Equivariant total ring of fractions and factoriality of rings generated by semiinvariants

Mitsuyasu Hashimoto

Nagoya University

September 1, 2011

The purpose

The purpose of this talk is two fold.

- Introducing an equivariant version of the total ring of fractions.
- Giving its applications to invariant theory. In particular, we give some new criteria on factoriality (the UFD property) of the rings of (semi)invariants.

Extending an action

R: a commutative ring.
F: an affine flat R-group scheme.
S: an F-algebra (i.e., an R-algebra on which F acts).

We sometimes want to extend the action of F on S to that on $Q(S)$, the total ring of fractions of S.

An action of an abstract group Γ on S is always extended to an action on $Q(S)$ via $g(a/b) = ga/gb$. But this does not apply to the (rational) action of F on S...

An example

Example 1

Let $R = k$ be a field, $F = \mathbb{G}_m$, and $S = k[x]$. F acts on S via $\deg x = 1$. Then $Q(S) = k(x)$ cannot be $\mathbb{Z}$-graded so that the inclusion $S \hookrightarrow Q(S)$ preserves grading.

The definition of the equivariant total ring of fractions

Let $\omega : S \to S \otimes R[F]$ be the coaction. As F is R-flat, ω is flat. So $\omega' : Q(S) \to Q(S \otimes R[F])$ is induced.
Set

$$\Omega := \{M \subset Q(S) \mid \omega'(M) \subset M \otimes R[F]\},$$

and define $Q_F(S) := \sum_{M \in \Omega} M$, and call $Q_F(S)$ the F-total ring of fractions of S.

Basic properties

- $Q_F(S)$ is an R-subalgebra of $Q(S)$.
- Letting $\omega' : Q_F(S) \to Q_F(S) \otimes R[F]$ be the coaction, $Q_F(S)$ is an F-algebra.
- S is an F-subalgebra of $Q_F(S)$.
- If $S \subset T \subset Q(S)$, T is an S-submodule of $Q(S)$, and T has an (F, S)-module structure such that $S \hookrightarrow T$ is F-linear, then $T \subset Q_F(S)$.
- $(\omega')^{-1}(Q(S) \otimes R[F]) = Q_F(S)$.

Another description in Noetherian case

Lemma 2

Let S be Noetherian. Then

- $Q_F(S) = \bigcup_I S :_{Q(S)} I$, where I runs through all the F-ideals of S containing a nonzerodivisor.
- $Q_F(S) = \varinjlim \Gamma(U, \mathcal{O}_{\operatorname{Spec} S})$, where U runs through all the F-stable open subsets such that $S \to \Gamma(U, \mathcal{O}_{\operatorname{Spec} S})$ are injective.

Corollary 3

Let S be Noetherian, and I and J be F-stable ideals of S. If J contains a nonzerodivisor, then $I :_{Q(S)} J$ is an (F, S)-submodule of $Q_F(S)$.

Normalization

Lemma 4

Let F be smooth over R, and S be Noetherian and reduced. Then the integral closure S' of S in $Q(S)$ is an F-subalgebra of $Q_F(S)$.

Some examples

Example 5

If S is Noetherian and F is finite over R, then $Q_F(S) = Q(S)$.

Example 6

Let $R = \mathbb{Z}$, $F = \mathbb{G}_m^n$, and S a domain. Then S is $\mathbb{Z}^n$-graded. We have $Q_F(S) = S_\Gamma$, where Γ is the set of nonzero homogeneous elements of S.

Example 7

Let $R = k$ be a field, V a finite dimensional k-vector space, $F = \mathrm{GL}(V)$, and $S = \mathrm{Sym}\, V$. If $\dim V \geq 2$, then $Q_F(S) = S$.

$Q_F(S)$ as a subintersection

Lemma 8

Let S be a Noetherian normal domain. Then

$$Q_F(S) = \bigcap_{P \in X^1(S),\ P^* = 0} S_P,$$

where $X^1(S)$ is the set of height one prime ideals of S, and P^* is the largest F-ideal of S contained in P. In particular, $Q_F(S)$ is a Krull domain.

$Q(S)^F$

Let $\iota : S \to S \otimes R[F]$ be the map given by $\iota(s) = s \otimes 1$. As it is flat, it induces $\iota' : Q(S) \to Q(S \otimes R[F])$. We define $Q(S)^F$ to be the kernel of the map $\iota' - \omega' : Q(S) \to Q(S \otimes R[F])$.

Remark 9

The notation $Q(S)^F$ does not mean that F acts on $Q(S)$.

$Q(S)^F$ and $Q_F(S)^F$

The following are easy.

- $Q(S)^F$ is a subring of $Q(S)$.
- $Q(S)^F \cap Q(S)^\times = (Q(S)^F)^\times$. In particular, if S is a domain, then $Q(S)^F$ is a subfield of $Q(S)$.
- If R is a field, F is of finite type over R, and $F(k)$ is Zariski dense in F, then $Q(S)^F = Q(S)^{F(k)}$.
- $Q(S)^F = Q_F(S)^F$.

Application to invariant theory

We give an application of $Q_F(S)$ to invariant theory.

Factoriality of invariant subrings

From now on, until the end of this talk, let k be a field, G an affine algebraic group (smooth of finite type) over k, and S a G-algebra.

Question 10

When is S^G a UFD?

The first cohomology group and the factoriality

Lemma 11

Let B be a UFD on which an abstract group Γ acts. If the first cohomology group $H^1(\Gamma, B^\times)$ vanishes, then B^Γ is a UFD.

Corollary 12

Let B be a UFD on which an abstract group Γ acts. If $B^\times \subset B^\Gamma$, and if there is no nontrivial group homomorphism $\Gamma \to B^\times$, then B^Γ is a UFD.

Algebraic group over an algebraically closed field

Theorem 13 (Popov)

Let k be algebraically closed, S a UFD, and the character group $X(G)$ be trivial. Assume either

(a) S is finitely generated and G is connected; or

(b) $S^\times \subset S^G$.

Then S^G is a UFD.

The ring of semiinvariants

Let χ be a character (that is, one-dimensional G-module) of G. Let V be a G-module. We define

$$V^\chi := \{v \in V \mid \omega_V(v) = v \otimes \chi\} = \sum_{\phi \in \mathrm{Hom}_G(\chi, V)} \mathrm{Im}\,\phi,$$

where we identify
$\chi \in \mathrm{Hom}_{\mathrm{Alggrp}}(G, \mathbb{G}_m) \subset \mathrm{Hom}_{\mathrm{Sch}\,/k}(G, \mathbb{A}^1 \setminus \{0\}) = k[G]^\times$. Note that $S_G := \bigoplus_{\chi \in X(G)} S^\chi$ is a k-subalgebra of S. It is $X(G)$-graded, where $X(G)$ is the character group of G. A homogeneous element of S_G is called a semiinvariant of S. The degree zero component S_G is S^G.

Notation

Let B be a domain, and $f \in B$. There is a unique largest open subset U of $\operatorname{Spec} B$ such that $f \in \Gamma(U, \mathcal{O}_{\operatorname{Spec} B})$. We call U the domain of definition of f, and denote it by $U(f)$.

Then $f : U(f) \to \mathbb{A}^1_{\mathbb{Z}}$ is a morphism. Let $(\mathbb{A}^1_{\mathbb{Z}})^* := \mathbb{A}^1_{\mathbb{Z}} \setminus 0$, where $0 \cong \operatorname{Spec} \mathbb{Z}$ is the origin. We denote $f^{-1}((\mathbb{A}^1_{\mathbb{Z}})^*)$ by $U^*(f)$.

A generalization of a theorem of Popov and Kamke (1)

Lemma 14

Let G be connected. Let S be a G-algebra domain of finite type over k. Let K be the integral closure of k in $Q(S)$. Assume that $X(G) \to X(K \otimes_k G)$ is surjective. Then for $f \in Q(S)$, the following are equivalent.

- $f \in Q_G(S)$, and f is a semiinvariant of $Q_G(S)$.
- $U^*(f)$ is a G-stable open subset of $\operatorname{Spec} S$.
- $Sf \subset Q_G(S)$ is a G-submodule.

In particular, any unit of S is a homogeneous unit of S_G.

Similar lemmas for disconnected G (1)

Lemma 15

Let S be a domain, and K denote the integral closure of k in $Q(S)$. Assume that

- $S^\times \subset S^G$;
- $G(K)$ is dense in $K \otimes_k G$;
- Sf is a G-submodule of $Q_G(S)$;
- $X(G) \to X(K \otimes_k G)$ is surjective.

Then f is a semiinvariant of $Q_G(S)$. If, moreover, $X(G)$ is trivial, then $f \in Q(S)^G$.

Similar lemmas for disconnected G (2)

Lemma 16

Let S be a domain. Let $G(k)$ be dense in G. Assme that $S^\times = k^\times$. If Sf is a $G(k)$-submodule of $Q(S)$, then $f \in Q_G(S)$, and f is a semiinvariant.

Groups with trivial character groups

If $X(G)$ is trivial, then a semiinvariant is an invariant.

Remark 17

Let k be algebraically closed.

- If N is a normal subgroup of G and $X(N)$ is trivial, then $X(G/N) \cong X(G)$.
- The canonical map $X(G/[G,G]) \to X(G)$ is an isomorphism.
- If G is unipotent, then $X(G)$ is trivial.
- If G is semisimple, then $G = [G,G]$, and $X(G)$ is trivial.

A generalization of Popov's theorem (1)

Theorem 18

Let G be connected. Let S be a finitely generated G-algebra domain over k. Let K be the integral closure of k in $Q(S)$. Assume that $X(G) \to X(K \otimes_k G)$ is surjective. Set $A := S_G$. Assume that if P is a G-stable height one prime ideal of S such that $P \cap A$ is a minimal prime of some nonzero principal ideal, then P is a principal ideal. Then

- If P is a G-stable height one prime ideal of S such that $P \cap A$ is a minimal prime of a nonzero principal ideal, then $P = Sf$ for some homogeneous prime element f of A.
- A is a UFD.
- Any homogeneous prime element of A is a prime element of S.
- If, moreover, $X(G)$ is trivial, then $S^G = A$ is a UFD.

A generalization of Popov's theorem (2)

Proposition 19

Let G be connected. Let S be a G-algebra. Assume that S is a UFD. Assume that $X(G) \to X(K \otimes_k G)$ is surjective, where K is the integral closure of k in S. Then $A := S_G$ is a UFD. Any homogeneous prime element of A is a prime element of S. If, moreover, $X(G)$ is trivial, then $S^G = A$ is a UFD.

Remark 20

In the proposition, we need not assume that S is finitely generated.

A generalization of Popov's theorem (3)

Lemma 21

Let S be a G-algebra which is a UFD. Assume that $G(K)$ is dense in $K \otimes_k G$, where K is the integral closure of k in S. Assume that $X(K \otimes_k G)$ is trivial. Assume also that $S^\times \subset A = S^G$. Then A is a UFD.

Corollary 22

Let S be a G-algebra which is a UFD. Assume that $S^\times = k^\times$. If $G(k)$ is dense in G and $X(G)$ is trivial, then S^G is a UFD.

The Italian problem

Problem 23 (Mukai)

When we have $Q(S)^G = Q(S^G)$?

The problem is called the Italian problem.

A generalization of a theorem of Popov and Kamke (2)

Proposition 24

Let G be connected. Let S be a G-algebra which is a Krull domain. Assume also that any G-stable height one prime ideal of S is principal (e.g., S is a UFD). Moreover, assume that $X(G) \to X(K \otimes_k G)$ is surjective, where K is the integral closure of k in S. Then $Q_G(S)_G = Q_T(A)$, where $T = \operatorname{Spec} kX(G)$. If, moreover, $X(G)$ is trivial, then $Q(S)^G = Q(S^G)$.

Geometric approach (1)

Let S be a finitely generated G-algebra domain. Set $X := \operatorname{Spec} S$. Let

$$s := \max\{\dim Gx \mid x \in X\} = \dim G - \min\{\dim G_x \mid x \in X\}.$$

Proposition 25

We have

$$s = \dim S - \operatorname{trans.deg}_k Q(S)^G.$$

Geometric approach (2)

Let S be a finitely generated G-algebra domain. Set $r := \dim S - \operatorname{trans.deg}_k Q(S^G)$.

Lemma 26

If S is normal, then $Q(S^G) = Q(S)^G$ if and only if $r = s$.

Example

Let $G = \mathbb{G}_m$ act on $\mathbb{A}^2 = \operatorname{Spec} k[x, y]$ via $\deg x = \deg y = 1$. Then $r = 2$ and $s = 1$. $Q(S)^G = k(x/y)$ and $Q(S^G) = k$.

The main theorem

Theorem 27

Let S be a finitely generated G-algebra which is a normal domain. Assume that G is connected. Assume that $X(G) \to X(K \otimes_k G)$ is surjective, where K is the integral closure of k in S. Let $X_G^1(S)$ be the set of height one G-stable prime ideals of S. Let $M(G)$ be the subgroup of the class group $\mathrm{Cl}(S)$ of S generated by the image of $X_G^1(S)$. Let Γ be a subset of $X_G^1(S)$ whose image in $M(G)$ generates $M(G)$. Set $A := S_G$. Assume that $Q_G(S)_G \subset Q(A)$. Assume that if $P \in \Gamma$, then either the height of $P \cap A$ is not one or $P \cap A$ is principal. Then A is a UFD. If, moreover, $X(G)$ is trivial, then $S^G = A$ is a UFD.

Example (1)

Example 28

An example of Theorem 27. Let $n \geq m \geq t \geq 2$ be positive integers, $V := k^m$, $W := k^n$, and $M := V \otimes W$. Let $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ be the standard bases of V and W, respectively. Let $S := (\operatorname{Sym} M)/I_t$, where $I_t = I_t(v_i \otimes w_j)$ is the determinantal ideal. Let G be the subgroup of the unipotent upper triangular matrices in $\mathrm{GL}_m = \mathrm{GL}(V)$. Then $A = S^G$ is a UFD.

Example (2)

The sketch of the proof of Example 28. Let P be the ideal of S generated by the $(t-1)$-minors of the first $(t-1)$ rows of the matrix $(v_i \otimes w_j)$. P is G-invariant, and generates $\mathrm{Cl}(S) = M(G) \cong \mathbb{Z}$. We set $\Gamma := \{P\}$. It is easy to check that

- $\dim S = (t-1)(m+n-t+1)$;
- S^G is finitely generated, and $\dim S^G = (t-1)(n+1-t/2)$;
- $\dim S^G/P^G = (t-2)(n+1-(t-1)/2)$;
- $\operatorname{ht} P^G = n-t+2 \geq 2$.

Examples (3)

Note that S is normal and $K = k$. As G is unipotent, $X(G)$ is trivial. To apply the theorem, it remains to show that $Q_G(S)_G \subset Q(S_G)$. As $X(G)$ is trivial. This is equivalent to $Q(S)^G = Q(S^G)$. So it suffices to show that $r = s$. Clearly $r = \dim S - \dim S^G = (t-1)(m-t/2)$. On the other hand, the orbit Gx, where

$$x = \begin{pmatrix} E_{t-1} & O \\ O & O \end{pmatrix} \in (\operatorname{Spec} S)(k),$$

is $(t-1)(m-t/2)$-dimensional, as can be seen easily. So $r = s$, as desired.

Another Example

Example 29

A finite group G acting on a UFD S such that there is no nontrivial homomorphism $G \to S^\times$, but S^G is not a UFD.

$G := \mathbb{Z}/3\mathbb{Z} = \langle \sigma \rangle$, k an algebraically closed field of characteristic 3. $S := k[A^{\pm 1}, B^{\pm 1}]$, and G acts on S via $\sigma A = B$ and $\sigma B = (AB)^{-1}$. Then S is a UFD. $\operatorname{Spec} S \to \operatorname{Spec} S^G$ is étale in codimension one. So by Fossum's theorem, $\operatorname{Cl}(S^G) \cong H^1(G, S^\times) \cong \mathbb{Z}/3\mathbb{Z}$.

Yet another example (1)

Example 30

S is a finitely generated UFD over k, G is connected, $X(G)$ is trivial, but S^G is not a UFD.

$k = \mathbb{R}$,

$$G = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a^2 + b^2 = 1 \right\} \subset \mathrm{GL}_2(k).$$

Let G act on $S := \mathbb{C}[x, y, s, t]$ by

Yet another example (2)

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} x = (a + b\sqrt{-1})x, \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix} y = (a + b\sqrt{-1})y,$$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} s = (a - b\sqrt{-1})s, \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix} t = (a - b\sqrt{-1})t$$

(G acts trivially on $\mathbb{C}$). Then S is a finitely generated UFD over $\mathbb{R}$, G is connected, $X(G)$ is trivial, but $S^G = \mathbb{C}[xs, xt, ys, yt]$ is not a UFD.

Some remarks on graded UFD

Lemma 31

Let B be a $\mathbb{Z}^n$-graded Krull domain. If any nonzero homogeneous element is either a unit or divisible by a prime element, then B is a UFD.

Proposition 32

Let B be a $\mathbb{Z}^n$-graded domain. If any nonzero homogeneous element of B is either a unit or a product of prime elements, then B is a UFD.

The equivariant class group

Let S be a Noetherian normal domain. Let us denote the set of isomorphism classes of (G, S)-modules which are reflexive of rank one as S-modules by $\mathrm{Cl}(G, S)$. We denote the class of M in $\mathrm{Cl}(G, S)$ by $[M]$. Defining the product by

$$[M] \cdot [M'] := [(M \otimes_S M')^{**}],$$

$\mathrm{Cl}(G, S)$ is an abelian group, where $(-)^* = \mathrm{Hom}_S(-, S)$. The class group $\mathrm{Cl}(S^G)$ can be viewed as the group of isomorphism classes of divisorial fractional ideals of S^G modulo the group of principal fractional ideals.

The relationship between $\mathrm{Cl}(G, S)$ and $\mathrm{Cl}(S^G)$

Proposition 33

Assume that S is a Noetherian normal domain. Then the map $\Phi_S : \mathrm{Cl}(S^G) \to \mathrm{Cl}(G, S)$ given by

$$\Phi_S([I]) = [(S \otimes_{S^G} I)^{**}]$$

is an injective group homomorphism.

Corollary 34

If S is a polynomial ring over k, then $\mathrm{Cl}(S^G)$ is isomorphic to a subgroup of the character group $X(G)$. So it is finitely generated.

Proof.

This is because $\mathrm{Cl}(G, S) \cong X(G)$. □

Principal G-bundle

Let G act on a k-scheme X. We say that a morphism $\varphi : X \to Y$ is a principal fiber bundle (with the group G) if

1. G acts trivially on Y, and φ is a G-morphism.
2. φ is faithfully flat and quasi-compact.
3. The map $\Phi : G \times X \to X \times_Y X$ given by $\Phi(g, x) = (gx, x)$ is an isomorphism.

Almost principal G-bundle

Let G act on a k-scheme X. We say that a morphism $\varphi : X \to Y$ is an almost principal fiber bundle if

1. G acts trivially on Y, and φ is a G-morphism.
2. φ is affine, and the canonical map $\mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^G$ is an isomorphism.
3. X is Noetherian normal.
4. There exists some closed subset Z of Y such that
    1. $\operatorname{codim}_Y Z \geq 2$;
    2. $\operatorname{codim}_X \varphi^{-1}(Z) \geq 2$;
    3. $\varphi|_{\varphi^{-1}(Y \setminus Z)} : \varphi^{-1}(Y \setminus Z) \to Y \setminus Z$ is a fiber bundle.

Equivalence on reflexive sheaves

Proposition 35

Let $\varphi : X \to Y$ be an almost principal fiber bundle with the group G. Assume that Y is Noetherian. Then the functor $\alpha : \mathrm{Rx}(Y) \to \mathrm{Rx}(G, X)$ given by

$$\alpha(\mathcal{M}) = (\varphi^* \mathcal{M})^{**}$$

is an equivalence with a quasi-inverse β given by

$$\beta(\mathcal{N}) = (\varphi_* \mathcal{N})^G,$$

where $\mathrm{Rx}(Y)$ is the category of reflexive coherent sheaves on Y, and $\mathrm{Rx}(G, X)$ is the category of cohernt $(G, \mathcal{O}_X)$-modules which are reflexive as $\mathcal{O}_X$-modules.

Isomorphism of class groups

Corollary 36

Assume further that X is integral. Then α and β induce isomorphisms between $\mathrm{Cl}(Y)$ and $\mathrm{Cl}(G, X)$.

Corollary 37

If, moreover, $X = \operatorname{Spec} k[x_1, \ldots, x_n]$, then $Cl(Y) \cong X(G)$.

Example of finite groups in SL_2

Example 38

Let $S = \mathbb{C}[[x, y]]$ on which $\mathrm{SL}_2(\mathbb{C})$ acts in a natural way. Let G be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. Then $\varphi : X = \operatorname{Spec} S \to \operatorname{Spec} S^G = Y$ is an almost principal fiber bundle, since G does not have a pseudo-reflection. So

1. $\mathrm{Cl}(S^G) \cong X(G)$. In particular, if G is the binary icosahedral group (a central extension of the alternating group A_5 of order 120), then S^G is a UFD.
2. $\mathcal{P}(S * G) = \mathrm{Rx}(G, S) \to \mathrm{Rx}(S^G) = \mathrm{CM}(S^G)$ given by $N \mapsto N^G$ is an equivalence, where $\mathcal{P}(S * G)$ is the category of finitely generated projective left modules over the twisted group algebra $S * G$, and $\mathrm{CM}(S^G)$ is the category of maximal Cohen–Macaulay S^G-modules. In particular, S^G is of finite representation type.

Determinantal rings

Example 39

Let $n \geq m \geq t \geq 2$, and $V := k^n$, $W := k^m$, and $E := k^{t-1}$. Set $X := \mathrm{Hom}(E, W) \times \mathrm{Hom}(V, E)$ and $Y := \{f \in \mathrm{Hom}(V, W) \mid \mathrm{rank}\, f < t\}$. $G = \mathrm{GL}(E)$ acts on X by $g(\varphi, \psi) = (\varphi g^{-1}, g\psi)$. Define $\pi : X \to Y$ by $\pi(\varphi, \psi) = \varphi\psi$. Then π is an almost principal fiber bundle (De Concini–Procesi, H—). Hence, as is well known, $\mathrm{Cl}(Y) \cong X(G) \cong \mathbb{Z}$.

Thank you.

This slide will soon be available at
http://www.math.nagoya-u.ac.jp/~hasimoto/