F-rationality of the ring of modular invariants

Mitsuyasu Hashimoto

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March 22, 2016

Partially Joint with P. Symonds

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Let $k = \bar{k}$ be an algebraically closed field of characteristic $p > 0$. Let $V = k^d$, and G be a finite subgroup of $\mathrm{GL}(V) = \mathrm{GL}_d$. We say that $g \in \mathrm{GL}(V)$ is a pseudo-reflection if $\operatorname{rank}(1_V - g) = 1$. Let $B = \operatorname{Sym} V = k[v_1, \ldots, v_d]$, where $v_1, \ldots, v_d$ is a basis of V, and $A = B^G$.

Question 1

Assume that G does not have a pseudo-reflection.

1. When is $A = B^G$ strongly F-regular?
2. When is $A = B^G$ F-rational?

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Broer–Yasuda theorem

Theorem 2 (Broer, Yasuda)

Assume that G does not have a pseudo-reflection. The following are equivalent.

1. $A = B^G$ is strongly F-regular.
2. A is a direct summand subring of B.
3. p does not divide the order $\#G$ of G.

2⇒1 is simply because strong F-regularity is inherited by a direct summand. **1⇒2** is because a weakly F-regular ring is a splinter (Hochster–Huneke). **3⇒2** is by the existence of the Reynolds operator. Broer and Yasuda proved **2⇒3**.

Today we consider the problem for F-rationality.

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Graded (G, B)-modules

Let $V = k^d$, and G be a finite subgroup of $\mathrm{GL}(V) = \mathrm{GL}_d$. Let $B = \mathrm{Sym}\, V$ and $A = B^G$.

A G-module B-module M is called a (G, B)-module if $g(bm) = (gb)(gm)$ holds. This is the same as a module over the twisted group algebra $B * G$.
Let $\mathcal{M}$ be the category of $\mathbb{Z}[1/p]$-graded (G, B)-modules. Let $\mathcal{F}$ be its full subcategory consisting of B-finite B-free objects.
The Frobenius twist ${}^e(?)$ is an endofunctor of $\mathcal{M}$, and ${}^e\mathcal{F} \subset \mathcal{F}$.
If $M \in \mathcal{M}$ and $m \in M$ is of degree d, then ${}^e m \in {}^e M$ is of degree d/p^e.

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Frobenius twists of objects of $\mathcal{F}$

Lemma 3 (Symonds–H)

There exists some $e_0 \geq 1$ such that for any $E \in \mathcal{F}$ of rank f, there exists a direct summand E_0 of ${}^{e_0}E$ in $\mathcal{F}$ such that $E_0 \cong (B \otimes_k kG)^f$ as (G, B)-modules.

Lemma 4 (Symonds–H)

${}^{e}(B \otimes_k kG) \cong (B \otimes_k kG)^{p^{de}}$ as (G, B)-modules.

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Asymptotic behavior of Frobenius twists

Theorem 5 (Symonds–H)

There exist some $c > 0$ and $0 < \alpha < 1$ such that for any $E \in \mathcal{F}$ of rank f and any $e \geq 1$, there exists some decomposition

$$ {}^{e}E \cong E_{0,e} \oplus E_{1,e} $$

in $\mathcal{F}$ such that $E_{0,e}$ is a direct sum of copies of $B \otimes_k kG$ as a (G, B)-module, and $E_{1,e}$ is an object of $\mathcal{F}$ whose rank is less than or equal to $fcp^{de}\alpha^e$.

Some observations on $E_{0,e}$ and $E_{1,e}$

- $\lim_{e\to\infty} \frac{1}{p^{de}} \operatorname{rank} E_{1,e} = 0$. Hence $\lim_{e\to\infty} \frac{1}{p^{de}} \operatorname{rank} E_{0,e} = f$.
- Since $(B \otimes_k kG)^G \cong B$ as A-modules, we have $\lim_{e\to\infty} \frac{1}{p^{de}} \mu_{\hat{A}}(\hat{E}^G_{0,e}) = f\mu_{\hat{A}}(\hat{B})/|G| = fe_{\mathrm{HK}}(\hat{A})$ (by Watanabe–Yoshida theorem, as $[Q(\hat{B}) : Q(\hat{A})] = |G|$), where $\hat{A}$ and $\hat{B}$ are the completions of A and B, respectively.
- As $\lim_{e\to\infty} \frac{1}{p^{de}} \mu_{\hat{A}}({}^e\hat{E}^G) = e_{\mathrm{HK}}({}^e\hat{E}^G) = fe_{\mathrm{HK}}(\hat{A})$, we have

Corollary 6 (Symonds–H)

$$\lim_{e\to\infty} \frac{1}{p^{de}} \mu_{\hat{A}}(\hat{E}^G_{1,e}) = 0.$$

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Interpretation to A-modules

Let $k = V_0, V_1, \ldots, V_n$ be the list of simple G-modules. Let P_i be the projective cover of V_i. Set $M_i := (B \otimes_k P_i)^G$.

Theorem 7 (Symonds–H)

There exists some sequence of non-negative integers $\{a_e\}$ such that

1. $\lim_{e\to\infty} a_e/p^{de} = 1/|G|$; and
2. For each B-finite B-free $\mathbb{Z}$-graded (G, B)-module E of rank f and $e \geq 1$, there is a decomposition

$$^eE^G \cong \bigoplus_{i=0}^{n} M_i^{\oplus f a_e \dim V_i} \oplus M_{E,e}$$

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Sannai's dual F-signature

Let $(R, \mathfrak{m}, k)$ be a d-dimensional reduced F-finite local ring of prime characteristic p with k perfect. For finite R-modules M and N, define

$$\operatorname{surj}_R(M, N) := \max\{r \in \mathbb{Z}_{\geq 0} \mid \exists \text{ a surjection } M \to N^{\oplus r}\}.$$

We define

$$s(M) := \limsup_{e \to \infty} \frac{\operatorname{surj}_R({}^e M, M)}{p^{de}},$$

and call it the dual F-signature of M (Sannai). $s(R)$ is nothing but the F-signature of the ring R (defined by Huneke–Leuschke).

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Characterizations of F-regularity and F-rationality

Theorem 8

Let $(R, \mathfrak{m}, k)$ be a reduced F-finite local ring with k perfect.

1. (Tucker) $s(R) := \limsup_{e\to\infty} \frac{\mathrm{surj}({}^eR, R)}{p^{de}} = \lim_{e\to\infty} \frac{\mathrm{surj}({}^eR, R)}{p^{de}}$.
2. (Aberbach–Leuschke) R is strongly F-regular if and only if $s(R) > 0$.
3. (Gabber) R is a homomorphic image of a regular local ring.
4. (Sannai) R is F-rational if and only if R is Cohen–Macaulay and $s(\omega_R) > 0$, where ω_R is the canonical module of R.

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The group $[\mathcal{C}]$

Let $\mathcal{C}$ be an additive category. We define

$$[\mathcal{C}] := (\bigoplus_{M \in \mathcal{C}} \mathbb{Z} \cdot M)/(M - M_1 - M_2 \mid M \cong M_1 \oplus M_2).$$

The class of M in the group $[\mathcal{C}]$ is denoted by $[M]$.
The vector space $\mathbb{R} \otimes_{\mathbb{Z}} [\mathcal{C}]$ is denoted by $[\mathcal{C}]_{\mathbb{R}}$. If $\mathcal{C}$ is Krull–Schmidt and $\mathcal{C}_0$ is a complete set of representatives of $\operatorname{Ind} \mathcal{C}$, then $\{[M] \mid M \in \mathcal{C}_0\}$ is an $\mathbb{R}$-basis of $[\mathcal{C}]_{\mathbb{R}}$.

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The metric of $[\mathrm{mod}(R)]$

Let R be a Henselian local ring, and $\mathcal{C} := \mathrm{mod}(R)$. For $\alpha \in [\mathcal{C}]_{\mathbb{R}}$, we can write

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We define $\|\alpha\| := \sum_M |c_M| \mu_R(M)$. Then $([\mathcal{C}]_{\mathbb{R}}, \|\cdot\|)$ is a normed space. So it is a metric space by the metric

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$\mu_R : [\mathcal{C}]_{\mathbb{R}} \to \mathbb{R}$ is a short map. That is, $|\mu_R(\alpha) - \mu_R(\beta)| \leq \|\alpha - \beta\|$. Similarly for $\mathrm{sum}_N : [\mathcal{C}]_{\mathbb{R}} \to \mathbb{R}$ for $N \in \mathcal{C}_0$. In particular, they are uniformly continuous.

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Let $\alpha = \sum_{M \in \mathcal{C}_0} c_M[M] \in [\mathcal{C}]_{\mathbb{R}}$. If $FL(\alpha)$ exists, then

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Theorem 11 (Symonds–H)

For each B-finite B-free $\mathbb{Z}$-graded (G, B)-module E of rank f,

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From now, unless otherwise stated explicitly (in an example), assume that G has no pseudo-reflection.

Theorem 13 (Watanabe–Peskin–Broer–Braun)

Let $\det = \det_V$ denote the one-dimensional representation $\bigwedge^d V$ of G. Then

1. $\omega_A \cong (B \otimes_k \det)^G$.
2. Hence $B \otimes_k \det \cong (B \otimes_A \omega_A)^{**}$.
3. In particular, A is quasi-Gorenstein if and only if $\det \cong k$ as a G-module (or equivalently, $G \subset \mathrm{SL}(V)$).

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The F-signature $s(\hat{A})$ of $\hat{A}$ is zero if p divides $|G|$, and is $1/|G|$ otherwise.

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$s(\hat{A}) = FS_{\hat{A}}(\hat{A}) = |G|^{-1} \sum_{i=0}^{n} (\dim V_i) \operatorname{sum}_{\hat{A}}([\hat{M}_i])$. □

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Representation theoretic characterization of $s(\omega_{\hat{A}}) > 0$

Let ν be the number such that $V_\nu \cong \det$.

Theorem 15 (Main Theorem)

Assume that A is not strongly F-regular (or equivalently, p divides $|G|$). Then the following are equivalent.

1. $s(\omega_{\hat{A}}) > 0$;
2. The canonical map $M_\nu \to \omega_A$ is surjective.
3. $H^1(G, B \otimes_k \operatorname{rad} P_\nu) = 0$.
4. For any non-projective indecomposable G-summand M of B, M does not contain $\det^{-1}$ (the k-dual of det).

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Let $k = V_0, V_1, \ldots, V_n$ be the list of simple G-modules. Let P_i be the projective cover of V_i. Set $M_i := (B \otimes_k P_i)^G$.

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$$FL([\omega_{\hat{A}}]) = \frac{1}{|G|}[\hat{B}] = \frac{1}{|G|}\bigoplus_{i=0}^{n}(\dim V_i)[\hat{M}_i],$$

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The proof of $\mathbf{2} \Rightarrow \mathbf{1}$

As we assume that there is a surjection $M_\nu \to \omega_A$, $\operatorname{surj}(\hat{M}_\nu, \omega_{\hat{A}}) \geq 1$. By Theorem 11 (applied to $E = B \otimes \det$),

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The proof of $\mathbf{1}\Rightarrow\mathbf{2}$ (1)

By Theorem 11, we have that $\operatorname{asn}([\hat{B}], \omega_{\hat{A}}) > 0$. Or equivalently, there is a surjection $h : \hat{B}^r \to \omega_{\hat{A}}$ for $r \gg 0$. By the equivalence $\gamma = (\hat{B} \otimes_{\hat{A}} ?)^{**} : \operatorname{Ref}(\hat{A}) \to \operatorname{Ref}(G, \hat{B})$, there corresponds

$$\tilde{h} = \gamma(h) : (\hat{B} \otimes_k kG)^r \to \hat{B} \otimes_k \det .$$

As $\hat{B} \otimes_k kG$ is a projective object in the category of $(G, \hat{B})$-modules, $\tilde{h}$ factors through the surjection

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Returning to the category Ref $\hat{A}$, the surjection $h : \hat{B}^r \to \omega_{\hat{A}}$ factors through

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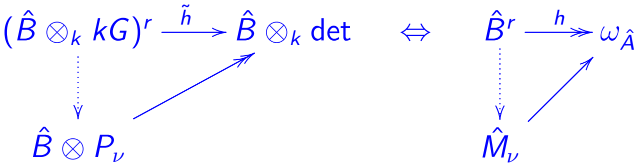

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Let $\det^{-1}$ denote the dual representation of $\det$. Assume that p divides $|G|$. If $s(\omega_{\hat{A}}) > 0$, then $\det^{-1}$ is not a direct summand of B.

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Note that the one-dimensional representation $\det^{-1}$ is not projective. The result follows from $\mathbf{1} \Rightarrow \mathbf{4}$ of the theorem. ☐

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Let M and N be in $\mathrm{Ref}(G, B)$. There is a natural isomorphism

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A is F-rational if and only if the following three conditions hold.

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k is a direct summand of B, and $H^1(G, B) = 0$. □

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An example (1)

- Let p be an odd prime number.
- Let us identify $\mathrm{Map}(\mathbb{F}_p, \mathbb{F}_p)^\times$ with the symmetric group S_p.
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- As $\tau\sigma\tau^{-1} = \sigma^\alpha$, Γ normalizes Q. Set $G = Q\Gamma$. $C_G(Q) = Q$.
- $G = \{\phi \in S_p \mid \exists a \in \mathbb{F}_p^\times\ \exists b \in \mathbb{F}_p\ \forall x \in \mathbb{F}_p\ \phi(x) = ax + b\} \subset S_p$.
- $\#G = p(p-1)$.

An example (2)

- The only involution of Γ is $\tau^{(p-1)/2} = ((-1)\cdot) = (1\ (p-1))(2\ (p-2))\cdots((p-1)/2\ (p+1)/2)$, which is a transposition if and only if $p = 3$.
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- $G \subset S_p$ acts on $P = k^p = \langle w_0, w_1, \dots, w_{p-1} \rangle$ by $\phi w_i = w_{\phi(i)}$ for $\phi \in G$ and $i \in \mathbb{F}_p$.
- Let $r \geq 1$, and set $V = P^{\oplus r}$.
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- Let $\lambda \in \mathbb{Z}^p$, and let $w^\lambda = w_0^{\lambda_0} \cdots w_{p-1}^{\lambda_{p-1}}$ be the corresponding monomial of S.
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- Now consider $V = P^{\oplus r}$ and $B := \operatorname{Sym} V \cong S^{\otimes r}$.
- Let k^- be the sign representation of G. As $\tau \in G$ is an odd permutation, $k^- \not\cong k$.
- $\det_V = (\det P)^{\otimes r} = (k^-)^{\otimes r} \cong \det_V^{-1}$. This is k if r is even and k^- if r is odd.
- If M is a projective G-module and N a G-module, then $M \otimes N$ is projective. So $B = S^{\otimes r}$ is again a direct sum of projectives and copies of k.
- If $r = 1$ and $p = 3$, then $A := B^G = k[e_1, e_2, e_3]$, the polynomial ring generated by the elementary symmetric polynomials.
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Kemper's theorem

Let k be a field of characteristic $p > 0$, and G be a subgroup of the symmetric group of S_d acting on $B = k[v_1, \ldots, v_d]$ by permutation. Let Q be a Sylow p-subgroup of G. Assume that $|Q| = p$. Let $N = N_G(Q)$ be the normalizer. Let $X_1, \ldots, X_c$ be the Q-orbits of $\{v_1, \ldots, v_d\}$. Set

$$H := \{\sigma \in N \mid \forall i \ \sigma(X_i) \subset X_i\}.$$

Then Q is a normal subgroup of H. Set $m := [H : C_H(Q)]$.

Theorem 21 (Kemper)

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- For our G, Q, and V, $H = N = G$. $C_H(Q) = Q$.
- So $m = p - 1$, and $c = r$.
- So $\operatorname{depth} A = \min\{2p - 2 + r, rp\}$ and $\dim A = d = rp$.
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Conclusion

Theorem 22

Let $p \geq 3$, r, G, V, $B = \mathrm{Sym}\, V$, and $A = B^G$ be as above.

1. $\# G = p(p-1)$.
2. If $p = 3$ and $r = 1$, then G is a reflection group and A is a polynomial ring. Otherwise, G does not have a pseudo-reflection, and A is not F-regular.
3. If $p \geq 5$ and $r = 1$, then A is F-rational but not F-regular.
4. If $r = 2$, then A is Gorenstein, but not F-rational.
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Thank you

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http://www.math.okayama-u.ac.jp/˜hashimoto/