G-prime and G-primary G-ideals on G-schemes

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Abstract

Let G be a flat finite-type group scheme over a scheme S, and X a noetherian S-scheme on which G-acts. We define and study G-prime and G-primary G-ideals on X and study their basic properties. In particular, we prove the existence of minimal G-primary decomposition and the well-definedness of G-associated G-primes. We also prove a generalization of Matijevic–Roberts type theorem. In particular, we prove Matijevic–Roberts type theorem on graded rings for F-regular and F-rational properties.

1. Introduction

In this introduction, let R be a noetherian base ring, G a flat group scheme of finite type over R, and consider a noetherian R-algebra A with a G-action, for simplicity.

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Let H be a finitely generated abelian group, $R = \mathbb{Z}$, and $W = RH$ the group algebra. Letting each $h \in H$ group-like, W is a finitely generated flat commutative Hopf algebra over R, and hence $G = \operatorname{Spec} W$ is an affine flat R-group scheme of finite type. It is well-known that a G-algebra is nothing but an H-graded ring, and for a G-algebra A, a (G, A)-module is nothing but a graded A-module. This is the most typical and important case, and actually many of our ideas and results in this paper for general G already appeared in [6], [7], [17], [25], and [23] as those for H-graded rings. Note that in [17] and [25], non-finitely generated H is treated (until section 5, our G need not be of finite type, and so the case that H is not finitely generated is also treated in our discussion).

Let $\mathfrak{p}$ be a prime ideal of a $\mathbb{Z}^n$-graded ring A. Then $\mathfrak{p}^*$, the largest homogeneous ideal contained in $\mathfrak{p}$, is again a prime ideal. An associated prime of a homogeneous ideal of a noetherian $\mathbb{Z}^n$-graded ring is again homogeneous. These well-known facts on graded rings can be generalized to results on actions of smooth groups with connected fibers, see Corollary 6.25.

However, this is not true any more for more general group scheme actions. For example, these results fail to be true for torsion-graded rings. But even for general G-algebra for a group scheme G, the ideal of the form $\mathfrak{p}^*$ is very special among other G-ideals, where $\mathfrak{p}^*$ is defined to be the largest G-ideal contained in $\mathfrak{p}$. In general, we define that a G-ideal P is *G-prime* if $P = \mathfrak{p}^*$ for some prime ideal $\mathfrak{p}$ of A. It is not difficult to show that a G-ideal P of A is G-prime if and only if the following holds: $P \neq A$, and if I and J are G-ideals and $IJ \subset P$, then either $I \subset P$ or $J \subset P$, see Lemma 5.3. Thus our definition is a straightforward generalization of Kamoi's definition of an H-prime [17, Definition 1.2].

A G-primary G-ideal is defined similarly, and our definition is a natural generalization of that of [25] and [23]. The purpose of this paper is to define G-prime and G-primary G-ideal and study basic properties of these G-ideals. G-radicals of G-ideals are also defined and studied. We also define and study basic properties of G-radical and G-maximal G-ideals.

One of the most important motivation of our study is a generalization of so called Matijevic–Roberts type theorem. This type of theorem asserts that if A is a noetherian ring with a G-action, P a prime ideal of A, and if A_{P^*} enjoys the property $\mathbb{P}$, then A_P has the same property $\mathbb{P}$, where $\mathbb{P}$ is either 'Cohen–Macaulay,' 'Gorenstein,' 'complete intersection,' or 'regular.' The theorem was originally conjectured by Nagata [21] for $\mathbb{Z}$-graded rings and Cohen–Macaulay property. The theorem for $\mathbb{Z}^n$-graded rings was proved by

Hochster–Ratliff [16], Matijevic–Roberts [20], Aoyama–Goto [1], Matijevic [19], Goto–Watanabe [7], Cavaliere–Niesi [4], and Avramov–Achilles [3]. The theorem was then generalized to the action of affine smooth G with connected fibers [11, Theorem II.2.4.2]. The affine assumption was recently removed by M. Ohtani and the first author (unpublished). Note that A_{P^*} makes sense in these cases because P^* is a prime ideal. On the other hand, Kamoi [17, Theorem 2.13] proved the theorem for graded rings, graded by general H, for Cohen–Macaulay and Gorenstein properties. As P^* is not a prime any more, he modified the statement of the theorem.

Although we need to assume that either G is smooth or $R = k$ is a perfect field for the property 'regular,' we prove this theorem for general G, see Corollary 7.7. Assuming that A is locally excellent and also assuming that G is smooth or $R = k$ is a perfect field, we also prove the theorem for F-regularity and F-rationality. It seems that this assertion has not been known even as a theorem on $\mathbb{Z}^n$-graded rings. Note that P^* is not a prime ideal in the general settings, and as in Kamoi's work, we need to modify the statement. That is, we replace the condition 'A_{P^*} enjoys $\mathbb{P}$' by 'for some minimal prime $\mathfrak{p}$ of P^*, $A_{\mathfrak{p}}$ enjoys $\mathbb{P}$.' We can prove that "for some minimal prime" is equivalent to "for any minimal prime," see Corollary 7.8.

We also discuss G-primary decomposition. It is an analogue of primary decomposition. Our main results are the existence and the uniqueness of the G-primary decomposition. The case for H-graded rings is treated in [25] and [23]. We also prove that a G-primary G-ideal does not have an embedded prime (see Corollary 6.2). There is a deep connection between primary decompositions of a G-ideal I and those of 'G-primary components' of I, see Theorem 6.10. For related results on graded rings, see [17, Proposition 2.2] and [23, Corollary 4.2].

When we consider a group action, considering only affine schemes is sometimes too restrictive, even if a goal is a theorem on rings. We treat a group scheme action on a noetherian scheme X.

We also prove some scheme theoretic properties on G-schemes such that 0 is G-primary. If X is a noetherian G-scheme such that 0 is G-primary, then the dimension of the fiber of the second projection $p_2 : G \times X \to X$ is constant (Proposition 6.27). If, moreover, X is of finite type over a field or $\mathbb{Z}$ (more generally, X is *Ratliff*, see (6.28)), then X is equidimensional (Proposition 6.35). We say that a ring B is *Ratliff* if B is noetherian, universally catenary, Hilbert, and B/P satisfies the first chain condition for any minimal prime P of B. A field and $\mathbb{Z}$ are Ratliff. A finite-type algebra over a Ratliff ring is Ratliff (Lemma 6.33). We say that a scheme Y is Ratliff if Y has a finite open covering consisting of prime spectra of Ratliff rings.

Section 2 is preliminaries. We review basics on scheme theoretic image, Fitting ideals, and $(G, \mathcal{O}_X)$-modules. In section 3, we study primary decompositions of ideal sheaves over a noetherian scheme. In section 4, we define and study G-prime and G-radical G-ideals. We also study some basic properties of $\mathfrak{n}^*$ (the largest quasi-coherent $(G, \mathcal{O}_X)$-module contained in $\mathfrak{n}$) used later. In section 5, assuming that the G-scheme X is noetherian, we define and study G-primary G-ideals on X and G-primary decomposition. We prove that G-associated G-prime is well-defined. In section 6, assuming that G is of finite type (more generally, the second projection $p_2 : G \times X \to X$ is of finite type), we study further properties of G-prime and G-primary G-ideals. In section 7, we prove a generalization of Matijevic–Roberts type theorem as explained above.

After the former version of this paper was posted on the arXiv, we got aware of the important paper of Perling and Trautmann [24]. Although they treat only linear algebraic (smooth) groups over an algebraically closed field, there are some overlaps with [24] and ours. In particular, [24, Theorem 4.11] for X of finite type follows from our Lemma 5.11, Corollary 6.24, and Corollary 6.25. Note that disconnected G is also treated in [24]. Our paper is basically written independently of [24], but we added a paragraph (6.41) which shows how [24, Theorem 4.18] (for the case that X is quasi-compact) is proved from our approach. The largest difference of our paper from [24] is that we define and studied G-primary $(G, \mathcal{O}_X)$-module for a general group scheme.

2. Preliminaries

(2.1) A scheme is not required to be separated in general.

Let Y be a scheme and $\mathcal{I}$ a quasi-coherent ideal of $\mathcal{O}_Y$. Then we denote the closed subscheme defined by $\mathcal{I}$ by $V(\mathcal{I})$. For an ideal $\mathcal{J}$ of $\mathcal{O}_Y$, the sum of all quasi-coherent ideals of $\mathcal{O}_Y$ contained in $\mathcal{J}$ is denoted by $\mathcal{J}^\star$. Note that $\mathcal{J}^\star$ is the largest quasi-coherent ideal contained in $\mathcal{J}$.

Let $\varphi : Y \to Z$ be a morphism of schemes. Then $V(\mathrm{Ker}(\mathcal{O}_Z \to \varphi_* \mathcal{O}_Y)^\star)$ is denoted by $\mathrm{SIm}\,\varphi$, and called the *scheme theoretic image* (or the closed image) of φ [8, (9.5.3)]. Let $\psi : Y \to \mathrm{SIm}\,\varphi$ be the induced map. If W is a separated scheme, then

$$\psi^* : \mathrm{Hom}_{\underline{\mathrm{Sch}}}(\mathrm{SIm}\,\varphi, W) \to \mathrm{Hom}_{\underline{\mathrm{Sch}}}(Y, W)$$

is injective, where Sch denotes the category of schemes [8, (9.5.6)].

(2.2) By definition, $\mathrm{SIm}\,\varphi$ is the smallest closed subscheme of Z through which φ factors. So it is easy to see that if Y is reduced, then the closure of the image $\varphi(Y)$ of Y by φ with the reduced structure is $\mathrm{SIm}\,\varphi$ [10, Exercise II.3.11 (d)]. In particular, $\mathrm{SIm}\,\varphi$ is reduced if Y is reduced.

(2.3) Let Y be a scheme, and Z a subscheme of Y. The scheme theoretic image $\mathrm{SIm}\,\iota$ is called the *closure* of Z in Y, and is denoted by $\bar{Z}$, where $\iota : Z \hookrightarrow Y$ is the inclusion.

(2.4) Let $\varphi : Y \to Z$ be a faithfully flat morphism. Then for an $\mathcal{O}_Z$-module $\mathcal{M}$, the unit of adjunction $u : \varphi^*\mathcal{M} \to \varphi^*\varphi_*\varphi^*\mathcal{M}$ is a split mono. Since φ^* is faithful exact, $u : \mathcal{M} \to \varphi_*\varphi^*\mathcal{M}$ is also monic. In particular, $\mathcal{O}_Z \to \varphi_*\mathcal{O}_Y$ is a mono, and hence $\mathrm{SIm}\,\varphi = Z$.

(2.5) Let

$$Y \xrightarrow{\varphi} Z \xrightarrow{\psi} W$$

be a sequence of morphisms of schemes. Let $\iota : \mathrm{SIm}\,\varphi \to Z$ be the inclusion. Then $\mathrm{SIm}(\psi\varphi) = \mathrm{SIm}(\psi\iota)$ [8, (9.5.5)].

In particular, if $\varphi : Y \to Z$ is a flat finite-type morphism between noetherian schemes, then $\mathrm{SIm}\,\varphi = \overline{\mathrm{Im}\,\varphi}$.

(2.6) If $\varphi : Y \to Z$ is a quasi-compact morphism, then $\mathrm{Ker}(\mathcal{O}_Z \to \varphi_*\mathcal{O}_Y)$ is quasi-coherent, and hence $\mathrm{SIm}(\varphi) = V(\mathrm{Ker}(\mathcal{O}_Z \to \varphi_*\mathcal{O}_Y))$. If this is the case, $\mathrm{SIm}(\varphi)$ agrees with the closure of the image $\varphi(Y)$ of φ, set-theoretically.

To verify this, we may assume that $Z = \mathrm{Spec}\,A$ is affine, and hence Y is quasi-compact. There is a finite affine open covering (U_i) of Y. Let $U := \coprod_i U_i \to Y$ be the canonical map. Replacing Y by U, we may assume that $Y = \mathrm{Spec}\,B$ is also affine. Then $\varphi_*\mathcal{O}_Y$ is quasi-coherent, and hence $\mathrm{Ker}(\mathcal{O}_Z \to \varphi_*\mathcal{O}_Y)$ is also quasi-coherent. It remains to show that if $A \hookrightarrow B$ is an inclusion of a subring, then the associated map $\varphi : \mathrm{Spec}\,B \to \mathrm{Spec}\,A$ is dominating. Note that for $P \in \mathrm{Spec}\,A$, there exists some minimal prime $Q \in \mathrm{Spec}\,A$ such that $Q \subset P$ (this follows from Zorn's lemma). As $\kappa(Q) = A_Q \to B_Q = \kappa(Q) \otimes_A B$ is injective, $\mathrm{Spec}(\kappa(Q) \otimes_A B)$ is non-empty. This shows $Q \in \varphi(\mathrm{Spec}\,B)$, and we are done.

Note that in general, if φ is quasi-compact quasi-separated, then $\varphi_*\mathcal{O}_Y$ is quasi-coherent [8, (9.2.1)].

If φ is not quasi-compact, then $\mathrm{SIm}(\varphi)$ need not agree with the closure of $\varphi(Y)$. For example, let $Z = \operatorname{Spec} R$, where R is a DVR with a uniformizing parameter t. Let $Y = \coprod_{i\geq 1} \operatorname{Spec} R/(t^i)$, and $\varphi : Y \to Z$ be the canonical map. Then the closure of the image of φ is the closed point of Z set-theoretically, but $\mathrm{SIm}(\varphi)$ is the whole Z.

(2.7) (cf. [8, (9.5.8)]) Let

$$\begin{array}{ccc} Y' & \xrightarrow{\varphi'} & Z' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\varphi} & Z \end{array}$$

be a cartesian square of schemes. Assume that φ is quasi-compact quasi-separated, and f is flat. Then $\mathrm{SIm}\,\varphi' = f^{-1}(\mathrm{SIm}\,\varphi)$. Indeed, if

$$0 \to \mathcal{I} \to \mathcal{O}_Z \to \varphi_*\mathcal{O}_Y$$

is exact, then applying the exact functor f^* to it,

$$0 \to f^*\mathcal{I} \to \mathcal{O}_{Z'} \to (\varphi')_*\mathcal{O}_{Y'}$$

is exact.

(2.8) Let Y be a scheme. For a closed subscheme W of Y, we denote the defining ideal sheaf of W by $\mathcal{I}(W)$. For a morphism $f : Y \to Z$ of schemes and an ideal sheaf $\mathcal{I}$ of $\mathcal{O}_Y$, the ideal sheaf $(\eta^{-1}(f_*\mathcal{I}))^\star$ is denoted by $\mathcal{I} \cap \mathcal{O}_Z$, where $\eta : \mathcal{O}_Z \to f_*\mathcal{O}_Y$ is the canonical map. In other words, $\mathrm{SIm}(V(\mathcal{I}) \hookrightarrow Y \to Z) = V(\mathcal{I} \cap \mathcal{O}_Z)$.

If $Y \xrightarrow{\varphi} Z \xrightarrow{\psi} W$ is a sequence of morphisms and $\mathcal{I}$ is an ideal of $\mathcal{O}_Y$, then $(\mathcal{I} \cap \mathcal{O}_Z) \cap \mathcal{O}_W = \mathcal{I} \cap O_W$ by (2.5).

For a morphism $f : Y \to Z$ and an ideal $\mathcal{J}$ of $\mathcal{O}_Z$, the image of $f^*\mathcal{J} \to f^*\mathcal{O}_Z \to \mathcal{O}_Y$ is denoted $\mathcal{J}\mathcal{O}_Y$. If $\mathcal{J}$ is a quasi-coherent ideal, then so is $\mathcal{J}\mathcal{O}_Y$, and $V(\mathcal{J}\mathcal{O}_Y) = f^{-1}(V(\mathcal{J}))$. Note that $\mathcal{J} \subset \mathcal{J}\mathcal{O}_Y \cap \mathcal{O}_Z$ and $\mathcal{I} \supset (\mathcal{I} \cap \mathcal{O}_Z)\mathcal{O}_Y$.

If $f : Y \to Z$ is a quasi-compact immersion, then $(\mathcal{I} \cap \mathcal{O}_Z)\mathcal{O}_Y$ is $\mathcal{I}$.

(2.9) Let R be a commutative ring. Let $\varphi : F \to E$ be a map of R-free modules with E finite, and n be an integer. If $n \leq 0$, we define $I_n(\varphi) = R$. If $n \geq 1$, we define $I_n(\varphi)$ to be the image of $\varphi_n : \bigwedge^n F \otimes \bigwedge^n E^* \to R$ given by

$$\varphi_n(f_1 \wedge \cdots \wedge f_n \otimes \epsilon_1 \wedge \cdots \wedge \epsilon_n) = \det(\epsilon_i(\varphi(f_j))).$$

Let M be a finitely generated R-module. Take a presentation

$$F \xrightarrow{\varphi} R^r \to M \to 0 \tag{1}$$

with F being R-free (not necessarily finite). For $j \in \mathbb{Z}$, $I_{r-j}(\varphi)$ is independent of the choice of the presentation (1). We denote $I_{r-j}(\varphi)$ by $\mathrm{Fitt}_j(M)$, and call it the jth Fitting ideal of M, see [5, (20.2)].

The construction of Fitting ideals commutes with base change [5, Corollary 20.5]. So for a scheme Y and a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{M}$ of finite type and $j \in \mathbb{Z}$, the quasi-coherent ideal $\underline{\mathrm{Fitt}}_j(\mathcal{M})$ of $\mathcal{O}_Y$ is defined in an obvious way. If $\mathcal{M}$ is locally of finite presentation, then $\underline{\mathrm{Fitt}}_j(\mathcal{M})$ is of finite type for any j.

2.10 Lemma. *Let $f : Y \to Z$ be a morphism of schemes, $\mathcal{M}$ a quasi-coherent $\mathcal{O}_Z$-module of finite type, and $j \in \mathbb{Z}$. Then $(\underline{\mathrm{Fitt}}_j(\mathcal{M}))\mathcal{O}_Y = \underline{\mathrm{Fitt}}_j(f^*(\mathcal{M}))$.*

Proof. Follows immediately by [5, Corollary 20.5]. □

As in [5, Proposition 20.8], we can prove the following easily.

2.11 Lemma. *Let Y be a scheme, $\mathcal{M}$ a quasi-coherent $\mathcal{O}_Y$-module of finite type, and $r \geq 0$. If $\underline{\mathrm{Fitt}}_r(\mathcal{M}) = \mathcal{O}_Y$ and $\underline{\mathrm{Fitt}}_{r-1}(\mathcal{M}) = 0$, then $\mathcal{M}$ is locally free of well-defined rank r.*

(2.12) Throughout the article, S denotes a scheme, and G denotes an S-group scheme. Throughout, X denotes a G-scheme (i.e., an S-scheme with a left G-action). We always assume that the second projection $p_2 : G \times_S X \to X$ is flat.

(2.13) As in [12, section 29], let $B_G^M(X)$ be the diagram

$$G \times G \times X \;\overset{1_G \times a}{\underset{p_{23}}{\overset{\mu \times 1_X}{\rightrightarrows}}}\; G \times X \;\overset{a}{\underset{p_2}{\rightrightarrows}}\; X\,,$$

where $a : G \times X \to X$ is the action, $\mu : G \times G \to G$ is the product, and $p_2 : G \times X \to X$ and $p_{23} : G \times G \times X \to G \times X$ are appropriate projections. Note that $B_G^M(X)$ has flat arrows. To see this, it suffices to see that $a : G \times X \to X$ is flat. But since $a = p_2 b$ with b an isomorphism, where $b(g, x) = (g, gx)$, this is trivial.

By definition, a $(G, \mathcal{O}_X)$-module is an $\mathcal{O}_{B_G^M(X)}$-module. It is said to be equivariant, quasi-coherent or coherent, if it is so as an $\mathcal{O}_{B_G^M(X)}$-module, see

[12]. The category of equivariant (resp. quasi-coherent, coherent) $(G, \mathcal{O}_X)$-modules is denoted by $\mathrm{EM}(G, X)$ (resp. $\mathrm{Qch}(G, X)$, $\mathrm{Coh}(G, X)$). In general, $\mathrm{Qch}(G, X)$ is closed under kernels, small colimits (in particular, cokernels), and extensions in the category of $(G, \mathcal{O}_X)$-modules. In particular, it is an abelian category with the (AB5) condition ([12, Lemma 7.6]).

(2.14) Let X be as above. We say that $(\mathcal{M}, \Phi)$ is a G-linearized $\mathcal{O}_X$-module if $\mathcal{M}$ is an $\mathcal{O}_X$-module, and $\Phi : a^*\mathcal{M} \to p_2^*\mathcal{M}$ an isomorphism of $\mathcal{O}_{G\times X}$-modules such that

$$(\mu \times 1_X)^*\Phi : (\mu \times 1_X)^* a^*\mathcal{M} \to (\mu \times 1_X)^* p_2^*\mathcal{M}$$

agrees with the composite map

$$(\mu \times 1_X)^* a^*\mathcal{M} \xrightarrow{d} (1_G \times a)^* a^*\mathcal{M} \xrightarrow{\Phi} (1_G \times a)^* p_2^*\mathcal{M} \\ \xrightarrow{d} p_{23}^* a^*\mathcal{M} \xrightarrow{\Phi} p_{23}^* p_2^*\mathcal{M} \xrightarrow{d} (\mu \times 1_X)^* p_2^*\mathcal{M},$$

where d's are canonical isomorphisms. We call Φ the G-linearization of $\mathcal{M}$.

A morphism $\varphi : (\mathcal{M}, \Phi) \to (\mathcal{N}, \Psi)$ of G-linearized $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-linear map $\varphi : \mathcal{M} \to \mathcal{N}$ such that $\Psi a^*\varphi = p_2^*\varphi\Phi$. We denote the category of G-linearized $\mathcal{O}_X$-modules by $\mathrm{Lin}(G, X)$. The full subcategory of G-linearized quasi-coherent $\mathcal{O}_X$-modules is denoted by $\mathrm{LQ}(G, X)$.

For $\mathcal{M} \in \mathrm{EM}(G, X)$, $(\mathcal{M}_0, \alpha_{\delta_1(1)}^{-1}\alpha_{\delta_0(1)})$ is in $\mathrm{Lin}(G, X)$, and this correspondence gives an equivalence. With this equivalence, $\mathrm{Qch}(G, X)$ is equivalent to $\mathrm{LQ}(G, X)$. See the proof of [12, Lemma 9.4].

(2.15) If X and $G \times X$ are quasi-compact quasi-separated, then $\mathrm{Qch}(G, X)$ is Grothendieck. If, moreover, X is noetherian, then $\mathrm{Qch}(G, X)$ is locally noetherian, and $\mathcal{M} \in \mathrm{Qch}(G, X)$ is a noetherian object if and only if $\mathcal{M}_0$ is coherent ([12, Lemma 12.8]).

(2.16) The restriction functor $(?)_0 : \mathrm{Qch}(G, X) \to \mathrm{Qch}(X)$ is faithful exact. With this reason, we sometimes let $\mathcal{N} \in \mathrm{Qch}(X)$ mean $\mathcal{M} \in \mathrm{Qch}(G, X)$ if $\mathcal{N} = \mathcal{M}_0$. For example, $\mathcal{O}_X$ means the quasi-coherent $(G, \mathcal{O}_X)$-module $\mathcal{O}_{B_G^M(X)}$, since $(\mathcal{O}_{B_G^M(X)})_0 = \mathcal{O}_X$. Note that $\mathcal{M} \in \mathrm{Qch}(G, X)$ is coherent (i.e., it is in $\mathrm{Coh}(G, X)$) if and only if $\mathcal{N} = \mathcal{M}_0 \in \mathrm{Coh}(X)$, and no confusion will occur.

Let $\mathcal{M}'$ be a subobject of $\mathcal{M} \in \mathrm{Qch}(G, X)$. Then $\mathcal{M}'_0 \subset \mathcal{M}_0$ and the $(G, \mathcal{O}_X)$-module structure of $\mathcal{M}$ together determine the $(G, \mathcal{O}_X)$-submodule structure of $\mathcal{M}'$ uniquely. This is similar to the fact that for a ring A and an A-module M and its A-submodule N, the subset $N \subset M$ and the A-module structure of M together determine the A-submodule structure of N uniquely. So by abuse of notation, we sometimes say that $\mathcal{M}'_0$ is a quasi-coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}_0$, instead of saying that $\mathcal{M}'$ is a quasi-coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$. Applying this abuse to $\mathcal{O}_X$, we sometimes say that $\mathcal{I} \subset \mathcal{O}_X$ is a quasi-coherent G-ideal of $\mathcal{O}_X$ (i.e., a quasi-coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{O}_X$).

(2.17) Let $(\mathcal{M}, \Phi)$ be a G-linearized $\mathcal{O}_X$-module, and $\mathcal{N}$ an $\mathcal{O}_X$-submodule. We identify $p_2^*\mathcal{N}$ by its image in $p_2^*\mathcal{M}$, since p_2 is flat. Similarly, a is also flat, and we identify $a^*\mathcal{N}$ by its image in $a^*\mathcal{M}$. Then $\mathcal{N}$ is a $(G, \mathcal{O}_X)$-submodule if and only if $\Phi(a^*\mathcal{N}) = p_2^*\mathcal{N}$, since then $\Phi : a^*\mathcal{N} \to p_2^*\mathcal{N}$ is an isomorphism.

So for a G-equivariant $\mathcal{O}_X$-module $\mathcal{M}$ and an $\mathcal{O}_X$-submodule $\mathcal{N}$ of $\mathcal{M}_0$, $\mathcal{N}$ is a $(G, \mathcal{O}_X)$-submodule if and only if the image of $a^*\mathcal{N}$ by the map $\alpha : a^*\mathcal{M}_0 \to \mathcal{M}_1$ agrees with the image of $p_2^*\mathcal{N}$ by the map $\alpha : p_2^*\mathcal{M}_0 \to \mathcal{M}_1$.

2.18 Lemma. *Let $\mathcal{M}$ be a quasi-coherent $(G, \mathcal{O}_X)$-module of finite type, and $j \in \mathbb{Z}$. Then the Fitting ideal $\underline{\mathrm{Fitt}}_j\, \mathcal{M}$ is a G-ideal of $\mathcal{O}_X$.*

Proof. By Lemma 2.10, the two extended ideals $(\underline{\mathrm{Fitt}}_j\, \mathcal{M})\mathcal{O}_{G\times X}$ via $a : G \times X \to X$ and $p_2 : G \times X \to X$ agree, since the former one is $\underline{\mathrm{Fitt}}_j(a^*\mathcal{M})$, the latter one is $\underline{\mathrm{Fitt}}_j(p_2^*\mathcal{M})$, and $a^*\mathcal{M} \cong p_2^*\mathcal{M}$. $\square$

(2.19) The restriction $(?)_0 : \mathrm{Qch}(G, X) \to \mathrm{Qch}(X)$ has a right adjoint, if the second projection $p_2 : G \times X \to X$ is quasi-compact quasi-separated ([12, Lemma 12.11]).

3. Primary decompositions over noetherian schemes

Let Y be a scheme. An ideal of $\mathcal{O}_Y$ means a quasi-coherent ideal sheaves, unless otherwise specified. An $\mathcal{O}_Y$-module means a quasi-coherent module, unless otherwise specified.

(3.1) An ideal $\mathcal{P}$ of $\mathcal{O}_Y$ is said to be a prime if $V(\mathcal{P})$ is integral. An ideal $\mathcal{P}$ of $\mathcal{O}_Y$ is said to be a quasi-prime if $\mathcal{P} \neq \mathcal{O}_Y$, and if $\mathcal{I}$ and $\mathcal{J}$ are ideals of $\mathcal{O}_Y$ such that $\mathcal{I}\mathcal{J} \subset \mathcal{P}$, then $\mathcal{I} \subset \mathcal{P}$ or $\mathcal{J} \subset \mathcal{P}$ holds.

3.2 Lemma. *Let $\mathcal{P}$ be an ideal of $\mathcal{O}_Y$. If $\mathcal{P}$ is a prime, then $\mathcal{P}$ is quasi-prime. If Y is quasi-compact quasi-separated and $\mathcal{P}$ is quasi-prime, then $\mathcal{P}$*

is a prime.

Proof. Replacing Y by $V(\mathcal{P})$, $\mathcal{P}$ by 0, $\mathcal{I}$ by $\mathcal{I}\mathcal{O}_{V(\mathcal{P})}$, and $\mathcal{J}$ by $\mathcal{J}\mathcal{O}_{V(\mathcal{P})}$, we may assume that $\mathcal{P} = 0$.

We prove the first part. Since Y is integral, it is irreducible and hence is non-empty. Thus $\mathcal{O}_Y \neq 0 = \mathcal{P}$.

Let η be the generic point of Y. Since $\mathcal{I}\mathcal{J} = 0$, $\mathcal{I}_\eta\mathcal{J}_\eta = 0$. Since $\mathcal{O}_{Y,\eta}$ is an integral domain, $\mathcal{I}_\eta = 0$ or $\mathcal{J}_\eta = 0$. If $\mathcal{I}_\eta = 0$, then

$$\mathcal{I} \subset \mathcal{I}_\eta \cap \mathcal{O}_Y = 0 \cap \mathcal{O}_Y.$$

This is zero by (2.2), applied to $\operatorname{Spec} \mathcal{O}_{Y,\eta} \to Y$. Similarly, $\mathcal{J}_\eta = 0$ implies $\mathcal{J} = 0$.

We prove the second part. Since Y is quasi-compact, it has a finite affine open covering $(U_i)_{i=1}^n$. We may assume that $U_1, \ldots, U_s$ are reduced, and U_i is not reduced for $i > s$. Let $\mathcal{I}_i = 0_{U_i} \cap \mathcal{O}_Y$ be the pull-back of zero for $i \leq s$. For $i > s$, there is a non-zero ideal $\mathcal{J}_i$ of $\mathcal{O}_{U_i}$ such that $\mathcal{J}_i^2 = 0$, since U_i is affine and non-reduced. Set $\mathcal{I}_i = (\mathcal{J}_i \cap \mathcal{O}_Y)^2$.

Since the inclusion $U_i \hookrightarrow Y$ is quasi-compact, $\mathcal{I}_i|_{U_i} = 0$. Thus $\mathcal{I}_1 \cdots \mathcal{I}_n = 0$. By assumption, there exists some i such that $\mathcal{I}_i = 0$. If $i > s$, then $\mathcal{J}_i \cap \mathcal{O}_Y = 0$ by assumption again. So $\mathcal{J}_i = (\mathcal{J}_i \cap \mathcal{O}_Y)|_{U_i} = 0$, and this is a contradiction. So $i \leq s$. Thus Y is the scheme theoretic image of the inclusion $U_i \hookrightarrow Y$. By (2.2), Y is reduced.

Since $\mathcal{O}_Y \neq 0$, Y is non-empty. Assume that Y is not irreducible. Then $Y = Y_1 \cup Y_2$ for some closed subsets $Y_i \neq Y$. Let us consider the reduced structure of Y_i, and set $\mathcal{K}_i = \mathcal{I}(Y_i)$. Then $\mathcal{K}_1 \cap \mathcal{K}_2 = 0$. By assumption, $\mathcal{K}_1 = 0$ or $\mathcal{K}_2 = 0$. This contradicts $Y_i \neq Y$. So Y must be irreducible, and Y is integral. □

(3.3) An ideal $\mathcal{M}$ of $\mathcal{O}_Y$ is said to be maximal, if $\mathcal{M}$ is a maximal element of the set of proper (i.e., not equal to $\mathcal{O}_Y$) ideals of $\mathcal{O}_Y$. The defining ideal of a closed point (with the reduced structure) is maximal. Since a non-empty quasi-compact T_0-space has a closed point, if Y is non-empty and quasi-compact, then $\mathcal{O}_Y$ has a maximal ideal. It is easy to see that a maximal ideal is a prime ideal.

3.4 Lemma. *If $f : Y \to Z$ is a morphism and $\mathcal{P}$ is a prime ideal of $\mathcal{O}_Y$, then $\mathcal{P} \cap \mathcal{O}_Z$ is a prime ideal.*

Proof. This is nothing but the restatement of the fact that the scheme theoretic image of the composite

$$V(\mathcal{P}) \hookrightarrow Y \xrightarrow{f} Z$$

is integral, see (2.2). $\square$

(3.5) Let $\mathcal{I}$ be an ideal of $\mathcal{O}_Y$. For an affine open subset U of Y, we define $\Gamma(U, \sqrt{\mathcal{I}}) := \sqrt{\Gamma(U, \mathcal{I})}$. This defines a quasi-coherent ideal $\sqrt{\mathcal{I}}$ of $\mathcal{O}_Y$. We call $\sqrt{\mathcal{I}}$ the *radical* of $\mathcal{I}$. The formation of $\sqrt{\mathcal{I}}$ is local (that is, for any open subset U of Y, $\sqrt{\mathcal{I}}|_U = \sqrt{\mathcal{I}|_U}$), and $V(\mathcal{I})$ is reduced if and only if $\mathcal{I} = \sqrt{\mathcal{I}}$. Hence $\mathcal{P} = \sqrt{\mathcal{P}}$ for a prime ideal $\mathcal{P}$ of $\mathcal{O}_Y$. Note also that $\sqrt{\mathcal{I}} = \mathcal{O}_X$ if and only if $\mathcal{I} = \mathcal{O}_X$, since the formation of $\sqrt{\mathcal{I}}$ is local.

3.6 Lemma. *For an ideal $\mathcal{I}$ of $\mathcal{O}_Y$,*

$$\sqrt{\mathcal{I}} = \left(\bigcap \mathcal{P}\right)^\star,$$

where the intersection is taken over all prime ideals $\mathcal{P}$ containing $\mathcal{I}$.

Proof. Note that the assertion is well-known for affine schemes. Set the right hand side to be $\mathcal{J}$. Since $\mathcal{I} \subset \mathcal{P}$, we have $\sqrt{\mathcal{I}} \subset \sqrt{\mathcal{P}} = \mathcal{P}$. So $\sqrt{\mathcal{I}} \subset \mathcal{J}$ is obvious.

To prove $\sqrt{\mathcal{I}} \supset \mathcal{J}$, it suffices to prove that $\sqrt{\mathcal{I}}|_U \supset \mathcal{J}|_U$ for any affine open subset U of Y. Let $\mathcal{Q}$ be a prime ideal of $\mathcal{O}_U$ such that $\mathcal{Q} \supset \mathcal{I}|_U$. Then $\mathcal{Q} \cap \mathcal{O}_Y$ is a prime by Lemma 3.4, and $\mathcal{Q} \cap \mathcal{O}_Y \supset \mathcal{I}|_U \cap \mathcal{O}_Y \supset \mathcal{I}$. Hence $\mathcal{Q} \cap \mathcal{O}_Y \supset \mathcal{J}$. Thus $\mathcal{Q} \supset (\mathcal{Q} \cap \mathcal{O}_Y)|_U \supset \mathcal{J}|_U$. Hence $\mathcal{J}|_U \subset \bigcap \mathcal{Q} = \sqrt{\mathcal{I}|_U} = \sqrt{\mathcal{I}}|_U$. $\square$

(3.7) An ideal $\mathcal{I}$ of $\mathcal{O}_Y$ is said to be a *radical ideal* if $\sqrt{\mathcal{I}} = \mathcal{I}$.

3.8 Lemma. *Let $(\mathcal{I}_\lambda)_{\lambda \in \Lambda}$ be a family of radical ideals of $\mathcal{O}_Y$. Then $(\bigcap_\lambda \mathcal{I}_\lambda)^\star$ is a radical ideal.*

Proof. Let $Y_\lambda := V(\mathcal{I}_\lambda)$, and $Y' := \coprod_\lambda Y_\lambda$. Note that Y' is reduced. So the scheme theoretic image of the canonical map $\varphi : Y' \to Y$ is also reduced by (2.2). On the other hand, the defining ideal of $\mathrm{SIm}(\varphi)$ is $(\bigcap_\lambda \mathcal{I}_\lambda)^\star$, and we are done. $\square$

(3.9) From now, until the end of this section, let Y be noetherian. For an $\mathcal{O}_Y$-module $\mathcal{M}$, the subset

$$\{y \in Y \mid \operatorname{Hom}_{\mathcal{O}_{Y,y}}(\kappa(y), \mathcal{M}_y) \neq 0\}$$

of Y is denoted by $\operatorname{Ass}(\mathcal{M})$. A point y in $\operatorname{Ass}(\mathcal{M})$ is called an associated point of $\mathcal{M}$. The closed subscheme $V(\overline{\{y\}})$ with $y \in \operatorname{Ass}(\mathcal{M})$ is called an associated component. The defining ideal of an associated component is called an associated prime.

3.10 Lemma. *Let $\mathcal{M}$ be a coherent $\mathcal{O}_Y$-module, and $\mathcal{N}$ its submodule. Then the following are equivalent.*

(i) $\operatorname{Ass}(\mathcal{M}/\mathcal{N}) = \{y\}$ *is a singleton.*

(ii) $\mathcal{M} \neq \mathcal{N}$*, and if* $\mathcal{L}$ *is a submodule of* $\mathcal{M}$*,* $\mathcal{I}$ *an ideal of* $\mathcal{O}_Y$*,* $\mathcal{I}\mathcal{L} \subset \mathcal{N}$*, and* $\mathcal{L} \not\subset \mathcal{N}$*, then* $\mathcal{I} \subset \sqrt{\mathcal{N} : \mathcal{M}}$*.*

If this is the case, $\sqrt{\mathcal{N} : \mathcal{M}}$ *is a prime, and* y *is the generic point of* $V(\sqrt{\mathcal{N} : \mathcal{M}})$.

Proof. Replacing $\mathcal{M}$ by $\mathcal{M}/\mathcal{N}$, we may assume that $\mathcal{N} = 0$. Moreover, replacing Y by $V(\underline{\operatorname{ann}}\,\mathcal{M})$, we may assume that $\underline{\operatorname{ann}}\,\mathcal{M} = 0$.

(i)⇒(ii). Since $\operatorname{Ass}\mathcal{M}$ is non-empty, $\mathcal{M} \neq 0$.

Note that for any non-empty affine open set U of Y, $\operatorname{Ass}\mathcal{M}|_U = \{y\} \cap U \neq \emptyset$. So y must be the generic point of Y. In particular, Y is irreducible. The last assertion is now obvious.

Let $\mathcal{I}\mathcal{L} = 0$ and $\mathcal{L} \neq 0$. Then there is a non-empty affine open subset U of Y such that $\mathcal{L}|_U \neq 0$. Since $\mathcal{I}|_U\mathcal{L}|_U = 0$, $\operatorname{Ass}(\mathcal{M}) = \{y\}$, $\mathcal{I}|_U^n = 0$ for some $n \geq 1$ by the usual commutative ring theory. Assume that $\mathcal{I}^n \neq 0$, and take $x \in \operatorname{Supp}\mathcal{I}^n$. Let V be an affine open neighborhood of x. Then $\mathcal{I}^n|_V \neq 0$, and $\operatorname{Supp}\mathcal{I}^n|_V \subset V \setminus U$. Set $\mathcal{K} := \operatorname{ann}(\mathcal{I}^n|_V)$. Then $V(\mathcal{K}) = \operatorname{Supp}(\mathcal{I}^n|_V) \subset V \setminus U \subsetneq V$. So $\mathcal{K} \not\subset \sqrt{0}$. Since $\mathcal{K}\mathcal{I}^n|_V = 0$ and the ideal 0 of $\Gamma(\mathcal{O}_V, V)$ is a primary ideal, $\mathcal{I}^n|_V = 0$, and this is a contradiction. Hence $\mathcal{I}^n = 0$.

(ii)⇒(i). Since $\mathcal{M} \neq 0$, $Y \neq \emptyset$. First, we prove that $\sqrt{0}$ is a prime. Since $Y \neq \emptyset$, we have that $\sqrt{0} \neq \mathcal{O}_Y$. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals of $\mathcal{O}_Y$, and assume that $\mathcal{I}\mathcal{J} \subset \sqrt{0}$ and $\mathcal{J} \not\subset \sqrt{0}$. Then there exists some $n \geq 1$ such that $\mathcal{I}^n\mathcal{J}^n = 0$. Since $\mathcal{J}^n \neq 0$ and $\underline{\operatorname{ann}}\,\mathcal{M} = 0$, $\mathcal{J}^n\mathcal{M} \neq 0$. By assumption, $\mathcal{I}^n \subset \sqrt{0}$. So $\mathcal{I} \subset \sqrt{0}$. So $\sqrt{0}$ is a prime, and Y is irreducible. Let y denote the generic point of Y.

Next, let U be a non-empty affine open subscheme of Y. Let $\mathcal{I}$ be an ideal of $\mathcal{O}_U$, and $\mathcal{L}$ be a nonzero submodule of $\mathcal{M}|_U$ such that $\mathcal{I}\mathcal{L} = 0$. Let $\mathcal{L}'$ be the kernel of $\mathcal{M} \to i_* i^* \mathcal{M} \to i_* i^*(\mathcal{M}/\mathcal{L})$, where $i : U \hookrightarrow Y$ is the inclusion. Note that $\mathcal{L}' \neq 0$, since $\mathcal{L}'|_U = \mathcal{L}$. Note also that $\operatorname{Supp}((\mathcal{I} \cap \mathcal{O}_Y)\mathcal{L}') \subset Y \setminus U$. Set $\mathcal{K} := \underline{\operatorname{ann}}((\mathcal{I} \cap \mathcal{O}_Y)\mathcal{L}')$. Then $\mathcal{K} \not\subset \sqrt{0}$. Since $\mathcal{K}(\mathcal{I} \cap \mathcal{O}_Y)\mathcal{L}' = 0$, we have $(\mathcal{I} \cap \mathcal{O}_Y)\mathcal{L}' = 0$ by assumption. Since $\mathcal{L}' \neq 0$, there exists some $n \geq 1$ such that $(\mathcal{I} \cap \mathcal{O}_Y)^n = 0$. Hence

$$\mathcal{I}^n = ((\mathcal{I} \cap \mathcal{O}_Y)|_U)^n = ((\mathcal{I} \cap \mathcal{O}_Y)^n)|_U = 0.$$

As U is affine, this shows that $\Gamma(U, \mathcal{M})$ is $\sqrt{0}$-primary, and hence $\operatorname{Ass}(\mathcal{M}|_U) = \{y\}$. Since U is arbitrary, it is easy to see that $\operatorname{Ass} \mathcal{M} = \{y\}$. □

(3.11) Let $\mathcal{M}$ be a coherent $\mathcal{O}_Y$-module and $\mathcal{N}$ its submodule. If the equivalent conditions of the lemma is satisfied, then we say that $\mathcal{N}$ is a *primary submodule* of $\mathcal{M}$. If $\mathcal{M} = \mathcal{O}_Y$, then we say that $\mathcal{N}$ is a *primary ideal*. If 0 is a primary submodule of $\mathcal{M}$, then we say that $\mathcal{M}$ is a *primary module*. If $\mathcal{O}_Y$ is a primary module, then we say that Y is *primary*.

If $\mathcal{N}$ is a primary submodule of $\mathcal{M}$, $\mathcal{M}$ is coherent, and $\operatorname{Ass}(\mathcal{M}/\mathcal{N}) = \{y\}$, then we say that $\mathcal{N}$ is y-primary. We also say that $\mathcal{N}$ is $\sqrt{\mathcal{N} : \mathcal{M}}$-primary.

Note that if $\mathcal{N}$ is a primary submodule of $\mathcal{M}$, then $\mathcal{N} : \mathcal{M}$ is a primary ideal (easy).

We consider that Y is an ordered set with respect to the order given by $y \leq y'$ if and only if y is a generalization of y'. For a coherent $\mathcal{O}_Y$-module $\mathcal{M}$, any minimal element of $\operatorname{Supp} \mathcal{M}$ is a member of $\operatorname{Ass}(\mathcal{M})$. The set of minimal elements of $\operatorname{Supp} \mathcal{M}$ is denoted by $\operatorname{Min}(\mathcal{M})$. If $\mathcal{M}$ is coherent, then $\operatorname{Ass}(\mathcal{M})$ is a finite set. If $\mathcal{M}$ is quasi-coherent, then $\operatorname{Ass}(\mathcal{M}) = \emptyset$ if and only if $\mathcal{M} = 0$. If $\mathcal{M}$ is quasi-coherent, then $\operatorname{Supp}(\mathcal{M}) \supset \operatorname{Ass}(\mathcal{M}) \supset \operatorname{Min}(\mathcal{M})$.

(3.12) Coherent prime ideals of $\mathcal{O}_Y$ is in one to one correspondence with integral closed subschemes of Y ($\mathcal{P}$ corresponds to $V(\mathcal{P})$). So they are also in one to one correspondences with points in Y ($V(\mathcal{P})$ corresponds to its generic point). So for a coherent $\mathcal{O}_Y$-module $\mathcal{M}$, $\operatorname{Supp} \mathcal{M}$, $\operatorname{Ass} \mathcal{M}$, and $\operatorname{Min} \mathcal{M}$ are sometimes considered as sets of prime ideals of Y. An element of $\operatorname{Ass} \mathcal{M} \setminus \operatorname{Min} \mathcal{M}$ is called an embedded prime of $\mathcal{M}$.

(3.13) Let $\mathcal{M}$ be a coherent sheaf over Y, $\operatorname{Ass}(\mathcal{M}) = \{y_1, \ldots, y_r\}$, and $y_1, \ldots, y_r$ be distinct. Let $0 = M_{i,1} \cap \cdots \cap M_{i,r_i}$ be a minimal primary decomposition of $0 \subset \mathcal{M}_{y_i}$. Since $\operatorname{depth} \mathcal{M}_{y_i} = 0$, by reordering if necessary, we may assume that $M_{i,1}$ is $\mathfrak{m}_{y_i}$-primary. We set $L_i = M_{i,1}$. Note that

$H^0_{\mathfrak{m}_{y_i}}(\mathcal{M}_{y_i}) = \bigcap_{j\geq 2} M_{i,j}$. Hence $H^0_{\mathfrak{m}_{y_i}}(\mathcal{M}_{y_i}) \cap L_i = 0$ and $\mathcal{M}_{y_i}/L_i$ has a finite length.

Let $\varphi_i : \operatorname{Spec} \mathcal{O}_{y_i} \to Y$ be the canonical map. Let $\mathcal{N}_i$ be the kernel of the composite

$$\mathcal{M} \to (\varphi_i)_*\mathcal{M}_{y_i} \to (\varphi_i)_*(\mathcal{M}_{y_i}/L_i)$$

so that there is a monomorphism $\mathcal{M}/\mathcal{N}_i \hookrightarrow (\varphi_i)_*(\mathcal{M}_{y_i}/L_i)$. It is easy to see that $\operatorname{Ass}(\mathcal{M}/\mathcal{N}_i) = \{y_i\}$.

Since $(\mathcal{N}_i)_{y_i} = L_i$ and $H^0_{\mathfrak{m}_{y_i}}(L_i) = 0$, we have that $y_i \notin \operatorname{Ass}(\mathcal{N}_i)$. In particular, $\operatorname{Ass}(\mathcal{N}_1 \cap \cdots \cap \mathcal{N}_r) = \emptyset$.

So there is a decomposition

$$0 = \mathcal{N}_1 \cap \cdots \cap \mathcal{N}_r$$

such that $\operatorname{Ass}(\mathcal{M}/\mathcal{N}_i) = \{y_i\}$. We call such a decomposition a minimal primary decomposition of 0.

If $\mathcal{N}$ is a coherent $\mathcal{O}_Y$-submodule of $\mathcal{M}$ and $\operatorname{Ass}(\mathcal{M}/\mathcal{N}) = \{y_1, \ldots, y_r\}$, ($y_1, \ldots, y_r$ are distinct). then there is a decomposition

$$\mathcal{N} = \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_r \tag{2}$$

such that $\operatorname{Ass}(\mathcal{M}/\mathcal{M}_i) = \{y_i\}$. We call such a decomposition a minimal primary decomposition of $\mathcal{N}$. Such a decomposition is not unique in general. If a coherent $\mathcal{O}_X$-submodule $\mathcal{W}$ agrees with some $\mathcal{M}_i$ for some minimal primary decomposition (2), we say that $\mathcal{W}$ is a primary component of $\mathcal{N}$. Note that a primary component $\mathcal{M}_i$ for $y_i \in \operatorname{Min}(\mathcal{M}/\mathcal{N})$ is known to be unique. If $\mathcal{M} = \mathcal{O}_X$ and $\mathcal{N}$ is a radical ideal, then $\mathcal{O}_X/\mathcal{N}$ does not have an embedded prime, and $\mathcal{M}_i$ is the unique prime ideal such that the generic point of $V(\mathcal{M}_i)$ is y_i.

3.14 Lemma. *Let $\mathcal{M}$ be a coherent $\mathcal{O}_Y$-module, and $\mathcal{N}$ a coherent submodule of $\mathcal{M}$. Let* (2) *be a minimal primary decomposition of $\mathcal{N}$. If $\mathcal{M}/\mathcal{N}$ does not have an embedded prime, then*

$$\mathcal{N} : \mathcal{M} = \bigcap_{i=1}^{r} \mathcal{M}_i : \mathcal{M} \tag{3}$$

is a minimal primary decomposition. In particular, $\operatorname{Ass} \mathcal{M}/\mathcal{N} = \operatorname{Ass} \mathcal{O}_X/(\mathcal{N} : \mathcal{M})$.

Proof. It is obvious that the equation (3) holds, and it is a primary decomposition.

Since (2) is minimal, $\sqrt{\mathcal{M}_i : \mathcal{M}}$ are distinct, and there is no incidence relation each other. The minimality of (3) follows easily. □

(3.15) Let $\mathcal{M}$ be a coherent sheaf over Y. Note that $\mathcal{M}$ satisfies Serre's (S_1)-condition (see [9, (5.7.2)]) if and only if $\mathcal{M}$ has no embedded prime. So $\mathcal{M}$ is primary if and only if $\mathcal{M}$ satisfies the (S_1) and $\operatorname{Supp}\mathcal{M}$ is irreducible.

3.16 Lemma. *Let U be an open subset of Y, and $i : U \hookrightarrow Y$ be the inclusion. Let $\mathcal{M}$ be a coherent sheaf over Y.*

(i) $\operatorname{Ass}(i^*\mathcal{M}) = \operatorname{Ass}(\mathcal{M}) \cap U$.

(ii) $\operatorname{Ass}(\underline{H}^0_{Y\setminus U}(\mathcal{M})) = \operatorname{Ass}(\mathcal{M}) \setminus U$, *where $\underline{H}^0_{Y\setminus U}(\mathcal{M})$ is the kernel of the canonical map $\mathcal{M} \to i_*i^*\mathcal{M}$.*

(iii) *The following are equivalent:*

- **(a)** $\operatorname{Ass}(\mathcal{M}) \subset U$;
- **(b)** $\operatorname{Ass}(\mathcal{M}) = \operatorname{Ass}(i^*\mathcal{M})$;
- **(c)** $\underline{H}^0_{Y\setminus U}(\mathcal{M}) = 0$;
- **(d)** $\mathcal{M} \to i_*i^*\mathcal{M}$ *is monic,*

(iv) *Assume that $U \cap \operatorname{Supp}\mathcal{M}$ is dense in $\operatorname{Supp} M$. If $\mathcal{M}$ satisfies (S_1), then $\mathcal{M} \to i_*i^*\mathcal{M}$ is monic.*

Proof. **(i)** This is because $(i^*\mathcal{M})_x \cong \mathcal{M}_x$ for $x \in U$.

(ii) Note that $\underline{H}^0_{Y\setminus U}\mathcal{M}$ is coherent. Let $x \in Y \setminus U$. The image of the injective map $(\underline{H}^0_{Y\setminus U}\mathcal{M})_x \to \mathcal{M}_x$ is identified with $H^0_{\mathcal{I}_x}(\mathcal{M}_x)$, where $\mathcal{I}$ is any ideal such that $V(\mathcal{I}) = Y \setminus U$ set theoretically. As any map $\kappa(x) \to \mathcal{M}_x$ factors through $H^0_{\mathcal{I}_x}(\mathcal{M}_x)$, the assertion follows.

(iii) **(a)**$\Leftrightarrow$**(b)** follows from **(i)**. **(c)** is equivalent to $\operatorname{Ass}(\underline{H}^0_{Y\setminus U}(\mathcal{M})) = \emptyset$. By **(ii)**, **(a)**$\Leftrightarrow$**(c)** follows. **(c)**$\Leftrightarrow$**(d)** is trivial.

(iv) Note that any associated point of $\mathcal{M}$ in $Y \setminus U$ is embedded by assumption. As $\mathcal{M}$ does not have an embedded prime by assumption, $\operatorname{Ass}\mathcal{M} \subset U$. By **(iii)**, $\mathcal{M} \to i_*i^*\mathcal{M}$ is monic. □

3.17 Corollary. *Let $f : Z \to Y$ be a flat morphism of finite type. If the image of f is dense in Y and Y satisfies (S_1), then $\operatorname{SIm} f = Y$.*

Proof. Replacing Z by $\operatorname{Im} f$, we may assume that f is an open immersion. The assertion follows immediately by the lemma applied to $\mathcal{O}_Y$. $\square$

3.18 Lemma. *Let $f : Z \to Y$ be a morphism of noetherian schemes. Let $\mathcal{M}$ be a primary coherent $\mathcal{O}_Z$-module. If $\mathcal{N}$ is a coherent $\mathcal{O}_Y$-submodule of $f_*\mathcal{M}$, then $\mathcal{N}$ is either zero or primary. In particular, if Z is primary, then* $\operatorname{SIm} f$ *is primary.*

Proof. First, note that if $A \to B$ is an injective homomorphism between noetherian rings, M a primary finitely generated faithful B-module, and N a finitely generated A-submodule of M, then either $N = 0$ or N is primary and $\operatorname{Supp} N = \operatorname{Spec} A$, set theoretically.

Indeed, if $n \in N \setminus 0$, then $0 :_B n \subset \sqrt{0_B}$. So $0 :_A n \subset \sqrt{0_A}$.

To prove the lemma, replacing Z by $V(\underline{\operatorname{ann}}\,\mathcal{M})$, we may assume that $\underline{\operatorname{ann}}\,\mathcal{M} = 0$. By Lemma 3.14, Z is primary. In particular, Z is irreducible. Next, replacing Y by $\operatorname{SIm} f$, we may assume that $Y = \operatorname{SIm} f$. In particular, Y is irreducible, and $\mathcal{O}_Y \to f_*\mathcal{O}_Z$ is monic. Clearly, we may assume that $\mathcal{N} \neq 0$. Let $U = \operatorname{Spec} A$ be an open subset of Y such that $i^*\mathcal{N} \neq 0$, where $i : U \hookrightarrow Y$ is the inclusion. Let $V = \operatorname{Spec} B$ be a non-empty open subset of $f^{-1}(U)$. By Lemma 3.16,

$$i^*\mathcal{N} \to i^* f_*\mathcal{M} \cong g_* j^*\mathcal{M} \to g_* k_* k^* j^*\mathcal{M} = (gk)_*(jk)^*\mathcal{M}$$

is monic, where $j : f^{-1}(U) \to Z$ is the inclusion, $k : V \to f^{-1}(U)$ is the inclusion, and $g : f^{-1}(U) \to U$ is the restriction of f. By the affine case above, $\operatorname{Ass}(i^*\mathcal{N}) = \{\eta\}$, where η is the generic point of Y.

The diagram

$$\begin{array}{ccc} \mathcal{N} & \longrightarrow & i_* i^*\mathcal{N} \\ \downarrow & & \downarrow \\ f_*\mathcal{M} & \longrightarrow & f_*(jk)_*(jk)^*\mathcal{M} = i_*(gk)_*(jk)^*\mathcal{M} \end{array}$$

is commutative. Thus the canonical map $\mathcal{N} \to i_* i^*\mathcal{N}$ is monic by Lemma 3.16 applied to $\mathcal{M}$. By Lemma 3.16 applied to $\mathcal{N}$, $\operatorname{Ass}\mathcal{N} = \operatorname{Ass}(i^*\mathcal{N}) = \{\eta\}$, and $\mathcal{N}$ is primary, as desired. $\square$

4. G-prime and G-radical G-ideals

Let S be a scheme, G an S-group scheme, and X a G-scheme. We say that X is a p-flat G-scheme if the second projection $p_2 : G \times X \to X$ is flat. If

G is flat over S, then any G-scheme is p-flat. We always assume that X is p-flat. Although we do not assume that G is S-flat, the sheaf theory as in [12] and [13] goes well, since we assume that X is p-flat and hence $B_G^M(X)$ has flat arrows.

In the rest of the paper, an $\mathcal{O}_X$-module and an ideal of $\mathcal{O}_X$ are required to be quasi-coherent, unless otherwise specified. A $(G,\mathcal{O}_X)$-module and a G-ideal of $\mathcal{O}_X$ are also required to be quasi-coherent unless otherwise specified.

(4.1) Let $\mathcal{M}$ be a $(G,\mathcal{O}_X)$-module. Note that the sum $\sum_\lambda \mathcal{M}_\lambda$ of quasi-coherent $(G,\mathcal{O}_X)$-submodules $\mathcal{M}_\lambda$ is a quasi-coherent $(G,\mathcal{O}_X)$-submodule. If $\mathcal{N}$ and $\mathcal{L}$ are quasi-coherent $(G,\mathcal{O}_X)$-submodules, then $\mathcal{N}\cap\mathcal{L}$ is again a quasi-coherent $(G,\mathcal{O}_X)$-submodule. For a quasi-coherent G-ideal $\mathcal{I}$, $\mathcal{I}\mathcal{N}$ is a quasi-coherent $(G,\mathcal{O}_X)$-submodule. If, moreover, $\mathcal{I}$ is coherent, then being the kernel of the canonical map

$$\mathcal{M} \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{I}, \mathcal{M}/\mathcal{N}), \tag{4}$$

$\mathcal{N}:\mathcal{I}$ is also a quasi-coherent $(G,\mathcal{O}_X)$-submodule, see [12, (7.11)] and [12, (7.6)]. More generally,

4.2 Lemma. *Let Y be a scheme, $\mathcal{M}$ a quasi-coherent $\mathcal{O}_Y$-module, $\mathcal{N}$ a quasi-coherent $\mathcal{O}_Y$-submodule of $\mathcal{M}$, and $\mathcal{I}$ a quasi-coherent ideal of $\mathcal{O}_Y$. If $\mathcal{I}$ is of finite type, then $\mathcal{N}:\mathcal{I}$, the kernel of* (4)*, is a quasi-coherent submodule of $\mathcal{M}$. If X is a G-scheme, $\mathcal{M}$ a quasi-coherent $(G,\mathcal{O}_X)$-module, $\mathcal{N}$ its quasi-coherent $(G,\mathcal{O}_X)$-submodule, and $\mathcal{I}$ a quasi-coherent G-ideal of finite type, then $\mathcal{N}:\mathcal{I}$ is a quasi-coherent $(G,\mathcal{O}_X)$-submodule of $\mathcal{M}$.*

Proof. We prove the first assertion. For an affine open subset U of Y, $\Gamma(U,\mathcal{N}:\mathcal{I})$ is the kernel of $M\to\mathrm{Hom}_A(I,M/N)$, where $A:=\Gamma(U,\mathcal{O}_X)$, $I:=\Gamma(U,\mathcal{I})$, $M:=\Gamma(U,\mathcal{M})$, and $N:=\Gamma(U,\mathcal{N})$. So $\Gamma(U,\mathcal{N}:\mathcal{I})=N:I$. Since $(N:I)B=N\otimes_A B:IB$ for a flat A-algebra B, the formation of a colon module (for of finite type $\mathcal{I}$) is compatible with the localization. So $\mathcal{N}:\mathcal{I}$ is quasi-coherent.

Next, we prove the second assertion. By the reason above, formation of a colon module (for of finite type $\mathcal{I}$) is compatible with a flat base change. So

$$\Phi(a^*(\mathcal{N}:\mathcal{I}))=\Phi(a^*\mathcal{N}:a^*\mathcal{I})=\Phi(a^*\mathcal{N}):a^*\mathcal{I}=p_2^*\mathcal{N}:p_2^*\mathcal{I}=p_2^*(\mathcal{N}:\mathcal{I}).$$

This shows that $\mathcal{N}:\mathcal{I}$ is a $(G,\mathcal{O}_X)$-submodule of $\mathcal{M}$. □

Similarly, if $\mathcal{L}$ is a quasi-coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$ of finite type, then $\mathcal{N} : \mathcal{L}$ is a quasi-coherent G-ideal of $\mathcal{O}_X$.

For an $\mathcal{O}_X$-submodule $\mathfrak{m}$, quasi-coherent or not, of $\mathcal{M}$, the sum of all quasi-coherent $(G, \mathcal{O}_X)$-submodules of $\mathcal{M}$ contained in $\mathfrak{m}$ is the largest quasi-coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$ contained in $\mathfrak{m}$. We denote this by $\mathfrak{m}^*$. If $\mathfrak{a}$ is a quasi-coherent ideal of $\mathcal{O}_X$ and $Y = V(\mathfrak{a})$, then we denote $V(\mathfrak{a}^*)$ by Y^*. Y^* is the smallest closed G-subscheme of X containing Y.

For a morphism $f : Y \to X$, $\mathrm{Ker}(\mathcal{O}_X \to f_*\mathcal{O}_Y)^*$ defines the smallest closed G-subscheme Y' of X such that $f^{-1}(Y') = Y$. We call Y' the G-scheme theoretic image of Y by f, and denote it by $G\mathrm{SIm}(f)$. Clearly, $G\mathrm{SIm}(f) \supset \mathrm{SIm}(f)$ and $G\mathrm{SIm}(f) = \mathrm{SIm}(f)^*$. Note that for a closed subscheme Y of X, Y^* is the G-scheme theoretic image of the inclusion $Y \hookrightarrow X$. It is easy to verify that, for a closed subscheme Y of X, the G-scheme theoretic image of the action $G \times Y \to X$ $((g, y) \mapsto gy)$ is Y^*.

If $f : Y \to X$ is a quasi-compact quasi-separated G-morphism of G-schemes, then $\mathrm{Ker}(\mathcal{O}_X \to f_*\mathcal{O}_Y)$ is a quasi-coherent G-ideal. So $G\mathrm{SIm}(f) = \mathrm{SIm}(f) = V(\mathrm{Ker}(\mathcal{O}_X \to f_*\mathcal{O}_Y))$.

4.3 Lemma. *Let $f : V \to X$ be a G-morphism of G-schemes. Let Y be a closed subscheme of V. Then*

$$G\mathrm{SIm}(Y^* \hookrightarrow V \to X) = \mathrm{SIm}(Y \hookrightarrow V \to X)^*.$$

Proof.

$$G\mathrm{SIm}(Y^* \hookrightarrow V \to X) \supset G\mathrm{SIm}(Y \hookrightarrow V \to X) = \mathrm{SIm}(Y \hookrightarrow V \to X)^*.$$

We prove the opposite inclusion. $f^{-1}(\mathrm{SIm}(Y \hookrightarrow V \to X)^*)$ is a G-closed subscheme of V containing Y. So it also contains Y^* by the minimality of Y^*. By the minimality of $G\mathrm{SIm}(Y^* \hookrightarrow V \to X)$, we have $G\mathrm{SIm}(Y^* \hookrightarrow V \to X) \subset \mathrm{SIm}(Y \hookrightarrow V \to X)^*$. □

4.4 Lemma. *Let $(\mathfrak{m}_\lambda)_{\lambda \in \Lambda}$ be a family of $\mathcal{O}_X$-submodules of $\mathcal{M}$. Then $(\bigcap_\lambda \mathfrak{m}_\lambda^*)^* = (\bigcap_\lambda \mathfrak{m}_\lambda)^*$.*

Proof. Since $\mathfrak{m}_\lambda^* \subset \mathfrak{m}_\lambda$ for each λ, we have $(\bigcap_\lambda \mathfrak{m}_\lambda^*)^* \subset (\bigcap_\lambda \mathfrak{m}_\lambda)^*$.

On the other hand, since $\mathfrak{m}_\lambda^* \supset (\bigcap_\lambda \mathfrak{m}_\lambda)^*$ for each λ, we have $\bigcap \mathfrak{m}_\lambda^* \supset (\bigcap_\lambda \mathfrak{m}_\lambda)^*$. By the maximality of $(\bigcap \mathfrak{m}_\lambda^*)^*$, we have $(\bigcap \mathfrak{m}_\lambda^*)^* \supset (\bigcap_\lambda \mathfrak{m}_\lambda)^*$. □

4.5 Corollary. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be $\mathcal{O}_X$-submodules of $\mathcal{M}$. Then $\mathfrak{m}^* \cap \mathfrak{n}^* = (\mathfrak{m} \cap \mathfrak{n})^*$.*

Proof. Follows immediately from the lemma, since $\mathfrak{m}^* \cap \mathfrak{n}^* = (\mathfrak{m}^* \cap \mathfrak{n}^*)^*$. □

4.6 Lemma. *Let $\mathfrak{m}$ be an $\mathcal{O}_X$-submodule of $\mathcal{M}$. If $\mathcal{M}$ is of finite type, then $(\mathfrak{m} : \mathcal{M})^* = \mathfrak{m}^* : \mathcal{M}$.*

Proof. Set $\mathcal{I} := (\mathfrak{m} : \mathcal{M})^*$. Then $\mathcal{I}\mathcal{M} \subset \mathfrak{m}$ and $\mathcal{I}\mathcal{M}$ is a quasi-coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$. Hence $\mathcal{I}\mathcal{M} \subset \mathfrak{m}^*$ by the maximality, and $\mathcal{I} \subset \mathfrak{m}^* : \mathcal{M}$.

On the other hand, $\mathfrak{m}^* : \mathcal{M} \subset \mathfrak{m} : \mathcal{M}$. Hence $\mathfrak{m}^* : \mathcal{M} \subset \mathcal{I}$ by the maximality. □

4.7 Lemma. *Let $\mathfrak{m}$ be an $\mathcal{O}_X$-submodule of $\mathcal{M}$, and $\mathcal{J}$ a finite-type G-ideal of $\mathcal{O}_X$. Then $(\mathfrak{m} : \mathcal{J})^* = \mathfrak{m}^* : \mathcal{J}$.*

Proof. Similar. □

(4.8) We denote the scheme X with the trivial G action by X'. Thus $G \times X'$ is the principal G-bundle (i.e., the G-scheme with the G-action given by $g(g', x) = (gg', x)$).

(4.9) Let us consider the diagram

$$G \times G \times G \times X \overset{\xrightarrow{1_G \times \mu \times 1_X}}{\underset{\xrightarrow{p_{234}}}{\xrightarrow{\mu \times 1_{G \times X}}}} G \times G \times X \underset{\xrightarrow{p_{23}}}{\xrightarrow{\mu \times 1}} G \times X \xrightarrow{p_2} X$$

on the finite category Δ_M^+ (see for the definition, [12, (9.1)]). For $\mathcal{M} \in \mathrm{Qch}(X)$, $\mathbb{A}\mathcal{M}$ is in $\mathrm{Qch}(G, X)$, where $\mathbb{A} = (?)_{\Delta_M} \circ L_{[-1]}$ is the ascent functor [12, (12.9)]. Thus we may say that $p_2^*\mathcal{M}$ is a quasi-coherent $(G, \mathcal{O}_X)$-module, since $(\mathbb{A}\mathcal{M})_0 = p_2^*\mathcal{M}$. The G-linearization of $p_2^*\mathcal{M}$ is the canonical isomorphism $d : (\mu \times 1)^* p_2^*\mathcal{M} \to p_{23}^* p_2^*\mathcal{M}$.

(4.10) Let $a : G \times X' \to X$ be the action. Then a is a G-morphism. Thus $a^*\mathcal{M}$ is a quasi-coherent G-linearized $\mathcal{O}_X$-module for $\mathcal{M} \in \mathrm{Qch}(G, X)$. The G-linearization is the composite map

$$(\mu \times 1)^* a^*\mathcal{M} \xrightarrow{d} (1 \times a)^* a^*\mathcal{M} \xrightarrow{\Phi} (1 \times a)^* p_2^*\mathcal{M} \xrightarrow{d} p_{23}^* a^*\mathcal{M}.$$

Since $\Phi : a^*\mathcal{M} \to p_2^*\mathcal{M}$ is a G-linearization, $\Phi : a^*\mathcal{M} \to p_2^*\mathcal{M}$ is an isomorphism of G-linearized $\mathcal{O}_X$-modules. In particular, the composite map

$$\omega : \mathcal{M} \xrightarrow{u} a_* a^*\mathcal{M} \xrightarrow{\Phi} a_* p_2^*\mathcal{M}$$

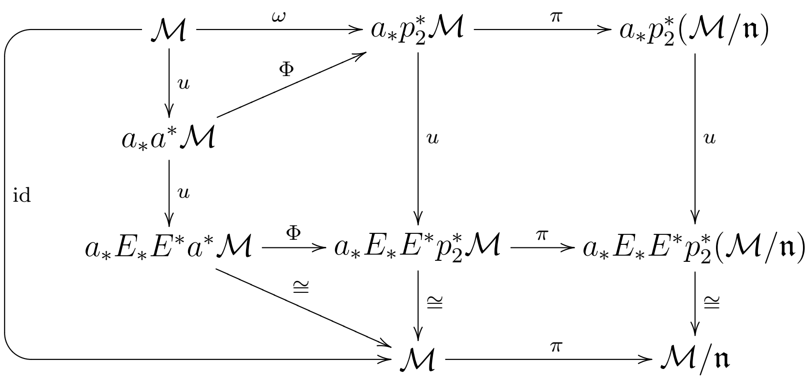
is $(G, \mathcal{O}_X)$-linear.

4.11 Lemma. *Let $\mathcal{M}$ be a quasi-coherent $(G, \mathcal{O}_X)$-module, and $\mathfrak{n}$ a quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{M}$. Assume that the second projection $G \times X \to X$ is quasi-compact quasi-separated. Then $\mathfrak{n}^*$ agrees with the kernel of the composite map*

$$\mathcal{M} \xrightarrow{\omega} a_* p_2^* \mathcal{M} \xrightarrow{\pi_{\mathfrak{n}}} a_* p_2^* (\mathcal{M}/\mathfrak{n}),$$

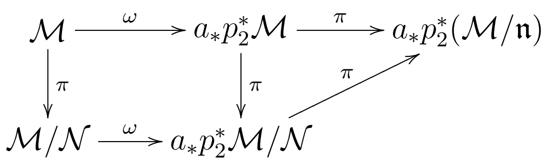
where $\pi_{\mathfrak{n}} : \mathcal{M} \to \mathcal{M}/\mathfrak{n}$ is the projection.

Proof. The diagram



is commutative, where $e : \operatorname{Spec} S \to G$ is the unit element, and $E = e \times 1 : X \to G \times X$. Thus $\operatorname{Ker}(\pi\omega) \subset \operatorname{Ker}(u\pi\omega) = \operatorname{Ker}\pi = \mathfrak{n}$. Moreover, $\operatorname{Ker}(\pi\omega) \subset \mathcal{M}$ is a quasi-coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$, since $\pi\omega$ is $(G, \mathcal{O}_X)$-linear.

So it suffices to show that any $(G, \mathcal{O}_X)$-submodule $\mathcal{N}$ of $\mathcal{M}$ contained in $\mathfrak{n}$ is also contained in $\operatorname{Ker}(\pi\omega)$. This is trivial, since the diagram



is commutative. □

4.12 Definition. We say that X is G-integral (resp. G-reduced) if there is an integral (resp. reduced) closed subscheme Y of X such that $Y^* = X$. A G-ideal $\mathcal{P}$ of $\mathcal{O}_X$ is said to be G-prime (resp. G-radical), if $V(\mathcal{P})$ is G-integral (resp. G-reduced).

4.13 Lemma. *Let $f : V \to X$ be a G-morphism of G-schemes. If V is G-integral* (*resp. G-reduced*)*, then $G\mathrm{SIm}(f)$ is G-integral* (*resp. G-reduced*)*.*

Proof. There is an integral (resp. reduced) closed subscheme Y of V such that $Y^* = V$. Then $Z := \mathrm{SIm}(Y \hookrightarrow V \to X)$ is integral (resp. reduced), see (2.2). Then $G\mathrm{SIm}(f) = Z^*$ by Lemma 4.3, and we are done. $\square$

4.14 Corollary. *Let $f : V \to X$ be a G-morphism of G-schemes. If $\mathcal{I}$ is a G-prime (resp. G-radical) ideal of $\mathcal{O}_V$, then $(\mathcal{I} \cap \mathcal{O}_X)^*$ is G-prime (resp. G-radical).*

Proof. This is because $(\mathcal{I} \cap \mathcal{O}_X)^*$ defines $G\mathrm{SIm}(V(\mathcal{I}) \hookrightarrow V \to X)$, which is G-integral (resp. G-reduced). $\square$

4.15 Lemma. *Let $f : W \to V$ and $g : V \to X$ be G-morphisms of G-schemes, and let $\iota : G\mathrm{SIm}\, f \hookrightarrow V$ be the inclusion. Then $G\mathrm{SIm}(gf) = G\mathrm{SIm}(g\iota)$.*

Proof. Similar to [8, (9.5.5)]. $\square$

4.16 Lemma. *For a family $(\mathcal{I}_\lambda)$ of G-radical G-ideals of $\mathcal{O}_X$, $(\bigcap_\lambda \mathcal{I}_\lambda)^*$ is G-radical.*

Proof. There exists a family $(\mathcal{J}_\lambda)$ of radical ideals of $\mathcal{O}_X$ such that $\mathcal{J}_\lambda^* = \mathcal{I}_\lambda$. By Lemma 4.4, we have

$$(\bigcap_\lambda \mathcal{I}_\lambda)^* = (\bigcap_\lambda \mathcal{J}_\lambda)^* = ((\bigcap_\lambda \mathcal{J}_\lambda)^\star)^*.$$

By Lemma 3.8, $(\bigcap_\lambda \mathcal{J}_\lambda)^\star$ is a radical ideal. So $(\bigcap_\lambda \mathcal{I}_\lambda)^*$ is G-radical. $\square$

4.17 Corollary. *The intersection of finitely many G-radical G-ideals is G-radical.* $\square$

4.18 Lemma. *Let Y be an S-scheme which is integral (resp. reduced). Assume that the principal G-bundle $G \times Y$ is p-flat. Then $G \times Y$ is G-integral (resp. G-reduced).*

Proof. Let us consider the closed subscheme $\{e\} \times Y$ of $G \times Y$, where e is the unit element. Then

$$G \times Y \cong G \times \{e\} \times Y \hookrightarrow G \times G \times Y \xrightarrow{\mu \times 1} G \times Y$$

is the identity, and its G-scheme theoretic image is $(\{e\} \times Y)^*$. Hence $(\{e\} \times Y)^* = G \times Y$. Since $\{e\} \times Y \cong Y$ is integral (resp. reduced), $G \times Y$ is G-integral (resp. G-reduced). $\square$

4.19 Corollary. *Let $f : Y \to X$ be an S-morphism. Assume that Y is integral (resp. reduced). Set g to be the composite*

$$G \times Y \xrightarrow{1_G \times f} G \times X \xrightarrow{a} X.$$

Then $G\mathrm{SIm}\, g$ is G-integral (resp. G-reduced).

Proof. Follows from Lemma 4.18 and Lemma 4.13. $\square$

4.20 Lemma. *Assume that the second projection $p_2 : G \times X \to X$ is quasi-compact quasi-separated. If X is G-integral (resp. G-reduced) and U is a non-empty G-stable open subscheme of X, then U is G-integral (resp. G-reduced).*

Proof. Take an integral closed subscheme Y of X such that $Y^* = X$. Then

$$\begin{array}{ccc} G \times (Y \cap U) & \xrightarrow{a} & U \\ \downarrow & & \downarrow \\ G \times Y & \xrightarrow{a} & X \end{array}$$

is a fiber square. By (2.7), $U = (Y \cap U)^*$. Since $Y \cap U$ is an integral (reduced) closed subscheme of U, U is G-integral (resp. G-reduced). $\square$

4.21 Lemma. *Assume that the second projection $p_2 : G \times X \to X$ is quasi-compact quasi-separated. Let $\varphi : X' \to X$ be a reduced G-morphism between locally noetherian G-schemes. If X is G-reduced, then X' is G-reduced.*

Proof. Similar. $\square$

(4.22) A G-ideal $\mathcal{P}$ of $\mathcal{O}_X$ is said to be *G-quasi-prime* if $\mathcal{P} \neq \mathcal{O}_X$, and if $\mathcal{I}$ and $\mathcal{J}$ are G-ideals of $\mathcal{O}_X$ such that $\mathcal{I}\mathcal{J} \subset \mathcal{P}$, then $\mathcal{I} \subset \mathcal{P}$ or $\mathcal{J} \subset \mathcal{P}$ holds.

4.23 Lemma. *If $\mathcal{P}$ is a G-prime ideal of $\mathcal{O}_X$, then $\mathcal{P}$ is G-quasi-prime.*

Proof. Let $\mathcal{P} = \mathfrak{p}^*$ for a prime ideal $\mathfrak{p}$ of $\mathcal{O}_X$. Since $\mathcal{P} \subset \mathfrak{p} \neq \mathcal{O}_X$, $\mathcal{P} \neq \mathcal{O}_X$. Let $\mathcal{I}$ and $\mathcal{J}$ be G-ideals of $\mathcal{O}_X$ such that $\mathcal{I}\mathcal{J} \subset \mathcal{P}$. Then $\mathcal{I}\mathcal{J} \subset \mathfrak{p}$. Since $\mathfrak{p}$ is a prime, we have $\mathcal{I} \subset \mathfrak{p}$ or $\mathcal{J} \subset \mathfrak{p}$. Since $\mathcal{I}$ and $\mathcal{J}$ are G-ideals, we have $\mathcal{I} \subset \mathfrak{p}^* = \mathcal{P}$ or $\mathcal{J} \subset \mathfrak{p}^* = \mathcal{P}$. $\square$

(4.24) For a G-ideal $\mathcal{I} \subset \mathcal{O}_X$, we denote the set of G-prime G-ideals containing $\mathcal{I}$ by $V_G(\mathcal{I})$. The set $V_G(0)$ is denoted by $\operatorname{Spec}_G(X)$. We define the *G-radical* of $\mathcal{I}$ by

$$\sqrt[G]{\mathcal{I}} := (\bigcap_{\mathcal{P} \in V_G(\mathcal{I})} \mathcal{P})^*,$$

where the right hand side is defined to be $\mathcal{O}_X$ if $V_G(\mathcal{I}) = \emptyset$.

4.25 Definition. For a G-ideal $\mathcal{I}$, we define

$$\Omega(\mathcal{I}) = \{\mathcal{J} \mid \mathcal{J} \text{ is a quasi-cohernt } G\text{-ideal of } \mathcal{O}_X, \text{ and } \mathcal{I} \subset \mathcal{J} \neq \mathcal{O}_X\}.$$

A maximal element of $\Omega(0)$ is said to be *G-maximal*.

4.26 Lemma. *A G-maximal G-ideal is G-quasi-prime. If X is quasi-compact, then a G-maximal G-ideal of $\mathcal{O}_X$ is of the form $\mathfrak{m}^*$ for some maximal ideal $\mathfrak{m}$. In particular, it is G-prime.*

Proof. We prove the first assertion. Let $\mathcal{M}$ be a G-maximal G-ideal of $\mathcal{O}_X$, and $\mathcal{I}$ and $\mathcal{J}$ be G-ideals such that $\mathcal{I}\mathcal{J} \subset \mathcal{M}$. Assume that $\mathcal{I} \not\subset \mathcal{M}$ and $\mathcal{J} \not\subset \mathcal{M}$. Then $\mathcal{I} + \mathcal{M} = \mathcal{O}_X$ and $\mathcal{J} + \mathcal{M} = \mathcal{O}_X$ by the G-maximality. So

$$\mathcal{O}_X = (\mathcal{I} + \mathcal{M})(\mathcal{J} + \mathcal{M}) \subset \mathcal{M} + \mathcal{I}\mathcal{J} \subset \mathcal{M}.$$

This contradicts $\mathcal{M} \neq \mathcal{O}_X$.

We prove the second assertion. Let $\mathcal{M}$ be a G-maximal G-ideal of $\mathcal{O}_X$. Since X is quasi-compact, $V(\mathcal{M})$ is quasi-compact, and is clearly non-empty. So there is a maximal ideal $\mathfrak{m}$ of $\mathcal{O}_X$ containing $\mathcal{M}$. Since $\mathcal{M} \subset \mathfrak{m}$, we have $\mathcal{M} \subset \mathfrak{m}^*$. By the maximality, $\mathcal{M} = \mathfrak{m}^*$. Since $\mathfrak{m}$ is a prime, $\mathcal{M}$ is G-prime. □

4.27 Lemma. *Let $\mathcal{I}$ be a G-ideal of $\mathcal{O}_X$. If $\mathcal{I} \neq \mathcal{O}_X$ and X is quasi-compact, then $\Omega(\mathcal{I})$ has a maximal element. In particular, a non-empty quasi-compact p-flat G-scheme has a G-maximal G-ideal.*

Proof. Since $\mathcal{I} \in \Omega(\mathcal{I})$, $\Omega(\mathcal{I})$ is non-empty. So it suffices to show that, by Zorn's lemma, for any non-empty chain (i.e., a totally ordered family) of elements $(\mathcal{J}_\lambda)_{\lambda \in \Lambda}$ of $\Omega(\mathcal{I})$, $\mathcal{J} := \sum_\lambda \mathcal{J}_\lambda$ is again in $\Omega(\mathcal{I})$. It is obvious that $\mathcal{J}$ is a quasi-coherent G-ideal and $\mathcal{J} \supset \mathcal{I}$. It suffices to show that $\mathcal{J} \neq \mathcal{O}_X$.

Assume the contrary. Let $X = \bigcup_{i=1}^n U_i$ be a finite affine open covering of X, which exists. Let $J_{\lambda,i} := \Gamma(U_i, \mathcal{J}_\lambda)$, and $J_i = \Gamma(U_i, \mathcal{J}) = \sum_\lambda J_{\lambda,i}$. As $1 \in J_i$, there exists some μ_i such that $1 \in J_{\mu_i,i}$. When we set $\mu = \max(\mu_1, \ldots, \mu_n)$, then $\mathcal{J}_\mu = \mathcal{O}_X$. This is a contradiction. □

4.28 Lemma. *Let $\mathcal{I}$, $\mathcal{J}$, $\mathcal{P}$, and $\mathcal{I}_\lambda$ $(\lambda \in \Lambda)$ be G-ideals of $\mathcal{O}_X$. Then the following hold:*

(i) $\sqrt[G]{\mathcal{I}} \supset \mathcal{I}$.

(ii) *If $\mathcal{I} \supset \mathcal{J}$, then $V_G(\mathcal{I}) \subset V_G(\mathcal{J})$. In particular, $\sqrt[G]{\mathcal{I}} \supset \sqrt[G]{\mathcal{J}}$.*

(iii) *$V_G(\sqrt[G]{\mathcal{I}}) = V_G(\mathcal{I})$. In particular, $\sqrt[G]{\sqrt[G]{\mathcal{I}}} = \sqrt[G]{\mathcal{I}}$.*

(iv) *$V_G(\mathcal{I}\mathcal{J}) = V_G(\mathcal{I} \cap \mathcal{J}) = V_G(\mathcal{I}) \cup V_G(\mathcal{J})$. So $\sqrt[G]{\mathcal{I}\mathcal{J}} = \sqrt[G]{\mathcal{I} \cap \mathcal{J}} = \sqrt[G]{\mathcal{I}} \cap \sqrt[G]{\mathcal{J}}$.*

(v) *$V_G(\sum_\lambda \mathcal{I}_\lambda) = \bigcap_\lambda V_G(\mathcal{I}_\lambda)$.*

(vi) *For $n \geq 1$, $\sqrt[G]{\mathcal{I}^n} = \sqrt[G]{\mathcal{I}}$.*

(vii) *If $\mathcal{P}$ is G-prime, then $\sqrt[G]{\mathcal{P}} = \mathcal{P}$.*

(viii) *If there exists some $n \geq 1$ such that $\mathcal{J}^n \subset \sqrt[G]{\mathcal{I}}$, then $\mathcal{J} \subset \sqrt[G]{\mathcal{I}}$.*

Proof. **(i)** If $\mathcal{P} \in V_G(\mathcal{I})$, then $\mathcal{P} \supset \mathcal{I}$. So $\bigcap_{\mathcal{P} \in V_G(\mathcal{I})} \mathcal{P} \supset \mathcal{I}$. By the maximality,

$$\sqrt[G]{\mathcal{I}} = (\bigcap_{\mathcal{P} \in V_G(\mathcal{I})} \mathcal{P})^* \supset \mathcal{I}.$$

(ii) If $\mathcal{I} \supset \mathcal{J}$ and $\mathcal{P} \in V_G(\mathcal{I})$, then $\mathcal{P} \supset \mathcal{I} \supset \mathcal{J}$. So $\mathcal{P} \in V_G(\mathcal{J})$. Thus $V_G(\mathcal{I}) \subset V_G(\mathcal{J})$. Hence

$$\sqrt[G]{\mathcal{I}} = (\bigcap_{\mathcal{P} \in V_G(\mathcal{I})} \mathcal{P})^* \supset (\bigcap_{\mathcal{P} \in V_G(\mathcal{J})} \mathcal{P})^* = \sqrt[G]{\mathcal{J}}.$$

(iii) By **(i)** and **(ii)**, $V_G(\sqrt[G]{\mathcal{I}}) \subset V_G(\mathcal{I})$. If $\mathcal{P} \in V_G(\mathcal{I})$, then

$$\mathcal{P} \supset (\bigcap_{\mathcal{Q} \in V_G(\mathcal{I})} \mathcal{Q})^* = \sqrt[G]{\mathcal{I}}.$$

Hence $\mathcal{P} \in V_G(\sqrt[G]{\mathcal{I}})$. So $V_G(\sqrt[G]{\mathcal{I}}) \supset V_G(\mathcal{I})$, and $V_G(\sqrt[G]{\mathcal{I}}) = V_G(\mathcal{I})$ holds. This shows

$$\sqrt[G]{\sqrt[G]{\mathcal{I}}} = (\bigcap_{\mathcal{P} \in V_G(\sqrt[G]{\mathcal{I}})} \mathcal{P})^* = (\bigcap_{\mathcal{P} \in V_G(\mathcal{I})} \mathcal{P})^* = \sqrt[G]{\mathcal{I}}.$$

(iv) By **(ii)**, $V_G(\mathcal{I}\mathcal{J}) \supset V_G(\mathcal{I} \cap \mathcal{J}) \supset V_G(\mathcal{I}) \cup V_G(\mathcal{J})$ is trivial. If $\mathcal{P} \in V_G(\mathcal{I}\mathcal{J})$, then since $\mathcal{P} \supset \mathcal{I}\mathcal{J}$ and $\mathcal{P}$ is G-quasi-prime, $\mathcal{P} \supset \mathcal{I}$ or $\mathcal{P} \supset \mathcal{J}$ holds. So $\mathcal{P} \in V_G(\mathcal{I}) \cup V_G(\mathcal{J})$. Thus $V_G(\mathcal{I}\mathcal{J}) = V_G(\mathcal{I} \cap \mathcal{J}) = V_G(\mathcal{I}) \cup V_G(\mathcal{J})$. Hence $\sqrt[G]{\mathcal{I}\mathcal{J}} = \sqrt[G]{\mathcal{I} \cap \mathcal{J}} = \sqrt[G]{\mathcal{I}} \cap \sqrt[G]{\mathcal{J}}$.

(v) $\mathcal{P} \in V_G(\sum_\lambda \mathcal{I}_\lambda) \iff \forall \lambda\ \mathcal{P} \supset \mathcal{I}_\lambda \iff \mathcal{P} \in \bigcap_\lambda V_G(\mathcal{I}_\lambda)$.

(vi) $\sqrt[G]{\mathcal{I}^n} = \sqrt[G]{\mathcal{I}\mathcal{I}\cdots\mathcal{I}} = \sqrt[G]{\mathcal{I} \cap \mathcal{I} \cap \cdots \cap \mathcal{I}} = \sqrt[G]{\mathcal{I}}$.

(vii) If $\mathcal{P}$ is G-prime, then $\mathcal{P}$ is the minimum element of $V_G(\mathcal{P})$. So $\sqrt[G]{\mathcal{P}} = (\bigcap_{\mathcal{Q} \in V_G(\mathcal{P})} \mathcal{Q})^* = \mathcal{P}^* = \mathcal{P}$.

(viii) $\mathcal{J} \subset \sqrt[G]{\mathcal{J}} = \sqrt[G]{\mathcal{J}^n} \subset \sqrt[G]{\sqrt[G]{\mathcal{I}}} = \sqrt[G]{\mathcal{I}}$. □

4.29 Lemma. *Let $\mathcal{I}$ be a G-ideal of $\mathcal{O}_X$. Then $\sqrt[G]{\mathcal{I}} = \sqrt{\mathcal{I}}^*$. In particular, $\mathcal{I} \subset \sqrt[G]{\mathcal{I}} \subset \sqrt{\mathcal{I}}$ and hence $\sqrt{\sqrt[G]{\mathcal{I}}} = \sqrt{\mathcal{I}}$. If, moreover, X is noetherian, then there exists some $n \geq 1$ such that $\sqrt[G]{\mathcal{I}}^n \subset \mathcal{I}$.*

Proof. In view of Lemma 4.28, **(i)**, it suffices to prove the first assertion. By Lemma 4.4,

$$\sqrt[G]{\mathcal{I}} = (\bigcap_{\mathcal{P} \in V_G(\mathcal{I})} \mathcal{P})^* = (\bigcap_{\mathfrak{p} \in V(\mathcal{I})} \mathfrak{p}^*)^* = (\bigcap_{\mathfrak{p} \in V(\mathcal{I})} \mathfrak{p})^* = \sqrt{\mathcal{I}}^*.$$

□

4.30 Corollary. *Let $\mathcal{I}$ be a G-ideal of $\mathcal{O}_X$. Then $\mathcal{I}$ is G-radical if and only if $\mathcal{I} = \sqrt[G]{\mathcal{I}}$.*

Proof. Assume that $\mathcal{I}$ is G-radical so that $\mathcal{I} = \mathfrak{a}^*$ for a radical ideal $\mathfrak{a}$. Since $\mathcal{I} = \mathfrak{a}^* \subset \mathfrak{a}$, $\sqrt{\mathcal{I}} \subset \sqrt{\mathfrak{a}} = \mathfrak{a}$. So $\mathcal{I} \subset \sqrt[G]{\mathcal{I}} = \sqrt{\mathcal{I}}^* \subset \mathfrak{a}^* = \mathcal{I}$, and hence $\mathcal{I} = \sqrt[G]{\mathcal{I}}$.

Conversely, assume that $\mathcal{I} = \sqrt[G]{\mathcal{I}}$. Then $\mathcal{I} = \sqrt[G]{\mathcal{I}} = \sqrt{\mathcal{I}}^*$, and since $\sqrt{\mathcal{I}}$ is a radical ideal, $\mathcal{I}$ is G-radical. □

4.31 Corollary. *For a G-ideal $\mathcal{I}$ of $\mathcal{O}_X$, $\mathcal{I} = \mathcal{O}_X$ if and only if $\sqrt[G]{\mathcal{I}} = \mathcal{O}_X$ if and only if $V_G(\mathcal{I}) = \emptyset$.*

Proof. $V_G(\mathcal{O}_X) = \emptyset$ is trivial. If $V_G(\mathcal{I}) = \emptyset$, then $\sqrt[G]{\mathcal{I}} = \mathcal{O}_X$ by definition. If $\sqrt[G]{\mathcal{I}} = \mathcal{O}_X$, then $\sqrt{\mathcal{I}} = \mathcal{O}_X$, and hence $\mathcal{I} = \mathcal{O}_X$. □

4.32 Lemma. *Assume that X is noetherian. If $\mathfrak{a}$ is an ideal of $\mathcal{O}_X$, then $\sqrt[G]{\mathfrak{a}^*} = \sqrt{\mathfrak{a}}^*$.*

Proof. Since $\sqrt[G]{\mathfrak{a}^*} \subset \sqrt{\mathfrak{a}^*} \subset \sqrt{\mathfrak{a}}$ and $\sqrt[G]{\mathfrak{a}^*}$ is a quasi-coherent G-ideal, we have $\sqrt[G]{\mathfrak{a}^*} \subset \sqrt{\mathfrak{a}}^*$.

Next, set $\mathcal{J} := \sqrt{\mathfrak{a}}^*$. Since $\mathcal{J} \subset \sqrt{\mathfrak{a}}$, there exists some $n \geq 1$ such that $\mathcal{J}^n \subset \mathfrak{a}$. Then $\mathcal{J}^n \subset \mathfrak{a}^* \subset \sqrt[G]{\mathfrak{a}^*}$. Hence $\mathcal{J} \subset \sqrt[G]{\mathfrak{a}^*}$. □

4.33 Definition. We say that X is a *G-point* if X has exactly two quasi-coherent G-ideals of $\mathcal{O}_X$. In other words, X is a G-point if and only if 0 is a G-maximal G-ideal.

4.34 Corollary. *Let X be a G-point, and $\mathcal{M}$ a quasi-coherent $(G, \mathcal{O}_X)$-module of finite type. Then $\mathcal{M}$ is locally free of a well-defined rank.*

Proof. Let r be the smallest integer such that $\underline{\mathrm{Fitt}}_r(\mathcal{M}) \neq 0$. Since $\underline{\mathrm{Fitt}}_r \mathcal{M}$ is a nonzero G-ideal, $\underline{\mathrm{Fitt}}_r \mathcal{M} = \mathcal{O}_X$. By Lemma 2.11, $\mathcal{M}$ is locally free of rank r. □

5. G-primary G-ideals

(5.1) As in the last section, let S be a scheme, G an S-group scheme, and X a p-flat G-scheme. In this section, we always assume that X is noetherian. Let $\mathcal{M}$ be a coherent $(G, \mathcal{O}_X)$-module. Unless otherwise specified, a submodule of a coherent $\mathcal{O}_X$-module means a coherent submodule. In particular, an ideal of $\mathcal{O}_X$ means a coherent ideal. Let $\mathcal{N}$ be a (coherent) $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$.

5.2 Lemma. *Let $\mathcal{P}$ be a G-quasi-prime G-ideal of $\mathcal{O}_X$. Then $\mathcal{P} = \sqrt{\mathcal{P}}^*$.*

Proof. Since $\mathcal{P} \subset \sqrt{\mathcal{P}}$, we have $\mathcal{P} \subset \sqrt{\mathcal{P}}^*$.

Now we set $\mathcal{J} := \sqrt{\mathcal{P}}^*$ and we prove $\mathcal{J} \subset \mathcal{P}$. Since X is noetherian, there exists some $n \geq 1$ such that $\sqrt{\mathcal{P}}^n \subset \mathcal{P}$. Then $\mathcal{J}^n \subset \sqrt{\mathcal{P}}^n \subset \mathcal{P}$. Since $\mathcal{P}$ is G-quasi-prime, $\mathcal{J} \subset \mathcal{P}$, as desired. □

5.3 Lemma. *Let $\mathcal{P}$ be a G-ideal of $\mathcal{O}_X$. Then $\mathcal{P}$ is a G-prime if and only if $\mathcal{P}$ is a G-quasi-prime. If this is the case, $\mathcal{P} = \mathfrak{p}^*$ for some minimal prime $\mathfrak{p}$ of $\mathcal{P}$.*

Proof. The 'only if' part is Lemma 4.23.

Assume that $\mathcal{P}$ is a quasi-prime. Let

$$\sqrt{\mathcal{P}} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$$

be a minimal prime decomposition so that each $\mathfrak{p}_i$ is a minimal prime of $\mathcal{P}$. Then by Lemma 5.2,

$$\mathcal{P} = \sqrt{\mathcal{P}}^* = \mathfrak{p}_1^* \cap \cdots \cap \mathfrak{p}_r^*.$$

As $\mathcal{P}$ is a G-quasi-prime, there exists some i such that $\mathfrak{p}_i^* \subset \mathcal{P}$. So $\mathfrak{p}_i^* = \mathcal{P}$. □

5.4 Definition. A $(G, \mathcal{O}_X)$-submodule $\mathcal{N}$ is said to be a *G-primary* submodule of $\mathcal{M}$ if

(i) $\mathcal{M} \neq \mathcal{N}$, and

(ii) For a $(G, \mathcal{O}_X)$-submodule $\mathcal{L}$ of $\mathcal{M}$ and for a G-ideal $\mathcal{I}$ of $\mathcal{O}_X$, if $\mathcal{I}\mathcal{L} \subset \mathcal{N}$ and $\mathcal{L} \not\subset \mathcal{N}$, then $\mathcal{I} \subset \sqrt[G]{\mathcal{N} : \mathcal{M}}$.

A G-primary $(G, \mathcal{O}_X)$-submodule of $\mathcal{O}_X$ is said to be a G-primary G-ideal. If 0 is a G-primary submodule of $\mathcal{M}$, then we say that $\mathcal{M}$ is a G-primary $(G, \mathcal{O}_X)$-module. If $\mathcal{O}_X$ is a G-primary module, then we say that X is G-primary.

5.5 Lemma. *Let $\mathcal{Q}$ be a G-primary ideal of $\mathcal{O}_X$. Then $\sqrt[G]{\mathcal{Q}}$ is G-prime.*

Proof. It suffices to prove that $\sqrt[G]{\mathcal{Q}}$ is a G-quasi-prime.

Since $\mathcal{Q} \neq \mathcal{O}_X$, we have $\sqrt[G]{\mathcal{Q}} \neq \mathcal{O}_X$ by Corollary 4.31.

Let $\mathcal{I}$ and $\mathcal{J}$ be quasi-coherent G-ideals. Assume that $\mathcal{I}\mathcal{J} \subset \sqrt[G]{\mathcal{Q}}$ and $\mathcal{J} \not\subset \sqrt[G]{\mathcal{Q}}$. Then there exists some $n \geq 1$ such that $\mathcal{I}^n\mathcal{J}^n \subset \mathcal{Q}$ and $\mathcal{J}^n \not\subset \sqrt[G]{\mathcal{Q}}$. Hence $\mathcal{I}^n \subset \mathcal{Q}$. This shows that $\mathcal{I} \subset \sqrt[G]{\mathcal{Q}}$. □

5.6 Lemma. *Let $\mathcal{N}$ be a G-primary coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$. Then $\mathcal{N} : \mathcal{M}$ is G-primary. In particular, $\sqrt[G]{\mathcal{N} : \mathcal{M}}$ is G-prime.*

Proof. Let $\mathcal{I}$ and $\mathcal{J}$ be coherent G-ideals, and assume that $\mathcal{I}\mathcal{J} \subset \mathcal{N} : \mathcal{M}$ and that $\mathcal{J} \not\subset \mathcal{N} : \mathcal{M}$. Then $\mathcal{J}\mathcal{M} \not\subset \mathcal{N}$ and $\mathcal{I}\mathcal{J}\mathcal{M} \subset \mathcal{N}$. Hence $\mathcal{I} \subset \sqrt[G]{\mathcal{N} : \mathcal{M}}$. The last assertion follows from Lemma 5.5. □

(5.7) If $\mathcal{N}$ is G-primary and $\sqrt[G]{\mathcal{N} : \mathcal{M}} = \mathcal{P}$, then we say that $\mathcal{N}$ is $\mathcal{P}$-G-primary.

5.8 Lemma. *If $\mathfrak{m}$ is a $\mathfrak{p}$-primary submodule of $\mathcal{M}$, then $\mathfrak{m}^*$ is a $\mathfrak{p}^*$-G-primary $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$.*

Proof. By Lemma 4.32 and Lemma 4.6,

$$\mathfrak{p}^* = \sqrt{\mathfrak{m} : \mathcal{M}}^{\,*} = \sqrt[G]{(\mathfrak{m} : \mathcal{M})^*} = \sqrt[G]{\mathfrak{m}^* : \mathcal{M}}.$$

Let $\mathcal{I}$ be a coherent G-ideal of $\mathcal{O}_X$ and $\mathcal{L}$ a coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$. Assume that $\mathcal{I}\mathcal{L} \subset \mathfrak{m}^*$ and $\mathcal{L} \not\subset \mathfrak{m}^*$. Then $\mathcal{I}\mathcal{L} \subset \mathfrak{m}$ and $\mathcal{L} \not\subset \mathfrak{m}$. Hence $\mathcal{I} \subset \sqrt{\mathfrak{m} : \mathcal{M}}$. Hence

$$\mathcal{I} \subset \sqrt{\mathfrak{m} : \mathcal{M}}^{\,*} = \mathfrak{p}^* = \sqrt[G]{\mathfrak{m}^* : \mathcal{M}}.$$

Since $\mathfrak{m} \neq \mathcal{M}$, we have $\mathfrak{m}^* \neq \mathcal{M}$. □

5.9 Lemma. *Let $\mathcal{N}$ and $\mathcal{N}'$ be $\mathcal{P}$-G-primary coherent $(G, \mathcal{O}_X)$-submodules of $\mathcal{M}$. Then $\mathcal{N} \cap \mathcal{N}'$ is also $\mathcal{P}$-G-primary.*

Proof. $(\mathcal{N} : \mathcal{M}) \cap (\mathcal{N}' : \mathcal{M}) = (\mathcal{N} \cap \mathcal{N}') : \mathcal{M}$. So

$$\sqrt[G]{(\mathcal{N} \cap \mathcal{N}') : \mathcal{M}} = \sqrt[G]{(\mathcal{N} : \mathcal{M}) \cap (\mathcal{N}' : \mathcal{M})}$$
$$= \sqrt[G]{\mathcal{N} : \mathcal{M}} \cap \sqrt[G]{\mathcal{N}' : \mathcal{M}} = \mathcal{P} \cap \mathcal{P} = \mathcal{P}.$$

Let $\mathcal{I}$ be a coherent G-ideal, and $\mathcal{L}$ a coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$ such that $\mathcal{I}\mathcal{L} \subset \mathcal{N} \cap \mathcal{N}'$ and $\mathcal{L} \not\subset \mathcal{N} \cap \mathcal{N}'$. Then $\mathcal{L} \not\subset \mathcal{N}$ or $\mathcal{L} \not\subset \mathcal{N}'$. If $\mathcal{L} \not\subset \mathcal{N}$, then $\mathcal{I} \subset \sqrt[G]{\mathcal{N} : \mathcal{M}} = \mathcal{P}$. If $\mathcal{L} \not\subset \mathcal{N}'$, then $\mathcal{I} \subset \sqrt[G]{\mathcal{N}' : \mathcal{M}} = \mathcal{P}$. □

5.10 Definition. Let $\mathcal{M}$ be a coherent $(G, \mathcal{O}_X)$-module, and $\mathcal{N}$ a coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$. An expression

$$\mathcal{N} = \mathcal{N}_1 \cap \cdots \cap \mathcal{N}_r \tag{5}$$

is called a *G-primary decomposition* if this equation holds, and each $\mathcal{N}_i$ is G-primary. The G-primary decomposition (5) is said to be *irredundant* if for each i, $\bigcap_{j \neq i} \mathcal{N}_j \neq \mathcal{N}$. It is said to be *minimal* if it is irredundant and $\sqrt[G]{\mathcal{N}_i : \mathcal{M}} \neq \sqrt[G]{\mathcal{N}_j : \mathcal{M}}$ for $i \neq j$.

5.11 Lemma. *Let $\mathcal{M}$ be a coherent $(G, \mathcal{O}_X)$-module, and $\mathcal{N}$ a coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$. Then $\mathcal{N}$ has a minimal G-primary decomposition.*

Proof. Let $\mathcal{N} = \mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_s$ be a primary decomposition, which exists by (3.13). Then

$$\mathcal{N} = \mathcal{N}^* = (\mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_s)^* = \mathfrak{n}_1^* \cap \cdots \cap \mathfrak{n}_s^*.$$

This is a G-primary decomposition by Lemma 5.8. Omitting redundant terms, we get an irredundant decomposition. Let

$$\mathcal{N} = \mathcal{N}_1 \cap \cdots \cap \mathcal{N}_t$$

be the decomposition so obtained. We say that $i \sim j$ if $\sqrt[G]{\mathcal{N}_i : \mathcal{M}} = \sqrt[G]{\mathcal{N}_j : \mathcal{M}}$. Let $E_1, \ldots, E_r$ be the equivalence classes with respect to the equivalence relation $\sim$. Then letting $\mathcal{E}_i = \bigcap_{j \in E_i} \mathcal{N}_j$,

$$\mathcal{N} = \mathcal{E}_1 \cap \cdots \cap \mathcal{E}_r$$

is a minimal G-primary decomposition by Lemma 5.9. □

5.12 Lemma. *Let $\mathcal{M}$ be a coherent $(G, \mathcal{O}_X)$-module, $\mathcal{N}$ a coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$, and $\mathcal{P}$ a coherent G-ideal of $\mathcal{O}_X$. Then the following are equivalent.*

(i) *$\mathcal{N}$ is $\mathcal{P}$-G-primary.*

(ii) *The following three conditions hold:*

- **(a)** *$\mathcal{N} \neq \mathcal{M}$.*
- **(b)** *$\mathcal{P} \subset \sqrt[G]{\mathcal{N} : \mathcal{M}}$.*
- **(c)** *If $\mathcal{L}$ is a coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$, $\mathcal{J}$ is a G-ideal, $\mathcal{L} \not\subset \mathcal{N}$, and $\mathcal{J} \not\subset \mathcal{P}$, then $\mathcal{J}\mathcal{L} \not\subset \mathcal{N}$.*

Proof. **(i)**$\Rightarrow$**(ii)** is clear.

(ii)$\Rightarrow$**(i)** Set $\mathcal{K} := \sqrt[G]{\mathcal{N} : \mathcal{M}}$. We show $\mathcal{P} = \mathcal{K}$. There exists some $n \geq 1$ $\mathcal{K}^n\mathcal{M} \subset \mathcal{N}$ and $\mathcal{K}^{n-1}\mathcal{M} \not\subset \mathcal{N}$ by **(a)**. Then by **(c)**, $\mathcal{K} \subset \mathcal{P}$. So $\mathcal{K} = \mathcal{P}$ by **(b)**. By **(a)** and **(c)**, we have that $\mathcal{N}$ is $\mathcal{P}$-G-primary. □

5.13 Lemma. *Let $\mathcal{M}$ be a coherent $(G, \mathcal{O}_X)$-module, and $\mathcal{N}$ a G-primary coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$. Set $\mathcal{P} = \sqrt[G]{\mathcal{N} : \mathcal{M}}$. Let $\mathcal{L}$ be a coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$, and $\mathcal{I}$ a coherent G-ideal of $\mathcal{O}_X$. Then the following hold.*

(i) *If $\mathcal{L} \subset \mathcal{N}$, then $\mathcal{N} : \mathcal{L} = \mathcal{O}_X$.*

(ii) *If $\mathcal{L} \not\subset \mathcal{N}$, then $\mathcal{N} : \mathcal{L}$ is $\mathcal{P}$-G-primary.*

(iii) *If $\mathcal{I} \subset \mathcal{N} : \mathcal{M}$, then $\mathcal{N} : \mathcal{I} = \mathcal{M}$.*

(iv) *If $\mathcal{I} \not\subset \mathcal{N} : \mathcal{M}$, then $\mathcal{N} : \mathcal{I}$ is $\mathcal{P}$-G-primary.*

(v) *If $\mathcal{I} \not\subset \mathcal{P}$, then $\mathcal{N} : \mathcal{I} = \mathcal{N}$.*

Proof. **(i)** and **(iii)** are trivial.

(ii) $\mathcal{N} : \mathcal{L} \neq \mathcal{O}_X$ is trivial. $\sqrt[G]{\mathcal{N} : \mathcal{L}} \supset \sqrt[G]{\mathcal{N} : \mathcal{M}} = \mathcal{P}$. Let $\mathcal{J}$ and $\mathcal{K}$ be coherent G-ideals of $\mathcal{O}_X$ such that $\mathcal{J} \not\subset \mathcal{N} : \mathcal{L}$ and $\mathcal{K} \not\subset \mathcal{P}$. Then $\mathcal{J}\mathcal{L} \not\subset \mathcal{N}$, and $\mathcal{K} \not\subset \mathcal{P}$. So $\mathcal{J}\mathcal{K}\mathcal{L} \not\subset \mathcal{N}$. This shows $\mathcal{J}\mathcal{K} \not\subset \mathcal{N} : \mathcal{L}$. By Lemma 5.12, $\mathcal{N} : \mathcal{L}$ is $\mathcal{P}$-G-primary.

(iv) Since $\mathcal{I}\mathcal{M} \not\subset \mathcal{N}$, $\mathcal{M} \neq \mathcal{N} : \mathcal{I}$. We have

$$\sqrt[G]{(\mathcal{N} : \mathcal{I}) : \mathcal{M}} = \sqrt[G]{\mathcal{N} : \mathcal{I}\mathcal{M}} \supset \sqrt[G]{\mathcal{N} : \mathcal{M}} = \mathcal{P}.$$

Let $\mathcal{L}$ be a coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$, and $\mathcal{J}$ be a coherent G-ideal of $\mathcal{O}_X$ such that $\mathcal{L} \not\subset \mathcal{N} : \mathcal{I}$ and $\mathcal{J} \not\subset \mathcal{P}$. Since $\mathcal{I}\mathcal{L} \not\subset \mathcal{N}$ and $\mathcal{J} \not\subset \mathcal{P}$, we have that $\mathcal{I}\mathcal{J}\mathcal{L} \not\subset \mathcal{N}$. This shows $\mathcal{J}\mathcal{L} \not\subset \mathcal{N} : \mathcal{I}$. By Lemma 5.12, $\mathcal{N} : \mathcal{I}$ is $\mathcal{P}$-G-primary.

(v) If $\mathcal{N} : \mathcal{I} \not\subset \mathcal{N}$, then as $\mathcal{I} \not\subset \mathcal{P}$, we have that $\mathcal{I}(\mathcal{N} : \mathcal{I}) \not\subset \mathcal{N}$. This is a contradiction. □

(5.14) Let $\mathcal{M}$ be a coherent $(G, \mathcal{O}_X)$-module, and $\mathcal{N}$ a coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$. Let

$$\mathcal{N} = \mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_r \tag{6}$$

be a minimal G-primary decomposition, which exists by Lemma 5.11. Set $\mathcal{M}_i = \bigcap_{j \neq i} \mathcal{Q}_j$, and $\mathcal{P}_i = \sqrt[G]{\mathcal{Q}_i : \mathcal{M}}$.

5.15 Theorem. *We have*

$$\{\mathcal{P}_1, \ldots, \mathcal{P}_r\} = \{\mathcal{N} : \mathcal{L} \mid \mathcal{L} \textit{ is a coherent } (G, \mathcal{O}_X)\textit{-submodule of } \mathcal{M}, \textit{ and } \mathcal{N} : \mathcal{L} \textit{ is } G\textit{-prime}\}.$$

In particular, this set depends only on $\mathcal{M}/\mathcal{N}$, and independent of the choice of minimal G-primary decomposition of $\mathcal{N}$.

Proof. Since the decomposition (6) is irredundant,

$$\mathcal{N} : \mathcal{M}_i = \bigcap_{j=1}^{r} (\mathcal{Q}_j : \mathcal{M}_i) = \mathcal{Q}_i : \mathcal{M}_i \neq \mathcal{O}_X.$$

Thus $\mathcal{N} : \mathcal{M}_i$ is $\mathcal{P}_i$-G-primary by Lemma 5.13, **(ii)**. Take the minimum $n \geq 1$ such that $\mathcal{P}_i^n \subset \mathcal{N} : \mathcal{M}_i$, and set $\mathcal{L} := \mathcal{P}_i^{n-1}\mathcal{M}_i$. Since $\mathcal{P}_i^{n-1} \not\subset \mathcal{N} : \mathcal{M}_i$, $\mathcal{N} : \mathcal{L} = \mathcal{N} : \mathcal{P}_i^{n-1}\mathcal{M}_i = (\mathcal{N} : \mathcal{M}_i) : \mathcal{P}_i^{n-1}$ is $\mathcal{P}_i$-G-primary. In particular, $\mathcal{N} : \mathcal{L} \subset \sqrt[G]{\mathcal{N} : \mathcal{L}} = \mathcal{P}_i$. On the other hand,

$$(\mathcal{N} : \mathcal{L}) : \mathcal{P}_i = \mathcal{N} : \mathcal{P}_i^n\mathcal{M}_i = (\mathcal{N} : \mathcal{M}_i) : \mathcal{P}_i^n = \mathcal{O}_X.$$

Thus $\mathcal{P}_i \subset \mathcal{N} : \mathcal{L}$. Hence $\mathcal{N} : \mathcal{L} = \mathcal{P}_i$. Thus each $\mathcal{P}_i$ is of the form $\mathcal{N} : \mathcal{L}$ for some $\mathcal{L}$.

Conversely, let $\mathcal{L} \subset \mathcal{M}$, and assume that $\mathcal{N} : \mathcal{L}$ is G-prime. Set $\mathcal{P} = \mathcal{N} : \mathcal{L}$. We show that $\mathcal{P} = \mathcal{P}_i$ for some i. Renumbering if necessary, we may assume that $\mathcal{L} \not\subset \mathcal{Q}_i$ if and only if $i \leq s$. Then by Lemma 5.13, **(ii)**,

$$\mathcal{P} = \sqrt[G]{\mathcal{P}} = \sqrt[G]{\mathcal{N} : \mathcal{L}} = \sqrt[G]{\bigcap_{i=1}^{s}(\mathcal{Q}_i : \mathcal{L})} = \bigcap_{i=1}^{s} \sqrt[G]{\mathcal{Q}_i : \mathcal{L}} = \bigcap_{i=1}^{s} \mathcal{P}_i.$$

So $s \geq 1$, and there exists some i such that $\mathcal{P}_i = \mathcal{P}$. □

5.16 Definition. We set $\operatorname{Ass}_G(\mathcal{M}/\mathcal{N}) = \{\mathcal{P}_1, \ldots, \mathcal{P}_r\}$. Note that $\operatorname{Ass}_G(\mathcal{M}/\mathcal{N})$ depends only on $\mathcal{M}/\mathcal{N}$. An element of $\operatorname{Ass}_G(\mathcal{M}/\mathcal{N})$ is called a G-associated G-prime of $\mathcal{M}/\mathcal{N}$ (however, also called a G-associated G-prime of the submodule $\mathcal{N}$). The set of minimal elements in $\operatorname{Ass}_G(\mathcal{M}/\mathcal{N})$ is denoted by $\operatorname{Min}_G(\mathcal{M}/\mathcal{N})$. An element of $\operatorname{Min}_G(\mathcal{M}/\mathcal{N})$ is called a minimal G-prime of $\mathcal{M}/\mathcal{N}$. An element of $\operatorname{Ass}_G(\mathcal{M}/\mathcal{N}) \setminus \operatorname{Min}_G(\mathcal{M}/\mathcal{N})$ is called an embedded G-prime.

5.17 Proposition. *Let Ω be a poset ideal of $\{\mathcal{P}_1, \ldots, \mathcal{P}_r\}$ with respect to the incidence relation. Then $\bigcap_{\mathcal{P}_j \in \Omega} \mathcal{Q}_j$ is independent of the choice of minimal G-primary decomposition.*

Proof. Set $\mathcal{J} := \bigcap_{\mathcal{P}_i \notin \Omega} \mathcal{P}_i$. It suffices to prove that $\bigcap_{\mathcal{P}_j \in \Omega} \mathcal{Q}_j = \mathcal{N} : \mathcal{J}^n$ for $n \gg 0$.

For $\mathcal{P}_i \notin \Omega$, $\mathcal{J} \subset \mathcal{P}_i = \sqrt[G]{\mathcal{Q}_i : \mathcal{M}}$. Hence there exists some n_0 such that $\mathcal{J}^{n_0} \subset \mathcal{Q}_i : \mathcal{M}$ for all i such that $\mathcal{P}_i \notin \Omega$. Take n so that $n \geq n_0$. Then $\mathcal{Q}_i : \mathcal{J}^n = \mathcal{M}$, since $\mathcal{J}^n\mathcal{M} \subset \mathcal{Q}_i$, for $\mathcal{P}_i \notin \Omega$.

If $\mathcal{P}_i \notin \Omega$ and $\mathcal{P}_j \in \Omega$, then $\mathcal{P}_i \not\subset \mathcal{P}_j$ by assumption. Hence $\mathcal{J} \not\subset \mathcal{P}_j$. Hence $\mathcal{J}^n \not\subset \mathcal{P}_j$. Thus $\mathcal{Q}_j : \mathcal{J}^n = \mathcal{Q}_j$ for $\mathcal{P}_j \in \Omega$ by Lemma 5.13, **(v)**.

So $\mathcal{N} : \mathcal{J}^n = \bigcap_i \mathcal{Q}_i : \mathcal{J}^n = \bigcap_{\mathcal{P}_j \in \Omega} \mathcal{Q}_j$. □

5.18 Lemma. *Let $\mathcal{M}$ be a coherent $(G, \mathcal{O}_X)$-module, and $\mathcal{N}$ a coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$. If $\mathcal{P} \in \operatorname{Ass}_G(\mathcal{M}/\mathcal{N})$ and $\mathfrak{p} \in \operatorname{Ass}(\mathcal{O}_X/\mathcal{P})$, then $\mathfrak{p} \in \operatorname{Ass}(\mathcal{M}/\mathcal{N})$.*

Proof. There exists some coherent $(G, \mathcal{O}_X)$-submodule $\mathcal{L}$ of $\mathcal{M}$ such that $\mathcal{N} : \mathcal{L} = \mathcal{P}$. Moreover, there exists some coherent ideal $\mathfrak{a}$ of $\mathcal{O}_X$ such that $\mathfrak{p} = \mathcal{P} : \mathfrak{a}$. Then

$$\mathfrak{p} = \mathcal{P} : \mathfrak{a} = (\mathcal{N} : \mathcal{L}) : \mathfrak{a} = \mathcal{N} : \mathfrak{a}\mathcal{L}.$$

Thus $\mathfrak{p} \in \operatorname{Ass}(\mathcal{M}/\mathcal{N})$. □

5.19 Lemma. *Let $\mathcal{M}$ be a coherent $(G, \mathcal{O}_X)$-module, and $\mathcal{N}$ a coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$. Then we have $V_G(\mathcal{N} : \mathcal{M}) = \bigcup_{\mathcal{P} \in \operatorname{Min}_G(\mathcal{M}/\mathcal{N})} V_G(\mathcal{P})$. In particular, for a quasi-coherent G-prime G-ideal $\mathcal{P}$ of $\mathcal{O}_X$, $\mathcal{P} \in \operatorname{Min}_G(\mathcal{M}/\mathcal{N})$ if and only if $\mathcal{P}$ is a minimal element of $V_G(\mathcal{N} : \mathcal{M})$. Moreover, we have $\bigcap_{\mathcal{Q} \in V_G(\mathcal{N}:\mathcal{M})} \mathcal{Q} = \bigcap_{\mathcal{P} \in \operatorname{Min}_G(\mathcal{M}/\mathcal{N})} \mathcal{P}$.*

Proof. We may assume that $\mathcal{N} = 0$. Let

$$0 = \mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_r$$

be a minimal primary decomposition of 0 in $\mathcal{M}$. Set $\mathcal{P}_i := \sqrt[G]{\mathcal{Q}_i : \mathcal{M}}$. Then

$$\mathcal{P}_i \supset \mathcal{Q}_i : \mathcal{M} \supset 0 : \mathcal{M}.$$

In particular, $V_G(\mathcal{P}_i) \subset V_G(0 : \mathcal{M})$. On the other hand, $\mathcal{M}$ is a submodule of $\bigoplus_{i=1}^r \mathcal{M}/\mathcal{Q}_i$. So $0 : \mathcal{M} \supset 0 : \bigoplus_i \mathcal{M}/\mathcal{Q}_i$. So

$$V_G(0 : \mathcal{M}) \subset V_G(0 : \bigoplus_i \mathcal{M}/\mathcal{Q}_i) = V_G(\bigcap_i (\mathcal{Q}_i : \mathcal{M})) = \bigcup_i V_G(\mathcal{P}_i) \subset V_G(0 : \mathcal{M}).$$

So $V_G(0 : \mathcal{M}) = \bigcup_i V_G(\mathcal{P}_i)$. If $\mathcal{P}_i$ is an embedded G-prime, then $V_G(\mathcal{P}_i)$ in the union is redundant, and we have $V_G(0 : \mathcal{M}) = \bigcup_{\mathcal{P} \in \operatorname{Min}_G \mathcal{M}} V_G(\mathcal{P})$. The rest of the assertions are now obvious. □

5.20 Corollary. *For an ideal $\mathcal{I}$ of $\mathcal{O}_X$, $\sqrt[G]{\mathcal{I}} = \bigcap_{\mathcal{P} \in V_G(\mathcal{I})} \mathcal{P} = \bigcap_{\mathcal{P} \in \operatorname{Min}_G(\mathcal{O}_X/\mathcal{I})} \mathcal{P}$.*

Proof. By Lemma 5.19,

$$\sqrt[G]{\mathcal{I}} = (\bigcap_{\mathcal{P} \in V_G(\mathcal{I})} \mathcal{P})^* = (\bigcap_{\mathcal{P} \in \operatorname{Min}_G(\mathcal{O}_X/\mathcal{I})} \mathcal{P})^* = \bigcap_{\mathcal{P} \in \operatorname{Min}_G(\mathcal{O}_X/\mathcal{I})} \mathcal{P} = \bigcap_{\mathcal{P} \in V_G(\mathcal{I})} \mathcal{P},$$

since $\operatorname{Min}_G(\mathcal{O}_X/\mathcal{I})$ is a finite set and the intersection of finitely many quasi-coherent G-ideals is a quasi-coherent G-ideal. □

5.21 Lemma. *Let $\mathcal{M}$ be a coherent $(G, \mathcal{O}_X)$-module, and $\mathcal{N}$ a G-primary coherent $(G, \mathcal{O}_X)$-submodule. Let*

$$\mathcal{N} = \mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_r$$

be a minimal primary decomposition. If $\sqrt{\mathfrak{n}_1 : \mathcal{M}}$ is a minimal prime, then $\mathfrak{n}_1^* = \mathcal{N}$. In particular, $\sqrt{\mathfrak{n}_1 : \mathcal{M}}^* = \sqrt[G]{\mathcal{N} : \mathcal{M}}$.

Proof. First assume that $\mathcal{N} : \mathfrak{n}_1^* \subset \sqrt[G]{\mathcal{N} : \mathcal{M}}$. Then

$$\bigcap_{i \geq 2} (\mathfrak{n}_i : \mathcal{M}) \subset \bigcap_{i \geq 1} (\mathfrak{n}_i : \mathfrak{n}_1^*) = \mathcal{N} : \mathfrak{n}_1^* \subset \sqrt[G]{\mathcal{N} : \mathcal{M}} \subset \sqrt{\mathcal{N} : \mathcal{M}} \subset \sqrt{\mathfrak{n}_1 : \mathcal{M}}.$$

This contradicts the minimality of $\sqrt{\mathfrak{n}_1 : \mathcal{M}}$. Hence $\mathcal{N} : \mathfrak{n}_1^* \not\subset \sqrt[G]{\mathcal{N} : \mathcal{M}}$. Since $\mathfrak{n}_1^*(\mathcal{N} : \mathfrak{n}_1^*) \subset \mathcal{N}$ and $\mathcal{N}$ is G-primary, $\mathfrak{n}_1^* \subset \mathcal{N}$. As $\mathfrak{n}_1^* \supset \mathcal{N}$ is trivial, $\mathfrak{n}_1^* = \mathcal{N}$.

Hence,

$$\sqrt{\mathfrak{n}_1 : \mathcal{M}}^* = \sqrt[G]{(\mathfrak{n}_1 : \mathcal{M})^*} = \sqrt[G]{\mathfrak{n}_1^* : \mathcal{M}} = \sqrt[G]{\mathcal{N} : \mathcal{M}}.$$

□

5.22 Corollary. *Let $\mathcal{P}$ be a G-prime G-ideal. For any minimal prime $\mathfrak{p}$ of $\mathcal{P}$, we have $\mathfrak{p}^* = \mathcal{P}$.*

Proof. Let $\mathcal{P} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ be a minimal primary decomposition such that $\sqrt{\mathfrak{q}_1} = \mathfrak{p}$. Then, $\mathfrak{p}^* = \sqrt{\mathfrak{q}_1}^* = \sqrt[G]{\mathfrak{q}_1^*} = \sqrt[G]{\mathcal{P}} = \mathcal{P}$. □

5.23 Corollary. *Let $\mathcal{M}$ be a coherent $(G, \mathcal{O}_X)$-module, and $\mathcal{N}$ a $(G, \mathcal{O}_X)$-submodule. Then $\mathcal{N}$ is a G-primary submodule of $\mathcal{M}$ if and only if $\mathcal{N} = \mathfrak{n}^*$ for some primary submodule $\mathfrak{n}$ of $\mathcal{M}$.*

Proof. The 'if' part is Lemma 5.8. The 'only if' part follows from Lemma 5.21. □

5.24 Lemma. *For a G-ideal $\mathcal{I}$ of $\mathcal{O}_X$, the following are equivalent.*

(i) *$\mathcal{I}$ is G-radical.*

(ii) *There are finitely many G-prime ideals $\mathcal{P}_1, \ldots, \mathcal{P}_r$ of $\mathcal{O}_X$ such that $\mathcal{I} = \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r$.*

Proof. **(i)⇒(ii)** follows from Corollary 5.20 and Corollary 4.30.

(ii)⇒(i) $\sqrt[G]{\mathcal{I}} = \sqrt[G]{\mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r} = \sqrt[G]{\mathcal{P}_1} \cap \cdots \cap \sqrt[G]{\mathcal{P}_r} = \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r = \mathcal{I}$. □

6. Group schemes of finite type

In this section, S is a scheme, G an S-group scheme, X a p-flat noetherian G-scheme, and $\mathcal{M}$ a coherent $(G, \mathcal{O}_X)$-module. In this section, we assume that $p_2 : G \times X \to X$ is of finite type.

Let $\mathcal{N}$ be a coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$.

6.1 Lemma. *Let*

$$\mathcal{N} = \mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_r \cap \mathfrak{n}_{r+1} \cap \cdots \cap \mathfrak{n}_{r+s}$$

be a minimal primary decomposition such that

$$\operatorname{Min}(\mathcal{M}/\mathcal{N}) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\},$$

where $\mathfrak{p}_i = \sqrt{\mathfrak{n}_i : \mathcal{M}}$. *Then*

(i) *The* (S_1)*-locus of* $\mathcal{M}/\mathcal{N}$ *is* $X \setminus \bigcup_{i=1}^{s} \operatorname{Supp}(\mathcal{M}/\mathfrak{n}_{i+r})$.

(ii) *The* (S_1)*-locus of* $\mathcal{M}/\mathcal{N}$ *is a* G*-stable open subset of* X.

(iii) $\mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_r$ *is a coherent* $(G, \mathcal{O}_X)$*-submodule of* $\mathcal{M}$.

Proof. Replacing $\mathcal{M}$ by $\mathcal{M}/\mathcal{N}$, we may assume that $\mathcal{N} = 0$. Since $X \setminus \operatorname{Supp} \mathcal{M}$ is G-stable open, replacing X by $V(\underline{\operatorname{ann}}\, \mathcal{M})$, we may assume that $\underline{\operatorname{ann}}\, \mathcal{M} = 0$.

(i) Note that a coherent $\mathcal{O}_X$-module $\mathcal{L}$ satisfies Serre's (S_1)-condition at $x \in X$ if and only if $\mathcal{L}_x$ does not have an embedded prime. The assertion follows from this.

(ii) Let U be the (S_1) locus $X \setminus \bigcup_{i=1}^{s} \operatorname{Supp}(\mathcal{M}/\mathfrak{n}_{i+r})$ of $\mathcal{M}$. It is an open subset. Since the action $a : G \times X \to X$ and the second projection $p_2 : G \times X \to X$ are Cohen–Macaulay morphisms by [12, Lemma 31.14], both $a^{-1}(U)$ and $p_2^{-1}(U) = G \times U$ are the (S_1)-locus of $a^*\mathcal{M} \cong p_2^*\mathcal{M}$ by [9, (6.4.1)]. So $a^{-1}(U) = G \times U$, and U is G-stable.

(iii) Let $\iota : U \hookrightarrow X$ be the inclusion. It suffices to show that $\underline{\Gamma}_{X,U}(\mathcal{M}) := \operatorname{Ker}(\mathcal{M} \to \iota_*\iota^*\mathcal{M})$ agrees with $\mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_r$, see for the notation, [13, (3.1)].

Since the composite

$$\mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_r \hookrightarrow \mathcal{M} \to \bigoplus_{i=1}^{s} \mathcal{M}/\mathfrak{n}_{i+r}$$

is a mono, $\iota^*(\mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_r) = 0$. Hence $\mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_r \subset \underline{\Gamma}_{X,U}(\mathcal{M})$. It suffices to show that $\underline{\Gamma}_{X,U}(\mathcal{M}/\mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_r) = 0$. (If so, $\mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_r = \underline{\Gamma}_{X,U}(\mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_r) = \underline{\Gamma}_{X,U}(\mathcal{M})$). As $\mathcal{M}/\mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_r \subset \bigoplus_{i=1}^r \mathcal{M}/\mathfrak{n}_i$, it suffices to show that $\underline{\Gamma}_{X,U}(\mathcal{M}/\mathfrak{n}_i) = 0$ for $i \leq r$. Assume the contrary. Then since $\mathrm{Ass}(\underline{\Gamma}_{X,U}(\mathcal{M}/\mathfrak{n}_i)) \subset \mathrm{Ass}(\mathcal{M}/\mathfrak{n}_i)$, $\mathrm{Ass}(\mathcal{M}/\mathfrak{n}_i)$ contains a point in $X \setminus U$. On the other hand, $\mathrm{Ass}(\mathcal{M}/\mathfrak{n}_i)$ is a singleton, and its point is a generic point of an irreducible component of X. As U is dense in X, this is a contradiction. □

6.2 Corollary. *If $\mathcal{N}$ is G-primary, then $\mathcal{M}/\mathcal{N}$ does not have an embedded prime.*

Proof. We may assume that $\mathcal{N} = 0$ and $\underline{\mathrm{ann}}\,\mathcal{M} = 0$. Let

$$0 = \mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_r \cap \mathfrak{n}_{r+1} \cap \cdots \cap \mathfrak{n}_{r+s}$$

be a minimal primary decomposition such that $\mathrm{Min}(\mathcal{M}) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$, where $\mathfrak{p}_i = \sqrt{\mathfrak{n}_i : \mathcal{M}}$. Set $\mathcal{L} = \mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_r$. It suffices to show that $\mathcal{L} = 0$. Set $\mathcal{J} := \underline{\mathrm{ann}}\,\mathcal{L}$. Note that $\mathcal{L}$ is a coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$ by the lemma, and $\mathcal{J}$ is a coherent G-ideal. Since $V(\mathcal{J}) \subset X \setminus U$, where $U = X \setminus \bigcup_{i=1}^s \mathrm{Supp}(\mathcal{M}/\mathfrak{n}_{i+r})$, $\mathcal{J} \not\subset \sqrt[G]{0}$. Since $\mathcal{J}\mathcal{L} = 0$ and 0 is G-primary, $\mathcal{L} = 0$, as desired. □

6.3 Corollary. *If $\mathcal{N}$ is G-primary and*

$$\mathcal{N} = \mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_r$$

is a minimal primary decomposition, then $\mathfrak{n}_i^* = \mathcal{N}$ for $i = 1, \ldots, r$.

Proof. Follows immediately from Corollary 6.2 and Lemma 5.21. □

6.4 Corollary. *If $\mathcal{P}$ is a G-prime G-ideal of $\mathcal{O}_X$, then for any associated prime $\mathfrak{p}$ of $\mathcal{P}$, $\mathfrak{p}^* = \mathcal{P}$.*

Proof. Follows immediately from Corollary 6.2 and Corollary 5.22. □

(6.5) Assume that X satisfies the (S_1) condition (i.e., $\mathcal{O}_X$ satisfies the (S_1) condition). Let

$$0 = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r \tag{7}$$

be the minimal primary decomposition. Set $X_i := V(\mathfrak{q}_i)$, and $Y_i := X_i \setminus \bigcup_{j \neq i} X_j$. Define $G_{ij} = p_2^{-1} Y_i \cap a^{-1} Y_j$. We say that $i \to j$ if $X_i^* \supset X_j$.

6.6 Lemma. *Let the notation be as above. For* $1 \leq i, j \leq r$*, the following are equivalent.*

(i) $i \to j$.

(ii) $G_{ij} \neq \emptyset$.

Proof. **(i)⇒(ii)** Since the closure of $G \times Y_i$ in $G \times X$ is $G \times X_i$, the scheme theoretic image of the action $a|_{G\times Y_i} : G \times Y_i \to X$ is X_i^*. Since $X_i^* \supset X_j$, $G \times Y_i$ intersects $a^{-1}(Y_j)$. Namely, $G_{ij} \neq \emptyset$.

(ii)⇒(i) Applying Corollary 3.17, the scheme theoretic image of $a|_{G_{ij}} : G_{ij} \to X_j$ is X_j. This shows $X_i^* \supset X_j$. □

6.7 Corollary. $\to$ *is an equivalence relation of* $\{1, \ldots, r\}$.

Proof. Since $X_i^* \supset X_i$, $i \to i$.

Consider the isomorphism $\Theta : G \times X \to G \times X$ given by $\Theta(g, x) = (g^{-1}, gx)$. Then $\Theta(G_{ij}) = G_{ji}$. Thus $i \to j$ if and only if $j \to i$.

Assume that $i \to j$ and $j \to k$. Then $X_i^* = X_i^{**} \supset X_j^* \supset X_k$. Hence $i \to k$. □

6.8 Lemma. *Assume that* $\mathcal{M}/\mathcal{N}$ *does not have an embedded prime. Let*

$$\mathcal{N} = \mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_r \tag{8}$$

be a minimal primary decomposition. Then

$$\mathfrak{n}_i^* = \bigcap_{i \to j} \mathfrak{n}_j,$$

where we say that $i \to j$ *if* $\underline{\mathrm{ann}}(\mathcal{M}/\mathfrak{n}_i)^* \subset \underline{\mathrm{ann}}(\mathcal{M}/\mathfrak{n}_j)$.

Proof. Replacing $\mathcal{M}$ by $\mathcal{M}/\mathcal{N}$ and $\mathfrak{n}_j$ by $\mathfrak{n}_j/\mathcal{N}$, we may assume that $\mathcal{N} = 0$. Replacing X by $\operatorname{Supp} \mathcal{M} := V(\underline{\mathrm{ann}}\, \mathcal{M})$, we may assume that $\underline{\mathrm{ann}}\, \mathcal{M} = 0$. Set $\mathfrak{q}_j := \underline{\mathrm{ann}}(\mathcal{M}/\mathfrak{n}_j)$ and $X_j := V(\mathfrak{q}_j)$. Note that (7) is a minimal primary decomposition by Lemma 3.14, and X satisfies the (S_1) condition.

Now we define Y_j and G_{ij} as in (6.5). The definition of $i \to j$ is consistent with that in (6.5).

Let $\rho_i : Y_i \to X$ be the inclusion, $\rho : \coprod_{i \to j} Y_j \to X$ be the inclusion, $\varphi : \coprod_{i \to j} G_{ij} \to G \times X$ be the inclusion, $a_0 : \coprod_{i \to j} G_{ij} \to \coprod_{i \to j} Y_j$ be the restriction of a, and $p_0 : \coprod_{i\to j} G_{ij} \to Y_i$ be the restriction of p_2, so that the diagrams

$$\begin{CD} \coprod_{i\to j} G_{ij} @>{\varphi}>> G\times X \\ @V{a_0}VV @VV{a}V \\ \coprod_{i\to j} Y_j @>{\rho}>> X \end{CD} \qquad \begin{CD} \coprod_{i\to j} G_{ij} @>{\varphi}>> G\times X \\ @V{p_0}VV @VV{p_2}V \\ Y_i @>{\rho_i}>> X \end{CD}$$

are commutative.

Since $a(G\times Y_i) \subset \bigcup_{i\to j} X_j$ and $\coprod_{i\to j} Y_j$ is dense in $\bigcup_{i\to j} X_j$, $\coprod_{i\to j} G_{ij}$ is dense in $G\times Y_i$ by (2.7). As $G\times Y_i$ is dense in $G\times X_i$, $\coprod_{i\to j} G_{ij}$ is also dense in $G\times X_i$. Since $\operatorname{Supp} p_2^*(\mathcal{M}/\mathfrak{n}_i) = G\times X_i$ and $p_2^*(\mathcal{M}/\mathfrak{n}_i)$ satisfies the (S_1) condition, $u : a_*p_2^*(\mathcal{M}/\mathfrak{n}_i) \to a_*\varphi_*\varphi^*p_2^*(\mathcal{M}/\mathfrak{n}_i)$ is a monomorphism by Lemma 3.16.

Similarly, $u : \rho_*\rho^*\mathcal{M} \to \rho_*(a_0)_*a_0^*\rho^*\mathcal{M}$ is a monomorphism.

Since the diagram

$$\begin{CD} \rho_*\rho^*\mathcal{M} @>{u}>> \rho_*(a_0)_*a_0^*\rho^*\mathcal{M} @>{\cong}>> a_*\varphi_*\varphi^*a^*\mathcal{M} \\ @A{u}AA @. @VV{\cong}V \\ \mathcal{M} @>{\omega}>> a_*p_2^*\mathcal{M} @>{u}>> a_*\varphi_*\varphi^*p_2^*\mathcal{M} \\ @. @V{\pi_{\mathfrak{n}_i}}VV @VV{\cong}V \\ @. a_*p_2^*(\mathcal{M}/\mathfrak{n}_i) @. a_*\varphi_*p_0^*\rho_i^*\mathcal{M} \\ @. @V{u}VV @VV{\cong}V \\ @. a_*\varphi_*\varphi^*p_2^*(\mathcal{M}/\mathfrak{n}_i) @>{\cong}>> a_*\varphi_*p_0^*\rho_i^*(\mathcal{M}/\mathfrak{n}_i) \end{CD}$$

is commutative,

$$\mathfrak{n}_i^* = \operatorname{Ker}(\mathcal{M} \xrightarrow{\omega} a_*p_2^*\mathcal{M} \xrightarrow{\pi_{\mathfrak{n}_i}} a_*p_2^*(\mathcal{M}/\mathfrak{n}_i)) = \operatorname{Ker}(u : \mathcal{M} \to \rho_*\rho^*\mathcal{M}) = \bigcap_{i\to j} \mathfrak{n}_j$$

by Lemma 4.11. □

6.9 Corollary. *Let* $\mathcal{N}$ *be* G*-primary in* Lemma 6.8. *Then for* $i, j \in \{1, \ldots, r\}$, $i \to j$.

Proof. By Corollary 6.3 and Lemma 6.8, $\mathcal{N} = \mathfrak{n}_i^* = \bigcap_{i\to j} \mathfrak{n}_j$. Since the decomposition (8) is irredundant, $i \to j$ holds for all $j \in \{1, \ldots, r\}$. □

6.10 Theorem. *Let*

$$\mathcal{N} = \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_s \tag{9}$$

be a minimal G-primary decomposition, and let

$$\mathcal{M}_l = \mathfrak{m}_{l,1} \cap \cdots \cap \mathfrak{m}_{l,r_l} \tag{10}$$

be a minimal primary decomposition. Then

$$\mathcal{N} = \bigcap_{l=1}^{s} (\mathfrak{m}_{l,1} \cap \cdots \cap \mathfrak{m}_{l,r_l})$$

is a minimal primary decomposition.

Proof. Note that $\sqrt{\mathfrak{m}_{l,j} : \mathcal{M}}$ are distinct. Indeed, we have

$$\sqrt{\mathfrak{m}_{l,j} : \mathcal{M}}^* = \sqrt[G]{(\mathfrak{m}_{l,j} : \mathcal{M})^*} = \sqrt[G]{\mathfrak{m}_{l,j}^* : \mathcal{M}} = \sqrt[G]{\mathcal{M}_l : \mathcal{M}}.$$

Since (9) is minimal, $\sqrt{\mathfrak{m}_{l,j} : \mathcal{M}}$ is different if l is different. On the other hand, if l is the same and j is different, then $\sqrt{\mathfrak{m}_{l,j} : \mathcal{M}}$ is different, since (10) is minimal.

So it suffices to prove that each $\sqrt{\mathfrak{m}_{l,j} : \mathcal{M}}$ is an associated prime of $\mathcal{N}$. By Lemma 5.6, $\mathcal{M}_l : \mathcal{M}$ is G-primary, and $\sqrt[G]{\mathcal{M}_l : \mathcal{M}}$ is a G-prime. So neither $\mathcal{M}_l : \mathcal{M}$ nor $\sqrt[G]{\mathcal{M}_l : \mathcal{M}}$ has an embedded prime by Corollary 6.2. So

$$\operatorname{Ass}(\mathcal{O}_X/(\mathcal{M}_l : \mathcal{M})) = \operatorname{Ass}(\mathcal{O}_X/(\sqrt[G]{\mathcal{M}_l : \mathcal{M}})) = \operatorname{Ass}(\mathcal{O}_X/\sqrt{\mathcal{M}_l : \mathcal{M}}).$$

By Lemma 3.14,

$$\mathcal{M}_l : \mathcal{M} = \bigcap_j \mathfrak{m}_{l,j} : \mathcal{M}$$

is a minimal primary decomposition. So $\sqrt[G]{\mathcal{M}_l : \mathcal{M}} \in \operatorname{Ass}_G \mathcal{M}/\mathcal{N}$ and $\sqrt{\mathfrak{m}_{l,j} : \mathcal{M}} \in \operatorname{Ass}(\mathcal{O}_X/\sqrt[G]{\mathcal{M}_l : \mathcal{M}})$. By Lemma 5.18, $\sqrt{\mathfrak{m}_{l,j} : \mathcal{M}} \in \operatorname{Ass}(\mathcal{M}/\mathcal{N})$, as desired. □

6.11 Corollary. *A prime ideal $\mathfrak{p}$ of $\mathcal{O}_X$ is an associated prime of some coherent $(G, \mathcal{O}_X)$-module $\mathcal{M}$ if and only if $\mathfrak{p}$ is a minimal prime of $\mathfrak{p}^*$.*

Proof. The 'if' part is trivial. We prove the converse. Take a minimal G-primary decomposition (9) and minimal primary decompositions (10). We may assume that $\mathfrak{p} = \sqrt{\mathfrak{m}_{1,1} : \mathcal{M}}$. Then $\mathfrak{p}^* = \sqrt[G]{\mathcal{M}_1 : \mathcal{M}}$. As $\mathcal{M}/\mathcal{M}_1$ does not have an embedded prime, $\mathfrak{m}_{1,1} : \mathcal{M}$ is a primary component of $\mathcal{M}_1 : \mathcal{M}$ corresponding to a minimal prime, and hence $\mathfrak{p}$ is a minimal prime of $\mathcal{M}_1 : \mathcal{M}$. Since $\mathcal{M}_1 : \mathcal{M}$ and $\mathfrak{p}^* = \sqrt[G]{\mathcal{M}_1 : \mathcal{M}}$ have the same radical, we are done. □

6.12 Corollary. *Let* (9) *be a minimal G-primary decomposition. Then*

$$\mathrm{Ass}(\mathcal{M}/\mathcal{N}) = \coprod_{l=1}^{s} \mathrm{Ass}(\mathcal{M}/\mathcal{M}_l) = \coprod_{\mathcal{P}\in \mathrm{Ass}_G(\mathcal{M}/\mathcal{N})} \mathrm{Ass}(\mathcal{O}_X/\mathcal{P})$$

and

$$\mathrm{Ass}_G(\mathcal{M}/\mathcal{N}) = \{\mathfrak{p}^* \mid \mathfrak{p} \in \mathrm{Ass}(\mathcal{M}/\mathcal{N})\}.$$

Proof. Follows immediately by Theorem 6.10. □

6.13 Corollary. *We have*

$$\mathrm{Min}(\mathcal{M}/\mathcal{N}) = \coprod_{\mathcal{P}\in \mathrm{Min}_G(\mathcal{M}/\mathcal{N})} \mathrm{Ass}(\mathcal{O}_X/\mathcal{P})$$

and

$$\mathrm{Min}_G(\mathcal{M}/\mathcal{N}) = \{\mathfrak{p}^* \mid \mathfrak{p} \in \mathrm{Min}(\mathcal{M}/\mathcal{N})\}$$

Proof. Assume that $\mathfrak{p} \in \mathrm{Min}(\mathcal{M}/\mathcal{N})$ and $\mathfrak{p}^* \notin \mathrm{Min}_G(\mathcal{M}/\mathcal{N})$. Then there exists some $\mathcal{P} \in \mathrm{Min}_G(\mathcal{M}/\mathcal{N})$ such that $\mathcal{P} \subsetneq \mathfrak{p}^*$. If

$$\sqrt{\mathcal{P}} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$$

is a minimal prime decomposition, then each $\mathfrak{p}_i$ is an element of $\mathrm{Ass}(\mathcal{M}/\mathcal{N})$ by Corollary 6.12. Since

$$\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s = \sqrt{\mathcal{P}} \subset \sqrt{\mathfrak{p}^*} \subset \sqrt{\mathfrak{p}} = \mathfrak{p},$$

there exists some i such that $\mathfrak{p}_i \subset \mathfrak{p}$. Since $\mathfrak{p}_i^* = \mathcal{P} \neq \mathfrak{p}^*$, we have $\mathfrak{p}_i \subsetneq \mathfrak{p}$. This contradicts the minimality of $\mathfrak{p}$. So $\mathfrak{p} \in \mathrm{Min}(\mathcal{M}/\mathcal{N})$ implies $\mathfrak{p}^* \in \mathrm{Min}_G(\mathcal{M}/\mathcal{N})$. By Corollary 6.11, $\mathfrak{p} \in \mathrm{Min}(\mathcal{M}/\mathcal{N})$ implies that $\mathfrak{p}$ is an associated prime of $\mathfrak{p}^*$. So the $\subset$ direction of the first equation and $\supset$ direction of the second equation have been proved.

Conversely, assume that $\mathcal{P} \in \operatorname{Min}_G(\mathcal{M}/\mathcal{N})$, and $\mathfrak{p} \in \operatorname{Ass}(\mathcal{O}_X/\mathcal{P})$. By Lemma 5.18, $\mathfrak{p} \in \operatorname{Ass}(\mathcal{M}/\mathcal{N})$. Assume that $\mathfrak{p}$ is not minimal. Then there exists some $\mathfrak{p}' \in \operatorname{Min}(\mathcal{M}/\mathcal{N})$ such that $\mathfrak{p}' \subsetneq \mathfrak{p}$. Then $(\mathfrak{p}')^* \subset \mathfrak{p}^* = \mathcal{P}$. By the minimality of $\mathcal{P}$, $(\mathfrak{p}')^* = \mathfrak{p}^* = \mathcal{P}$. So $\mathfrak{p}' \in \operatorname{Ass}(\mathcal{O}_X/\mathcal{P})$ by Corollary 6.11. As $\mathcal{O}_X/\mathcal{P}$ does not have an embedded prime by Corollary 6.2, this contradicts $\mathfrak{p}' \subsetneq \mathfrak{p}$. So $\mathfrak{p}$ must be minimal. This proves the $\supset$ direction of the first equation. As $\mathcal{P} = \mathfrak{p}^*$ with $\mathfrak{p} \in \operatorname{Min}(\mathcal{M}/\mathcal{N})$, the $\subset$ direction of the second equation has also been proved. □

6.14 Corollary. *We have* $\operatorname{Ass}(\mathcal{M}/\mathcal{N}) = \operatorname{Min}(\mathcal{M}/\mathcal{N})$ *if and only if* $\operatorname{Ass}_G(\mathcal{M}/\mathcal{N}) = \operatorname{Min}_G(\mathcal{M}/\mathcal{N})$.

Proof. Obvious by Corollary 6.12 and Corollary 6.13. □

6.15 Corollary. *A G-radical G-ideal does not have an embedded prime.*

Proof. Obvious by Corollary 6.14. □

6.16 Lemma. *If X is G-integral and $\mathcal{I}$ a nonzero G-ideal, then $V(\mathcal{I})$ is nowhere dense in X.*

Proof. Assume the contrary. Then there exists some minimal prime $\mathfrak{p}$ of 0 such that $\mathcal{I} \subset \mathfrak{p}$. Then $\mathcal{I} = \mathcal{I}^* \subset \mathfrak{p}^* = 0$, since 0 is a G-prime. This is a contradiction. □

6.17 Lemma. *Let X be G-integral, and $\mathcal{M}$ a coherent $(G, \mathcal{O}_X)$-module. Then there exists some r and some dense G-stable open subset U of X such that for $x \in X$, it holds $x \in U$ if and only if $\mathcal{M}_x \cong \mathcal{O}_{X,x}^r$. In this case, $\mathcal{M}|_U$ is locally free of rank r.*

Proof. Let r be the smallest integer such that $\underline{\operatorname{Fitt}}_r \mathcal{M} \neq 0$. Then $r \geq 0$, and letting $U := X \setminus V(\underline{\operatorname{Fitt}}_r \mathcal{M})$, U is G-stable open, and for $x \in X$, it holds $x \in U$ if and only if $\mathcal{M}_x \cong \mathcal{O}_{X,x}^r$ by [5, Proposition 20.8].

U is dense, since $V(\underline{\operatorname{Fitt}}_r \mathcal{M})$ is nowhere dense by Lemma 6.16 and closed. □

6.18 Lemma. *Let X be G-reduced, and $\mathcal{M}$ a coherent $(G, \mathcal{O}_X)$-module. Then $U := \{x \in X \mid \mathcal{M}_x$ is projective$\}$ is a dense G-stable open subset of X.*

Proof. It is easy to see that $U = \bigcup_{r \geq 0}(X \setminus (V(\underline{\mathrm{Fitt}}_r \mathcal{M}) \cup \operatorname{Supp} \underline{\mathrm{Fitt}}_{r-1} \mathcal{M}))$ is G-stable open, and if $x \in U$, then $\mathcal{M}_x$ is projective. Conversely, if $\mathcal{M}_x$ is projective of rank r, then $\mathrm{Fitt}_r(\mathcal{M}_x) = \mathcal{O}_{X,x}$ and $\mathrm{Fitt}_{r-1}(\mathcal{M}_x) = 0$ by [5, Proposition 20.8], and hence $x \in U$.

It remains to show that U is dense. Let $\mathfrak{p}$ be any associated (or equivalently, minimal, by Corollary 6.15) prime of 0. We need to show that the generic point ξ of $V(\mathfrak{p})$ is in U.

Let $0 = \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r$ be a minimal G-prime decomposition. Then we may assume that $\mathfrak{p}$ is a minimal prime of $\mathcal{P}_1$. As X is G-reduced, $\mathfrak{p} \not\supset \mathcal{P}_i$ for $i \geq 2$. Hence $\xi \in Y := X \setminus (\bigcup_{i \geq 2} V(\mathcal{P}_i))$. As Y is a non-empty G-stable open subscheme of $V(\mathcal{P}_1)$, it is G-integral by Lemma 4.20, and ξ is a generic point of an irreducible component of Y. Since $Y \cap U$ is dense in Y by Lemma 6.17 and its proof, $\xi \in U$, as desired. □

6.19 Corollary. *Let X be G-reduced, and $\mathcal{L}$ a quasi-coherent $(G, \mathcal{O}_X)$-module. Then for the generic point ξ of an irreducible component of X, $\mathcal{L}_\xi$ is $\mathcal{O}_{X,\xi}$-flat.*

Proof. Since $\mathcal{L}$ is a filtered inductive limit $\varinjlim \mathcal{M}_\lambda$ of its coherent $(G, \mathcal{O}_X)$-submodules $\mathcal{M}_\lambda$ by (2.15) and $(\mathcal{M}_\lambda)_\xi$ is a free module by Lemma 6.18, $\mathcal{L}_\xi$ is $\mathcal{O}_{X,\xi}$-flat. □

6.20 Corollary. *Let X be G-reduced, and $f : V \to X$ an affine G-morphism. Let $v \in V$, and assume that $f(v)$ is a generic point of X, then f is flat at v.*

Proof. This is because $(f_* \mathcal{O}_V)_{f(v)}$ is $\mathcal{O}_{X,f(v)}$-flat, and $\mathcal{O}_{V,v}$ is a localization of $(f_* \mathcal{O}_V)_{f(v)}$. □

6.21 Lemma. *Let $p_2 : G \times X \to X$ have regular fibers. Then*

(i) *If $\mathfrak{a}$ is a radical quasi-coherent ideal of $\mathcal{O}_X$, then $\mathfrak{a}^*$ is also radical.*

(ii) *A G-radical G-ideal of $\mathcal{O}_X$ is radical.*

Proof. Clearly, **(ii)** follows from **(i)**. We prove **(i)**.

Set $Y := V(\mathfrak{a})$. Then Y is reduced. By assumption, $G \times Y$ is reduced. So the scheme theoretic image Y^* of the action $a : G \times Y \to X$ is also reduced. Since $Y^* = V(\mathfrak{a}^*)$, we have that $\mathfrak{a}^*$ is radical. □

6.22 Corollary. *Let $p_2 : G \times X \to X$ have regular fibers. If $\mathcal{I}$ is a G-ideal of $\mathcal{O}_X$, then $\sqrt{\mathcal{I}} = \sqrt[G]{\mathcal{I}}$ is a G-radical G-ideal.*

Proof. Note that $\sqrt[G]{\mathcal{I}}$ is G-radical, and hence is radical by the lemma. Hence

$$\sqrt{\mathcal{I}} = \sqrt{\sqrt[G]{\mathcal{I}}} = \sqrt[G]{\mathcal{I}}$$

is a G-radical G-ideal. □

6.23 Lemma. *Let $p_2 : G \times X \to X$ have connected fibers. Then*

(i) *If $\mathfrak{n}$ is a primary submodule of $\mathcal{M}$, then $\mathfrak{n}^*$ is also primary.*

(ii) *A G-primary G-submodule of $\mathcal{M}$ is a primary submodule.*

Proof. Since **(ii)** follows from **(i)** and Corollary 6.3, we only need to prove **(i)**.

By assumption, $\mathcal{M}/\mathfrak{n}$ is primary, and hence it satisfies (S_1). As $p_2 : G \times X \to X$ is flat with Cohen–Macaulay fibers (see e.g., [12, (31.14)]), $p_2^*(\mathcal{M}/\mathfrak{n})$ also satisfies (S_1) by [9, (6.4.1)]. As p_2 is faithfully flat, $\operatorname{Supp} p_2^*(\mathcal{M}/\mathfrak{n}) = p_2^{-1}(\operatorname{Supp}(\mathcal{M}/\mathfrak{n}))$, which is irreducible by the irreducibility of $\operatorname{Supp}(\mathcal{M}/\mathfrak{n})$, and the assumption that p_2 has connected (or equivalently, geometrically irreducible) fibers. Thus $p_2^*(\mathcal{M}/\mathfrak{n})$ is a primary module. By Lemma 4.11, there is a monomorphism $\mathcal{M}/\mathfrak{n}^* \hookrightarrow a_* p_2^*(\mathcal{M}/\mathfrak{n})$. By Lemma 3.18, $\mathfrak{n}^*$ is a primary submodule, as desired. □

6.24 Corollary. *Let $p_2 : G \times X \to X$ have connected fibers. If $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module and $\mathcal{N}$ is a coherent $(G, \mathcal{O}_X)$-submodule, then a minimal G-primary decomposition of $\mathcal{N}$ is also a minimal primary decomposition of $\mathcal{N}$.*

Proof. Follows from Lemma 6.23 and Theorem 6.10. □

6.25 Corollary. *Let $p_2 : G \times X \to X$ have regular and connected fibers. Then*

(i) *If $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_X$, then $\mathfrak{p}^*$ is also a prime ideal.*

(ii) *A G-prime G-ideal of $\mathcal{O}_X$ is prime.*

(iii) *For a coherent $(G, \mathcal{O}_X)$-module $\mathcal{M}$ of $\mathcal{O}_X$, $\operatorname{Ass}_G(\mathcal{M}) = \operatorname{Ass}(\mathcal{M})$ and $\operatorname{Min}_G(\mathcal{M}) = \operatorname{Min}(\mathcal{M})$. In particular, any associated prime of a coherent $(G, \mathcal{O}_X)$-module is a G-prime G-ideal.*

Proof. **(i)** and **(ii)** follow immediately from Lemma 6.21 and Lemma 6.23. **(iii)** follows from **(ii)**, Corollary 6.12, and Corollary 6.13. □

6.26 Lemma. *Let S be locally noetherian and G be flat of finite type over S. Let π be the structure map $G \to S$. Then $h(s) := \dim(\pi^{-1}(s))$ is a locally constant function on S.*

Proof. We may assume that $S = \operatorname{Spec} A$ is affine and reduced. Then it suffices to show that h is constant on each irreducible component. So we may further assume that S is irreducible. Let σ be the generic point of S. The fiber $\pi^{-1}(\sigma)$ is a finite-type group scheme over the field $\kappa(\sigma)$, and is equidimensional of dimension $h(\sigma)$.

Let $s \in S$. Note that π is an open map (follows easily from [9, (1.10.4)]). By [9, (14.2.4)], each irreducible component of $\pi^{-1}(s)$ is $h(\sigma)$-dimensional. Hence, $h(s) = h(\sigma)$, and this value is independent of s. □

6.27 Proposition. *Assume that $\operatorname{Min}_G(0)$ is a singleton, where 0 is the zero ideal of $\mathcal{O}_X$. Then the dimension of the fiber of $p_2 : G \times X \to X$ is constant. In particular, if X is G-primary, then the dimension of the fiber of $p_2 : G \times X \to X$ is constant.*

Proof. Set $h(x) = \dim p_2^{-1}(x)$. We want to prove that h is constant. By Lemma 6.26, h is locally constant. Let $0 = \mathcal{Q}_1 \cap \mathcal{Q}_2 \cap \cdots \cap \mathcal{Q}_r$ be a minimal G-primary decomposition where $\sqrt[G]{\mathcal{Q}_1}$ is a minimal G-prime. Then $Y = V(\mathcal{Q}_2 \cap \cdots \cap \mathcal{Q}_r)$ is nowhere dense in X. By the local constantness, replacing X by $X \setminus Y$, we may assume that X is G-primary. Let $0 = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ be the minimal primary decomposition of 0 in $\mathcal{O}_X$. Set $X_i := V(\mathfrak{q}_i)$, and let ξ_i be the generic point of X_i. Let $\pi : X \to S$ be the structure map. Let $Y_i := X_i \setminus \bigcup_{j \neq i} X_j$.

It suffices to show that for $1 \leq i, j \leq s$, $h(\xi_i) = h(\xi_j)$. Note that $G_{ij} = a^{-1}Y_j \cap p_2^{-1}Y_i$ is non-empty by Corollary 6.9 and Lemma 6.6. Let γ be the generic point of an irreducible component of G_{ij}. By flatness, $a(\gamma) = \xi_j$ and $p_2(\gamma) = \xi_i$. Hence $\sigma := \pi(\xi_j) = \pi a(\gamma) = \pi p_2(\gamma) = \pi(\xi_i)$. Since $\{\xi_j\} \to \{\sigma\}$ associates with a field extension, and is faithfully flat quasi-compact, and $G \times \{\xi_j\}$ is of finite type over $\{\xi_j\}$, we have that $G \times \{\sigma\}$ is of finite type over $\{\sigma\}$ by [9, (2.7.1)]. It is easy to see that $G \times \{\sigma\}$ and $G \times \{\xi_j\}$ have the same dimension, $h(\xi_j)$. By the same reason, $G \times \{\sigma\}$ is $h(\xi_i)$-dimensional. Hence $h(\xi_i) = h(\xi_j)$. □

(6.28) A commutative ring A is said to be *Hilbert* if any prime ideal P of A equals the intersection $\bigcap_{\mathfrak{m}} \mathfrak{m}$, where the intersection is taken over the maximal ideals containing P. We say that A satisfies the *first chain condition*

(FCC for short) if each maximal chain of prime ideals in A has the length equals to $\dim A$. We say that A is *Ratliff* if A is noetherian, universally catenary, Hilbert, and for any minimal prime P of A, A/P satisfies the FCC.

6.29 Lemma. *A noetherian ring A is Ratliff if and only if A_{red} is.*

Proof. By assumption, both A and A_{red} are noetherian. Almost by definition, A is Hilbert if and only if A_{red} is. A finite-type algebra over A_{red} is of finite type over A. So plainly, if A is universally catenary, then A_{red} is universally catenary. Let B be a finite type algebra over A. Then B is catenary if and only if B_{red} is, and B_{red} is a finite-type algebra over A_{red}. So if A_{red} is universally catenary, then so is A. Note that $\mathrm{Min}(A_{\mathrm{red}}) = \{PA_{\mathrm{red}} \mid P \in \mathrm{Min}(A)\}$, and $A/P \cong A_{\mathrm{red}}/PA_{\mathrm{red}}$. So A/P satisfies the FCC for any $P \in \mathrm{Min}(A)$ if and only if $A_{\mathrm{red}}/\bar{P}$ satisfies the FCC for any $\bar{P} \in \mathrm{Min}(\bar{A})$. $\square$

(6.30) An artinian ring is Ratliff. A one-dimensional noetherian domain with infinitely many prime ideals is Ratliff. For example, $\mathbb{Z}$ is Ratliff.

6.31 Lemma. *Let A be a catenary noetherian ring such that for each minimal prime P of A, A/P satisfies the FCC. Then for any prime ideal Q of A, A/Q satisfies the FCC.*

Proof. Easy. $\square$

6.32 Corollary. *A homomorphic image of a Ratliff ring is Ratliff.*

Proof. Follows from Lemma 6.31. $\square$

6.33 Lemma. *If A is Ratliff and $A \to B$ is of finite type, then B is Ratliff.*

Proof. Note that B is noetherian by Hilbert's basis theorem. B is universally catenary, since a finite-type algebra C over B is also of finite type over A. It is well-known that a finite-type algebra over a Hilbert ring is Hilbert [22, Chapter 6, Theorem 1]. It remains to show that for any minimal prime P, $\bar{B} := B/P$ satisfies the FCC. Since we know that $\bar{B}$ is catenary, it suffices to show that $\dim \bar{B}_{\mathfrak{m}}$ is independent of the choice of a maximal ideal $\mathfrak{m}$ of $\bar{B}$. Let $\mathfrak{p} := P \cap A$, and set $\bar{A} := A/\mathfrak{p}$. Note that $\bar{A}$ is Ratliff. Note also that $\mathfrak{n} := \mathfrak{m} \cap \bar{A}$ is a maximal ideal of $\bar{A}$ [22, Chapter 6, Theorem 2], and $\dim \bar{A}_{\mathfrak{n}} = \dim \bar{A}$ is independent of $\mathfrak{n}$. Clearly, $\kappa(\mathfrak{m})$ is an algebraic extension of $\kappa(\mathfrak{n})$. By the dimension formula [9, (5.6.1)], $\dim \bar{B}_{\mathfrak{m}} = \dim \bar{A} + \mathrm{trans.deg}_{\bar{A}} \bar{B}$ is independent of $\mathfrak{m}$. $\square$

(6.34) We say that a scheme Y is Ratliff if Y has a finite affine open covering (U_i) such that each U_i is isomorphic to the prime spectrum of a Ratliff ring. Y is Ratliff if and only if Y is quasi-compact, and each point of Y has an affine Ratliff open neighborhood. A Ratliff scheme Y has a finite dimension.

If Y is Ratliff and $Z \to Y$ is of finite type, then Z is Ratliff.

A noetherian scheme Y is said to be *equidimensional*, if the irreducible components of Y have the same dimension. If Y is Ratliff and equidimensional, then $\dim Y = \dim \mathcal{O}_{Y,y}$ for any closed point y of Y. Let Z be a closed subset of a Ratliff scheme Y. Let W be the set of closed points of Z. Then the closure of W is Z. It follows that any non-empty open subscheme U of a Ratliff scheme Y contains a closed point of Y. In particular, if Y is Ratliff equidimensional and U is a non-empty open subset, then $\dim U = \dim Y$. In particular, for any point y of Y, $\dim_y Y = \dim Y$.

Let $f : Z \to Y$ be of finite type and dominating, Y be Ratliff, and Z and Y be irreducible. Then $\dim Z = \dim Y + \text{trans.deg}_{\kappa(\eta)} \kappa(\zeta)$, where η and ζ are respectively the generic points of Y and Z.

6.35 Proposition. *Assume that X is Ratliff, and assume that* $\text{Min}_G(0)$ *is a singleton, where* 0 *is the zero ideal of* $\mathcal{O}_X$. *Then X is equidimensional.*

Proof. Discarding a nowhere dense G-stable closed subscheme form X, we may assume that X is G-primary.

Let X_i, Y_i, and G_{ij} be as in the proof of Proposition 6.27. Let g be a closed point of G_{ij}. Then $p_2(g)$ is a closed point of Y_i, and we have $\dim \mathcal{O}_{G,g} = \dim Y_i + h$, where h is the dimension of the fibers of $p_2 : G \times X \to X$, see Proposition 6.27. Since $a(g)$ is a closed point of Y_j, $\dim \mathcal{O}_{G,g} = \dim Y_j + h$ (note that the dimension of the fibers of a also have the constant value h). Hence, $\dim X_j = \dim Y_j = \dim Y_i = \dim X_i$, as desired. □

6.36 Example. Let R be a DVR, and t a prime element of R, and $K = R[t^{-1}]$ the field of fractions of R. Set $S = \operatorname{Spec} R$. Let $\tilde{G} = S_0 \coprod S_1$ be the constant group $\mathbb{Z}/2\mathbb{Z}$ over S, where $S = S_0 = S_1$, and S_0 corresponds to the unit element, and S_1 corresponds to the other element of $\mathbb{Z}/2\mathbb{Z}$. Then $G = S_0 \coprod \operatorname{Spec} K$ is a flat of finite type group scheme over S. Letting $X = G$, the left regular action, X is G-primary. But X is not equidimensional.

This example also shows that the following statement is *false*. Let $\mathcal{P}$ and $\mathcal{Q}$ be G-primes of $\mathcal{O}_X$ such that $\mathcal{Q} \supset \mathcal{P}$. Then for any minimal prime $\mathfrak{p}$ of $\mathcal{P}$, there exists some minimal prime $\mathfrak{q}$ of $\mathcal{Q}$ such that $\mathfrak{q} \supset \mathfrak{p}$. Indeed, let $\mathcal{P} = 0$, and $\mathcal{Q}$ be the defining ideal of the closed point of S_0. Then the component $\operatorname{Spec} K \subset S_1$ of $V(\mathcal{P})$ does not contain a component of $V(\mathcal{Q})$.

6.37 Example. Let (R, t) be a DVR of mixed characteristic p. Then $\mu_{p,R} = \operatorname{Spec} R[x]/(x^p - 1)$ is a local scheme with two or more irreducible components.

6.38 Example. A primary ideal $\mathfrak{q}$ of $\mathcal{O}_X$ which is a primary component of some G-ideal of $\mathcal{O}_X$, but not a primary component of $\mathfrak{q}^*$.

Construction. Let $S = \operatorname{Spec} k$ with k a field. Let $G = \mathbb{G}_m^2$, and $X = \operatorname{Spec} A$, $A = k[x, y]$. G acts on A with $\deg x = (1, 0)$ and $\deg y = (0, 1)$. Let $\mathfrak{q} = (x^4, yx^3, y^2x^2 + y^3x, y^4)$. It is a primary component of the homogeneous ideal $I = (x^4, yx^3) = (x^3) \cap \mathfrak{q}$. But $\mathfrak{q}$ is not a primary component of $\mathfrak{q}^* = (x^4, yx^3, y^3x^2, y^4)$. $\square$

6.39 Lemma. *Let $\mathcal{N}$ be a $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$. If $\mathfrak{m}$ is the primary component of $\mathcal{N}$ corresponding to a minimal prime of $\mathcal{N}$, then $\mathfrak{m}$ is the primary component of $\mathfrak{m}^*$ corresponding to a minimal prime of $\mathfrak{m}^*$.*

Proof. Let (9) be a minimal G-primary decomposition of $\mathcal{N}$, and (10) be a minimal primary decomposition of $\mathcal{M}_l$. By the uniqueness of the primary component for minimal primes, we may assume that $\mathfrak{m}_{1,1} = \mathfrak{m}$. Then $\mathfrak{m}$ is a primary component of $\mathcal{M}_1 = \mathfrak{m}^*$ corresponding to a minimal prime by Corollary 6.3 and Corollary 6.2. $\square$

6.40 Example. Even if $\mathfrak{m}$ is a maximal ideal of $\mathcal{O}_X$, $\mathfrak{m}^*$ may not be a G-maximal G-ideal. Let $S = \operatorname{Spec} k$ with k a field, $G = \mathbb{G}_m$, and $X = \mathbb{A}^1$ on which G acts by multiplication. For any maximal ideal $\mathfrak{m}$ of $k[X] = k[t]$ not corresponding to the origin, $\mathfrak{m}^* = 0$ is not G-maximal.

(6.41) Let us consider the case that $S = \operatorname{Spec} k$, where k is an algebraically closed field. Let G be a linear (smooth) algebraic group over k. Let G° denote the identity component of G. Then there exists some $h_0, h_1, \ldots, h_m \in G(k)$ such that $h_0 = e$ is the unit element of G, and $G = h_0 G^\circ \coprod h_1 G^\circ \coprod \cdots \coprod h_m G^\circ$. Let $\mathcal{N}$ be a coherent $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$. Let $\mathfrak{p} \in \operatorname{Ass}(\mathcal{M}/\mathcal{N})$. Then $\mathfrak{p}$ is G°-stable by Corollary 6.25. Let $Y_{\mathfrak{p},0} = V(\mathfrak{p})$, and $Y_{\mathfrak{p},i} = h_i Y_{\mathfrak{p},0}$ for $i = 1, \ldots, m$. Then

$$Y_{\mathfrak{p}} = \bigcup_{i=0}^{m} Y_{\mathfrak{p},i} \tag{11}$$

with the reduced structure is G-stable. So $Y^*_{\mathfrak{p},i} = Y_\mathfrak{p}$ for each i, as can be seen easily. So $\operatorname{Ass}_G(\mathcal{M}/\mathcal{N}) = \{\mathcal{I}(Y_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Ass}(\mathcal{M}/\mathcal{N})\}$ by Corollary 6.12, where $\mathcal{I}(Y_\mathfrak{p})$ is the defining ideal of $Y_\mathfrak{p}$.

We can show that for a coherent $(G, \mathcal{O}_X)$-module $\mathcal{L}$, $\mathcal{L}$ is $\mathcal{I}(Y_\mathfrak{p})$-$G$-primary if and only if $\operatorname{Ass}\mathcal{L} = \{\mathcal{I}(Y_{\mathfrak{p},i}) \mid i = 0, \ldots, m\}$. To verify this, note that $Y_{\mathfrak{p},i}$ in (11) are isomorphic one another, and (11) is an irreducible decomposition without embedded component (but there may be some redundancy). The 'only if' part follows from the fact that $\mathcal{L}$ does not have an embedded prime by Corollary 6.2, and $V(\sqrt{0 : \mathcal{L}}) = Y_\mathfrak{p}$. The if part follows from the fact that $\mathcal{I}(Y_{\mathfrak{p},i})^* = \mathcal{I}(Y_\mathfrak{p})$ for $i = 0, \ldots, m$.

In particular, the existence of G-primary decomposition of $\mathcal{N}$ gives another proof of [24, Thoerem 4.18] for the case that X is quasi-compact.

7. Matijevic–Roberts type theorem

In this section, S, G, X, and $\mathcal{M}$ are as in the last section. As in the last section, we assume that $p_2 : G \times X \to X$ is of finite type.

(7.1) A flat homomorphism of noetherian rings $\varphi : A \to B$ is said to be l.c.i. (local complete intersection) (resp. regular) if for any prime ideal P of A, the fiber ring B_P/PB_P is l.c.i. (resp. geometrically regular over the field A_P/PA_P). By the openness of l.c.i. locus [11, (I.2.12.4)] (resp. smooth locus [9, (6.8.7)]) for a finite-type morphism, a flat local homomorphism essentially of finite type $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ is l.c.i. (resp. regular) if and only if the closed fiber $B/\mathfrak{m}B$ is a complete intersection (resp. geometrically regular).

7.2 Theorem. *Let y be a point of X, and Y the integral closed subscheme of X whose generic point is y. Let η be the generic point of an irreducible component of Y^*. Then there are a noetherian local ring A and flat l.c.i. local homomorphisms essentially of finite type $\varphi : \mathcal{O}_{X,y} \to A$ and $\psi : \mathcal{O}_{X,\eta} \to A$, such that* $\dim A = \dim \mathcal{O}_{X,y}$ *and that* $A \otimes_{\mathcal{O}_{X,\eta}} \mathcal{L}_\eta \cong A \otimes_{\mathcal{O}_{X,y}} \mathcal{L}_y$ *for any quasi-coherent $(G, \mathcal{O}_X)$-module $\mathcal{L}$ of X. If, moreover, $p_2 : G \times X \to X$ is smooth, then φ and ψ can be taken to be regular.*

Proof. The action $a : G \times Y \to Y^*$ is dominating, so there is a point $z \in G \times Y$ such that $a(z) = \eta$. Let ζ be the generic point of an irreducible component of $G \times Y$ containing z. Then $a(\zeta)$ is a generalization of η, and hence $a(\zeta) = \eta$ by the choice of η. Since the second projection $p_2 : G \times Y \to Y$ is flat, $p_2(\zeta) = y$. Set $A := \mathcal{O}_{G\times X,\zeta}$. Let $\varphi : \mathcal{O}_{X,y} \to A$ and

$\psi : \mathcal{O}_{X,\eta} \to A$ be the homomorphisms induced by $p_2 : G \times X \to X$ and $a : G \times X \to X$, respectively. As p_2 and a are finite-type flat local complete intersection morphisms [12, (31.14)], φ and ψ are essentially of finite type flat homomorphisms with complete intersection fibers. As ζ is the generic point of a component of $p_2^{-1}(y)$, it is easy to see that $\dim A = \dim \mathcal{O}_{X,y}$. Moreover,

$$A \otimes_{\mathcal{O}_{X,\eta}} \mathcal{L}_\eta \cong (a^*\mathcal{L})_\zeta \cong (p_2^*\mathcal{L})_\zeta \cong A \otimes_{\mathcal{O}_{X,y}} \mathcal{L}_y.$$

The last assertion is trivial. □

7.3 Corollary. *Let $S = \operatorname{Spec} k$, with k a perfect field, and let G be of finite type over S. Let y, Y, and η be as in the theorem. Then there exist A, φ and ψ as in the theorem such that φ and ψ are regular.*

Proof. Replacing G by G_{red} if necessary, we may assume that G is k-smooth. The assertion follows immediately by the theorem. □

7.4 Corollary. *Let y and η be as in the theorem. Then* $\dim \mathcal{O}_{X,y} \geq \dim \mathcal{O}_{X,\eta}$.

Proof. $\dim \mathcal{O}_{X,y} = \dim A \geq \dim \mathcal{O}_{X,\eta}$. □

7.5 Corollary. *Let Y be as in the theorem, and η_1 and η_2 be the generic points of irreducible components of Y^*. Then $\dim \mathcal{O}_{X,\eta_1} = \dim \mathcal{O}_{X,\eta_2}$. There are a noetherian local ring A such that $\dim A = \dim \mathcal{O}_{X,\eta_1}$ and flat l.c.i. local homomorphisms $\varphi_i : \mathcal{O}_{X,\eta_i} \to A$ essentially of finite type such that for any quasi-coherent $(G, \mathcal{O}_X)$-module $\mathcal{L}$, $A \otimes_{\mathcal{O}_{X,\eta_1}} \mathcal{L}_{\eta_1} \cong A \otimes_{\mathcal{O}_{X,\eta_2}} \mathcal{L}_{\eta_2}$. If, moreover, $p_2 : G \times X \to X$ is smooth, or $S = \operatorname{Spec} k$ with k a perfect field and G is of finite type over S, then φ_i can be taken to be regular.*

Proof. Let $\mathfrak{p}$ be the defining ideal of Y. Let $\mathfrak{q}_i$ be the defining ideal of Z_i, where Z_i is the closed integral subscheme of X whose generic point is η_i, for $i = 1, 2$. By Corollary 6.13, $\mathfrak{q}_i^* \in \operatorname{Min}_G(\mathcal{O}_X/\mathfrak{p}^*) = \{\mathfrak{p}^*\}$. Applying Corollary 7.4 to $y = \eta_1$ and $\eta = \eta_2$, $\dim \mathcal{O}_{X,\eta_1} \geq \dim \mathcal{O}_{X,\eta_2}$. Similarly, $\dim \mathcal{O}_{X,\eta_2} \geq \dim \mathcal{O}_{X,\eta_1}$, and hence $\dim \mathcal{O}_{X,\eta_1} = \dim \mathcal{O}_{X,\eta_2}$. The other assertions are clear by Theorem 7.2. □

7.6 Corollary. *Let $\mathcal{C}$ and $\mathcal{D}$ be classes of noetherian local rings, and $\mathbb{P}(A, M)$ a property of a finite module M over a noetherian local ring A. Assume that:*

(i) *If $A \in \mathcal{C}$, M a finite A-module with $\mathbb{P}(A, M)$, and $A \to B$ a flat l.c.i. (resp. regular) local homomorphism essentially of finite type, then $B \in \mathcal{D}$ and $\mathbb{P}(B, B \otimes_A M)$ holds.*

(ii) *If $A \to B$ is a flat l.c.i. (resp. regular) local homomorphism essentially of finite type of noetherian local rings, M is a finite A-module, and if $B \in \mathcal{D}$ and $\mathbb{P}(B, B \otimes_A M)$ holds, then $A \in \mathcal{D}$ and $\mathbb{P}(A, M)$ holds.*

If $\mathcal{O}_{X,\eta} \in \mathcal{C}$ and $\mathbb{P}(\mathcal{O}_{X,\eta}, \mathcal{M}_\eta)$ holds (resp. $\mathbb{P}(\mathcal{O}_{X,\eta}, \mathcal{M}_\eta)$ holds and either $p_2 : G \times X \to X$ is smooth, or $S = \operatorname{Spec} k$ with k a perfect field and G is of finite type over S), then $\mathcal{O}_{X,y} \in \mathcal{D}$ and $\mathbb{P}(\mathcal{O}_{X,y}, \mathcal{M}_y)$ holds. Conversely, if $\mathcal{O}_{X,y} \in \mathcal{C}$ and $\mathbb{P}(\mathcal{O}_{X,y}, \mathcal{M}_y)$ holds (resp. $\mathbb{P}(\mathcal{O}_{X,y}, \mathcal{M}_y)$ holds and either $p_2 : G \times X \to X$ is smooth, or $S = \operatorname{Spec} k$ with k a perfect field and G is of finite type over S), then $\mathcal{O}_{X,\eta} \in \mathcal{D}$ and $\mathbb{P}(\mathcal{O}_{X,\eta}, \mathcal{M}_\eta)$ holds.

Proof. Because of $\psi : \mathcal{O}_{X,\eta} \to A$, $A \in \mathcal{D}$ and $\mathbb{P}(A, A \otimes_{\mathcal{O}_{X,\eta}} \mathcal{M}_\eta)$ holds, by the condition **(i)**. Since $A \otimes_{\mathcal{O}_{X,\eta}} \mathcal{M}_\eta \cong A \otimes_{\mathcal{O}_{X,y}} \mathcal{M}_y$ and we have $\varphi : \mathcal{O}_{X,y} \to A$, $\mathcal{O}_{X,y} \in \mathcal{D}$ and $\mathbb{P}(\mathcal{O}_{X,y}, \mathcal{M}_y)$ by the condition **(ii)**.

The proof of the converse is similar. □

7.7 Corollary. *Let y and η be as in the theorem, and m, n, and g be non-negative integers or ∞. Then*

(i) *If $\mathcal{M}_\eta$ is maximal Cohen–Macaulay (resp. of finite injecctive dimension, projective dimension m, $\dim - \operatorname{depth} = n$, torsionless, reflexive, G-dimension g, zero) as an $\mathcal{O}_{X,\eta}$-module if and only if $\mathcal{M}_y$ is so as an $\mathcal{O}_{X,y}$-module.*

(ii) *If $\mathcal{O}_{X,\eta}$ is a complete intersection, then so is $\mathcal{O}_{X,y}$.*

(iii) *Assume that $p_2 : G \times X \to X$ is smooth, or $S = \operatorname{Spec} k$ with k a perfect field and G is of finite type over S. If $\mathcal{O}_{X,\eta}$ is regular, then so is $\mathcal{O}_{X,y}$.*

(iv) *Assume that $p_2 : G \times X \to X$ is smooth, or $S = \operatorname{Spec} k$ with k a perfect field and G is of finite type over S. Let p be a prime number, and assume that $\mathcal{O}_{X,\eta}$ is excellent. If $\mathcal{O}_{X,\eta}$ is weakly F-regular (resp. F-regular, F-rational) of characteristic p, then so is $\mathcal{O}_{X,y}$. If $\mathcal{O}_{X,y}$ is excellent and weakly F-regular of characteristic p, then $\mathcal{O}_{X,\eta}$ is weakly F-regular of characteristic p.*

Proof. **(i)** Let $\mathcal{C} = \mathcal{D}$ be the class of all noetherian local rings, and $\mathbb{P}(A, M)$ be the property "M is a maximal Cohen–Macaulay A-module" in Corollary 7.6. The assertion follows immediately by Corollary 7.6. Similarly for other properties.

(ii) Let $\mathcal{C} = \mathcal{D}$ be the class of complete intersection noetherian local rings, and $\mathbb{P}(A, M)$ be "always true."

(iii) Let $\mathcal{C} = \mathcal{D}$ be the class of regular local rings.

(iv) Let $\mathcal{C}$ be the class of excellent weakly F-regular local rings of characteristic p, and $\mathcal{D}$ be the class of weakly F-regular local rings of characteristic p. By [14], **(i)** and **(ii)** of Corollary 7.6 holds. Similarly for F-regularity, see [14]. For F-rationality, see [26]. □

7.8 Corollary. *Let Y be a G-primary closed subscheme of X. Let η_1 and η_2 be the generic points of irreducible components of Y. Let d, δ, m, and g be nonnegative integers or ∞.*

(i) *$\mathcal{M}_{\eta_1}$ is maximal Cohen–Macaulay (resp. of finite injective dimension, dimension d, depth δ, projective dimension m, torsionless, reflexive, G-dimension g, zero) as an $\mathcal{O}_{X,\eta_1}$-module if and only if the same is true of $\mathcal{M}_{\eta_2}$ as an $\mathcal{O}_{X,\eta_2}$-module.*

(ii) *$\mathcal{O}_{X,\eta_1}$ is a complete intersection if and only if $\mathcal{O}_{X,\eta_2}$ is so.*

(iii) *Assume that $p_2 : G \times X \to X$ is smooth, or $S = \operatorname{Spec} k$ with k a perfect field and G is of finite type over S. Then $\mathcal{O}_{X,\eta_1}$ is regular if and only if $\mathcal{O}_{X,\eta_2}$ is so. Assume further that X is locally excellent (that is, the all local rings of X are excellent). Then $\mathcal{O}_{X,\eta_1}$ is of characteristic p and weakly F-regular (resp. F-regular, F-rational) if and only if $\mathcal{O}_{X,\eta_2}$ is so.*

Proof. Follows immediately from Corollary 7.5. □

(7.9) Let $Y = V(\mathcal{Q})$ be as in the corollary. Then we say that $\mathcal{M}$ is maximal Cohen–Macaulay (resp. of finite injective dimension, dimension d, depth δ, projective dimension m, torsionless, reflexive, G-dimension g) along Y (or at $\mathcal{Q}$) if $\mathcal{M}_\eta$ is so for the generic point η of some (or equivalently, any) irreducible component of Y. We say that X is complete intersection along Y (or at $\mathcal{Q}$) if $\mathcal{O}_{X,\eta}$ is a complete intersection for some (or equivalently, any) η. Assume that $p_2 : G \times X \to X$ is smooth, or $S = \operatorname{Spec} k$ with k a perfect field and G is of finite type over S. We say that X is regular along Y (or at $\mathcal{Q}$) if $\mathcal{O}_{X,\eta}$ is a regular local ring for some (or equivalently, any) η. Assume further that X is a locally excellent $\mathbb{F}_p$-scheme. We say that X is weakly F-regular (resp. F-regular, F-rational) along Y (or at $\mathcal{Q}$) if $\mathcal{O}_{X,\eta}$ is so for some (or equivalently, any) η.

7.10 Corollary. *Let p be a prime number, and A a $\mathbb{Z}^n$-graded noetherian ring. Let P be a prime ideal of A, and P^* be the prime ideal of A generated by the homogeneous elements of P. If A_{P^*} is excellent of characteristic p and is weakly F-regular (resp. F-regular, F-rational), then A_P is weakly F-regular (resp. F-regular, F-rational). If A_P is excellent of characteristic p and is weakly F-regular, then A_{P^*} is weakly F-regular.*

Proof. Let $S = \operatorname{Spec} \mathbb{Z}$, $G = \mathbb{G}_m^n$, and $X = \operatorname{Spec} A$. If $y = P$, then η in the theorem is P^*. The assertion follows immediately by Corollary 7.7. □

7.11 Corollary (Cf. [15, (4.7)]). *Let A be a $\mathbb{Z}^n$-graded locally excellent ring of characteristic p. If $A_{\mathfrak{m}}$ is F-regular (resp. F-rational) for any graded maximal ideals (that is, G-maximal ideal for $G = \mathbb{G}_m^n$), then A is F-regular (resp. F-rational).*

Proof. If Q is a graded prime ideal, then it is contained in a graded maximal ideal. So A_Q is F-regular (resp. F-rational, see [14, (4.2)]).

Now consider any prime ideal $\mathfrak{p}$ of A. Then $A_{\mathfrak{p}^*}$ is F-regular (resp. F-rational), since $\mathfrak{p}^*$ is a graded prime ideal. Hence $A_{\mathfrak{p}}$ is so by Corollary 7.10. So A is F-regular (resp. F-rational). □

7.12 Remark. Let k be an (F-finite) field of characteristic p, and $A = \bigoplus_{n\geq 0} A_n$, $A_0 = k$, a positively graded finitely generated k-algebra. Set $\mathfrak{m} := \bigoplus_{n>0} A_n$. Lyubeznik and Smith [18] proved that if $A_{\mathfrak{m}}$ is weakly F-regular, then A is (strongly) F-regular.

7.13 Corollary. *Let $A = \bigoplus_{n\geq 0} A_n$ be an $\mathbb{N}$-graded locally excellent noetherian ring of characteristic p. Let $t \in A_+ := \bigoplus_{n>0} A_n$ be a nonzerodivisor of A. If A/tA is F-rational, then A is F-rational.*

Proof. Note that A/tA is Cohen–Macaulay [26, Proposition 0.10]. By Corollary 7.11, it suffices to show that $A_{\mathfrak{m}}$ is F-rational for any graded maximal ideal (i.e., G-maximal ideal for $G = \mathbb{G}_m^n$). It is easy to see that $\mathfrak{m} = \mathfrak{m}_0 + A_+$. Hence $t \in \mathfrak{m}$. Since $A_{\mathfrak{m}}/tA_{\mathfrak{m}}$ is Cohen–Macaulay F-rational, $A_{\mathfrak{m}}$ is also F-rational by [14, (4.2)]. □

7.14 Corollary. *Let A be a ring of characteristic p, and $(F_n)_{n\geq 0}$ a filtration of A. That is, $F_0 \subset F_1 \subset F_2 \subset \cdots \subset A$, $1 \in F_0$, $F_iF_j \subset F_{i+j}$, and $\bigcup_{n\geq 0} F_n = A$. Set $R = \bigoplus_{n\geq 0} F_n t^n \subset A[t]$, and $G = R/tR$. If G is (locally) excellent noetherian and F-rational, then A is also (locally) excellent noetherian and F-rational.*

Proof. There exist homogeneous elements $a_1t^{n_1}, \cdots, a_rt^{n_r}$ of R such that their images in G generate the ideal G_+, the irrelevant ideal of G. Then R_+ is generated by $t, a_1t^{n_1}, \ldots, a_rt^{n_r}$. Since $F_0 \cong G/G_+$, F_0 is (locally) excellent noetherian. Since R is generated by $t, a_1t^{n_1}, \ldots, a_rt^{n_r}$ as an F_0-algebra, R is also (locally) excellent noetherian. Then by Corollary 7.13, R is F-rational. Hence $R[t^{-1}] \cong A[t, t^{-1}]$ is (locally) excellent and F-rational. Hence A is (locally) excellent and F-rational. □

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