Good filtrations and F-purity of invariant subrings

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Abstract

Let k be an algebraically closed field of positive characteristic, G a reductive group over k, and V a finite dimensional G-module. Let B be a Borel subgroup of G, and U its unipotent radical. We prove that if $S = \operatorname{Sym} V$ has a good filtration, then S^U is F-pure.

Throughout this paper, p denotes a prime number. Let k be an algebraically closed field of characteristic p, and G a reductive group over k. Let B be a Borel subgroup of G, and U its unipotent radical. We fix a maximal torus T contained in B, and fix a base of the root system Σ of G so that B is negative. For any weight $\lambda \in X(T)$, we denote the induced module $\operatorname{ind}_B^G(\lambda)$ by $\nabla_G(\lambda)$. We denote the set of dominant weights by X^+. For $\lambda \in X^+$, we call $\nabla_G(\lambda)$ the dual Weyl module of highest weight λ. We say that a G-module W has a good filtration [1] if $H^1(G, W \otimes \nabla_G(\lambda)) = 0$ for any $\lambda \in X^+$.

Let V be a finite dimensional G-module, and $S = \operatorname{Sym} V$. The objective of this paper is to prove the following.

Theorem 1. *If S has a good filtration, then S^U is F-pure.*

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For a commutative ring R and an R-linear map $\varphi : M \to N$ of R-modules, we say that φ is pure or R-pure if $1_W \otimes \varphi : W \otimes_R M \to W \otimes_R N$ is injective for any R-module W. We say that φ is a split mono if there is an R-linear map $\psi : N \to M$ such that $\psi\varphi = 1_M$. A split mono is pure. A pure map $M \to N$ is a split mono if M/N is finitely presented [8, (5.2)]. A ring homomorphism $\varphi : R \to R'$ is said to be pure (resp. split) if it is pure (resp. a split mono) as an R-linear map. A commutative ring R of characteristic p is said to be F-pure (resp. F-finite) if the Frobenius map $F_R : R \to R$ given by $a \mapsto a^p$ is pure (resp. finite) as a ring homomorphism. F-purity was defined by Hochster–Roberts in 1970's [7] [8]. This notion is deeply connected with log-canonical singularity in characteristic zero [12]. An $\mathbb{F}_p$-scheme X is said to be Frobenius split if $\mathcal{O}_{X^{(1)}} \to F_*\mathcal{O}_X$ splits as an $\mathcal{O}_{X^{(1)}}$-linear map [10]. For the notation $X^{(1)}$, see [9, (I.9.2)]. For an F-finite ring R of characteristic p, R is F-pure if and only if $\operatorname{Spec} R$ is Frobenius split. If a nonnegatively graded F-finite noetherian ring R of characteristic p is F-pure, then $\operatorname{Proj} R$ is Frobenius split.

An F-finite noetherian ring R of characteristic p is said to be strongly F-regular if for any nonzerodivisor a of R, the $R^{(e)}$-linear map $aF^e : R^{(e)} \to R$ $(x \mapsto ax^{p^e})$ is $R^{(e)}$-split [4]. A strongly F-regular F-finite ring is F-regular in the sense of Hochster–Huneke [5], and hence it is Cohen–Macaulay normal ([6, (4.2)] and [11, (0.10)]). Under the same assumption as in Theorem 1, S^G is strongly F-regular [3], and in particular, Cohen–Macaulay.

Proof of Theorem 1. Let $\Gamma \to [G, G]$ be the universal covering. Then S has a good filtration as a Γ-module by [1, (3.1.3), (3.2.7)], and U is identified with the unipotent radical of a Borel subgroup of Γ. Thus without loss of generality, we may assume that G is semisimple and simply connected.

Let ρ denote the half sum of positive roots. Let St denote the first Steinberg module $\nabla_G((p-1)\rho)$. For a G-module W and $r \geq 0$, $W^{(r)}$ denotes the rth Frobenius twist of W [9, (I.9.10)]. Note that $W^{(r)} = W$ as an abelian group. $w \in W$, viewed as an element of $W^{(r)}$ is sometimes denoted by $w^{(r)}$. If R is a G-algebra, then $R^{(r)}$ is also a G-algebra, and the rth Frobenius map $F^r : R^{(r)} \to R$ is a G-algebra map [3].

Lemma 2. *There is a $(G, S^{(1)})$-linear splitting of the Frobenius map $1 \otimes F_S : St \otimes S^{(1)} \to St \otimes S$ given by $x \otimes s^{(1)} \mapsto x \otimes s^p$.*

Proof. The same proof as that of (4) in the proof of [3, Theorem 6] works ($r = 1$ and $d = 0$ there). □

Let $C := k[G]^U$, where U acts on $k[G]$ right regularly. Then $C = \bigoplus_{\lambda\in X^+} \nabla_G(\lambda)$, and it is an X^+-graded G-algebra.

Lemma 3. *There is a $(G, C^{(1)})$-linear splitting of the Frobenius map $1\otimes F_C : St\otimes C^{(1)} \to St\otimes C$ given by $x\otimes c^{(1)} \mapsto x\otimes c^p$.*

Proof. The product $St\otimes\nabla_G(\lambda)^{(1)} \to \nabla_G(p(\lambda+\rho)-\rho)$ is nonzero, since for $x\in St\setminus 0$ and $y\in\nabla_G(\lambda)\setminus 0$, $xy^p\neq 0$, as C is an integral domain. Since $St\otimes\nabla_G(\lambda)^{(1)} \cong \nabla_G(p(\lambda+\rho)-\rho)$ [9, (II.3.19)] and $\operatorname{End}_G(\nabla_G(\mu)) \cong k$ for each $\mu\in X^+$, the product $St\otimes\nabla_G(\lambda)^{(1)} \to \nabla_G(p(\lambda+\rho)-\rho)$ is an isomorphism. This shows that the product $m : St\otimes C^{(1)} \to C_{(p-1)\rho+pX^+}$ is an isomorphism, where for a subset Λ of X^+, $C_\Lambda := \bigoplus_{\lambda\in\Lambda}\nabla_G(\lambda)$.

Thus

$$St\otimes C \xrightarrow{\pi} St\otimes C_{pX^+} \xrightarrow{m'} C_{(p-1)\rho+pX^+} \xrightarrow{m^{-1}} St\otimes C^{(1)}$$

is the splitting of $1\otimes F_C$, where π is the projection, and m' is the product. ☐

Lemma 4. *There exists some $(G, (S\otimes C)^{(1)})$-linear splitting of the Frobenius map $1\otimes F : St\otimes (S\otimes C)^{(1)} \to St\otimes S\otimes C$.*

Proof. Follows immediately from Lemma 2 and Lemma 3. ☐

Proof of Theorem 1 (continued). Let $\Phi : St\otimes S\otimes C \to St\otimes (S\otimes C)^{(1)}$ be a $(G, (S\otimes C)^{(1)})$-linear splitting of $1\otimes F$ as in Lemma 4. Set $A := (S\otimes C)^G$, and consider the commutative diagram of $(G, A^{(1)})$-modules

$$\begin{array}{ccccc}
St\otimes A^{(1)} & \hookrightarrow & St\otimes (S\otimes C)^{(1)} & \xrightarrow{\mathrm{id}} & St\otimes (S\otimes C)^{(1)} \\
\downarrow{\scriptstyle 1\otimes F} & & \downarrow{\scriptstyle 1\otimes F} & \nearrow{\scriptstyle \Phi} & \\
St\otimes A & \hookrightarrow & St\otimes (S\otimes C). & &
\end{array}$$

Applying the functor $\operatorname{Hom}_G(St, ?)$ to this, we get a commutative diagram

$$\begin{array}{ccccc}
A^{(1)} & \xrightarrow{\mathrm{id}} & A^{(1)} & \xrightarrow{\mathrm{id}} & A^{(1)} \\
\downarrow{\scriptstyle F} & & \downarrow & \nearrow & \\
A & \longrightarrow & \operatorname{Hom}_G(St, St\otimes (S\otimes C)) & &
\end{array}$$

of $A^{(1)}$-modules, see [3, Proposition 1, **5**]. Thus $F : A^{(1)} \to A$ splits, and $A = (S\otimes C)^G$ is F-pure.

Finally, as in the proof of [2, (1.2)], $A = (S\otimes C)^G \cong S^U$ ($X = \operatorname{Spec} S$ and $H = U$ there). Thus S^U is F-pure. ☐

Corollary 5. *Under the same assumption as in Theorem 1,* $\operatorname{Proj} S^U$ *is Frobenius split.*

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