The Picard and the class groups of an invariant subring

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Main theorem

Theorem 1

Let k be a field, G a smooth k-group scheme of finite type, and X a normal variety over k on which G acts. Let $\varphi : X \to Y$ be a G-invariant morphism such that $\mathcal{O}_Y \cong (\varphi_* \mathcal{O}_X)^G$. Then

(1) If $\operatorname{Pic}(X)$ is a finitely generated abelian group, then so is $\operatorname{Pic}(Y)$.

(2) If $\operatorname{Cl}(X)$ is a finitely generated abelian group, then so is $\operatorname{Cl}(Y)$.

If $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} B^G$, and $\varphi : X \to Y$ is the canonical map, then the condition $\mathcal{O}_Y \cong (\varphi_* \mathcal{O}_X)^G$ is satisfied. Results similar to (2) for connected G are proved by Magid and Waterhouse.

The Zariski site $\mathrm{Zar}(X_\bullet)$ (1)

Let I be a small category, and $X_\bullet$ a contravariant functor from I to the category $\mathrm{Sch}\,/S$ of S-schemes. Then we define the site $\mathrm{Zar}(X_\bullet)$ by:

1. $\mathrm{Ob}(\mathrm{Zar}(X_\bullet)) = \{(i, U) \mid i \in \mathrm{Ob}(I),\ \ U \in \mathrm{Ob}(\mathrm{Zar}(X_i))\}$;
2. $$\mathrm{Hom}_{\mathrm{Zar}(X_\bullet)}((j, V), (i, U)) = \{(\phi, h) \mid \phi \in \mathrm{Hom}_I(i, j),$$
$$h \in \mathrm{Hom}_{\mathrm{Sch}\,/S}(V, U), \quad \begin{array}{ccc} V & \xrightarrow{h} & U \\ \downarrow & & \downarrow \\ X_j & \xrightarrow{X_\phi} & X_i \end{array} \text{ commutes}\};$$
3. $\{(\phi_\lambda, h_\lambda) : (i_\lambda, U_\lambda) \to (i, U)\}$ is a covering if $i_\lambda = i$, $\phi_\lambda = \mathrm{id}_i$ for any λ, and $U = \bigcup_\lambda h_\lambda(U_\lambda)$.

The Zariski site $\mathrm{Zar}(X_\bullet)$ (2)

Moreover, the sheaf of commutative rings $\mathcal{O}_{X_\bullet}$ is defined by

4. $\Gamma((i, U), \mathcal{O}_{X_\bullet}) = \Gamma(U, \mathcal{O}_{X_i})$,

and $(\mathrm{Zar}(X_\bullet), \mathcal{O}_{X_\bullet})$ is a ringed site.

Basic operations

As $\mathrm{Zar}(X_\bullet)$ is a ringed site, the tensor product $\otimes_{\mathcal{O}_{X_\bullet}}$ and the internal hom $\underline{\mathrm{Hom}}_{\mathcal{O}_{X_\bullet}}$ are readily defined.

If $f_\bullet : X_\bullet \to Y_\bullet$ is a morphism in the category $\mathrm{Func}(I^{\mathrm{op}}, \mathrm{Sch}\,/S)$ (that is, a natural transformation), then a continuous functor $f_\bullet^{-1} : \mathrm{Zar}(Y_\bullet) \to \mathrm{Zar}(X_\bullet)$ is given by $f_\bullet^{-1}((i, U)) = (i, f_i^{-1}(U))$. From this, the direct and the inverse images $(f_\bullet)_*$ and $f_\bullet^*$ are induced.

Under some Noetherian settings, the twisted inverse pseudo-functor $f_\bullet^!$ and the theory of dualizing complexes and canonical sheaves are obtained, as in the case of single schemes.

The category Δ and Δ^+

Let Ord be the category of ordered sets and order-preserving maps. Let Δ be the full subcategory of Ord with $\mathrm{Ob}(\Delta) = \{[0], [1], [2], \ldots\}$, where $[n] = \{0 < 1 < \cdots < n\}$. Let Δ^+ be the subcategory of Δ such that $\mathrm{Ob}(\Delta^+) = \mathrm{Ob}(\Delta)$ and
$\mathrm{Mor}(\Delta^+) = \{\phi \in \mathrm{Mor}(\Delta) \mid \phi \text{ is an injective map}\}$.
Thus Δ^+ looks like

$$[0] \;\substack{\xrightarrow{\delta_0^0} \\ \xrightarrow{\delta_1^0}}\; [1] \;\substack{\xrightarrow{\delta_0^1} \\ \xrightarrow{\delta_1^1} \\ \xrightarrow{\delta_2^1}}\; [2] \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \cdots,$$

where $\delta_i^n : [n] \to [n+1]$ is the unique injective monotone map such that $i \notin \mathrm{Im}\, \delta_i^n$.

$B_G^+(X)$ (1)

Let G be an S-group scheme acting on X. Then we associate $B_G^+(X) \in \mathrm{Func}((\Delta^+)^{\mathrm{op}}, \mathrm{Sch}/S)$ as

$$B_G^+(X) := X \underset{\xleftarrow{d_0^0}}{\xleftarrow{d_1^0}} G \times X \;\substack{\xleftarrow{d_2^1} \\ \xleftarrow{d_1^1} \\ \xleftarrow{d_0^1}}\; G \times G \times X \;\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow}\; \cdots,$$

where

$$d_i^n = B_G^+(X)_{\delta_i^n} : B_G^+(X)_{[n+1]} = G^{n+1} \times X \to G^n \times X = B_G^+(X)_{[n]}$$

is defined by:

$B_G^+(X)$ (2)

$$d_i^n(g_n \dots, g_0, x) = \begin{cases} (g_n, \dots, g_1, g_0 x) & (i = 0) \\ (g_n, \dots, g_i g_{i-1}, \dots, g_0, x) & (0 < i < n+1) \\ (g_{n-1}, \dots, g_0, x) & (i = n+1) \end{cases} .$$

The categories of modules $\mathrm{Mod}(\mathrm{Zar}(B_G^+(X)))$ and quasi-coherent modules $\mathrm{Qch}(\mathrm{Zar}(B_G^+(X)))$ are denoted by $\mathrm{Mod}(G, X)$ and $\mathrm{Qch}(G, X)$, respectively. An object of $\mathrm{Mod}(G, X)$ is called a $(G, \mathcal{O}_X)$-module.

If G is S-flat, then $\mathrm{Qch}(G, X)$ is closed under kernels, cokernels and extensions in $\mathrm{Mod}(G, X)$, and it is an abelian category and the inclusion $\mathrm{Qch}(G, X) \hookrightarrow \mathrm{Mod}(G, X)$ is exact.

Algebraic G-cohomology

Let $\mathcal{C}$ be a site. Let $\mathrm{Ps}(\mathcal{C})$ and $\mathrm{Sh}(\mathcal{C})$ denote the category of presheaves and sheaves over $\mathcal{C}$, respectively. For $\mathcal{M} \in \mathrm{Ps}(\mathcal{C})$ and $\mathcal{N} \in \mathrm{Sh}(\mathcal{C})$, we write $H_p^i(\mathcal{C}, \mathcal{M}) := \mathrm{Ext}^i_{\mathrm{Ps}(\mathcal{C})}(\underline{\mathbb{Z}}, \mathcal{M})$ and $H^i(\mathcal{C}, \mathcal{N}) := \mathrm{Ext}^i_{\mathrm{Sh}(\mathcal{C})}(a\,\underline{\mathbb{Z}}, \mathcal{N})$, where $\underline{\mathbb{Z}}$ is the constant presheaf and $a\,\underline{\mathbb{Z}}$ its sheafification.

For $\mathcal{M} \in \mathrm{Ps}(\mathrm{Zar}(B_G^+(X)))$, we denote $H_p^i(\mathrm{Zar}(B_G^+(X)), \mathcal{M})$ by $H^i_{\mathrm{alg}}(G, \mathcal{M})$, and call it the ith algebraic G-cohomology group of $\mathcal{M}$.

Explicit description of algebraic G-cohomology

Lemma 2

$H^i_{\mathrm{alg}}(G, \mathcal{M})$ is the cohomology group of the complex

$$0 \to \Gamma(([0], X), \mathcal{M}) \xrightarrow{d_0 - d_1} \Gamma(([1], G \times X), \mathcal{M}) \xrightarrow{d_0 - d_1 + d_2} \Gamma(([2], G \times G \times X), \mathcal{M}) \to \cdots .$$

The Picard group of a ringed site

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. An $\mathcal{O}$-module $\mathcal{L}$ is called an invertible sheaf if for any $c \in \mathrm{Ob}(\mathcal{C})$, there exists some covering $(c_\lambda \to c)$ of c such that for each λ, $\mathcal{L}|_{c_\lambda} \cong \mathcal{O}|_{c_\lambda}$, where $(?)|_{c_\lambda}$ is the restriction to $\mathcal{C}/c_\lambda$. An invertible sheaf is quasi-coherent.

The set of isomorphism classes of invertible sheaves on $\mathcal{C}$ is denoted by $\mathrm{Pic}(\mathcal{C})$, and called the Picard group of $\mathcal{C}$. It is an additive group by the addition

$$[\mathcal{L}] + [\mathcal{L}'] := [\mathcal{L} \otimes_{\mathcal{O}} \mathcal{L}'].$$

There is an isomorphism $\mathrm{Pic}(\mathcal{C}) \cong H^1(\mathcal{C}, \mathcal{O}^\times)$.

The G-equivariant Picard group

Definition 3

$\mathrm{Pic}(B_G^+(X))$ is denoted by $\mathrm{Pic}(G, X)$, and is called the *G-equivariant Picard group of X*.

There is an obvious map

$$\rho : \mathrm{Pic}(G, X) \to \mathrm{Pic}(X)$$

forgetting the G-action. The image of ρ is contained in

$$\mathrm{Pic}(X)^G := \mathrm{Ker}(\mathrm{Pic}(X) \xrightarrow{d_0 - d_1} \mathrm{Pic}(G \times X)) = \{[\mathcal{L}] \in \mathrm{Pic}(X) \mid a^*\mathcal{L} \cong p_2^*\mathcal{L}\},$$

where $a = d_0 : G \times X \to X$ is the action, and $p_2 = d_1 : G \times X \to X$ is the second projection.

A five-term exact sequence

From the five-term exact sequence

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \to E^2$$

of the Grothendieck spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{alg}}(G, \underline{H}^q(\mathcal{O}^\times)) \Rightarrow H^{p+q}(\mathrm{Zar}(B_G^+(X)), \mathcal{O}^\times),$$

we get

Lemma 4

There is an exact sequence

$$0 \to H^1_{\mathrm{alg}}(G, \mathcal{O}^\times) \to \mathrm{Pic}(G, X) \xrightarrow{\rho} \mathrm{Pic}(X)^G \to$$
$$H^2_{\mathrm{alg}}(G, \mathcal{O}^\times) \to H^2(\mathrm{Zar}(B_G^+(X)), \mathcal{O}^\times).$$

Main theorem

Theorem 5

Let k be a field, G a smooth k-group scheme of finite type, and X a reduced G-scheme which is quasi-compact and quasi-separated. Assume that there is a k-scheme Z of finite type and a dominating k-morphism $Z \to X$. Then $H^1_{\mathrm{alg}}(G, \mathcal{O}^\times) = \mathrm{Ker}(\rho : \mathrm{Pic}(G, X) \to \mathrm{Pic}(X))$ is a finitely generated abelian group.

Note that a reduced k-scheme X of finite type is reduced, quasi-compact and quasi-separated, admitting a dominating map from a k-scheme of finite type, that is, $\mathrm{id} : Z = X \to X$!

Finite generation of the Picard group of an invariant subring

Lemma 6

Let $\varphi : X \to Y$ be a G-invariant morphism such that $\mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^G$ is an isomorphism. Then there is an injective homomorphism $\operatorname{Pic}(Y) \hookrightarrow \operatorname{Pic}(G, X)$.

Corollary 7

Let k, G, X and $Z \to X$ be as in the theorem, and let $\varphi : X \to Y$ be a G-invariant morphism such that $\mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^G$ is an isomorphism. If $\operatorname{Pic}(X)$ is a finitely generated abelian group, then $\operatorname{Pic}(G, X)$ and $\operatorname{Pic}(Y)$ are also finitely generated.

Proof of the theorem (1) — the case of finite group action on a finite algebra

The case that G is a finite group, and $X = \operatorname{Spec} B$ is also finite.

(1) The case that $G \subset \operatorname{Aut}(B/k)$. Then $H^1_{\text{alg}}(G, \mathcal{O}^\times) = H^1(G, B^\times) = 0$ (Hilbert's Theorem 90).

(2) The case that the action of G on X is trivial. Then $H^1(G, B^\times)$ is the group of homomorphisms from G to $B^\times$. This is finite.

(3) General case. Let N be the kernel of the map $G \to \operatorname{GL}(B)$. Then there is an exact sequence

$$0 \to H^1(G/N, B^\times) \to H^1(G, B^\times) \to H^1(N, B^\times).$$

As $H^1(G/N, B^\times)$ and $H^1(N, B^\times)$ are finitely generated, $H^1(G, B^\times)$ is also finitely generated.

Proof of the theorem (2) — group to group scheme

Let G and X be finite (G is a finite group scheme, and is not a finite group in general). Let k' be a finite Galois extension of k such that $\Omega := k' \otimes_k G$ is a finite group (i.e., a disjoint union of $\operatorname{Spec} k'$). Let $\Gamma := \operatorname{Gal}(k'/k)$. Then there is an equivalence of categories

$$\operatorname{Mod}(G, B) \cong \operatorname{Mod}(\Theta, k' \otimes_k B),$$

where Θ is the semidirect product $\Gamma \ltimes \Omega$. Replacing G by Θ, the problem is reduced to the case of finite groups.

Proof of the theorem (3) — Affine case

The case that $G = \operatorname{Spec} H$ and $X = \operatorname{Spec} B$ are both affine. Let H_0 and B_0 be the integral closures of k in H and B, respectively. Then $G_0 := \operatorname{Spec} H_0$ is an affine k-group scheme acting on $X_0 := \operatorname{Spec} B_0$. Then the map of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_0^\times & \longrightarrow & (H_0 \otimes B_0)^\times & \longrightarrow & (H_0 \otimes H_0 \otimes B_0)^\times & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B^\times & \longrightarrow & (H \otimes B)^\times & \longrightarrow & (H \otimes H \otimes B)^\times & \longrightarrow & \cdots \end{array}$$

is an isomorphism in the quotient category $\mathcal{A} := \operatorname{Mod}(\mathbb{Z}) / \operatorname{mod}(\mathbb{Z})$ by the next lemma, and the problem is reduced to the finite case.

Rosenlicht's lemma

Lemma 8 (Rosenlicht, H—)

Let k be a field, and X be a reduced k-scheme. Assume that there is a k-scheme Z of finite type and a dominating k-morphism $Z \to X$. Then there is a short exact sequence of the form

$$1 \to K^\times \xrightarrow{\iota} \Gamma(X, \mathcal{O}_X)^\times \to \mathbb{Z}^r \to 0,$$

where K is the integral closure of k in $k[X] = H^0(X, \mathcal{O}_X)$, and ι is the inclusion.

Proof of the theorem (4) — the general case

Set $H = k[G]$, $G_1 = \operatorname{Spec} H$, $B = k[X]$, and $X_1 = \operatorname{Spec} B$. Then G_1 is an affine k-group scheme acting on X_1. The complex computing $H^i_{\mathrm{alg}}(G, \mathcal{O}_X^\times)$ and the one computing $H^i_{\mathrm{alg}}(G_1, \mathcal{O}_{X_1}^\times)$ are the same, and the problem is reduced to the affine case. □

Varieties with trivial unit groups

Lemma 9

Let k be a field, and G a quasi-compact quasi-separated k-group scheme such that $k[G]$ is geometrically reduced over k. Let X be a G-scheme. Assume that $\bar{k} \otimes_k X$ is integral, or X is quasi-compact quasi-separated and $\bar{k} \otimes_k k[X]$ is integral. If the unit group of $\bar{k} \otimes_k k[X]$ is $\bar{k}^\times$, then $H^i_{\mathrm{alg}}(G, \mathcal{O}_X^\times) \cong H^i_{\mathrm{alg}}(G, k^\times)$. In particular, $H^1_{\mathrm{alg}}(G, \mathcal{O}_X^\times) \cong \mathcal{X}(G) := \{\chi \in k[G]^\times \mid \chi(gg') = \chi(g)\chi(g')\}$.

Example 10

If a smooth k-group scheme G acts on the affine space $X = \mathbb{A}^n$, then $H^1_{\mathrm{alg}}(G, \mathcal{O}_X^\times) \cong \mathcal{X}(G) \cong \mathrm{Pic}(G, \operatorname{Spec} k) \cong \mathrm{Pic}(G, X)$.

Connected groups

Proposition 11

Let G be a connected smooth k-group scheme of finite type, and X a quasi-compact quasi-separated G-scheme such that $k[X]$ is reduced and k is integrally closed in $k[X]$. Then

$$H^n_{\mathrm{alg}}(G, \mathcal{O}_X^\times) = \begin{cases} (k[X]^G)^\times & (n = 0) \\ \mathcal{X}(G)/\mathcal{X}(G, X) & (n = 1) \\ 0 & (n \geq 2) \end{cases},$$

where

$$\mathcal{X}(G, X) := \{\chi \in \mathcal{X}(G) \mid \exists \alpha \in k[X]^\times \; \forall g \in G \; x \in X \; \alpha(gx) = \chi(g)\alpha(x)\}.$$

Some corollaries

Corollary 12 (Kamke, H—)

In the proposition, assume that G and $X = \operatorname{Spec} B$ are affine. If f is a nonzerodivisor of B and Bf is a G-ideal of B, then f is a semiinvariant. That is, there exists some $\chi \in \mathcal{X}(G)$ such that $f(gx) = \chi(g)f(x)$ for $x \in X$ and $g \in G$.

Corollary 13 (more or less well-known)

Under the assumption of the proposition,

$$\rho : \operatorname{Pic}(G, X) \to \operatorname{Pic}(X)^G$$

is surjective.

Krull domain

Let A be an integral domain with the field of fractions $K = Q(A)$.

Definition 14

We say that A is a Krull domain if there exists a set Λ of DVR's such that

1. For each $R \in \Lambda$, $R \subset K$ and $Q(R) = K$.
2. $A = \bigcap_{R \in \Lambda} R$.
3. For each $a \in K^\times$, there are only finitely many $R \in \Lambda$ such that $a \notin R^\times$.

If A is a Krull domain, then Λ can be taken to be

$$\{A_P \mid P \in X^1(A)\},$$

where $X^1(A)$ is the set of height-one prime ideals of A.

Noetherian normal domains versus Krull domains

Remark 15

1. A Noetherian normal domain is a Krull domain.
2. A Krull domain is normal, but may not be Noetherian.
3. (Fossum and others) Over Krull domains, we can imitate the theory of the class groups of Noetherian normal domains.
4. Let R be a domain, and $K \subset Q(R)$ a subfield. If R is Krull, then so is $K \cap R$. Even if $R = k[x_1, \ldots, x_n]$ for some subfield k of $K \cap R$, $K \cap R$ may not be Noetherian (Nagata).
5. If R is Krull and L is a finite extension field of $Q(R)$, the integral closure R' of R in L is again Krull. But even if R is Noetherian, R' may not be so (an example of bad Noetherian rings, Nagata).
6. If X is a normal variety over a field, then $H^0(X, \mathcal{O}_X)$ is Krull, but may not be Noetherian.

The class group

A locally Krull scheme is a shceme which is locally the prime spectrum of a Krull doain by definition.

Let Y be a quasi-compact locally Krull scheme. Let $X^1(Y)$ be the set of integral closed subschemes of codimension one. Let $Q(Y)$ be the total quotient ring of $H^0(U, \mathcal{O}_Y)$, where U is any dense affine open subscheme of Y (independent of the choice of U). For $F \in X^1(Y)$, let v_F be the normalized discrete valuation associated with the DVR $\mathcal{O}_{Y,F}$.

Let $\mathrm{Div}(Y)$ be the free abelian group with the basis $X^1(Y)$. For $f \in Q(Y)^\times$, we define $\operatorname{div} f = \sum_{F \in X^1(Y)} v_F(f)[F] \in \mathrm{Div}(Y)$. We define $\mathrm{Prin}(Y) := \{\operatorname{div} f \mid f \in Q(Y)^\times\} \cup \{0\}$ and $\mathrm{Cl}'(Y) := \mathrm{Div}(Y)/\mathrm{Prin}(Y)$. $\mathrm{Cl}'(Y)$ is called the class group of Y.

Reflexive modules and sheaves

Let A be a Krull domain. An A-module M is said to be reflexive (or divisorial), if M is a submodule of some finitely generated module, and the canonical map $M \to M^{**}$ is an isomorphism, where $(?)^* = \operatorname{Hom}_A(?, A)$.

Let Y be a locally Krull scheme. An $\mathcal{O}_Y$-module $\mathcal{M}$ is said to be reflexive if $\mathcal{M}$ is quasi-coherent, and $H^0(U, \mathcal{M})$ is a reflexive A-module for each affine open subset $U = \operatorname{Spec} A$ such that A is a Krull domain. If, moreover, $H^0(U, \mathcal{M})$ is of rank n for each U, then we say that $\mathcal{M}$ is of rank n.

A second definition of the class group

Let Y be a locally Krull scheme. We denote the set of isomorphism classes of rank-one reflexive sheaves over Y by $\mathrm{Cl}(Y)$ and call it the class group of Y (again!). Note that $\mathrm{Cl}(Y)$ is an additive group by the addition

$$[\mathcal{M}] + [\mathcal{M}'] = [(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{M}')^{**}].$$

Almost by definition, $\mathrm{Pic}(Y)$ is a subgroup by $\mathrm{Cl}(Y)$. If Y is a non-singular variety, then $\mathrm{Pic}(Y) = \mathrm{Cl}(Y)$.

If Y is quasi-compact, then the map $[D] \mapsto [\mathcal{O}_Y(D)]$ gives an isomorphism $\mathrm{Cl}'(Y) \to \mathrm{Cl}(Y)$.

Equivariant class group

Let G be S-flat and X be locally Krull. We say that a $(G, \mathcal{O}_X)$-module $\mathcal{M}$ is reflexive if $\mathcal{M}$ is quasi-coherent (as a $(G, \mathcal{O}_X)$-module), and is reflexive as an $\mathcal{O}_X$-module. The set of isomorphism classes of rank-one reflexive $(G, \mathcal{O}_X)$-modules is denoted by $\mathrm{Cl}(G, X)$, and we call it the G-equivariant class group of X.

Theorem 16 (H—)

Let G and X be as above, and $\mathcal{M}$ and $\mathcal{N}$ be reflexive $(G, \mathcal{O}_X)$-modules. Then

1. The $(G, \mathcal{O}_X)$-modules $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ and $(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})^{**}$ are reflexive, where $(?)^* = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(?, \mathcal{O}_X)$.
2. $\mathrm{Cl}(G, X)$ is an additive group with the sum

$$[\mathcal{M}] + [\mathcal{N}] = [(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})^{**}].$$

The α map and its kernel

There is an obvious map $\alpha : \mathrm{Cl}(G,X) \to \mathrm{Cl}(X)$, fogetting the G-action. We have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Ker}\rho & \longrightarrow & \operatorname{Pic}(G,X) & \xrightarrow{\rho} & \operatorname{Pic}(X)\,. \\ & & \downarrow\cong & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname{Ker}\alpha & \longrightarrow & \mathrm{Cl}(G,X) & \xrightarrow{\alpha} & \mathrm{Cl}(X) \end{array}$$

Removing closed subsets of codimension two or more

Lemma 17

Let G be a flat S-group scheme, and X be a locally Krull G-scheme. Let U be its G-stable open subset. Let $\varphi : U \hookrightarrow X$ be the inclusion. Assume that $\operatorname{codim}_X(X \setminus U) \geq 2$. Then
$\varphi^* : \mathrm{Ref}_n(G, X) \to \mathrm{Ref}_n(G, U)$ is an equivalence, and
$\varphi_* : \mathrm{Ref}_n(G, U) \to \mathrm{Ref}_n(G, X)$ is its quasi-inverse. In particular,
$\varphi^* : \mathrm{Cl}(G, X) \to \mathrm{Cl}(G, U)$ defined by $\varphi^*[\mathcal{M}] = [\varphi^*\mathcal{M}]$ is an isomorphism whose inverse is given by $\mathcal{N} \mapsto [\varphi_*\mathcal{N}]$.

Expressing the class group via the Picard groups

Proposition 18

Let Y be a quasi-compact locally Krull scheme. Then $\mathrm{Cl}(Y) \cong \varinjlim \mathrm{Pic}(U)$, where the inductive limit is taken over all open subsets U such that $\mathrm{codim}_Y(Y \setminus U) \geq 2$.

Invariant subring

Lemma 19

Let G be a flat S-group scheme. Let X be a quasi-compact quasi-separated locally Krull G-scheme, and let $\varphi : X \to Y$ be a G-invariant morphism such that $\mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^G$ is an isomorphism. Then Y is locally Krull, and the number of connected components of Y is finite. The class group $\mathrm{Cl}(Y)$ of Y is a subquotient of $\mathrm{Cl}(G, X)$.

Finite generation of the class group of an invariant subring

Theorem 20 (H—)

Let k be a field, G a smooth k-group scheme of finite type, and X a quasi-compact quasi-separated locally Krull G-scheme. Assume that there is a k-scheme Z of finite type and a dominating k-morphism $Z \to X$. Let $\varphi : X \to Y$ be a G-invariant morphism such that $\mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^G$ is an isomorphism. If $\mathrm{Cl}(X)$ is finitely generated, then $\mathrm{Cl}(G, X)$ and $\mathrm{Cl}(Y)$ are also finitely generated.

Even if X is a normal k-variety, Y may not be locally Noetherian.

Similar results for connected groups are proved by Magid and Waterhouse.

Thank you

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http://www.math.okayama-u.ac.jp/~hashimoto/