Cohen-Macaulay and Gorenstein properties of invariant subrings

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1 Introduction

Let k be an algebraically closed field, and G a reduced affine algebraic k-group such that G° is reductive and G/G° is linearly reductive, where G° denotes the connected component of G which contains the unit element. Let H be an affine algebraic k-group scheme, and S a $G \times H$-algebra of finite type over k, which is an integral domain. We set $A := S^G$, and we denote the corresponding morphism $X := \operatorname{Spec} S \to \operatorname{Spec} A =: Y$ by π. Note that π is an H morphism in a natural way.

Theorem 1 (Hilbert-Nagata-Haboush) *A is of finite type over k. If M is an S-finite (G, S)-module, then M^G is A-finite.*

For this theorem, we refer the reader to [20].

Question 2 Let the notation be as above. Let ω_S and ω_A be the canonical modules of S and A, respectively.

1 When A is Cohen-Macaulay, F-rational (type), or strongly F-regular (type)?

2 When $\omega_S^G \cong \omega_A$ as (H, A)-modules?

3 When A is Gorenstein?

Note that the question **3** is deeply related to **1** and **2**. The ring of invariants A is Gorenstein if and only if A is Cohen-Macaulay and ω_A is rank-one projective as an A-module.

2 Equivariant twisted inverse and canonical sheaves

Here we are assuming that ω_S and ω_A have natural equivariant structures. We briefly mention how these structures are introduced. Here we remark that any scheme in consideration is assumed to be separated.

Let G' be an affine k-group scheme of finite type. Let H be the coordinate ring $k[G']$ of G', and we denote its restricted dual Hopf algebra H° by U, see [1]. Note that any G'-module has a canonical U-module structure, and this gives a fully faithful exact functor $\phi : {}_{G'}\mathbb{M} \to {}_U\mathbb{M}$. See [10, I.4], for example.

Let X be a G'-scheme of finite type over k. We define the category $\mathcal{G}_\mathcal{X}$ by defining $\mathrm{ob}(\mathcal{G}_\mathcal{X})$ to be the set of G'-morphisms $f : Y \to X$ flat of finite type, and defining $\mathcal{G}_\mathcal{X}(\mathcal{Y}, \mathcal{Y}')$ to be the set of flat G'-morphisms from Y to Y' over X. Note that $\mathcal{G}_\mathcal{X}$ is a site with the fppf topology. Then, $\mathcal{O}_\mathcal{X}$ given by $\mathcal{O}_\mathcal{X}(\mathcal{Y}) = -(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ is a sheaf of G'-algebras. A $(U, \mathcal{O}_\mathcal{X})$-module and $(G', \mathcal{O}_\mathcal{X})$-module are defined in an appropriate way [10, II.2], and quasi-coherence and coherence of them are defined. Note that the category of quasi-coherent $(G', \mathcal{O}_\mathcal{X})$-modules $\mathrm{Qco}(G', X)$ is equivalent to the category of G'-linearlized quasi-coherent $\mathcal{O}_\mathcal{X}$-modules in [20], and is embedded in the category of quasi-coherent $(U, \mathcal{O}_\mathcal{X})$-modules $\mathrm{Qco}(U, X)$. Moreover, any quasi-coherent $(U, \mathcal{O}_\mathcal{X})$-module yields a quasi-coherent $\mathcal{O}_\mathcal{X}$-module in the usual Zariski topology (using the descent theory) in a natural way. We have an 'infinitesimally equivariant direct image' $f_* : \mathrm{Qco}(U, X') \to \mathrm{Qco}(U, X)$ for any G'-morphism of finite type, which is compatible with the forgetful functors $F' : \mathrm{Qco}(U, X') \to \mathrm{Qco}(X')$ and $F : \mathrm{Qco}(U, X) \to \mathrm{Qco}(X)$, i.e., $Ff_* \cong f_*F'$.

Let $p : X \to Y$ be a proper G'-morphism, with Y being of finite type over k.

(3) There is an exact left adjoint $\Phi : \mathrm{Qco}(Y) \to \mathrm{Qco}(U, Y)$ of F given by $\Phi(\mathcal{F})(\mathcal{Z}) = \mathcal{U} \otimes_{\|} -(\mathcal{Z}, \mathcal{F})$. Note that we have $\Phi_Y Rp_* = Rp_* \Phi_X$. This shows that $p^!$ is compatible with the forgetful functor: $p^! F = F p^!$, where $p^!$ is the right adjoint of Rp_*, which does exist by Neeman's theorem [23].

(4) If $y \in D^+(\mathrm{Qco}(U, Y))$, then $p^!(y) \in D^+(\mathrm{Qco}(U, X))$.

(5) Let $f : Y' \to Y$ be a flat G'-morphism of finite type. Then, the canonical natural transformation $(f')^* \circ p^! \to (p')^! \circ f^*$ is an isomorphism between the functors $D^+(\mathrm{Qco}(U, Y)) \to D^+(\mathrm{Qco}(U, X'))$, where $f' : X' \to X$ is the base change of f by p, and $p' : X' \to Y'$ is the base change of p by f. This is because of the compatibility with forgetful functors and the result of Verdier [27].

(6) We have that the canonical map

$$Rp_* R\,\underline{\mathrm{Hom}}_{\mathcal{O}_\mathcal{X}}(x, p^! y) \to R\,\underline{\mathrm{Hom}}_{\mathcal{O}_\mathcal{Y}}(Rp_* x, y)$$

is an isomorphism for any $y \in D^+(\mathrm{Qco}(U, Y))$ and any $x \in D^-(\mathrm{Coh}(U, X))$, where $\mathrm{Coh}(U, X)$ denotes the category of coherent $(U, \mathcal{O}_\mathcal{X})$-modules.

(7) If V is an G'-stable open subset of X such that $p|_V$ is smooth of relative dimension n, then $p^!(\mathcal{O}_\mathcal{Y})|_\mathcal{U} \cong \omega_{\mathcal{U}/\mathcal{Y}}[\backslash]$.

(8) If $y \in D^+(\mathrm{Qco}(U, Y))$ and if y lies in the essential image of the canonical functor $D^+(\mathrm{Qco}(G', Y)) \to D^+(\mathrm{Qco}(U, Y))$, then we have $H^i(p^!(y)) \in \mathrm{Qco}(G', X)$ for all $i \in \mathbb{Z}$.

(9) Assume that G-modules are closed under extensions in the category of U-modules. If $y \in D^+(\mathrm{Qco}(U, Y))$ and $H^i(y) \in \mathrm{Qco}(G', Y)$ for $i \in \mathbb{Z}$, then we have $H^i(p^!(y)) \in \mathrm{Qco}(G', X)$ for $i \in \mathbb{Z}$.

Let X be a G'-scheme of finite type over k. We say that X is G'-compactifiable if there is a G'-stable open immersion $i : X \hookrightarrow \bar{X}$ with $p : \bar{X} \to \operatorname{Spec} k$ being proper. Assuming that X is equi-dimensional, we define ω_X to be the lowest (leftmost) cohomology of $i^* p^!(k)$, which is independent of choice of factorization (see [27]). Note that $\omega_X \in \mathrm{Qco}(G', X)$. We call ω_X the (equivariant) canonical sheaf of X. In case $X = \operatorname{Spec} S$ is affine, ω_S is defined to be the global section of ω_X, which is a (G', S)-module. Note that any G'-stable open subset of $\operatorname{Spec} S$ is G'-compactifiable. Thus, ω_S, as an equivariant module, is defined. We remark that, if S is a normal domain of dimension s, then $\omega_S = (\bigwedge^s \Omega_{S/k})^{\star\star}$, where $(?)^\star$ denotes the S-dual $\mathrm{Hom}_S(?, S)$.

3 Known results

Here we list some of known results related to Question 2.

Semisimple group action on a UFD whose unit group is trivial Assume that G is (connected) semisimple, S is factorial, and $S^\times = k^\times$. Then, A is also factorial. Let $0 \neq f \in A$, and $f = f_1 \cdots f_r$ be the prime decomposition of f in S. As G acts on $V(f) \subset X$ and G is geometrically integral, G acts on each component $V(f_i)$. This shows that for each i and $g \in G(k)$, we have $gf_i = \chi_i(g) f_i$ for some $\chi_i(g) \in S^\times = k^\times$. It is easy to see that $\chi_i : G(k) \to k^\times$ is a character. On the other hand, $G(k)$ is perfect, i.e., $[G(k), G(k)] = G(k)$ [15, p.182]. This shows that χ_i is trivial, and $f_i \in A$. In particular, we have that A is factorial. Another consequence is that, we have $Q(S)^G = Q(A)$ under the same assumption, where $Q(?)$ denotes the fraction field.

Linearly reductive group Assume that G is a linearly reductive (i.e., $H^1(G, V) = 0$ for any G-module V) group.

a (Boutot [6]) If $\operatorname{char} k = 0$ and S has rational singularities, then so does A.

b If $\operatorname{char} k = p > 0$ and S is (strongly) F-regular, then so is A.

c (K.-i. Watanabe [30]) Even if $\operatorname{char} k = p > 0$, S is F-rational, A may not be F-rational.

d If $\operatorname{char} k = 0$, $S^\times = k^\times$ and S is factorial with rational singularities, then A is of strongly F-regular type.

For F-regularity and F-rationality, see [16]. The point of **a** and **b** are explained as follows. If G is linearly reductive, then any G-module V is uniquely decomposed into the direct sum of G-submodules $V = V^G \oplus U_V$. The corresponding projection $\phi_V : V \to V^G$ is called the *Reynolds operator*. It is easy to see that $\phi_S : S \to A$ is an A-linear splitting of the inclusion map $A \hookrightarrow S$. Hence, A is a direct summand subring of S. In particular, A is a pure subring of S. The assertions **a** and **b** are theorems for direct summand subrings and pure subrings. The assertion **d** is due to a theorem of N. Hara, a log-terminal singularity in characteristic zero is of strongly F-regular type [9]. Let $G_1 := [G^\circ, G^\circ]$ be the semisimple part of G. Then, by the last paragraph and **a**, we have that S^{G_1} is also factorial with rational singularities, in particular, log-terminal. For sufficiently general modulo p reductions, S^{G_1} is strongly F-regular, and G/G_1 is linearly reductive (as G/G_1 is an extension of a torus by a finite group, we can avoid primes which divides the order of the finite group), and we use **b**.

Finite case Let F be a linearly reductive k-finite group scheme, H an affine algebraic k-group scheme, and $1 \to F \to G' \to H \to 1$ be an exact sequence. Let S be a G'-algebra domain, and we set $A := S^F$. Then, S is module-finite over A, as is well-known. Moreover, A is a direct summand subring of S, as F is linearly reductive.

a If S is Cohen-Macaulay, then so is A.

b If S is F-rational, then so is A.

c (K.-i. Watanabe [28, 29]) $\omega_S^F \cong \omega_A$ as (H, A)-modules.

The statement **a** is trivial, because we have $H^i_{\mathfrak{m}}(A) \cong H^i_{\mathfrak{m}\mathfrak{S}}(S)^F = 0$ for $i \neq d$ and any maximal ideal $\mathfrak{m}$ of A, where $d := \dim S = \dim A$.

The statement **b** is also easy. For any parameter ideal $\mathfrak{q}$ of A, $\mathfrak{q}\mathfrak{S}$ is a parameter ideal of S because $A \hookrightarrow S$ is finite. As A is a pure subring of S, we have

$$\mathfrak{q}^* \subset (\mathfrak{q}\mathfrak{S})^* \cap \mathfrak{A} = \mathfrak{q}\mathfrak{S} \cap \mathfrak{A} = \mathfrak{q},$$

where $(?)^*$ denotes the tight closure.

The statement **c** is proved as follows. Note that $A = S^F$ is a G'-submodule of S because F is a normal subgroup of G'. This induces (H, A)-linear maps

$$\omega_S^F \cong \operatorname{Hom}_A(S, \omega_A)^F \to \operatorname{Hom}_A(S^F, \omega_A) = \omega_A.$$

As F is linearly reductive, the map in the middle must be an isomorphism.

Good linear action A G-module V is called *good* if for any dominant weight λ of G°, $\operatorname{Ext}^1_{G^\circ}(\Delta_{G^\circ}(\lambda), V) = 0$ holds, where $\Delta_{G^\circ}(\lambda)$ denotes the Weyl module of the heighest weight λ. See [17], [10] and references therein for informations on good modules.

Let V be a finite dimensional G-module, and $S := \operatorname{Sym} V$. If S is good and $\operatorname{char}(k) = p > 0$, then A is strongly F-regular. For the proof, see [11].

Torus linear action Let G be a torus, and $S = \operatorname{Sym} V$, with V a finite dimensional G-module. Stanley [24, Theorem 6.7] proved that if for any proper G-submodule $W \subsetneq V$ of V, $A \not\subset \operatorname{Sym} W$ (this is the essential case, because we may replace V by W, if $A \subset \operatorname{Sym} W$), then $\omega_A \cong \omega_S^G$ as A-modules.

Knop's theorem Assume that $\operatorname{char}(k) = 0$, S is factorial, $Q(S)^G = Q(A)$ (where $Q(?)$ denotes the fraction field), and $\operatorname{codim}_X(X - X^{(0)}) \geq 2$, where

$$X^{(0)} := \{x \in X \mid G_x \text{ is finite}\}.$$

Then, $((\omega_S \otimes_k \theta)^G)^{\vee\vee} \cong \omega_A$ as (H, A)-modules, where $\theta := \bigwedge^g \mathfrak{g}$, $\mathfrak{g} := \operatorname{Lie} \mathfrak{G}$, $g := \dim G$, and $(?)^\vee = \operatorname{Hom}_A(?, A)$. If, moreover, S has rational singularities, then $(\omega_S \otimes_k \theta)^G \cong \omega_A$, as (H, A)-modules. For the proof, see [18].

Note that θ is a one-dimensional representation of $G \times H$, on which $G^\circ \times H$ acts trivially. Hence, if G is connected, then we have $\theta \cong k$.

Examples Let $G = \mathbb{G}_m$, $S = k[x_1, \ldots, x_n]$ with $\deg x_i = 1$. Then, we have $\omega_S = S(-n)$. Hence, $\omega_S^G = 0 \neq k = A = \omega_A$. If $n \geq 2$, then we have $Q(S)^G = k(x_i/x_j) \neq k = Q(A)$. If $n = 1$, then $X - X^{(0)} = \{(0)\}$ has codimension one in X.

Next, we consider a less trivial example. Let us consider the case $S = \operatorname{Sym} V$, with V being an n-dimensional G-module. In this case, we have $\omega_S \cong \omega_{S/k} \cong S \otimes \bigwedge^n V$. Hence, the representation $\rho : G \to GL(V)$ factors through $SL(V)$ if and only if $S \cong \omega_S$ as a (G, S)-module. If these conditions are satisfied, then we have $A \cong S^G \cong \omega_S^G$. So assuming that A is Cohen-Macaulay (this is the case, if G is linearly reductive or S is good) and $S \cong \omega_S$, A is Gorenstein if and only if $\omega_S^G \cong \omega_A$ as A-modules. M. Hochster [12] conjectured that if G is linearly reductive, $S = \operatorname{Sym} V$, and $G \to GL(V)$ factors through $SL(V)$, then A is Gorenstein. We have seen that this conjecture is true if G is semisimple or finite. This is also true for the case G being a torus. We may choose a basis $\{x_1, \ldots, x_n\}$ of V so that $k \cdot x_i$ is a G-submodule of V for any i. As $x_1 \cdots x_n \in A$, it is easy to see that $A \not\subset \operatorname{Sym} W$ for any proper G-submodule W of V. Hence, we have $A \cong \omega_S^G \cong \omega_A$ by Stanley's theorem.

However, Hochster's conjecture is not true in general. Here is a counterexample essentially due to Knop (more is true, see [18, Satz 1]). Let $\operatorname{char}(k) = 0$, $W = k^2$, and set $G := SL(W) \times \mathbb{G}_m$. Let $V := W \oplus k^{\oplus 2} \oplus k^{\oplus 4}$, which is an $SL(W)$-module. We assign degree -1 to vectors of W and $k^{\oplus 2}$, and degree 1 to $k^{\oplus 4}$, which makes V a G-module. As $SL(W)$ is semisimple, and the sum of degrees of homogeneous basis elements of V is zero, we have that $G \to GL(V)$ factors through $SL(V)$. However, $A = S^G$ is not Gorenstein. Let x_1, x_2 be a basis of $k^{\oplus 2}$, and y_1, y_2, y_3, y_4 be a basis of $k^{\oplus 4}$. Then, as we have $(\operatorname{Sym} W)^{SL(W)} = k$,

$$S^G \cong ((\operatorname{Sym} W)^{SL(W)} \otimes k[x_1, x_2, y_1, y_2, y_3, y_4])^{\mathbb{G}_m} \\ = k[x_i y_j | 1 \leq i \leq 2,\ 1 \leq j \leq 4] \cong k[x_{ij}]/I_2(x_{ij}),$$

and S^G is not Gorenstein.

4 Knop's theorem in positive characteristic

In this section, we discuss the characteristic p version of Knop's theorem.

Theorem 10 *Let k be an algebraically closed field of characteristic $p > 0$, G a reduced affine algebraic group over k such that G° is reductive and G/G° is linearly reductive. Let H be an affine algebraic k-group scheme. Let S be a $G \times H$-algebra domain which is of finite type over k. We set $X := \operatorname{Spec} S$ and $A := S^G$. Assume*

(α) *S is factorial with $S^\times = k^\times$,*

(β) *$Q(S)^G = Q(A)$,*

(γ) *There exists some $c \geq 1$ such that $\operatorname{codim}_X(X - (X^{(0)} \cap X_c^{(00)})) \geq 2$, where*

$$X^{(0)} := \{x \in X \mid G_x \text{ is finite}\}$$

and

$$X_c^{(00)} := \{x \in X \mid (G_1)_x := [G^\circ, G^\circ]_x \text{ is finite étale over } \kappa(x) \\ \text{and } \dim_{\kappa(x)} \Gamma((G_1)_x, \mathcal{O}_{(\mathcal{G}_\infty)_\S}) = \rfloor\}.$$

Then, we have $((\omega_S \otimes \theta)^G)^{\vee\vee} \cong \omega_A$ as (H, A)-modules, where $\theta := \bigwedge^g \mathfrak{g}$, $\mathfrak{g} = \operatorname{Lie} \mathfrak{G}$, $g := \dim G$, and $(?)^\vee = \operatorname{Hom}_A(?, A)$. If, moreover, G° is semisimple or $S^{[G^\circ, G^\circ]}$ is F-rational, then we have $(\omega_S \otimes \theta)^G \cong \omega_A$ as (H, A)-modules.

The following questions seem to be natural to ask.

Question 11 Assume that S is good and F-rational in the theorem.

1 Is $S^{[G^\circ, G^\circ]}$ F-rational?

2 $X_c^{(00)} \supset X^{(0)}$?

As $[G^\circ, G^\circ]$ is semisimple and we are assuming (α), we have that $S^{[G^\circ, G^\circ]}$ is factorial. Hence, the F-rationality of $S^{[G^\circ, G^\circ]}$ is equivalent to the strong F-regularity of $S^{[G^\circ, G^\circ]}$, see [13].

Corollary 12 *Let G be a (connected) reductive group over a field k of positive characteristic, H an affine algebraic k-group scheme, and V a finite dimensional $G \times H$-module. We set $S := \operatorname{Sym} V$. Assume ($\beta$) and ($\gamma$) in the theorem, and assume also that S is good. Then,*

1 *A is strongly F-regular.*

2 *$\omega_S^G = \omega_A$ as (H, A)-modules.*

3 *If H is reductive and ω_S is $G \times H$-good, then ω_A is good as an H-module.*

4 *If $G \to GL(V)$ factors through $SL(V)$, then A is Gorenstein, and $a(A) = a(S) = -\dim V$, where a denotes the a-invariant.*

Before showing some examples, we briefly review what the conditions β and γ in the theorem mean.

Lemma 13 *Let k be an algebraically closed field, G be a reduced geometrically reductive algebraic group over k, and S an integral domain G-algebra of finite type over k. We set $A := S^G$, and let $\pi : X = \operatorname{Spec} S \to \operatorname{Spec} A = Y$ denote the associated morphism. We define $\Phi : G \times X \to X \times_Y X$ by $\Phi(g, x) = (gx, x)$. Moreover, we set $r := \dim X - \dim Y$, $g := \dim G$, and $s := \max\{dimGx \mid x \in X(k)\}$. Then, we have:*

1 *We have that the extension $Q(S)/Q(S)^G$ is a separable extension.*

2 *The following are equivalent for $x \in X(k)$.*

- **a** *G_x is finite (resp. finite and reduced).*
- **b** *Φ is quasi-finite (resp. unramified) at (g, x) for some $g \in G(k)$.*
- **c** *Φ is quasi-finite (resp. unramified) at (g, x) for any $g \in G(k)$.*

3 *We have $r \geq s$ and $g \geq s$.*

4 *Consider the following conditions.*

- **a** *There exists some non-empty open set U of X such that for any $x \in U(k)$, the orbit Gx is closed in X.*
- **b** *There exists some non-empty open set U of X such that for any $x \in U(k)$, $\overline{Gx} = \pi^{-1}(\pi(x))$, scheme theoretically.*
- **b'** *There exists some non-empty open set U of X such that for any $x \in U(k)$, $\overline{Gx} = \pi^{-1}(\pi(x))$, set theoretically.*
- **c** *$Q(S)^G = Q(A)$.*
- **d** *Φ is dominating (i.e., the image is dense in a topological sense) and there exists some $a \in A$, $a \neq 0$ such that $(S \otimes_A S)[1/a]$ is reduced.*
- **d'** *Φ is dominating.*
- **e** *$r = s$.*
- **f** *The extension $Q(S)^G/Q(A)$ is finite algebraic.*

Then, we have **b**$\Leftrightarrow$**c**$\Leftrightarrow$**d**$\Rightarrow$**b'**$\Leftrightarrow$**d'**$\Rightarrow$**e**$\Leftrightarrow$**f**. *If G is geometrically reductive, then* **a**$\Rightarrow$**b'**. *If S is normal, then we have* **f**$\Rightarrow$**c**.

5 *Assume that S is normal. If two of the following are true, then so is the third.*

- **a** *$Q(S)^G = Q(A)$, or equivalently, $r = s$.*

b $X^{(0)} \neq \emptyset$, *or equivalently,* $s = g$.

c $\dim X = \dim Y + \dim G$, *or equivalently,* $r = g$.

The lemma is more or less well-known, and some part of the lemma is proved in [22].
Proof **1** We use Artin's theorem [4]: Let G be a group, L a field on which G acts. If $e_1, \ldots, e_r$ is a sequence of elements in L which is linearly independent over L^G, then there exists some $g_1, \ldots, g_r$ such that $\det(g_i e_j) \neq 0$. It is easy to show that $Q(S)$ is linearly disjoint from $(Q(S)^G)^{1/p}$. **2** The fiber $\Phi^{-1}(\Phi(g,x)) = \Phi^{-1}(gx, x)$ agrees with $gG_x \times \{x\}$. As G_x is equidimensional, and G_x is either reduced or non-reduced at any point, we are done. **3** $g \geq s$ is obvious. As the dimension $\sigma(x)$ of the stabilizer G_x at $x \in X$ is upper-semicontinuous [20, p.7], s is the dimension of the general orbit. On the other hand, each orbit must be contained in the same fiber of π. This shows $r \geq s$. **4 a⇒b'** follows from the fact that if G is geometrically reductive, then each fiber of π contains exactly one closed orbit, see [20, Corollary A.1.3]. The implication **b⇒b'** is obvious. We show **d⇒b**. There exists some $b \in S \otimes_A S$ such that $b/1$ is a nonzerodivisor in $(S \otimes_A S)[1/a]$, and that $(k[G] \otimes S)[1/ab]$ is faithfully flat over $(S \otimes_A S)[1/ab]$, by generic freeness [14]. By the generic-freeness again, $(S \otimes_A S)/(b)$ is free over some non-empty open subset U of $X = \operatorname{Spec} S$. After replacing U by $U \cap \operatorname{Spec} S[1/a]$, we may assume that U is contained in $\operatorname{Spec} S[1/a]$. Then, for any $x \in U$, we have that as a function over $p_2^{-1}(x) = \pi^{-1}(\pi(x)) \times \{x\}$, b is a nonzerodivisor, because $x \in U$. For $x \in U$, off the locus of $b = 0$, $G \to \pi^{-1}(\pi(x))$ given by $g \mapsto gx$ is faithfully flat by the choice of a, U and b. Thus, after localizing by the nonzerodivizor b, $\pi^{-1}(\pi(x))$ is reduced. This shows $\pi^{-1}(\pi(x))$ is reduced. Another consequence is that, Gx is dense in $\pi^{-1}(\pi(x))$. This shows $\overline{Gx} = \pi^{-1}(\pi(x))$ for $x \in U$, as desired. The proof of **d'⇒b'** is similar and easier. We just take $b \in S \otimes_A S$ so that b is a non-zerodivisor in $(S \otimes_A S)_{\mathrm{red}}$ and $(k[G] \otimes S)[1/b]$ is $(S \otimes_A S)_{\mathrm{red}}[1/b]$-faithfully flat, and do the same trick. We show **b'⇒d'**. Let Z be the non-flat locus of $\pi : X \to Y$, and we set $V := \pi^{-1}(Y - \overline{\pi(Z)})$. As π is dominating and Y is integral, we have that V is a non-empty open set of X, which is obviously G-stable. Replacing U by GU (note that the action $G \times X \to X$ is universally open), we may and shall assume that U is G-stable. Replacing U by $U \cap V$, we may assume that π is flat at any point of U. Let $(u, u') \in (U \times_Y U)(k)$. Then, both Gu and Gu' are dense constructible sets in $\pi^{-1}(\pi(u)) = \pi^{-1}(\pi(u'))$. This shows $Gu \cap Gu' \neq \emptyset$, and $Gu = Gu'$. Namely, we have $(u, u') \in \operatorname{Im} \Phi$. Hence, $\Phi_U : G \times U \to U \times_Y U$ is surjective. This shows that Φ is dominating, set-theoretically. We now show **b⇒d**. We have $\pi^{-1}(\pi(u)) \cap U = Gu$ scheme-theoretically. As we are assuming that π is flat at any point of U, π is smooth at any point of U. Let us take $a \in A$, $a \neq 0$ so that $S[1/a]$ is $A[1/a]$-free. Then, $(S \otimes_A S)[1/a]$ is a subring of $Q(S) \otimes_{Q(A)} Q(S)$. As the field extension $Q(S)/Q(A)$ is separable, we are done. For **c⇒d**, see [22]. Next, we remark that when we invert some element $0 \neq a \in A$ such that $S[1/a]$ is $A[1/a]$-free, then r, s, $Q(A)$ and $Q(S)$ does not change. **d'⇒e** We may assume π is flat. Each component of $X \times_Y X$ is of dimension $\dim X + r$, and each component of $G \times X$ has dimension $\dim X + g$. The generic fiber of Φ has dimension $g - s$, and by assumption, we have $\dim X + r + g - s = \dim X + g$. Namely, $r = s$.

Let us consider the associated k-algebra map $\Phi' : S \otimes_A S \to k[G] \otimes S$ to Φ. When we denote by $\mu' : S \to k[G] \otimes S$ the associated ring homomorphism with the action $\mu : G \times X \to$

X, then we have $\Phi'(f \otimes f') = \mu'(f)(1 \otimes f')$. This induces a map $\Phi'' : L \otimes_{Q(A)} L \to k(G \times X)$, where $L = Q(S)$. It is easy to see that this map induces a map $\phi : L \otimes_{L^G} L \to k(G \times X)$. In fact, for $\alpha \in L^G$ and sufficiently general (g, x), we have $\Phi''(\alpha \otimes 1 - 1 \otimes \alpha)(g, x) = \alpha(gx) - \alpha(x) = 0$. This shows that $\Phi''(\alpha \otimes 1 - 1 \otimes \alpha) = 0$ in $k(X \times G)$, and Φ'' induces ϕ. So **d**$\Rightarrow$**c** is now obvious. Next we show that ϕ is injective. For this purpose, we may assume that $Q(A) = Q(S)^G$, as $Q(S)^G$ is a finitely generated field over k, and we may even assume that S is A-free. Then, the assertion follows from Luna's theorem **c**$\Rightarrow$**d**. Now we know that $Q(Q(S) \otimes_{Q(S)^G} Q(S))$ is the total quotient ring of the image of Φ'. As the generic fiber of Φ has dimension $g - s$, we have that $\operatorname{trans.deg}_{Q(S)^G} Q(S) = s$. On the other hand, we have that $\operatorname{trans.deg}_{Q(A)} Q(S) = r$. Hence, we have **e**$\Leftrightarrow$**f**.

Now assuming that S is normal, we show **f**$\Rightarrow$**c**. Let $\alpha \in Q(S)^G$. Then, by assumption, it is integral over $A[1/a]$, for some $0 \neq a \in A$. As α is integral over $S[1/a]$ and $S[1/a]$ is normal, we have $\alpha \in S[1/a] \cap Q(S)^G = A[1/a] \subset Q(A)$.

The assertion **5** is now obvious. $\square$

5 Examples

Let k be an algebraically closed field of arbitrary characteristic, and $m, n, t \in \mathbb{Z}$ with $2 \leq t \leq m, n$, and $E := k^{t-1}$, $F := k^n$ and $W := k^m$. We define

$$X := \operatorname{Hom}(E, W) \times \operatorname{Hom}(F, E) \xrightarrow{\pi} Y := \{\varphi \in \operatorname{Hom}(F, W) \mid \operatorname{rank} \varphi < t\}$$

by $(f_1, f_2) \mapsto f_1 \circ f_2$. Note that both $X = \operatorname{Spec} S$ and $Y = \operatorname{Spec} A$ are affine, where $S := k[x_{il}, \xi_{lj} \mid 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l < t]$ is the polynomial ring in $(t-1)(m+n)$-variables, and $A := k[y_{ij}]/I_t(y_{ij})$, where y_{ij} are variables, and $I_t(y_{ij})$ denotes the ideal of $k[y_{ij}]$ generated by all t-minors of the $m \times n$-matrix (y_{ij}). The morphism π is given by the k-algebra map $y_{ij} \mapsto \sum_{l=1}^{t-1} x_{il}\xi_{lj}$. We set $G := GL(E)$ and $H := GL(W) \times GL(F)$. The reductive group $G \times H$ acts on X and Y by

$$(g, h_1, h_2)(f_1, f_2) = (h_1 f_1 g^{-1}, g f_2 h_2^{-1}) \quad \text{and} \quad (g, h_1, h_2)\varphi = h_1 \varphi h_2^{-1}.$$

Note that the associated action of $G \times H$ on S is linear, and π is a $G \times H$-morphism.

The following is known.

(14) S is good as a $G \times H$-module.

(15) (De Concini-Procesi [8]) $S^G = A$. Namely, the k-algebra map $A \to S$ given above is injective, and induces an isomorphism $A \cong S^G$.

The assertion (14) follows from Akin-Buchsbaum-Weyman straightening formula (Cauchy formula) [2] and Donkin-Mathieu tensor product theorem [19], see also Boffi [5] and Andersen-Jantzen [3].

We check that this example enjoys the assumption of Corollary 12.

(16) Unless $\operatorname{rank} f_1 < t-1$ and $\operatorname{rank} f_2 < t-1$, we have that the G-orbit of (f_1, f_2) is isomorphic to G. This shows $\operatorname{codim}_X(X - (X^{(0)} \cap X_c^{(00)})) \geq 2$ with $c = 1$.

(17) Unless $\operatorname{rank} f_1 < t-1$ or $\operatorname{rank} f_2 < t-1$, the G-orbit of (f_1, f_2) is closed. By Lemma 13 **4**, we have $Q(S)^G = Q(A)$, as S is normal.

To verify (17), we may assume that

$$(f_1, f_2) = \left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & \\ & & & \end{bmatrix}, \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \end{bmatrix} \right),$$

and in this case, we have

$$(f_1 g^{-1}, g f_2) = \left(\begin{pmatrix} g^{-1} \\ 0 \end{pmatrix}, (g, 0) \right),$$

and the G-orbit is defined by a set of polynomial equations. The assertion (16) is proved similarly.

Now we have the following by Lemma 13 and Corollary 12.

a (Conca-Herzog [7]) A is strongly F-regular (type).

b (Akin-Buchsbaum-Weyman [2]) A is good as an H-module.

c $\omega_S^G \cong \omega_A$ as an (H, A)-module, and hence ω_A is good as an H-module.

d (Svanes [26], Lascoux [21]) If $m = n$, then A is Gorenstein, and $a(A) = a(S) = 2m(t-1)$ in this case.

The fact ω_A is good is proved in [10], and is used to prove the existence of resolution of determinantal ideals of certain type.

Next, we show that the assumption on $X_c^{(00)}$ in Theorem 10 is indispensable.

Example 18 Even if $S = \operatorname{Sym} V$, $Q(S)^G = Q(A)$, $\operatorname{codim}_X(X - X^{(0)}) \geq 2$, G is connected reductive, A is strongly F-regular and $\omega_S \cong S$ (i.e., $G \to GL(V)$ factors through $SL(V)$), A may not be Gorenstein (the assumption $\operatorname{codim}_X(X - X_c^{(00)}) \geq 2$ is missing).

Proof Let k be an algebraically closed field of characteristic $p > 0$. We set $W = k^2$, and $G := SL(W) \times \mathbb{G}_{\gtrdot}$. Giving degree 2, -1, -1 and -1 respectively on the $SL(W)$-modules W, $W^{(1)}$, k and k, we have a G-module structure on $V := W \oplus W^{(1)} \oplus k \oplus k$, where $W^{(1)}$ denotes the first Frobenius twisting of the vector representation W, see [17]. We take a basis x_1, x_2 of W, and we consider that $W^{(1)}$ is the k-span of $y_1 := x_1^p$ and $y_2 := x_2^p$ in $\operatorname{Sym}_p W$. We take a basis s, t of $k \oplus k$ so that x_1, x_2, y_1, y_2, s, t forms a basis of V. As the sum of degrees of these basis elements is zero, we have that the representation $G \to GL(V)$ factors through $SL(V)$. We set $S := \operatorname{Sym} V$. If $w_1^p \neq w_2$ and $(\alpha, \beta) \neq (0,0)$, then the stabilizer of $(w_1, w_2, \alpha, \beta) \in V^* = (\operatorname{Spec} S)(k)$ is finite (but not reduced). In fact, the stabilizer of $(x_1^*, w_2, \alpha, \beta)$ with $w_2 \neq (x_1^*)^p$ and $(\alpha, \beta) \neq (0,0)$ is

$$\begin{bmatrix} 1 & \alpha_p \\ 0 & 1 \end{bmatrix} \times \{1\},$$

where α_p denotes the first Frobenius kernel of the additive group $\mathbb{G}_a$. This shows $\operatorname{codim}_X(X - X^{(0)}) \geq 2$.

Let G_1 be the first Frobenius kernel of $SL(W)$. Then, $(\operatorname{Sym} W)^{G_1} = k[x_1, x_2]^{G_1}$ is contained in the constant ring of the derivations $e = x_2\partial_1$ and $f = x_1\partial_2$. Thus, we have $(\operatorname{Sym} W)^{G_1} \subset k[x_1^p, x_2^p]$. The opposite incidence is obvious, so we have $(\operatorname{Sym} W)^{G_1} = k[x_1^p, x_2^p]$. This shows,

$$\begin{aligned} A := S^G &= ((\operatorname{Sym} W)^{G_1} \otimes \operatorname{Sym}(W^{(1)} \oplus k \oplus k))^{(SL(W))/G_1 \times \mathbb{G}_m} \\ &= k[x_1^p y_2 - x_2^p y_1, s, t]^{\mathbb{G}_m} = k[rs^it^j \mid i + j = 2p - 1], \end{aligned}$$

where $r := x_1^p y_2 - x_2^p y_1$, which is of degree $2p - 1$. Hence, we have $\dim S^G = 2$, and $\dim S^G + \dim G = 2 + 4 = 6 = \dim S$. Hence, we have $Q(S)^G = Q(A)$. As A is a direct summand subring of the regular ring $k[r, s, t]$, A is strongly F-regular. However, by Stanley's theorem, ω_A is generated by $(rs^it^j \mid i + j = 2p - 1, i > 0, j > 0)$, which is not cyclic as an A-module. This shows A is not Gorenstein. □

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