

“Geometric quotients are algebraic schemes” based on Fogarty’s idea

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Abstract

Let S be a Noetherian scheme, $\varphi : X \rightarrow Y$ a surjective S -morphism of S -schemes, with X of finite type over S . We discuss what makes Y of finite type.

First, we prove that if S is excellent, Y is reduced, and φ is universally open, then Y is of finite type. We apply this to understand Fogarty’s theorem in “Geometric quotients are algebraic schemes, Adv. Math. **48** (1983), 166–171” for the special case that the group scheme G is flat over the Noetherian base scheme S and that the quotient map is universally submersive. Namely, we prove that if G is a flat S -group scheme of finite type acting on X and φ is its universal strict orbit space, then Y is of finite type (S need not be excellent. Geometric fibers of G can be disconnected and non-reduced).

Utilizing the technique used there, we also prove that Y is of finite type if φ is flat. The same is true if S is excellent, φ is proper, and Y is Noetherian.

1. Introduction

In [1], Fogarty proved the following theorem.

Theorem 1.1 (Fogarty). *Let S be an excellent scheme, G an S -group scheme of finite type with connected geometric fibers. Let X be a G -scheme of finite type over S . If (Y, φ) is a strict orbit space for the action of G on X ,*

then Y is of finite type over S . Moreover, if \mathcal{F} is a coherent (G, \mathcal{O}_X) -module (coherent G -linearized \mathcal{O}_X -module), then $(\varphi_*\mathcal{F})^G$ is a coherent \mathcal{O}_Y -module.

The main purpose of this paper is to try to understand this very important theorem in invariant theory. The author has not understood his proof yet. On the other hand, his idea is transparent, and if we assume that G is S -flat and φ is universally submersive, then one can keep track of his proof without much difficulty, and even remove some other assumptions. Namely, we prove

Theorem 1.2. *Let S be a Noetherian scheme, G a flat S -group scheme of finite type. Let X be a G -scheme of finite type over S . If (Y, φ) is a universal strict orbit space for the action of G on X , then Y is of finite type over S . Moreover, if \mathcal{F} is a coherent (G, \mathcal{O}_X) -module, then $(\varphi_*\mathcal{F})^G$ is a coherent \mathcal{O}_Y -module.*

Flatness assumption is important in our proof, since we use the assumption that the image and the kernel of a G -equivariant \mathcal{O}_X -module map between (G, \mathcal{O}_X) -modules are again (G, \mathcal{O}_X) -modules. Another merit in using flatness is the universal openness of universal orbit spaces [7, p.6].

We do not require that S is excellent. We do not require that G has connected geometric fibers. The theorem includes the case that $X = G$. So we do not claim that X_{red} is G -stable, or that irreducible components of X are G -stable.

The proof for the most essential case, the case that Y is reduced and S is excellent is purely ring-theoretic, see Theorem 2.3. The proof heavily depends on the idea of Fogarty [1] and Onoda [8]. To remove excellence assumption of the base S , we utilize Onoda's result [8, (2.20)].

Utilizing the technique used above, we also prove some finite generation results which do not have direct connection to group actions. Let S be a Noetherian scheme, and $\varphi : X \rightarrow Y$ a surjective morphism of S -schemes, with X of finite type. If φ is flat, then Y is of finite type (Corollary 2.6). If S is excellent, φ is proper, and Y is Noetherian, then Y is of finite type (Theorem 4.2).

Onoda [8] proved that if S is Nagata and all normal local rings that are essentially of finite type over S are analytically irreducible (e.g., S is excellent), Y is Noetherian normal, and the generic point of any irreducible component of X is mapped to the generic point of an irreducible component of Y , then Y is of finite type. Fogarty [2] proved the same result independently later.

Our argument more or less follows theirs, but none of our new assertions here does not cover their theorem.

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2. Main theorem — the reduced case

Throughout this section, let S be a Noetherian scheme, $\varphi : X \rightarrow Y$ a surjective S -morphism of S -schemes, with X of finite type over S . The following proposition is in [1]. We state and give a proof for readers' convenience.

Proposition 2.1. *Assume that $S = \text{Spec } R$ is affine, and $Y = \text{Spec } B$ is affine with B an integral domain. Then there is a finitely generated R -subalgebra A of B such that the induced morphism $\eta : Y = \text{Spec } B \rightarrow \text{Spec } A =: Z$ is birational and geometrically injective.*

Proof. Let (U_i) be a finite affine open covering of X . Then replacing X by $\coprod_i U_i$, we may assume that $X = \text{Spec } C$ is affine.

Let J be the kernel of the canonical map $C \otimes_R C \rightarrow C \otimes_B C$. Then there exist some $b_1, \dots, b_r \in B$ such that J is generated by

$$b_1 \otimes 1 - 1 \otimes b_1, \dots, b_r \otimes 1 - 1 \otimes b_r.$$

Set $A_0 = R[b_1, \dots, b_r]$. Let $y, y' \in Y(\xi)$ be distinct geometric points of Y , where ξ is an algebraically closed field. Since φ is surjective of finite type, there exist $x, x' \in X(\xi)$ such that $\varphi(x) = y$ and $\varphi(x') = y'$. Since $(x, x') \in (X \times X)(\xi) \setminus (X \times_Y X)(\xi)$, there exists some i such that $b_i(x) \neq b_i(x')$. This shows that the image of y and y' in $(\text{Spec } A_0)(\xi)$ are different. So $Y \rightarrow \text{Spec } A_0$ is geometrically injective. It is clear that there exist some $a_1, \dots, a_t \in B$ such that $A = A_0[a_1, \dots, a_t]$ is birational to B . Then A is the desired subalgebra, since $Y \rightarrow \text{Spec } A$ is still geometrically injective. \square

The following is [1, Lemma 3].

Lemma 2.2. *Let $\psi : U \rightarrow Z$ be an affine birational morphism between integral schemes. If Z is Noetherian normal and $\psi(U)$ is an open subset of Z , then ψ is an open immersion.*

We omit the proof.

The following is based on the ideas of Fogarty [1] and Onoda [8].

Theorem 2.3. *Let S be a universally catenary Nagata scheme (i.e., for any affine open subset $U = \text{Spec } R$ of S , R is universally catenary and Nagata), and $\varphi : X \rightarrow Y$ a surjective universally open S -morphism of S -schemes. If X is of finite type over S and Y is reduced, then Y is of finite type over S .*

Proof. Clearly, Y is quasi-compact. So the question is local on S and Y , and so we may assume that $S = \text{Spec } R$ and $Y = \text{Spec } B$ are affine. Let (U_i) be a finite affine open covering of X . Then replacing X by $\coprod_i U_i$, we may assume that $X = \text{Spec } C$ is also affine.

Since X has only finitely many irreducible components and φ is surjective, B has only finitely many minimal primes. Since $B \rightarrow \prod_{P \in \text{Min}(B)} B/P$ is injective and finite, it suffices to show that B/P is of finite type for $P \in \text{Min}(B)$. Replacing B by B/P , we may assume that B is a domain.

Take $A \hookrightarrow B$ as in Proposition 2.1 so that $\eta : Y = \text{Spec } B \rightarrow \text{Spec } A = Z$ is geometrically injective and birational, and Z is of finite type over R . Now let A' be the normalization of A , and $B' = B[A']$. Since R is Nagata, the associated morphism $\alpha : Z' = \text{Spec } A' \rightarrow \text{Spec } A = Z$ is finite. Let $Y' = \text{Spec } B'$ and $X' = X \times_Y Y'$.

$$\begin{array}{ccccc} X' & \xrightarrow{\varphi'} & Y' & \xrightarrow{\eta'} & Z' \\ \downarrow \gamma & & \downarrow \beta & & \downarrow \alpha \\ X & \xrightarrow{\varphi} & Y & \xrightarrow{\eta} & Z \end{array}$$

Note that Y' is a closed subscheme of $Y \times_Z Z'$. In particular, β is finite, since α and the closed immersion are. Similarly, η' is geometrically injective. Clearly, γ is finite and φ' is universally open.

Let $x' \in X'$. Set $y' = \varphi'(x')$, $z' = \eta'(y')$, $F = \mathcal{O}_{X',x'}$, $E = \mathcal{O}_{Y',y'}$, and $D = \mathcal{O}_{Z',z'}$. Then for any minimal prime Q of F , we have $Q \cap E = 0$, since φ' is open.

Since Z' is universally catenary, there exists some $n \geq 0$ such that

$$\begin{aligned} \dim E(t_1, \dots, t_n) - \dim D = \\ \text{trans.deg}_{R(Z')} R(Y')(t_1, \dots, t_n) - \text{trans.deg}_{\kappa(z')} \kappa(y')(t_1, \dots, t_n) \end{aligned}$$

by Onoda's dimension formula [8, (1.11)], where t_1, \dots, t_n are variables, and for a local ring (O, \mathfrak{m}) , $O(t_1, \dots, t_n)$ denotes the local ring $O[t_1, \dots, t_n]_{\mathfrak{m}[t_1, \dots, t_n]}$. Since η' is universally injective, $\kappa(y')$ is a purely inseparable algebraic extension of $\kappa(z')$ by [3, (3.5.8)]. Thus the right hand side is zero, since $R(Z') = R(Y')$.

Let P be a minimal prime of $\mathfrak{m}_{y'}F$ such that $\dim F \otimes_E \kappa(y') = \dim F/P$. Since $\varphi' \times 1 : X' \times \mathbb{A}^n \rightarrow Y' \times \mathbb{A}^n$ is an open map, we have

$$\text{ht } P = \text{ht } P[t_1, \dots, t_n] \geq \dim E(t_1, \dots, t_n) = \dim D.$$

So we have

$$\dim F \geq \dim D + \dim(F \otimes_E \kappa(y')) = \dim D + \dim(F \otimes_D \kappa(z'))$$

by the geometric injectivity of η' . Hence we have $\dim F = \dim D + \dim(F \otimes_D \kappa(z'))$ by [6, (15.1)].

For $r \geq 0$, define $X'(r)$ to be $(\bigcup X'_r) \setminus (\bigcup X'_{>r})$, where X'_r (resp. $X'_{>r}$) runs through all irreducible components (with reduced structures) of X' such that $\text{trans.deg}_{R(Y')} R(X'_r) = r$ (resp. $\text{trans.deg}_{R(Y')} R(X'_{>r}) > r$). Let $x' \in X'(r)$. By the dimension formula [5, (14.C)], we have that

$$\begin{aligned} \dim \mathcal{O}_{X'(r), x'} &= \dim \mathcal{O}_{X', x'} = \dim \mathcal{O}_{Z', z'} + \dim \mathcal{O}_{X', x'} \otimes_{\mathcal{O}_{Z', z'}} \kappa(z') \\ &\geq \dim \mathcal{O}_{Z', z'} + \dim \mathcal{O}_{X'(r), x'} \otimes_{\mathcal{O}_{Z', z'}} \kappa(z'), \end{aligned}$$

where $z' = (\eta'\varphi')(x')$. So we have

$$\dim \mathcal{O}_{X'(r), x'} = \dim \mathcal{O}_{Z', z'} + \dim \mathcal{O}_{X'(r), x'} \otimes_{\mathcal{O}_{Z', z'}} \kappa(z').$$

Since all local rings of $X'(r)$ are equidimensional by the dimension formula, we have that $\eta'\varphi'|_{X'(r)}$ is equidimensional for any r [4, (13.3.6)]. Since Z' is normal, $\eta'\varphi'|_{X'(r)}$ is universally open for any r , by Chevalley's criterion [4, (14.4.4)]. Since $X' = \bigcup_{r \geq 0} X'(r)$, we have that $\eta'\varphi'$ is universally open.

Since φ' is surjective, $\eta'(Y') = (\eta'\varphi')(X')$ is open in Z' . Since η' is affine birational and Z' is Noetherian normal, η' is an open immersion by Lemma 2.2. So Y' is of finite type. Since B' is finitely generated and $B \rightarrow B'$ is finite and injective, B is finitely generated. \square

Remark 2.4. We can also prove the following. Let S be a Nagata scheme, and $\varphi : X \rightarrow Y$ a surjective universally open equidimensional S -morphism of S -schemes. If X is of finite type over S and Y is reduced, then Y is of finite type over S . Indeed, since φ is universally open and equidimensional, φ' is so. Since φ' is equidimensional and η' is geometrically injective and birational, $\eta'\varphi'$ is also equidimensional, and the same argument works.

Corollary 2.5. *Let S be an excellent scheme, G a flat S -group scheme of finite type, and X a G -action of finite type over S . If $\varphi : X \rightarrow Y$ is a universal strict orbit space and Y is reduced, then Y is of finite type.*

Proof. φ is universally open and surjective, see [7, p.6]. So we are done. \square

Corollary 2.6. *Let S be a Noetherian scheme, and $\varphi : X \rightarrow Y$ a faithfully flat S -morphism of S -schemes. If X is of finite type over S , then Y is of finite type over S .*

Proof. We may assume that $S = \text{Spec } R$, $Y = \text{Spec } B$, and $X = \text{Spec } C$ are all affine. Since B is a pure subring of C and C is Noetherian, we have that B is also Noetherian. Since the nilradical of B is a finitely generated ideal, it suffices to show that B_{red} is of finite type. Replacing B by B_{red} and C by $C \otimes_B B_{\text{red}}$, we may assume that B is reduced. So it suffices to show that $\prod_{P \in \text{Min}(B)} B/P$ is of finite type. Replacing B by B/P , we may assume that B is a domain. As C is faithfully flat over the domain B , there exists some prime ideal Q of C such that $Q \cap B = 0$. By [8, (2.11) and (2.20)], we may assume that R is local. By [4, (2.7.1)], we may assume that R is a complete local ring. Replacing B again if necessary, we may still assume that B is a domain. Since R is excellent and B is a domain, B is of finite type by the theorem. \square

3. The general case

In this section, we prove Theorem 1.2. First we prove the following

Lemma 3.1. *Let S be a Noetherian scheme, and G a flat S -group scheme of finite type. Let X be a G -scheme of finite type over S . Assume that for any closed G -subscheme X_1 of X and its universal strict orbit space $\psi : X_1 \rightarrow Y_1$, we have that Y_1 is of finite type, provided Y_1 is reduced (e.g., S is excellent, see Corollary 2.5). If (Y, φ) is a universal strict orbit space for the action of G on X , then Y is of finite type over S . Moreover, if \mathcal{F} is a coherent (G, \mathcal{O}_X) -module, then $(\varphi_* \mathcal{F})^G$ is a Noetherian \mathcal{O}_Y -module.*

Proof. We may assume that $S = \text{Spec } R$ and $Y = \text{Spec } B$ are affine.

First, we prove the last assertion using the Noetherian induction on the coherent ideal sheaf $\text{ann } \mathcal{F} := \text{Ker}(\mathcal{O}_X \rightarrow \underline{\text{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}))$. We may assume that $\mathcal{F} \neq 0$. We also use the induction on $\nu(\mathcal{F}) := \sum_V \text{length}_{\mathcal{O}_{X,v}} \mathcal{F}_v$, where

V runs through the irreducible components of $\text{supp } \mathcal{F} = V(\underline{\text{ann}} \mathcal{F})$, and v is the generic point of V .

Let \mathcal{G} be the maximal coherent (G, \mathcal{O}_X) -submodule of \mathcal{F} such that $(\varphi_* \mathcal{G})^G$ is a Noetherian \mathcal{O}_Y -module. As $(\varphi_* ?)^G$ is left exact, we may assume that $\mathcal{G} = 0$, replacing \mathcal{F} by \mathcal{F}/\mathcal{G} . In particular, any nonzero coherent (G, \mathcal{O}_X) -subsheaf of \mathcal{F} has the same annihilator as that of \mathcal{F} .

If $(\varphi_* \mathcal{F})^G = 0$, then there is nothing to be proved. So we consider the case that $H^0(X, \mathcal{F})^G = (\varphi_* \mathcal{F})^G \neq 0$. Take $a \in H^0(X, \mathcal{F})^G \setminus \{0\} = \text{Hom}_{G, \mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \setminus \{0\}$. Then $a\mathcal{O}_X$ is a nonzero coherent (G, \mathcal{O}_X) -subsheaf of \mathcal{F} . So $\nu(a\mathcal{O}_X) \neq 0$. If $\nu(\mathcal{F}) > \nu(a\mathcal{O}_X)$, then $\mathcal{G} \supset a\mathcal{O}_X \neq 0$ by induction assumption, and this is a contradiction. So $\nu(\mathcal{F}) = \nu(a\mathcal{O}_X)$. So $\nu(\mathcal{F}/a\mathcal{O}_X) = 0$, which shows that $\text{supp}(\mathcal{F}/a\mathcal{O}_X) \subsetneq \text{supp } \mathcal{F}$. By induction assumption, $\varphi_*(\mathcal{F}/a\mathcal{O}_X)^G$ is Noetherian. So $\varphi_*(a\mathcal{O}_X)^G$ is not Noetherian, since we assume that $\mathcal{F} \neq 0 = \mathcal{G}$. Let \mathcal{J} be the kernel of $a : \mathcal{O}_X \rightarrow \mathcal{F}$, and Z be the G -stable closed subscheme of X defined by the coherent G -ideal \mathcal{J} of \mathcal{O}_X . We may assume that $\mathcal{F} = \mathcal{O}_Z$, from the beginning. If $H^0(X, \mathcal{O}_Z)^G$ is not reduced, then there exists some $b \in H^0(X, \mathcal{O}_Z)^G \setminus \{0\}$ such that $b^2 = 0$. Then $0 \neq b\mathcal{O}_Z \subset \mathcal{O}_Z$, and the annihilator of $b\mathcal{O}_Z$ is strictly larger than that of \mathcal{O}_Z . By induction assumption, this is a contradiction. So $B_1 = H^0(X, \mathcal{O}_Z)^G$ must be a reduced ring.

Set $Y_1 := \text{Spec } B_1$, B_0 to be the image of the canonical map $B \rightarrow B_1$, and $Y_0 = \text{Spec } B_0$ the scheme theoretic image of $Y_1 \rightarrow Y$. Let $\varphi_1 : Z \rightarrow Y_1$ and $\eta' : Y_1 \rightarrow Y_0$ be the canonical maps. Note that $Y_0 = \varphi(Z) = \text{Im}(\eta' \varphi_1)$ set-theoretically, since Z is G -stable closed, and φ is an orbit space. We have $\varphi^{-1}(Y_0) = Z$ set theoretically, so the inclusion $Z \hookrightarrow \varphi^{-1}(Y_0)$ is a universal homeomorphism. Hence $\eta' \varphi_1$ is surjective and universally open. It follows that $(Y_0, \eta' \varphi_1)$ is a universal strict orbit space. Since Y_0 is reduced, Y_0 is of finite type by assumption.

By [1, Proposition 1], (Y_1, φ_1) is a universal geometric quotient, and η' is a universal homeomorphism. In particular, Y_1 is of finite type by assumption. So $\eta' : Y_1 \rightarrow Y_0$ is a universal homeomorphism of finite type between Noetherian schemes. So η' is finite. Hence B_1 is a B_0 -finite module. Since B_0 is of finite type over R , B_1 is a Noetherian B_0 -module. Hence $(\varphi_* \mathcal{O}_Z)^G$ is a Noetherian \mathcal{O}_Y -module, as desired.

Next, we prove that $Y = \text{Spec } B$ is of finite type. We use Noetherian induction, and we may assume that for a G -stable closed subscheme $X_1 \subsetneq X$ and its universal strict orbit space $\varphi_1 : X_1 \rightarrow Y_1$, Y_1 is of finite type. If Y is reduced, then there is nothing to be proved. So assume that there exists

some $b \in B \setminus \{0\}$ such that $b^2 = 0$.

By assumption, $B \subset \tilde{B} := H^0(X, \mathcal{O}_X)^G$, and \tilde{B} is a Noetherian B -module by what we have already proved. Since B is a B -submodule of \tilde{B} , we have that B is a Noetherian ring. So we only need to prove that B_{red} is of finite type.

Set X_1 to be the G -stable closed subscheme of X defined by $b\mathcal{O}_X$, and let Y_1 be the scheme theoretic image of $X_1 \hookrightarrow X \rightarrow Y$. Then $X_1 \rightarrow Y_1$ is a universal strict orbit space, and hence Y_1 is of finite type by induction assumption. Hence $Y_{\text{red}} = (Y_1)_{\text{red}}$ is also of finite type, as desired. \square

Proof of Theorem 1.2. We only need to check the assumption of Lemma 3.1. So we may assume that Y is reduced, and it suffices to prove that Y is of finite type. We may assume that $S = \text{Spec } R$ and $Y = \text{Spec } B$ are affine.

Note that for any Noetherian flat R -algebra R' , the base change $\varphi' : X' \rightarrow Y'$ is again a universal strict orbit space. By Onoda's theorem [8, (2.11) and (2.20)], we may assume that R is local. Since \hat{R} is excellent, $\hat{R} \otimes_R Y$ is of finite type again by Lemma 3.1. By the descent argument [4, (2.7.1)], Y is of finite type. \square

4. Proper morphisms

Let R be a Noetherian ring, and $\varphi : X \rightarrow Y$ a surjective R -morphism of R -schemes with X of finite type.

Lemma 4.1. *Let R be universally catenary, $\varphi : X \rightarrow Y$ proper, and Y Noetherian. Then Y is universally catenary (i.e., all local rings of Y are universally catenary).*

Proof. Replacing Y by an integral scheme of finite type over Y , it suffices to show that under the assumption of the lemma, if $Y = \text{Spec } B$ is affine and integral, $Q, P \in \text{Spec } B$ with $Q \subset P$, $\text{ht}(P/Q) = 1$, then $\text{ht } P = \text{ht } Q + 1$. Replacing X by an irreducible component with the reduced structure that is surjectively mapped onto Y , we may assume that X is integral.

By Proposition 2.1, there exists some finitely generated R -subalgebra $A \subset B$ such that $\eta : Y = \text{Spec } B \rightarrow \text{Spec } A$ is birational and geometrically injective. Since φ is surjective, the dimension formula holds by [8, (1.11)], and we have $\text{ht } P = \text{ht}(P \cap A)$ and $\text{ht } Q = \text{ht}(Q \cap A)$.

As φ is a closed morphism, there is a sequence x_0, x_P, x_Q of points of X such that x_0 is a closed point of the generic fiber, x_Q is a specialization of x_0

and $f(x_Q) = Q$, and x_P is a specialization of x_Q and $f(x_P) = P$. Then by the dimension formula, we have

$$\begin{aligned}\dim \mathcal{O}_{X,x_P} &= \text{ht } P + \text{trans.deg}_{R(Y)} R(X) - \text{trans.deg}_{\kappa(P)} \kappa(x_P) \\ \dim \mathcal{O}_{X,x_Q} &= \text{ht } Q + \text{trans.deg}_{R(Y)} R(X) - \text{trans.deg}_{\kappa(Q)} \kappa(x_Q) \\ \dim \mathcal{O}_{\overline{x_Q},x_P} &= \text{ht}(P/Q) + \text{trans.deg}_{\kappa(Q)} \kappa(x_Q) - \text{trans.deg}_{\kappa(P)} \kappa(x_P).\end{aligned}$$

Since \mathcal{O}_{X,x_P} is catenary, we have that

$$\dim \mathcal{O}_{X,x_P} = \dim \mathcal{O}_{X,x_Q} + \dim \mathcal{O}_{\overline{x_Q},x_P}.$$

Hence $\text{ht } P = \text{ht } Q + \text{ht}(P/Q) = \text{ht } Q + 1$, as desired. \square

Theorem 4.2. *Let R be an excellent ring, and $f : X \rightarrow Y$ a surjective proper morphism of R -schemes. If X is of finite type over R and Y is a Noetherian scheme, then Y is of finite type over R .*

Proof. We may assume that $Y = \text{Spec } B$ is affine and integral. We may assume that X is integral. As in the proof of Lemma 4.1, we take a finitely generated R -subalgebra A of B such that $\text{Spec } B \rightarrow \text{Spec } A$ is geometrically injective. The dimension formula holds between A and B , and B is universally catenary. By [8, (4.9)], B is of finite type. \square

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