F-rationality of the ring of modular invariants

Mitsuyasu Hashimoto

Okayama University
Okayama 700–8530, JAPAN
mh@okayama-u.ac.jp

Dedicated to Professor Kei-ichi Watanabe

Abstract

Using the description of the F-limit of modules over (the completion of) the ring of invariants under a linear action of a finite group on a polynomial ring over an algebraically closed field of characteristic $p > 0$ developed by Symonds and the author, we give a characterization of the ring of invariants have a positive dual F-signature. Combining the result and Kemper's result on depths of the ring of invariants under an action of a permutation group, we give an example of an F-rational, but non F-regular ring of invariants under the action of a finite group.

1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$. Let $V = k^d$, and G a finite subgroup of $GL(V)$ without psuedo-reflections. Let $B = \operatorname{Sym} V$, the symmetric algebra of V, and $A = B^G$. Broer [Bro] proved that if p divides the order $|G|$ of G, then A is not a direct summand subring of B hence A is not weakly F-regular (as A is not a splinter). In this paper, we study when A is F-rational.

2010 *Mathematics Subject Classification*. Primary 13A50, 13A35. Key Words and Phrases. F-rational, F-regular, dual F-signature, F-limit.

Sannai [San] defined the dual F-signature $s(M)$ of a finite module M over an F-finite local ring R of characteristic p. He proved that R is F-rational if and only if R is Cohen–Macaulay and the dual F-signature $s(\omega_R)$ of the canonical module ω_R of R is positive. Utilizing the description of the F-limit of modules over $\hat{A}$ (the completion of A) by Symonds and the author, we give a characterization of V such that $s(\omega_{\hat{A}}) > 0$, see Theorem 4.4. The characterization is purely representation theoretic in the sense that the characterization depends only on the structure of B as a G-module, rather than a G-algebra.

Using the characterization and Kemper's result on the depth of the ring of invariants under the action of certain groups of permutations [Kem, (3.3)], we give an example of F-rational A for $p \geq 5$. We also give an example of Gorenstein and non-F-rational A for $p \geq 3$. We also get an example of A such that the dual F-signature $s_{\omega_{\hat{A}}}$ of the canonical module of the completion $\hat{A}$ is positive, but A (or equivalently, $\hat{A}$) is not Cohen–Macaulay. See Theorem 5.12.

In section 2, we introduce the invariant $\operatorname{asn}_N(M)$ for two finitely generated modules M and N ($N \neq 0$) over a Noetherian ring R. In section 3, using the definition and some basic results developed in section 2, we prove the formula $s(M) = \operatorname{asn}_M(\mathrm{FL}([M]))$, where FL denotes the F-limit defined in [HS]. Thus $s(M)$ depends only on $\mathrm{FL}([M])$. Using this, we give a characterization of a module M to have positive $s(M)$ in terms of $\mathrm{FL}([M])$ (Corollary 3.5).

Using this result and the description of the F-limits of certain modules over $\hat{A}$ proved in [HS], we give a characterization of V such that $s(\omega_{\hat{A}}) > 0$ in section 4.

In section 5, we give the examples.

Acknowledgments. The author is grateful to Professor Anurag Singh and Professor Kei-ichi Watanabe for valuable discussion.

2. Asymptotic surjective number

(2.1) This paper heavily depends on [HS].

(2.2) Let R be a Noetherian commutative ring. Let $\operatorname{mod} R$ denote the category of finite R-modules.

(2.3) For $M, N \in \operatorname{mod} R$, we set

$$\operatorname{surj}_N^R(M) = \operatorname{surj}_N(M) := \sup\{n \in \mathbb{Z}_{\geq 0} \mid \text{There is a surjective } R\text{-linear map } M \to N^{\oplus n}\},$$

and call $\operatorname{surj}_N(M)$ the surjective number of M with respect to N. If $N = 0$, this is understood to be ∞.

Lemma 2.4. *Let $M, M', N \in \operatorname{mod} R$. Then we have the following.*

1 *If R' is any Noetherian R-algebra, then*

$$\operatorname{surj}_N^R(M) \leq \operatorname{surj}_{R' \otimes_R N}^{R'}(R' \otimes_R M).$$

2 *If $(R, \mathfrak{m})$ is local and $N \neq 0$, then $\operatorname{surj}_N^R(M) \leq \mu_R(M)/\mu_R(N)$, where $\mu_R = \ell_R(R/\mathfrak{m} \otimes_R ?)$ denotes the number of generators.*

3 *If $N \neq 0$, then $\operatorname{surj}_N(M) < \infty$, and is a non-negative integer.*

4 *If $N \neq 0$, then $\operatorname{surj}_N(M) + \operatorname{surj}_N(M') \leq \operatorname{surj}_N(M \oplus M')$.*

5 *If $N \neq 0$ and $r \geq 0$, then $r \operatorname{surj}_N(M) \leq \operatorname{surj}_N(M^{\oplus r})$.*

Proof. **1** If there is a surjective R-linear map $M \to N^{\oplus n}$, then there is a surjective R'-linear map $R' \otimes_R M \to (R' \otimes_R N)^{\oplus n}$, and hence $n \leq \operatorname{surj}_{R' \otimes_R N}^{R'}(R' \otimes_R M)$.

2 By **1**, $\operatorname{surj}_N^R(M) \leq \operatorname{surj}_{N/\mathfrak{m}N}^{R/\mathfrak{m}}(M/\mathfrak{m}M) \leq \mu_R(M)/\mu_R(N)$ by dimension counting.

3 Take $\mathfrak{m} \in \operatorname{supp}_R N$. Then

$$\operatorname{surj}_N^R(M) \leq \operatorname{surj}_{N_\mathfrak{m}}^{R_\mathfrak{m}}(M_\mathfrak{m}) \leq \mu_{R_\mathfrak{m}}(M_\mathfrak{m})/\mu_{R_\mathfrak{m}}(N_\mathfrak{m}) < \infty.$$

4 Let $n = \operatorname{surj}_N(M)$ and $n' = \operatorname{surj}_N(M')$. Then there are surjective R-linear maps $M \to N^{\oplus n}$ and $M' \to N^{\oplus n'}$. Summing them, we get a surjective map $M \oplus M' \to N^{\oplus(n+n')}$.

5 follows from **4**. ∎

(2.5) Let $N, M \in \operatorname{mod} R$. Assume that N is nonzero. We define

$$\operatorname{nsurj}_N(M; r) := \frac{1}{r} \operatorname{surj}_N(M^{\oplus r})$$

for $r \geq 1$.

Lemma 2.6. *Let $r \geq 1$, and $M, M', N \in \operatorname{mod} R$. Assume that $N \neq 0$. Then*

1 $\operatorname{nsurj}_N(M; 1) = \operatorname{surj}_N(M)$.

2 $\operatorname{nsurj}_N(M; kr) \geq \operatorname{nsurj}_N(M; r)$ *for* $k \geq 0$.

3 $\operatorname{nsurj}_N(M; r) \geq \operatorname{surj}_N(M) \geq 0$.

4 $\operatorname{nsurj}_N(M; r) + \operatorname{nsurj}_N(M'; r) \leq \operatorname{nsurj}_N(M \oplus M'; r)$.

5 *If $R \to R'$ is a homomorphism of Noetherian rings, then* $\operatorname{nsurj}_N(M; r) \leq \operatorname{nsurj}_{R' \otimes_R N}(R' \otimes_R M; r)$.

6 *If $(R, \mathfrak{m})$ is local,* $\operatorname{nsurj}_N(M; r) \leq \mu_R(M)/\mu_R(N)$. *In general,* $\operatorname{nsurj}_N(M; r)$ *is bounded.*

Proof. **1** is by definition.

2. $kr \operatorname{nsurj}_N(M; kr) = \operatorname{surj}_N(M^{\oplus kr}) \geq k \operatorname{surj}_N(M^{\oplus r})$ by Lemma 2.4, **5**. Dividing by kr, we get the desired inequality.

3. This is immediate by **1** and **2**.

4 follows from Lemma 2.4, **4**.

5 follows from Lemma 2.4, **1**.

6 The first assertion is by Lemma 2.4, **2**. The second assertion follows from the first assertion and **5** applied to $R \to R' = R_\mathfrak{m}$, where $\mathfrak{m}$ is any element of $\operatorname{supp}_R N$. □

Lemma 2.7. *Let $M, N \in \operatorname{mod} R$. Assume that $N \neq 0$. Then the limit*

$$\lim_{r \to \infty} \operatorname{nsurj}_N(M; r) = \lim_{r \to \infty} \frac{1}{r} \operatorname{surj}_N(M^{\oplus r})$$

exists.

We call the limit the *asymptotic surjective number of M with respect to N*, and denote it by $\operatorname{asn}_N(M)$.

Proof. As $\operatorname{nsurj}_N(M;r)$ is bounded, $S = \limsup_{r\to\infty} \operatorname{nsurj}_N(M;r)$ and $I = \liminf_{r\to\infty} \operatorname{nsurj}_N(M;r)$ exist. Assume for contradiction that the limt does not exist. Then $S > I$. Set $\varepsilon = (S-I)/2 > 0$.

There exists some $r_0 \geq 1$ such that $\operatorname{nsurj}_N(M;r_0) > S - \varepsilon/2$. Take $n_0 \geq 1$ sufficiently large so that $\operatorname{nsurj}_N(M;r_0)/n_0 < \varepsilon/2$. Let $r \geq r_0 n_0$, and set $n := \lfloor r/r_0 \rfloor$. Note that $nr_0 \leq r < (n+1)r_0$ and $n \geq n_0$.

Then

$$\begin{aligned}\operatorname{nsurj}_N(M;r) &\geq \frac{1}{(n+1)r_0}\operatorname{surj}_N(M^{\oplus nr_0}) \geq \frac{n}{(n+1)r_0}\operatorname{surj}_N(M^{\oplus r_0}) \\ &= (1-\frac{1}{n+1})\operatorname{nsurj}_N(M;r_0) \geq \operatorname{nsurj}_N(M;r_0) - \varepsilon/2 > S - \varepsilon.\end{aligned}$$

Hence

$$I \geq \inf_{r \geq r_0 n_0} \operatorname{nsurj}_N(M;r) \geq S - \varepsilon > S - 2\varepsilon = I,$$

and this is a contradiction. □

Lemma 2.8. *Let $M, M', N \in \operatorname{mod} R$, and $N \neq 0$. Then*

1 $\operatorname{asn}_N(M^{\oplus r}) = r \operatorname{asn}_N(M)$.

2 $0 \leq \operatorname{surj}_N(M) \leq \operatorname{nsurj}_N(M;r) \leq \operatorname{asn}_N(M)$ *for any* $r \geq 1$.

3 $\operatorname{asn}_N(M) + \operatorname{asn}_N(M') \leq \operatorname{asn}_N(M \oplus M')$.

Proof. **1**.

$$r^{-1}\operatorname{asn}_N(M^{\oplus r}) = \lim_{r'\to\infty} \frac{1}{rr'}\operatorname{surj}_N(M^{\oplus rr'}) = \operatorname{asn}_N(M).$$

2. $0 \leq \operatorname{surj}_N(M) \leq \operatorname{nsurj}_N(M;r)$ is Lemma 2.6, **3**. So taking the limit, $\operatorname{surj}_N(M) \leq \operatorname{asn}_N(M)$. So $\operatorname{surj}_N(M^{\oplus r}) \leq \operatorname{asn}_N(M^{\oplus r}) = r\operatorname{asn}_N(M)$. Dividing by r, $\operatorname{nsurj}_N(M;r) \leq \operatorname{asn}_N(M)$. □

Lemma 2.9. *Let k be a field, and V a k-vector space, and $n \geq 0$. Assume that $\dim_k V \leq n$. Let Γ be a set of subspaces of V such that $\sum_{U\in\Gamma} U = V$. Then there exist some $U_1, \ldots, U_{n'} \in \Gamma$ with $n' \leq n$ such that $U_1 + \cdots + U_{n'} = V$.*

Proof. Trivial. □

Lemma 2.10. *Let k be a field, V a k-vector space, and Γ a set of subspaces of V. Let W and W' be subspaces of V such that $W + W' = V$. Assume that $W' \subset \sum_{U\in\Gamma} U$. If $\dim_k W' \leq n$, then there exist some $U_1, \dots, U_{n'} \in \Gamma$ with $n' \leq n$ such that $W + U_1 + \cdots + U_{n'} = V$.*

Proof. Apply Lemma 2.9 to the vector space V/W. □

Lemma 2.11. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $M, M', N \in \operatorname{mod} R$ with $N \neq 0$. Then*

$$\operatorname{surj}_N(M') \leq \operatorname{surj}_N(M \oplus M') - \operatorname{surj}_N(M) \leq \mu_R(M').$$

Proof. The first inequality is Lemma 2.4, **4**. We prove the second inequality. Let $m = \operatorname{surj}_N(M \oplus M')$ and $n = \mu_R(M')$. There is a surjective map $\varphi : M \oplus M' \to N^{\oplus m}$. Let $N_i = N$ be the ith summand of $N^{\oplus m}$. Let $\bar{?}$ denote the functor $R/\mathfrak{m}$. Set $V = \bar{N}^{\oplus m}$, $W = \bar{\varphi}(\bar{M})$, and $W' = \bar{\varphi}(\bar{M}')$. Then by Lemma 2.10, there exists some index set $I \subset \{1, 2, \dots, m\}$ such that $\#I \leq n$ and $W + \sum_{i\in I} \bar{N}_i = V$. By Nakayama's lemma, $\varphi(M) + \sum_{i\in I} N_i = N^{\oplus m}$. This shows that

$$M \hookrightarrow M \oplus M' \xrightarrow{\varphi} N^{\oplus m} \to N^{\oplus m}/\sum_{i\in I} N_i \cong N^{\oplus(m-\#I)}$$

is surjective. Hence $\operatorname{surj}_N(M) \geq m - \#I \geq m - n$, and the result follows. □

(2.12) Let $(R, \mathfrak{m})$ be a Henselian local ring. Let $\mathcal{C} := \operatorname{mod} R$. As in [HS], we define

$$[\mathcal{C}] := (\bigoplus_{M\in\mathcal{C}} \mathbb{Z} \cdot M)/(M - M_1 - M_2 \mid M \cong M_1 \oplus M_2),$$

and $[\mathcal{C}]_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} [\mathcal{C}]$. In [HS], $[\mathcal{C}]_{\mathbb{R}}$ is also written as $\Theta^\wedge(R)$ or $\Theta(R)$ (considering that R is trivially graded). In this paper, we write it as $\Theta(R)$. For $M \in \mathcal{C}$, we denote by $[M]$ the class of M in $\Theta(R)$. For an isomorphism class N of modules, $[N]$ is a well-defined element of $\Theta(R)$. Let $\operatorname{Ind}(R)$ denote the set of isomorphism classes of indecomposable modules in $\mathcal{C}$. The set $[\operatorname{Ind}(R)] := \{[M] \mid M \in \operatorname{Ind}(R)\}$ is an $\mathbb{R}$-basis of $\Theta(R) = [\mathcal{C}]_{\mathbb{R}}$. So $\alpha \in \Theta(R)$ can be written $\alpha = \sum_{M\in\operatorname{Ind}(R)} c_M[M]$ with $c_M \in \mathbb{R}$ uniquely. We say that $\alpha \geq 0$ if $c_M \geq 0$ for any $M \in \operatorname{Ind}(R)$. For $\alpha, \beta \in \Theta(R)$, we define $\alpha \geq \beta$ if $\alpha - \beta \geq 0$. This gives an ordering on $\Theta(R)$.

(2.13) For $\alpha = \sum_{M \in \mathrm{Ind}(R)} c_M[M] \in \Theta(R)$, we define

$$\langle \alpha \rangle := \sum_{M \in \mathrm{Ind}(R)} \max(0, \lfloor c_M \rfloor)[M].$$

So there exists some $M_\alpha \in \mathcal{C}$, unique up to isomorphisms, such that $\langle \alpha \rangle = [M_\alpha]$. For $N \in \mathrm{mod}\, R$ with $N \neq 0$, we define $\mathrm{surj}_N\, \alpha$ to be $\mathrm{surj}_N\, M_\alpha$.

(2.14) For $\alpha = \sum_{M \in \mathrm{Ind}(R)} c_M M \in \Theta(R)$, we define $\mathrm{supp}\, \alpha = \{M \in \mathrm{Ind}(R) \mid c_M > 0\}$. We define $Y(\alpha) = \bigoplus_{W \in \mathrm{supp}\, \alpha} W$ and $\nu(\alpha) := \mu_R(Y(\alpha))$.

Lemma 2.15. *Let $N \in \mathrm{mod}\, R$, $N \neq 0$, and $\alpha, \beta \in \Theta(R)$.*

1 *If $\alpha, \beta \geq 0$, then $0 \leq \mathrm{surj}_N\, \alpha \leq \mathrm{surj}_N(\alpha + \beta) - \mathrm{surj}_N\, \beta$.*

2 $|\mathrm{surj}_N\, \alpha - \mathrm{surj}_N\, \beta| \leq \|\alpha - \beta\| + \nu(\inf\{\alpha, \beta\})$.

Proof. **1**. As $\alpha, \beta \geq 0$, we have that $\langle \alpha \rangle + \langle \beta \rangle \leq \langle \alpha + \beta \rangle$. So by Lemma 2.4, **4**, $\mathrm{surj}_N\, \alpha + \mathrm{surj}_N\, \beta \leq \mathrm{surj}_N(\langle \alpha + \beta \rangle) \leq \mathrm{surj}_N(\alpha + \beta)$.

2. Replacing α by $\sup\{\alpha, 0\}$ and β by $\sup\{\beta, 0\}$, we may assume that $\alpha, \beta \geq 0$. Moreover, replacing α by $\sup\{\alpha, \beta\}$ and β by $\inf\{\alpha, \beta\}$, we may assume that $\alpha \geq \beta$. As we have $\langle \alpha \rangle - \langle \beta \rangle \leq \alpha - \beta + [Y(\beta)]$, by Lemma 2.11 we have that

$$\begin{aligned} \mathrm{surj}_N\, \alpha - \mathrm{surj}_N\, \beta &\leq \|\langle \alpha \rangle - \langle \beta \rangle\| \leq \|\alpha - \beta + [Y(\beta)]\| \\ &\leq \|\alpha - \beta\| + \|[Y(\beta)]\| = \|\alpha - \beta\| + \nu(\beta). \end{aligned}$$

This is what we wanted to prove. □

Lemma 2.16. *The limit*

$$\lim_{t \to \infty} \frac{1}{t} \mathrm{surj}_N(t\alpha)$$

exists for $N \in \mathrm{mod}\, R$, $N \neq 0$ and $\alpha \in \Theta(R)$.

We denote the limit by $\mathrm{asn}_N(\alpha)$.

Proof. Replacing α by $\sup\{0, \alpha\}$, we may assume that $\alpha \geq 0$. Let $\varepsilon > 0$. We can take $W \in \mathrm{mod}\, R$ and an integer $n > 0$ such that $\alpha - n^{-1}[W] \geq 0$ and $\|\alpha - n^{-1}[W]\| < \varepsilon/8$. As $\mathrm{asn}_N\, W$ exists, there exists some $r_0 \geq 1$ such that for any $r \geq r_0$, $|\mathrm{nsurj}_N(W; r) - \mathrm{asn}_N\, W| < n\varepsilon/8$. Set $R :=$

$\max\{r_0 n, 16\mu_R(W)/\varepsilon, 8n\|\alpha\|/\varepsilon\}$. Let $t > R$. Let $r := \lfloor t/n \rfloor$. Then $0 \leq t - rn < n$ and $r \geq r_0$. We have

$$\begin{aligned}
|t^{-1}\operatorname{surj}_N(t\alpha) - n^{-1}\operatorname{asn}_N W| &\leq t^{-1}|\operatorname{surj}_N(t\alpha) - \operatorname{surj}_N(W^{\oplus r})| \\
&\quad + ((rn)^{-1} - t^{-1})\operatorname{surj}_N(W^{\oplus r}) + |(rn)^{-1}\operatorname{surj}_N(W^{\oplus r}) - n^{-1}\operatorname{asn}_N W| \\
&< t^{-1}\|t\alpha - r[W]\| + t^{-1}\mu_R(W) + (rt)^{-1}\mu_R(W^{\oplus r}) + \varepsilon/8 \\
&\leq (n/t)\|\alpha\| + (nr/t)\|\alpha - n^{-1}[W]\| + \varepsilon/16 + \varepsilon/16 + \varepsilon/8 \\
&< \varepsilon/8 + \varepsilon/8 + \varepsilon/16 + \varepsilon/16 + \varepsilon/8 = \varepsilon/2.
\end{aligned}$$

So for $t_1, t_2 > R$,

$$|t_1^{-1}\operatorname{surj}_N(t_1\alpha) - t_2^{-1}\operatorname{surj}_N(t_2\alpha)| < \varepsilon,$$

and $\lim_{t\to\infty} t^{-1}\operatorname{surj}_N(t\alpha)$ exists, as desired. □

Lemma 2.17. *Let $\alpha, \beta \in \Theta(R)$ and $N \in \operatorname{mod} R$ with $N \neq 0$.*

1 *For $k \geq 0$, we have $\operatorname{asn}_N(k\alpha) = k\operatorname{asn}_N(\alpha)$.*

2 *For $k \geq 0$, $0 \leq \operatorname{surj}_N(k\alpha) \leq k\operatorname{asn}_N(\alpha) \leq k\|\alpha\|/\mu_R(N)$.*

3 *If $\alpha, \beta \geq 0$, then $\operatorname{asn}_N(\alpha + \beta) \geq \operatorname{asn}_N(\alpha) + \operatorname{asn}_N(\beta)$.*

4 $|\operatorname{asn}_N(\alpha) - \operatorname{asn}_N(\beta)| \leq \|\alpha - \beta\|$.

5 asn_N *is continuous.*

Proof. **1**. If $k = 0$, then both-hand sides are zero, and the assertion is clear. So we may assume that $k > 0$. Then

$$\operatorname{asn}_N(k\alpha) = \lim_{t\to\infty}\frac{1}{t}\operatorname{surj}(tk\alpha) = k\lim_{t\to\infty}\frac{1}{tk}\operatorname{surj}(tk\alpha) = k\operatorname{asn}_N(\alpha).$$

2. We may assume that $k > 0$. By **1**, replacing $k\alpha$ by α, we may assume that $k = 1$. Replacing α by $\sup\{0, \alpha\}$, we may assume that $\alpha \geq 0$. For $n \geq 0$, $n\langle\alpha\rangle \leq \langle n\alpha\rangle$. Hence, $n\operatorname{surj}_N(\alpha) \leq \operatorname{surj}_N(n\langle\alpha\rangle) \leq \operatorname{surj}_N(n\alpha)$. So $\operatorname{surj}_N(\alpha) \leq n^{-1}\operatorname{surj}_N(n\alpha)$. Passing to the limit, $\operatorname{surj}_N(\alpha) \leq \operatorname{asn}_N(\alpha)$. Similarly,

$$\frac{1}{n}\operatorname{surj}_N(n\alpha) \leq \frac{\|\langle n\alpha\rangle\|}{n\mu_R(N)} \leq \frac{\|n\alpha\|}{n\mu_R(N)} = \frac{\|\alpha\|}{\mu_R(N)}.$$

Passing to the limit, $\operatorname{asn}_N(\alpha) \leq \frac{\|\alpha\|}{\mu_R(N)}$, as desired.

3. By Lemma 2.15, **1**, for $t > 0$,

$$\frac{1}{t}\operatorname{surj}_N(t\alpha) + \frac{1}{t}\operatorname{surj}_N(t\beta) \leq \frac{1}{t}\operatorname{surj}_N(t(\alpha+\beta)).$$

Passing to the limit, $\operatorname{asn}_N(\alpha) + \operatorname{asn}_N(\beta) \leq \operatorname{asn}_N(\alpha+\beta)$.

4. By Lemma 2.15, **2**,

$$\begin{aligned}|\frac{1}{t}\operatorname{surj}_N(t\alpha) - \frac{1}{t}\operatorname{surj}_N(t\beta)| \leq \frac{1}{t}(\|t(\alpha-\beta)\| + \nu(\inf\{t\alpha, t\beta\}))\\ = \|\alpha-\beta\| + \nu(\inf\{\alpha,\beta\})/t.\end{aligned}$$

Passing to the limit, $|\operatorname{asn}_N(\alpha) - \operatorname{asn}_N(\beta)| \leq \|\alpha-\beta\|$, as desired.

5 is an immediate consequence of **4**. □

3. Sannai's dual F-signature

(3.1) In this section, let p be a prime number, and $(R, \mathfrak{m}, k)$ be an F-finite local ring of characteristic p of dimension d. Let $\mathfrak{d} = \log_p[k : k^p]$, and $\delta = d + \mathfrak{d}$.

(3.2) In [San], for $M \in \operatorname{mod} R$, Sannai defined the dual F-signature of M by

$$s_R(M) = s(M) := \limsup_{e\to\infty} \frac{\operatorname{surj}_M({}^eM)}{p^{\delta e}}.$$

$s(R)$ is the (usual) F-signature [HL], which is closely related to the strong F-regularity of R [AL]. While $s(\omega_R)$ measures the F-rationality of R, provided R is Cohen–Macaulay.

Theorem 3.3 ([San, (3.16)]). *R is F-rational if and only if R is Cohen–Macaulay and $s(\omega_R) > 0$.*

Now we connect the F-limit defined in [HS] with dual F-signature.

Theorem 3.4. *Let R be Henselian, and $M \in \operatorname{mod} R$. Assume that the F-limit*

$$\operatorname{FL}([M]) = \lim_{e\to\infty} \frac{1}{p^{\delta e}}[{}^eM] \in \Theta(R)$$

(*see* [HS]) *exists. Then*

$$s_R(M) = \lim_{e\to\infty} \frac{\operatorname{surj}_M({}^eM)}{p^{\delta e}} = \operatorname{asn}_M(\operatorname{FL}([M])).$$

Proof. By Lemma 2.15,

$$p^{-\delta e}|\operatorname{surj}_M(p^{\delta e}\operatorname{FL}([M])) - \operatorname{surj}_M([^e M])| \\ \leq \|\operatorname{FL}([M]) - p^{-\delta e}[^e M]\| + p^{-\delta e}\nu(\operatorname{supp}(\operatorname{FL}([M]))).$$

Taking the limit $e \to \infty$, we get the desired result. □

Corollary 3.5. *Let the assumption be as in the theorem. Then the following are equivalent.*

1 $s(M) > 0$.

2 *For any* $N \in \operatorname{mod} R$ *such that* $\operatorname{supp}([N]) = \operatorname{supp}(\operatorname{FL}(M))$, *there exists some* $r \geq 1$ *and a surjective* R*-linear map* $N^{\oplus r} \to M$.

3 *There exist some* $N \in \operatorname{mod} R$ *such that* $\operatorname{supp}([N]) \subset \operatorname{supp}(\operatorname{FL}(M))$ *and a surjective* R*-linear map* $N \to M$.

Proof. **1⇒2**. As $\operatorname{asn}_M(\operatorname{FL}(M)) > 0$, there exists some $t > 0$ such that $\operatorname{surj}_M(t\operatorname{FL}(M)) > 0$. By the choice of N, there exists some $r \geq 1$ such that $r[N] \geq t\operatorname{FL}(M)$ and so $\operatorname{surj}_M N^{\oplus r} \geq \operatorname{surj}_M(t\operatorname{FL}(M)) > 0$.

2⇒3. Let $N = W_1 \oplus \cdots \oplus W_r$, where $\{W_1, \ldots, W_r\} = \operatorname{supp}(\operatorname{FL}(M))$. Then there exists some $r \geq 1$ and a surjective R-linear map $N^{\oplus r} \to M$, and $\operatorname{supp}[N^{\oplus r}] \subset \operatorname{supp}(\operatorname{FL}(M))$.

3⇒1. By the choice of N, there exists some $k > 0$ such that $k\operatorname{FL}(M) \geq [N]$. Then $s(M) = \operatorname{asn}_M(\operatorname{FL}(M)) \geq k^{-1}\operatorname{asn}_M[N] \geq k^{-1}\operatorname{surj}_M[N] > 0$. □

4. The dual F-signature of the ring of invariants

Utilizing the result in [HS] and the last section, we give a criterion for the condition $s(\omega_{\hat{A}}) > 0$ for the ring of invariants A, where $\hat{A}$ is the completion.

(4.1) Let k be an algebraically closed field, $V = k^d$, G a finite subgroup of $GL(V)$. In this section, we assume that G does not have a pseudo-reflection, where we say that $g \in GL(V)$ is a pseudo-reflection if $\operatorname{rank}(g - 1_V) = 1$. Let $v_1, \ldots, v_d$ be a fixed k-basis of V. Let $B := \operatorname{Sym} V = k[v_1, \ldots, v_d]$, and $A = B^G$. Let $\mathfrak{m}$ and $\mathfrak{n}$ be the irrelevant ideals of A and B, respectively. Let $\hat{A}$ and $\hat{B}$ be the completion of A and B, respectively.

For a G-module W, we define $M_W := (B \otimes_k W)^G$. Let $k = V_0, V_1, \ldots, V_n$ be the irreducible representations of G. Let $P_i \to V_i$ be the projective cover.

Set $M_i := M_{P_i} = (B \otimes_k P_i)^G$. For a finite dimensional G-module W, $\det_W$ denote the determinant representation $\bigwedge^{\dim W} W$ of W. Let $V_\nu = \det_V$ be the determinant representation of V.

Lemma 4.2. *The canonical module ω_A of A is isomorphic to $M_\nu = M_{\det_V}$.*

Proof. See [Has2, (14.28)] and references therein. □

Lemma 4.3. *Let Λ be a selfinjective finite dimensional k-algebra, L a simple (left) Λ-module, and $h : P \to L$ its projective cover. Let M be a finitely generated indecomposable Λ-module. Then the following are equivalent.*

1 $\operatorname{Ext}^1_\Lambda(M, \operatorname{rad} P) = 0$.

2 $h_* : \operatorname{Hom}_\Lambda(M, P) \to \operatorname{Hom}_\Lambda(M, L)$ *is surjective.*

3 *M is either projective, or $M/\operatorname{rad} M$ does not contain L.*

Proof. **1**⇔**2**. This is because

$$\operatorname{Hom}_\Lambda(M, P) \xrightarrow{h_*} \operatorname{Hom}_\Lambda(M, L) \to \operatorname{Ext}^1_\Lambda(M, \operatorname{rad} P) \to \operatorname{Ext}^1_\Lambda(M, P)$$

is exact and $\operatorname{Ext}^1_\Lambda(M, P) = 0$ (since P is injective).

2⇒**3**. Assume the contrary. Then as $M/\operatorname{rad} M$ contains L, there is a surjective map $M \to L$. By assumption, this map lifts to $M \to P$, and this is surjective by Nakayama's lemma. As P is projective, this map splits. As M is indecomposable, $M \cong P$, and this is a contradiction.

3⇒**2**. If M is projective, then h_* is obviously surjective. If $M/\operatorname{rad} M$ does not contain L, then $\operatorname{Hom}_\Lambda(M, L) = 0$, and h_* is obviously surjective. □

Theorem 4.4. *Let p divide the order $|G|$ of G. Then the following are equivalent.*

1 $s(\omega_{\hat{A}}) > 0$.

2 *The canonical map $M_\nu \to M_{V_\nu} = \omega_A$ is surjective.*

3 $H^1(G, B \otimes_k \operatorname{rad} P_\nu) = 0$.

4 *For any non-projective finitely generated indecomposable G-summand M of B, M does not contain $\det_V^{-1}$, the k-dual of $\det_V$.*

If these conditions hold, then $s(\omega_{\hat{A}}) \geq 1/|G|$.

Proof. We prove the equivalence of **2** and **3** first. Let $B = \bigoplus_j N_j$ be a decomposition into finitely generated indecomposable G-modules. Such a decomposition exists, since B is a direct sum of finitely generated G-modules. The map $M_\nu \to M_{V_\nu}$ in **2** is the map

$$(B \otimes P_\nu)^G \to (B \otimes \det_V)^G$$

induced by the projective cover $P_\nu \to \det_V$. By the isomorphism $\operatorname{Ext}^i_G(N_j^*, ?) \cong H^i(G, N_j \otimes ?)$, this map can be identified with the sum of

$$\operatorname{Hom}_G(N_j^*, P_\nu) \to \operatorname{Hom}_G(N_j^*, \det_V).$$

On the other hand, **3** is equivalent to say that $\operatorname{Ext}^1_G(N_j^*, \operatorname{rad} P_\nu) = 0$ for any j. So the equivalence **2**⇔**3** follows from Lemma 4.3.

Similarly, **4** is equivalent to say that each N_j^* is injective (or equivalently, projective, as kG is selfinjective) or $N_j^*/\operatorname{rad} N_j^* \cong (\operatorname{soc} N_j)^*$ does not contain $\det_V$. This is equivalent to say that N_j is either projective, or N_j (or equivalently, $\operatorname{soc} N_j$) does not contain $\det^{-1}$. So **4**⇔**2** follows from Lemma 4.3.

We prove **2**⇒**1**. As there is a surjective map $M_\nu \to \omega_A$ and

$$\operatorname{FL}([\omega_{\hat{A}}]) = \frac{1}{|G|} \sum_{i=0}^{n} (\dim V_i)[\hat{M}_i]$$

by [HS, (5.1)], $s(\omega_{\hat{A}}) > 0$ by Corollary 3.5. Moreover,

$$s(\omega_{\hat{A}}) = \operatorname{asn}_{\omega_{\hat{A}}}(\operatorname{FL}([\omega_{\hat{A}}])) \geq \frac{\dim V_\nu}{|G|} \operatorname{asn}_{\omega_{\hat{A}}}(\hat{M}_\nu) \geq \frac{1}{|G|} \operatorname{surj}_{\omega_A}(M_\nu) \geq \frac{1}{|G|},$$

and the last assertion has been proved.

We prove **1**⇒**2**. By [HS, (4.16)],

$$\operatorname{FL}([\omega_{\hat{A}}]) = \frac{1}{|G|}[\hat{B}].$$

So by Corollary 3.5, there is some $r > 0$ and a surjective map $h : \hat{B}^r \to \omega_{\hat{A}}$. By the equivalence $\gamma = (\hat{B} \otimes_{\hat{A}} ?)^{**} : \operatorname{Ref}(\hat{A}) \to \operatorname{Ref}(G, \hat{B})$ (see [HasN, (2.4)] and [HS, (5.4)]), there corresponds

$$\tilde{h} = \gamma(h) : (\hat{B} \otimes_k kG)^r \to \hat{B} \otimes_k \det.$$

As $\hat{B}\otimes_k kG$ is a projective object in the category of (G,B)-modules, $\tilde{h}$ factors through the surjection

$$\hat{B}\otimes_k P_\nu \to \hat{B}\otimes_k \det .$$

Returning to the category Ref $\hat{A}$, h factors through $\hat{M}_\nu = (\hat{B}\otimes_{\hat{A}} P_\nu)^G \to \omega_{\hat{A}}$. So this map must be surjective, and **2** follows. ☐

Corollary 4.5. *Assume that p divides $|G|$. If $s(\omega_{\hat{A}})>0$, then $\det_V^{-1}$ is not a direct summand of B.*

Proof. Being a one-dimensional representation, $\det_V^{-1}$ is not projective by assumption. Thus the result follows from **1⇒4** of the theorem. ☐

Lemma 4.6. *Let M and N be in* $\mathrm{Ref}(G,B)$. *There is a natural isomorphism*

$$\gamma : \mathrm{Hom}_A(M^G,N^G)\to \mathrm{Hom}_B(M,N)^G.$$

Proof. This is simply because $\gamma = (B\otimes_A ?)^{**} : \mathrm{Ref}(A)\to\mathrm{Ref}(G,B)$ is an equivalence, and $\mathrm{Hom}_B(M,N)^G = \mathrm{Hom}_{G,B}(M,N)$. ☐

Theorem 4.7. *A is F-rational if and only if the following three conditions hold.*

1 *A is Cohen–Macaulay.*

2 $H^1(G,B)=0$.

3 *$(B\otimes_k(I/k))^G$ is a maximal Cohen–Macaulay A-module, where I is the injective hull of k.*

Proof. If the order $|G|$ of G is not divisible by p, then A is F-rational, and the three conditions hold. So we may assume that $|G|$ is divisible by p.

Assume that A is F-rational. Then A is Cohen–Macaulay. As $s(\omega_{\hat{A}})>0$, we have that $H^1(G,B\otimes_k \mathrm{rad}\,P_\nu)=0$, and

$$(1)\qquad 0\to (B\otimes \mathrm{rad}\,P_\nu)^G \to (B\otimes P_\nu)^G \to (B\otimes \det_V)^G\to 0$$

is exact. As $M_\nu = (B\otimes P_\nu)^G$ is a direct summand of $B=M_{kG}=(B\otimes kG)^G$, it is a maximal Cohen–Macaulay module. As $(B\otimes\det)^G=\omega_A$, it is also a maximal Cohen–Macaulay module. So the canonical dual of the exact sequence (1) is still exact. As there is an identification

$$\mathrm{Hom}_A((B\otimes_k ?)^G,\omega_A) = \mathrm{Hom}_B(B\otimes_k ?, B\otimes_k \det_V)^G = (B\otimes_k ?^*\otimes_k \det_V)^G,$$

we get the exact sequence of maximal Cohen–Macaulay A-modules

(2) $$0 \to A \to (B \otimes_k P_\nu^* \otimes_k \det_V)^G \to (B \otimes_k (\operatorname{rad} P_\nu)^* \otimes_k \det_V)^G \to 0.$$

As $(\operatorname{rad} P_\nu)^* \otimes \det_V \cong I/k$, $(B \otimes (I/k))^G$ is maximal Cohen–Macaulay. As I is an injective G-module, $B \otimes_k I$ is so as a G-module, and hence $H^1(G, B \otimes_k I) = 0$. By the long exact sequence of the G-cohomology, we get $H^1(G, B) = 0$.

The converse is similar. Dualizing (2), we have that (1) is exact. □

Corollary 4.8. *If A is F-rational, then $H^1(G, k) = 0$.*

Proof. k is a direct summand of B, and $H^1(G, B) = 0$. □

Example 4.9. If $p = 2$ and $G = S_2$ or S_3, the symmetric groups, then $H^1(G, k) \neq 0$. So A is not F-rational, provided G does not have a pseudo-reflection.

5. An example of F-rational ring of invariants which are not F-regular

(5.1) Let p be an odd prime number, and k an algebraically closed field of characteristic p.

(5.2) Let us identify $\operatorname{Map}(\mathbb{F}_p, \mathbb{F}_p)^\times$ with the symmetric group S_p. We write $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$. Define

$$\begin{aligned} G &:= \{\phi \in S_p \mid \exists a \in \mathbb{F}_p^\times\ \exists b \in \mathbb{F}_p\ \forall x \in \mathbb{F}_p\ \phi(x) = ax + b\} \subset S_p; \\ Q &:= \{\phi \in Q \mid \exists b \in \mathbb{F}_p\ \forall x \in \mathbb{F}_p\ \phi(x) = x + b\} \subset G; \\ \Gamma &:= \{\phi \in S_p \mid \exists a \in \mathbb{F}_p^\times\ \forall x \in \mathbb{F}_p\ \phi(x) = ax\} \subset G. \end{aligned}$$

G is a subgroup of S_p, Q is a normal subgroup of G, and Γ is a subgroup of G such that $G = Q \rtimes \Gamma$. Note that Q is cyclic of order p. Γ is cyclic of order $p - 1$. So G is of order $p(p-1)$.

(5.3) Let α be a primitive element of $\mathbb{F}_p$ (that is, a generator of the cyclic group $\mathbb{F}_p^\times$), and let $\tau \in \Gamma$ be the element given by $\tau(x) = \alpha x$. The only involution of Γ is $\tau^{(p-1)/2}$, the multiplication by -1. As a permutation, it is

$$(1\ (p-1))(2\ (p-2)) \cdots ((p-1)/2\ (p+1)/2),$$

which is a transposition if and only if $p = 3$. As Γ contains a Sylow 2-subgroup, a transposition of G, if any, is conjugate to an element of Γ, and it must be a transposition again. It follows that G has a transposition if and only if $p = 3$.

(5.4) Now let $G \subset S_p$ act on $P = k^p = \langle w_0, w_1, \ldots, w_{p-1} \rangle$ by the permutation action, that is, $\phi w_i = w_{\phi(i)}$ for $\phi \in G$ and $i \in \mathbb{F}_p$. $g \in G \subset GL(P)$ is a pseudo-reflection if and only if it is a transposition. So G has a pseudo-reflection if and only if $p = 3$.

Let $r \geq 1$, and set $V = P^{\oplus r}$. $G \subset GL(V)$ has a pseudo-reflection if and only if $p = 3$ and $r = 1$.

(5.5) Let $S = \operatorname{Sym} P$.

Lemma 5.6. *Let M be any finitely generated non-projective indecomposable G-summand of S. Then $M \cong k$.*

Proof. Let $\Omega = \{w^\lambda = w_0^{\lambda_0} \cdots w_{p-1}^{\lambda_{p-1}} \mid \lambda = (\lambda_0, \ldots, \lambda_{p-1}) \in \mathbb{Z}_{\geq 0}^p\}$ be the set of monomials of S. G acts on the set Ω. Let Θ be the set of orbits of this action of G on Ω. Let $Gw^\lambda \in \Theta$.

If $\lambda = (r, r, \ldots, r)$ for some $r \geq 0$, then $Gw^\lambda = \{w^\lambda\}$, and hence $(kG)w^\lambda \cong k$.

Otherwise, Q does not have a fixed point on the action on Gw^λ. As the order of Q is p, Q acts freely on Gw^λ. Hence $(kG)w^\lambda$ is kQ-free.

Since the order of $G/Q \cong \Gamma$ is $p-1$, the Lyndon–Hochschild–Serre spectral sequence collapses, and we have $H^i(G, M) \cong H^i(Q, M)^\Gamma$ for any G-module M. So a Q-injective (or equivalently, Q-projective) G-module is G-injective (or equivalently, G-projective).

As we have $S = \bigoplus_{\theta \in \Theta} k\theta$ as a G-module, S is a direct sum of G-projective modules and copies of k. Using Krull-Schmidt theorem, it is easy to see that $M \cong k$. $\square$

Lemma 5.7. *Let U and W be G-modules.*

1 *$kG \otimes_k W \cong kG \otimes_k W'$, where W' is the k-vector space W with the trivial G-action.*

2 *If U is G-projective, then $U \otimes_k W$ is G-projective.*

Proof. **1**. $g \otimes w \mapsto g \otimes g^{-1} w$ gives such an isomorphism.

2 follows from **1**. $\square$

(5.8) Let $B := \operatorname{Sym} V = \operatorname{Sym} P^{\oplus r} \cong S^{\otimes r}$.

Lemma 5.9. *Let M be any finitely generated non-projective indecomposable G-summand of B. Then $M \cong k$.*

Proof. Follows immediately from Lemma 5.6 and Lemma 5.7. □

Lemma 5.10. *Let k_- denote the sign representation. Then $\det_V \cong k_-$ if r is odd, and $\det_V \cong k$ if r is even. k_- is not isomorphic to k.*

Proof. As the determinant of a sign matrix is the signature of the permutation, $\det_P \cong k_-$. Hence $\det_V \cong (\det_P)^{\otimes r} \cong (k_-)^{\otimes r}$, and we get the desired result. The last assertion is clear, since $\tau = (x \mapsto \alpha x) \in \Gamma$ is a cyclic permutation of order $p-1$, and is an odd permutation. □

Theorem 5.11. *We have*

$$\operatorname{depth}_A = \min\{rp, 2(p-1)+r\}.$$

Hence A is Cohen–Macaulay if and only if $r \le 2$.

Proof. This is an immediate consequence of [Kem, (3.3)]. □

Theorem 5.12. *Let p, r, G, P, $V = P^{\oplus r}$, $B = \operatorname{Sym} V$ be as above, and $A := B^G$. Then*

1 *G is a finite subgroup of $GL(V)$ of order $p(p-1)$.*

2 *$G \subset GL(V)$ has a pseudo-reflection if and only if $p = 3$ and $r = 1$. If so, $G = S_3$ is the symmetric group acting regularly on $B = k[w_0, w_1, w_2]$ by permutations on w_0, w_1, w_2. The ring of invariants A is the polynomial ring. Otherwise, A is not weakly F-regular.*

3 *If $p \ge 5$ and $r = 1$, then A is F-rational, but not weakly F-regular.*

4 *If $r = 2$, then A is Gorenstein, but not F-rational.*

5 *If $r \ge 3$ and r is odd, then $s(\omega_{\hat{A}}) > 0$ but A is not Cohen–Macaulay.*

6 *If $r \ge 4$ and even, then A is quasi-Gorenstein, but not Cohen–Macaulay.*

Proof. We have already seen **1** and the first statement of **2**. If $p = 3$ and $r = 1$, then $G \subset S_3$ has order 6, and $G = S_3$. So A is the polynomial ring generated by the symmetric polynomials. Otherwise, as G does not have a pseudo-reflection and the order $|G|$ of G is divisible by p, A is not weakly F-regular, see [Bro], [Yas], and [HS, (5.8)].

The only non-projective finitely generated indecomposable G-summand of B is k by Lemma 5.9, and $\det_V^{-1} \subset k$ if and only if r is even by Lemma 5.10. Hence we have that $s(\omega_{\hat{A}}) > 0$ if and only if r is odd by Theorem 4.4.

3. A is not weakly F-regular by **2**. As $r = 1$ is odd, $s(\omega_{\hat{A}}) > 0$. On the other hand, A is Cohen–Macaulay by Theorem 5.11. Hence A is F-rational by Theorem 3.3.

4. By Theorem 5.11, A is Cohen–Macaulay. On the other hand, by Lemma 5.10, $\det_V \cong k$, and hence $\omega_A \cong (B \otimes_k \det_V)^G \cong B^G \cong A$ by Lemma 4.2. So A is Gorenstein. As A is Gorenstein but not weakly F-regular, it is not F-rational by [HH2, (4.7)].

5 and **6** are easy. □

References

[AL] I. Aberbach and G. Leuschke, The F-signature and strong F-regularity, *Math. Res. Lett.* **10** (2003), 51–56.

[Bro] A. Broer, The direct summand property in modular invariant theory, *Transform. Groups* **10** (2005), 5–27.

[Has2] M. Hashimoto, Equivariant class group. III. Almost principal fiber bundles, `arXiv:1503.02133v1`

[HasN] M. Hashimoto and Y. Nakajima, Generalized F-signature of invariant subrings, *J. Algebra* **443** (2015), 142–152.

[HS] M. Hashimoto and P. Symonds, The asymptotic behavior of Frobenius direct images of rings of invariants, `arXiv:1509.02592v1`

[HH2] M. Hochster and C. Huneke, F-regularity, test elements, and smooth base change, *Trans. Amer. Math. Soc.* **346** (1994), 1–62.

[HL] C. Huneke and G. Leuschke, Two theorems about maximal Cohen-Macaulay modules, *Math. Ann.* **324** (2002), 391–404.

[Kem] G. Kemper, The depth of invariant rings and cohomology, *J. Algebra* **245** (2001), 463–531.

[San] A. Sannai, Dual F-signature, *Internat. Math. Res. Notices* **162** (2013), 197–211.

[Yas] Pure subrings of regular local rings, endomorphism rings and Frobenius morphisms, *J. Algebra* **370** (2012), 15–31.