A pure subalgebra of a finitely generated algebra is finitely generated*

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Abstract

We prove the following. Let R be a Noetherian commutative ring, B a finitely generated R-algebra, and A a pure R-subalgebra of B. Then A is finitely generated over R.

In this paper, all rings are commutative. Let A be a ring and B an A-algebra. We say that $A \to B$ is pure, or A is a pure subring of B, if for any A-module M, the map $M = M \otimes_A A \to M \otimes_A B$ is injective. Considering the case $M = A/I$, where I is an ideal of A, we immediately have that $IB \cap A = I$.

There have been a number of cases where it has been shown that if B has a good property and A is a pure subring of B, then A has a good property. If B is a regular Noetherian ring containing a field, then A is Cohen-Macaulay [5], [4]. If k is a field of characteristic zero, A and B are essentially of finite type over k, and B has at most rational singularities, then A has at most rational singularities [1].

In this paper, we prove the following

Theorem 1. *Let R be a Noetherian ring, B a finitely generated R-algebra, and A a pure R-subalgebra of B. Then A is finitely generated over R.*

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The case that B is A-flat is proved in [3, Corollary 2.6]. This theorem is on the same line as the finite generation results in [3].

To prove the theorem, we need the following, which is a special case of a theorem of Raynaud-Gruson [7], [8].

Theorem 2. *Let $A \to B$ be a homomorphism of Notherian rings, and $\varphi\colon X \to Y$ the associated morphism of affine schemes. Let $U \subset Y$ be an open subset, and assume that $\varphi\colon \varphi^{-1}(U) \to U$ is flat. Then there exists some ideal I of A such that $V(I) \cap U = \emptyset$, and that the morphism $\Phi\colon \operatorname{Proj} R_B(BI) \to \operatorname{Proj} R_A(I)$, determined by the associated morphism of the Rees algebras $R_A(I) := A[tI] \to R_B(BI) := B[tBI]$, is flat.*

The morphism Φ in the theorem is called a flattening of φ.

Proof of Theorem 1. Note that for any A-algebra A', the homomorphism $A' \to B \otimes_A A'$ is pure.

Since B is finitely generated over R, it is Noetherian. Since A is a pure subring of B, A is also Noetherian. So if A_{red} is finitely generated, then so is A. Replacing A by A_{red} and B by $B \otimes_A A_{\mathrm{red}}$, we may assume that A is reduced.

Since $A \to \prod_{P \in \mathrm{Min}(A)} A/P$ is finite and injective, it suffices to prove that each A/P is finitely generated for $P \in \mathrm{Min}(A)$, where $\mathrm{Min}(A)$ denotes the set of minimal primes of A. By the base change, we may assume that A is a domain.

There exists some minimal prime P of B such that $P \cap A = 0$. Assume the contrary. Then take $a_P \in P \cap A \setminus \{0\}$ for each $P \in \mathrm{Min}(B)$. Then $\prod_P a_P$ must be nilpotent, which contradicts our assumption that A is a domain.

So by [6, (2.11) and (2.20)], A is a finitely generated R-algebra if and only if $A_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-algebra for each $\mathfrak{p} \in \operatorname{Spec} R$. So we may assume that R is a local ring.

By the descent argument [2, (2.7.1)], $\hat{R} \otimes_R A$ is a finitely generated $\hat{R}$-algebra if and only if A is a finitely generated R-algebra, where $\hat{R}$ is the completion of R. So we may assume that R is a complete local ring. We may lose the assumption that A is a domain (even if A is a domain, $\hat{R} \otimes_R A$ may not be a domain). However, doing the same reduction argument as above if necessary, we may still assume that A is a domain.

Let $\varphi\colon X \to Y$ be a morphism of affine schemes associated with the map $A \to B$. Note that φ is a morphism of finite type between Noetherian schemes. We denote the flat locus of φ by $\mathrm{Flat}(\varphi)$. Then $\varphi(X \setminus$

Flat(φ)) is a constructible set of Y not containing the generic point. So $U = Y \setminus \overline{\varphi(X \setminus \mathrm{Flat}(\varphi))}$ is a dense open subset of Y, and $\varphi\colon \varphi^{-1}(U) \to U$ is flat. By Theorem 2, there exists some nonzero ideal I of A such that $\Phi\colon \operatorname{Proj} R_B(BI) \to \operatorname{Proj} R_A(I)$ is flat.

If J is a homogeneous ideal of $R_A(I)$, then we have an expression $J = \bigoplus_{n\geq 0} J_n t^n$ ($J_n \subset I^n$). Since A is a pure subalgebra of B, we have $J_n B \cap I^n = J_n$ for each n. Since $JR_B(BI) = \bigoplus_{n\geq 0}(J_n B)t^n$, we have that $JR_B(BI) \cap R_A(I) = J$. Namely, any homogeneous ideal of $R_A(I)$ is contracted from $R_B(BI)$.

Let P be a homogeneous prime ideal of $R_A(I)$. Then there exists some minimal prime Q of $PR_B(BI)$ such that $Q \cap R_A(AI) = P$. Assume the contrary. Then for each minimal prime Q of $PR_B(BI)$, there exists some $a_Q \in (Q \cap R_A(AI)) \setminus P$. Then $\prod a_Q \in \sqrt{PR_B(BI)} \cap R_A(AI) \setminus P$. However, we have

$$\sqrt{PR_B(BI)} \cap R_A(I) = \sqrt{PR_B(BI) \cap R_A(I)} = \sqrt{P} = P,$$

and this is a contradiction. Hence $\Phi\colon \operatorname{Proj} R_B(BI) \to \operatorname{Proj} R_A(I)$ is faithfully flat.

Since $\operatorname{Proj} R_B(BI)$ is of finite type over R and Φ is faithfully flat, we have that $\operatorname{Proj} R_A(I)$ is of finite type by [3, Corollary 2.6]. Note that the blow-up $\operatorname{Proj} R_A(I) \to Y$ is proper surjective. Since R is excellent, Y is of finite type over R by [3, Theorem 4.2]. Namely, A is a finitely generated R-algebra. □

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