IB)^* \cap A = IB \cap A = I$.

In characteristic zero, the following holds.

3.8 Theorem (Boutot [5]). *Let k be a field of characteristic zero, B a k-algebra essentially of finite type, A a pure k-subalgebra of B which is essentially of finite type over k. If* $\operatorname{Spec} B$ *has rational singularities, then so does* $\operatorname{Spec} A$.

Nevertheless, F-rationality is not inherited by a pure subring in general, as was shown by an example by K.-i. Watanabe [37].

As we have seen, smooth base change of (weak) F-regularity and F-rationality holds. The flat descent is also true, and thus we have

3.9 Theorem ([20]). *Let R be a $\mathbb{Z}^n$-graded locally excellent noetherian ring of characteristic p, and $P \in \operatorname{Spec} R$. If R_{P^*} is weakly F-regular* (*resp. F-regular, F-rational*)*, then R_P is so.*

We give some applications of the theorem. Before that, we need some results on localizations.

3.10 Theorem ([24], [20]). *Let R be a noetherian ring of characteristic p, and S its multiplicatively closed subset. If R is Cohen–Macaulay F-rational* (*resp. F-regular, strongly F-regular, very strongly F-regular*)*, then so is R_S.*

But it is not known if weak F-regularity localizes (if so, then weak F-regularity is equivalent to F-regularity by definition).

3.11 Theorem ([23], [24], [20]). *Let R be a noetherian ring of characteristic p, If $R_{\mathfrak{m}}$ is Cohen–Macaulay F-rational (resp. weakly F-regular, F-regular, strongly F-regular) for any maximal ideal $\mathfrak{m}$ of R, then so is R.*

But it is not known if the similar statement holds for very strong F-regularity (if so, then strong F-regularity implies very strong F-regularity).

We also need the following result on deformation.

3.12 Theorem (Hochster–Huneke [24]). *Let $(R, \mathfrak{m})$ be a noetherian local ring of characteristic p, and $t \in \mathfrak{m}$ a nonzerodivisor. If R/tR is Cohen–Macaulay F-rational, then R is Cohen–Macaulay F-rational.*

Here is a corollary of Matijevic–Roberts type theorem for F-rationality.

3.13 Corollary (graded deformation [20]). *Let R be a locally excellent noetherian $\mathbb{N}$-graded ring of characteristic p, and $t \in R_+ := \bigoplus_{i>0} R_i$ a nonzerodivisor. If R/tR is F-rational, then R is F-rational.*

Proof. First, if $\mathfrak{M}$ is a *maximal ideal, then $\mathfrak{M} \supset R_+$, as R is $\mathbb{N}$-graded. So $t \in \mathfrak{M}$. As $R_{\mathfrak{M}}/tR_{\mathfrak{M}}$ is F-rational and $t \in \mathfrak{M}R_{\mathfrak{M}}$ is a nonzerodivisor, $R_{\mathfrak{M}}$ is F-rational. Note that $R_{\mathfrak{M}}$ is also Cohen–Macaulay, since R is locally excellent.

As any graded prime ideal is contained in some *maximal ideal and Cohen–Macaulay F-rational property localizes, $R_{\mathfrak{P}}$ is also F-rational for any graded prime ideal $\mathfrak{P}$.

Now take any prime P. Then R_{P^*} is F-rational, since P^* is a graded prime. So by Matijevic–Roberts type theorem, R_P is F-rational. It is also Cohen–Macaulay by locally excellent assumption. Thus R is F-rational. □

Here is another corollary.

3.14 Corollary. *Let A be a commutative ring of characteristic p, and $(F_t)_{t\geq 0}$ its filtration. That is, each F_i is an additive subgroup of A, $1 \in F_0 \subset F_1 \subset F_2 \subset \cdots$, $F_iF_j \subset F_{i+j}$ for $i, j \geq 0$, and $\bigcup_{i\geq 0} F_i = A$. Set $\mathcal{R} = \bigoplus_{i\geq 0} F_i t^i \subset A[t]$, and $\mathcal{G} = \mathcal{R}/t\mathcal{R}$. If $\mathcal{G}$ is noetherian locally excellent F-rational, then so is A.*

See for the proof, [20].

4. F-purity, strong F-regularity, and Cohen–Macaulay F-injectivity

We consider the Matijevic–Roberts type theorem for F-purity, strong F-regularity, and Cohen–Macaulay F-injectivity.

4.1 Definition. R is said to be *F-pure* if the Frobenius map $F_R : R \to {}^1R$ is pure.

A weakly F-regular ring is F-pure. As for the relationship with the characteristic zero singularities, the following is known.

4.2 Theorem (K.-i. Watanabe [38]). *Let A be a normal $\mathbb{Q}$-Gorenstein finite-type algebra over a field of characteristic zero. If A is of dense F-pure type, then* $\operatorname{Spec} A$ *is log canonical.*

In order to prove the smooth base change of F-purity, it is convenient to introduce the notion of F-purity of homomorphisms.

4.3 Definition. For a homomorphism $f : A \to B$ of commutative rings of characteristic p, we define

$$\Psi_e(f) = \Psi_e(A, B) : B \otimes_A {}^eA \to {}^eB$$

by $\Psi_e(f)(b \otimes {}^ea) = {}^e(b^{p^e}a)$, and call it the eth Radu–André homomorphism or the eth relative Frobenius map.

The following was proved by Radu [31] and André [4]. See also [7].

4.4 Theorem. *Let $f : A \to B$ be a homomorphism of noetherian rings of characteristic p. Then the following are equivalent.*

1. *f is regular.*
2. *$\Psi_e(f)$ is flat for some $e \geq 1$.*
3. *$\Psi_e(f)$ is flat for every $e \geq 1$.*

The absolute case (i.e., the case that $A = \mathbb{F}_p$) is due to Kunz [26].

4.5 Definition. A homomorphism $f : A \to B$ of commutative rings of characteristic p is said to be *F-pure* if $\Psi_e(f)$ is pure for every $e \geq 1$.

By the Radu–André theorem (Theorem 4.4), we immediately have that a regular homomorphism is F-pure.

We list some basic properties of F-pure homomorphisms.

4.6 Lemma. *Let $f : A \to B$ and $g : B \to C$ be homomorphisms between $\mathbb{F}_p$-algebras.*

1. *If f and g are F-pure, then so is gf.*

2. *A is F-pure if and only if the unique map $\mathbb{F}_p \to A$ is F-pure.*

3. *If gf is F-pure and g is pure, then f is F-pure.*

4. *If A is F-pure and f is F-pure, then B is F-pure.*

5. *A pure subring of an F-pure ring is F-pure.*

6. *Let A' be an A-algebra, and $B' = B \otimes_A A'$. If f is F-pure, then the base change $A' \to B'$ is also F-pure.*

7. *If $A \to A'$ is a pure homomorphism and $A' \to B'$ is F-pure, then f is F-pure.*

Thus the smooth base change and the flat descent are true for F-purity. Matijevic–Roberts type theorem is true for F-purity.

When the base ring is a field, F-purity of a homomorphism is described well as follows.

4.7 Lemma. *Let k be a field of characteristic p, and B a k-algebra. Then the following are equivalent.*

1. *$k \to B$ is F-pure, and B is noetherian.*

2. *For any $e > 0$, $B \otimes_k {}^e k$ is noetherian and F-pure.*

3. *There exists some $e > 0$ such that $B \otimes_k {}^e k$ is noetherian and F-pure.*

4. *B is noetherian, and B is geometrically F-pure over k, that is, for any finite algebraic extension L of k, $B \otimes_k L$ is F-pure.*

4.8 Remark. Thus an F-pure homomorphism has geometrically F-pure fibers. But Singh's example shows that even a flat homomorphism with geometrically F-pure fibers may not be F-pure.

4.9 Example (Singh [33]). There is a flat local homomorphism $f : A \to B$ essentially of finite type with A a DVR, f has geometrically F-regular fibers, but B is not F-pure.

It is natural to ask if an F-pure map is flat.

4.10 Definition. A homomorphism $f : A \to B$ of rings of characteristic p is said to be *Dumitrescu* if $\Psi_1(f) : B \otimes_A {}^1A \to {}^1B$ is ${}^1A$-pure.

4.11 Theorem (Dumitrescu [8]). *For a flat homomorphism $f : A \to B$ of noetherian rings of characteristic p, the following are equivalent.*

1. *f is Dumitrescu.*

2. *f is reduced.*

By definition, a pure homomorphism is Dumitrescu. We ask if a Dumitrescu map is flat. The following is relatively easy to prove.

4.12 Theorem. *Let $f : A \to B$ be a homomorphism of noetherian rings of characteristic p. If f is Dumitrescu and the image of $\operatorname{Spec} B \to \operatorname{Spec} A$ contains all maximal ideals of A, then f is pure.*

4.13 Corollary. *A Dumitrescu local homomorphism between noetherian local rings of characteristic p is pure. In particular, an F-pure local homomorphism is pure.*

For a homomorphism with finite fibers, Dumitrescu homomorphism is flat. Namely,

4.14 Theorem. *Let $f : A \to B$ be a homomorphism between noetherian rings of characteristic p. Assume that the fiber $B \otimes_A \kappa(P)$ is finite over $\kappa(P)$ for any $P \in \operatorname{Spec} A$. Then the following are equivalent.*

1. *f is F-pure.*

2. *f is Dumitrescu.*

3. *f is regular.*

4.15 Remark. The case that B is a domain and f is finite is due to K.-i. Watanabe.

Here is another sufficient condition for a Dumitrescu homomorphism flat.

4.16 Theorem. *Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a Dumitrescu local homomorphism between noetherian local rings of characteristic p. If $t \in \mathfrak{m}$, A is normally flat along tA, and $A/tA \to B/tB$ is flat, then f is flat.*

4.17 Corollary. *Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a Dumitrescu local homomorphism between noetherian local rings of characteristic p. If $t \in \mathfrak{m}$ is a nonzerodivisor of A and $A/tA \to B/tB$ is flat, then f is flat.*

4.18 Corollary. *Let* $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ *be a Dumitrescu local homomorphism between noetherian local rings of characteristic* p*. If* A *is regular, then* f *is flat.*

Stronger than the smooth base change, the "F-pure base change" of the strong F-regularity holds.

4.19 Theorem. *Let* $\varphi : A \to B$ *be a homomorphism of noetherian rings of characteristic* p*. Assume that* A *is a strongly* F*-regular domain. Assume that the generic fiber* $Q(A) \otimes_A B$ *is strongly* F*-regular, where* $Q(A)$ *is the field of fractions of* A*. If* φ *is* F*-pure and* B *is locally excellent, then* B *is strongly* F*-regular.*

Thus the smooth base change for strong F-regularity holds. Flat descent also holds, and thus Matijevic–Roberts type theorem for strong F-regularity holds.

Next we consider Cohen–Macaulay F-injectivity.

4.20 Definition. We say that a noetherian local ring of characteristic p, $(R, \mathfrak{m})$ is *F-injective* if for any $i \in \mathbb{N}$, the Frobenius map on the local cohomology $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}({}^1R)$ is injective. A noetherian ring of characteristic p is F-injective if its localizations at all maximal ideals are F-injective.

4.21 Lemma. *The following hold.*

1. (*Fedder* [12]) *An* F*-pure ring is* F*-injective.*
2. *An* F*-rational ring is* F*-injective.*
3. (*Fedder* [12]) *A Gorenstein* F*-injective ring is* F*-pure.*
4. (*Schwede* [32]) *A finite-type algebra over a field of characteristic zero is Du Bois if it is of dense* F*-injective type.*

The following statement, which is stronger than the smooth base change, was proved by Aberbach–Enescu [3]. See also [11].

4.22 Proposition. *Let* $A \to B$ *be a flat homomorphism with Cohen–Macaulay and geometrically* F*-injective fibers. If* A *is Cohen–Macaulay* F*-injective, then* B *is Cohen–Macaulay* F*-injective.*

4.23 Corollary. *Let* $(R, \mathfrak{m})$ *be a noetherian local ring of characteristic* p*, and* $t \in \mathfrak{m}$ *a nonzerodivisor. If* R/tR *is Cohen–Macaulay* F*-injective, then so is* R*.*

Proposition 4.22 shows that the smooth base change holds for Cohen–Macaulay F-injective property. The flat descent is easy, and Matijevic–Roberts type theorem holds for Cohen–Macaulay F-injective property.

So the following holds.

4.24 Theorem. *Let R be a $\mathbb{Z}^n$ noetherian ring of characteristic p, and $P \in \operatorname{Spec} R$. If R_{P^*} is F-pure (resp. excellent and strongly F-regular, Cohen–Macaulay F-injective), then R_P is F-pure (resp. strongly F-regular, Cohen–Macaulay F-injective).*

A localization of a Cohen–Macaulay F-injective ring is Cohen–Macaulay F-injective. Similarly to the F-rational property, we have the following, as corollaries to Matijevic–Roberts type theorem.

4.25 Corollary. *Let R be a $\mathbb{N}$-graded noetherian ring of characteristic p, and $t \in R_+ = \bigoplus_{i>0} R_i$ a nonzerodivisor. If R/tR is Cohen–Macaulay F-injective, then so is R.*

4.26 Corollary. *Let A be a commutative ring, and $(F_i)_{i\geq 0}$ be a filtration of A. Set $\mathcal{R} = \bigoplus_{i\geq 0} F_i t^t \subset A[t]$ and $\mathcal{G} = \mathcal{R}/t\mathcal{R}$. If $\mathcal{G}$ is noetherian and Cohen–Macaulay F-injective, then so is A.*

Finally, we introduce some results on the openness of loci of F-singularities. The following was proved using the technique of the Γ-construction developed by Hochster–Huneke [24].

4.27 Theorem (Vélez [36]). *Let R be a noetherian ring of characteristic p which is of finite type over an excellent local ring. Then the F-rational locus of R is open in* $\operatorname{Spec} R$.

Using the same technique, we have the following.

4.28 Theorem. *Let R be a noetherian ring of characteristic p which is either F-finite or essentially of finite type over an excellent local ring. Then the strongly F-regular locus and the Cohen–Macaulay F-injective locus of R is open in* $\operatorname{Spec} R$.

Recently, M. Hoshi proved the following, using the same technique.

4.29 Theorem. *Let R be a noetherian ring of characteristic p which is either F-finite or essentially of finite type over an excellent local ring. Then the F-pure locus of R is open in* $\operatorname{Spec} R$.

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