SALLY MODULES OF RANK ONE

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1. Introduction

This paper aims to give a structure theorem of Sally modules of rank one.

Let A be a Cohen-Macaulay local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. We assume the residue class field $k = A/\mathfrak{m}$ of A is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and choose a minimal reduction $Q = (a_1, a_2, \cdots, a_d)$ of I. Then we have integers $\{e_i = \mathrm{e}_i(I)\}_{0 \le i \le d}$ such that the equality

$$\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d$$

holds true for all $n \gg 0$. Let

$$R = \mathrm{R}(I) := A[It] \quad \text{and} \quad T = \mathrm{R}(Q) := A[Qt] \ \subseteq \ A[t]$$

denote, respectively, the Rees algebras of I and Q, where t stands for an indeterminate over A. We put

$$R' = \mathrm{R}'(I) := A[It, t^{-1}] \quad \text{and} \quad G = \mathrm{G}(I) := R'/t^{-1}R' \ \cong \ \bigoplus_{n \ge 0} I^n/I^{n+1}.$$

Let $B = T/\mathfrak{m}T$, which is the polynomial ring with d indeterminates over the field k. Following W. V. Vasconcelos [11], we then define

$$\mathrm{S}_Q(I) = IR/IT$$

and call it the Sally module of I with respect to Q. We notice that the Sally module $S = \mathrm{S}_Q(I)$ is a finitely generated graded T-module, since R is a module-finite extension of the graded ring T.

The Sally module S was introduced by W. V. Vasconcelos [11], where he gave an elegant review, in terms of his *Sally* module, of the works [8, 9, 10] of J. Sally about the structure of $\mathfrak{m}$-primary ideals I with interaction to the structure of the graded ring G and the Hilbert coefficients e_i's of I.

As is well-known, we have the inequality ([6])

$$e_1 \ge e_0 - \ell_A(A/I)$$

and C. Huneke [3] showed that $e_1 = e_0 - \ell_A(A/I)$ if and only if $I^2 = QI$ (cf. Corollary 2.3). When this is the case, both the graded rings G and $\mathrm{F}(I) = \bigoplus_{n\geq 0} I^n/\mathfrak{m}I^n$ are Cohen-Macaulay, and the Rees algebra R of I is also a Cohen-Macaulay ring, provided $d \geq 2$. Thus, the ideals I with $e_1 = e_0 - \ell_A(A/I)$ enjoy very nice properties.

J. Sally firstly investigated the second border, that is the ideals I satisfying the equality $e_1 = e_0 - \ell_A(A/I) + 1$ but $e_2 \neq 0$ (cf. [10, 11]). The present research is a continuation of [10, 11] and aims to give a complete structure theorem of the Sally module of an m-primary ideal I satisfying the equality $e_1 = e_0 - \ell_A(A/I) + 1$.

The main result of this paper is the following Theorem 1.1. Our contribution in Theorem 1.1 is the implication (1) $\Rightarrow$ (3), the proof of which is based on the new result that the equality $I^3 = QI^2$ holds true if $e_1 = e_0 - \ell_A(A/I) + 1$ (cf. Theorem 3.1).

Theorem 1.1. *The following three conditions are equivalent to each other.*

(1) $e_1 = e_0 - \ell_A(A/I) + 1$.

(2) $\mathfrak{m}S = (0)$ *and* $\operatorname{rank}_B S = 1$.

(3) $S \cong (X_1, X_2, \cdots, X_c)B$ *as graded* T*-modules for some* $0 < c \leq d$*, where* $\{X_i\}_{1\leq i\leq c}$ *are linearly independent linear forms of the polynomial ring* B.

When this is the case, $c = \ell_A(I^2/QI)$ *and* $I^3 = QI^2$*, and the following assertions hold true.*

(i) $\operatorname{depth} G \geq d - c$ *and* $\operatorname{depth}_T S = d - c + 1$.

(ii) $\operatorname{depth} G = d - c$*, if* $c \geq 2$.

(iii) *Suppose* $c < d$*. Then*

$$\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \binom{n+d-c-1}{d-c-1}$$

for all $n \geq 0$*. Hence*

$$e_i = \begin{cases} 0 & \text{if } i \neq c+1, \\ (-1)^{c+1} & \text{if } i = c+1 \end{cases}$$

for $2 \leq i \leq d$.

(iv) *Suppose* $c = d$*. Then*

$$\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1}$$

for all $n \geq 1$*. Hence* $e_i = 0$ *for* $2 \leq i \leq d$.

Thus Theorem 1.1 settles a long standing problem, although the structure of ideals I with $e_1 = e_0 - \ell_A(A/I) + 2$ or the structure of Sally modules S with $\mathfrak{m}S = (0)$ and $\operatorname{rank}_B S = 2$ remains unknown.

Let us now briefly explain how this paper is organized. We shall prove Theorem 1.1 in Section 3. In Section 2 we will pick up from the paper [1] some auxiliary results on Sally modules, all of which are known, but let us note them for the sake of the reader's convenience. In Section 4 we will construct one example in order to see the ubiquity of ideals I which satisfy condition (3) in Theorem 1.1.

In what follows, unless otherwise specified, let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim A > 0$. We assume that the field $k = A/\mathfrak{m}$ is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and let S be the Sally module of I with respect to a minimal reduction $Q = (a_1, a_2, \cdots, a_d)$ of I. We put $R = A[It], T = A[Qt], R' = A[It, t^{-1}]$, and $G = R'/t^{-1}R'$. Let

$$\tilde{I} = \bigcup_{n \geq 1} [I^{n+1} :_A I^n] = \bigcup_{n \geq 1} [I^{n+1} :_A (a_1^n, a_2^n, \cdots, a_d^n)]$$

denote the Ratliff-Rush closure of I, which is the largest $\mathfrak{m}$-primary ideal in A such that $I \subseteq \tilde{I}$ and $\mathrm{e}_i(\tilde{I}) = e_i$ for all $0 \leq i \leq d$ (cf. [7]). We denote by $\mu_A(*)$ the number of generators.

2. Auxiliary results

In this section let us firstly summarize some known results on Sally modules, which we need throughout this paper. See [1] and [11] for the detailed proofs.

The first two results are basic facts on Sally modules developed by Vasconcelos [11].

Lemma 2.1. *The following assertions hold true.*

(1) $\mathfrak{m}^\ell S = (0)$ *for integers* $\ell \gg 0$.

(2) *The homogeneous components* $\{S_n\}_{n \in \mathbb{Z}}$ *of the graded* T*-module* S *are given by*

$$S_n \cong \begin{cases} (0) & \text{if } n \leq 0, \\ I^{n+1}/IQ^n & \text{if } n \geq 1. \end{cases}$$

(3) $S = (0)$ *if and only if* $I^2 = QI$.

(4) *Suppose that* $S \neq (0)$ *and put* $V = S/MS$, *where* $M = \mathfrak{m}T + T_+$ *is the graded maximal ideal in* T. *Let* V_n $(n \in \mathbb{Z})$ *denote the homogeneous component of the finite-dimensional graded* T/M*-space* V *with degree* n *and put* $\Lambda = \{n \in \mathbb{Z} \mid V_n \neq (0)\}$. *Let* $q = \max \Lambda$. *Then we have* $\Lambda = \{1, 2, \cdots, q\}$ *and* $\mathrm{r}_Q(I) = q + 1$, *where* $\mathrm{r}_Q(I)$ *stands for the reduction number of* I *with respect to* Q.

(5) $S = TS_1$ *if and only if* $I^3 = QI^2$.

Proof. See [1, Lemma 2.1]. □

Proposition 2.2. *Let* $\mathfrak{p} = \mathfrak{m}T$. *Then the following assertions hold true.*

(1) $\mathrm{Ass}_T S \subseteq \{\mathfrak{p}\}$. *Hence* $\dim_T S = d$, *if* $S \neq (0)$.

(2) $\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - (e_0 - \ell_A(A/I))\cdot\binom{n+d-1}{d-1} - \ell_A(S_n)$ *for all* $n \geq 0$.

(3) *We have* $e_1 = e_0 - \ell_A(A/I) + \ell_{T_\mathfrak{p}}(S_\mathfrak{p})$. *Hence* $e_1 = e_0 - \ell_A(A/I) + 1$ *if and only if* $\mathfrak{m}S = (0)$ *and* $\mathrm{rank}_B S = 1$.

(4) *Suppose that* $S \neq (0)$. *Let* $s = \mathrm{depth}_T S$. *Then* $\mathrm{depth}\, G = s - 1$ *if* $s < d$. *S is a Cohen-Macaulay T-module if and only if* $\mathrm{depth}\, G \geq d - 1$.

Proof. See [1, Proposition 2.2]. □

Combining Lemma 2.1 (3) and Proposition 2.2, we readily get the following results of Northcott [6] and Huneke [3].

Corollary 2.3 ([3, 6]). *We have* $e_1 \geq e_0 - \ell_A(A/I)$. *The equality* $e_1 = e_0 - \ell_A(A/I)$ *holds true if and only if* $I^2 = QI$. *When this is the case,* $e_i = 0$ *for all* $2 \leq i \leq d$.

The following result is one of the keys for our proof of Theorem 1.1.

Theorem 2.4. *The following conditions are equivalent.*

(1) $\mathfrak{m}S = (0)$ *and* $\mathrm{rank}_B S = 1$.

(2) $S \cong \mathfrak{a}$ *as graded T-modules for some graded ideal* $\mathfrak{a}$ $(\neq B)$ *of* B.

Proof. We have only to show (1) ⇒ (2). Because $S_1 \neq (0)$ and $S = \sum_{n \geq 1} S_n$ by Lemma 2.1, we have $S \cong B(-1)$ as graded B-modules once S is B-free.

Suppose that S is not B-free. The B-module S is torsionfree, since $\mathrm{Ass}_T S = \{\mathfrak{m}T\}$ by Proposition 2.2 (1). Therefore, since $\mathrm{rank}_B S = 1$, we see $d \geq 2$ and $S \cong \mathfrak{a}(m)$ as graded B-modules for some integer m and some graded ideal $\mathfrak{a}$ $(\neq B)$ in B, so that we get the exact sequence

$$0 \to S(-m) \to B \to B/\mathfrak{a} \to 0$$

of graded B-modules. We may assume that $\mathrm{ht}_B\, \mathfrak{a} \geq 2$, since $B = k[X_1, X_2, \cdots, X_d]$ is the polynomial ring over the field $k = A/\mathfrak{m}$. We then have $m \geq 0$, since $\mathfrak{a}_{m+1} = [\mathfrak{a}(m)]_1 \cong S_1 \neq (0)$ and $\mathfrak{a}_0 = (0)$. We want to show $m = 0$.

Because $\dim B/\mathfrak{a} \leq d - 2$, the Hilbert polynomial of $B/\mathfrak{a}$ has degree at most $d - 3$. Hence

$$\begin{aligned} \ell_A(S_n) &= \ell_A(B_{m+n}) - \ell_A([B/\mathfrak{a}]_{m+n}) \\ &= \binom{m+n+d-1}{d-1} - \ell_A([B/\mathfrak{a}]_{m+n}) \\ &= \binom{n+d-1}{d-1} + m\binom{n+d-2}{d-2} + (\text{lower terms}) \end{aligned}$$

for $n \gg 0$. Consequently

$$
\begin{aligned}
\ell_A(A/I^{n+1}) &= \mathrm{e}_0\binom{n+d}{d} - (\mathrm{e}_0 - \ell_A(A/I))\cdot\binom{n+d-1}{d-1} - \ell_A(S_n) \\
&= \mathrm{e}_0\binom{n+d}{d} - (\mathrm{e}_0 - \ell_A(A/I)+1)\cdot\binom{n+d-1}{d-1} - m\binom{n+d-2}{d-2} \\
&\quad +(\text{lower terms})
\end{aligned}
$$

by Proposition 2.2 (2), so that we get $\mathrm{e}_2 = -m$. Thus $m = 0$, because $\mathrm{e}_2 \geq 0$ by Narita's theorem ([5]). $\square$

The following result will enable us to reduce the proof of Theorem 1.1 to the proof of the fact that $I^3 = QI^2$ if $e_1 = e_0 - \ell_A(A/I) + 1$.

Proposition 2.5. *Suppose $e_1 = e_0 - \ell_A(A/I) + 1$ and $I^3 = QI^2$. Let $c = \ell_A(I^2/QI)$. Then the following assertions hold true.*

(1) $0 < c \leq d$ *and* $\mu_B(S) = c$.
(2) $\operatorname{depth} G \geq d - c$ *and* $\operatorname{depth}_B S = d - c + 1$.
(3) $\operatorname{depth} G = d - c$, *if* $c \geq 2$.
(4) *Suppose $c < d$. Then* $\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \binom{n+d-c-1}{d-c-1}$ *for all* $n \geq 0$. *Hence*

$$e_i = \begin{cases} 0 & \text{if } i \neq c+1 \\ (-1)^{c+1} & \text{if } i = c+1 \end{cases}$$

for $2 \leq i \leq d$.
(5) *Suppose $c = d$. Then* $\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1}$ *for all* $n \geq 1$. *Hence* $e_i = 0$ *for* $2 \leq i \leq d$.

Proof. We have $\mathfrak{m}S = (0)$ and $\operatorname{rank}_B S = 1$ by Proposition 2.2 (3), while $S = TS_1$ since $I^3 = QI^2$ (cf. Lemma 2.1 (5)). Therefore by Theorem 2.4 we have $S \cong \mathfrak{a}$ as graded B-modules where $\mathfrak{a} = (X_1, X_2, \cdots, X_c)$ is an ideal in B generated by linear forms $\{X_i\}_{1 \leq i \leq c}$. Hence $0 < c \leq d$, $\mu_B(S) = c$, and $\operatorname{depth}_B S = d - c + 1$, so that assertions (1), (2), and (3) follow (cf. Proposition 2.2 (4)). Considering the exact sequence

$$0 \to S \to B \to B/\mathfrak{a} \to 0$$

of graded B-modules, we get

$$
\begin{aligned}
\ell_A(S_n) &= \ell_A(B_n) - \ell_A([B/\mathfrak{a}]_n) \\
&= \binom{n+d-1}{d-1} - \binom{n+d-c-1}{d-c-1}
\end{aligned}
$$

for all $n \geq 0$ (resp. $n \geq 1$), if $c < d$ (resp. $c = d$). Thus assertions (4) and (5) follow (cf. Proposition 2.2 (2)). $\square$

3. Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1. See Proposition 2.2 (3) for the equivalence of conditions (1) and (2) in Theorem 1.1. The implication (3) $\Rightarrow$ (2) is clear. So, we must show the implication (1) $\Rightarrow$ (3) together with the last assertions in Theorem 1.1. Suppose that $e_1 = e_0 - \ell_A(A/I) + 1$. Then, thanks to Theorem 2.4, we get an isomorphism

$$\varphi : S \to \mathfrak{a}$$

of graded B-modules, where $\mathfrak{a} \subsetneq B$ is a graded ideal of B. Notice that once we are able to show $I^3 = QI^2$, the last assertions of Theorem 1.1 readily follow from Proposition 2.5. On the other hand, since $\mathfrak{a} \cong S = BS_1$ (cf. Lemma 2.1 (5)), the ideal $\mathfrak{a}$ of B is generated by linearly independent linear forms $\{X_i\}_{1 \leq i \leq c}$ $(0 < c \leq d)$ of B and so, the implication (1) $\Rightarrow$ (3) in Theorem 1.1 follows. We have $c = \ell_A(I^2/QI)$, because $\mathfrak{a}_1 \cong S_1 = I^2/QI$ (cf. Lemma 2.1 (2)). Thus our Theorem 1.1 has been proven modulo the following theorem.

Theorem 3.1. *Suppose that $e_1 = e_0 - \ell_A(A/I) + 1$. Then $I^3 = QI^2$.*

Proof. We proceed by induction on d. Suppose that $d = 1$. Then S is B-free of rank one (recall that the B-module S is torsionfree; cf. Proposition 2.2 (1)) and so, since $S_1 \neq (0)$ (cf. Lemma 2.1 (3)), $S \cong B(-1)$ as graded B-modules. Thus $I^3 = QI^2$ by Lemma 2.1 (5).

Let us assume that $d \geq 2$ and that our assertion holds true for $d - 1$. Since the field $k = A/\mathfrak{m}$ is infinite, without loss of generality we may assume that a_1 is a superficial element of I. Let

$$\overline{A} = A/(a_1), \quad \overline{I} = I/(a_1), \text{ and } \quad \overline{Q} = Q/(a_1).$$

We then have $\mathrm{e}_i(\overline{I}) = e_i$ for all $0 \leq i \leq d-1$, whence

$$\mathrm{e}_1(\overline{I}) = \mathrm{e}_0(\overline{I}) - \ell_{\overline{A}}(\overline{A}/\overline{I}) + 1.$$

Therefore the hypothesis of induction on d yields $\overline{I}^3 = \overline{Q}\,\overline{I}^2$. Hence, because the element $a_1 t$ is a nonzerodivisor on G if $\operatorname{depth} G > 0$, we have $I^3 = QI^2$ in that case.

Assume that $\operatorname{depth} G = 0$. Then, thanks to Sally's technique ([10]), we also have $\operatorname{depth} \mathrm{G}(\overline{I}) = 0$. Hence $\ell_{\overline{A}}(\overline{I}^2/\overline{Q}\,\overline{I}) = d - 1$ by Proposition 2.5 (2), because $\mathrm{e}_1(\overline{I}) =$

$\mathrm{e}_0(\overline{I}) - \ell_{\overline{A}}(\overline{A}/\overline{I}) + 1$. Consequently, $\ell_A(S_1) = \ell_A(I^2/QI) \geq d-1$, because $\overline{I}^2/\overline{Q}\ \overline{I}$ is a homomorphic image of I^2/QI. Let us take an isomorphism

$$\varphi : S \to \mathfrak{a}$$

of graded B-modules, where $\mathfrak{a} \subsetneq B$ is a graded ideal of B. Then, since

$$\ell_A(\mathfrak{a}_1) = \ell_A(S_1) \geq d-1,$$

the ideal $\mathfrak{a}$ contains $d-1$ linearly independent linear forms, say $X_1, X_2, \cdots, X_{d-1}$ of B, which we enlarge to a basis $X_1, \cdots, X_{d-1}, X_d$ of B_1. Hence

$$B = k[X_1, X_2, \cdots, X_d],$$

so that the ideal $\mathfrak{a}/(X_1, X_2, \cdots, X_{d-1})B$ in the polynomial ring

$$B/(X_1, X_2, \cdots, X_{d-1})B = k[X_d]$$

is principal. If $\mathfrak{a} = (X_1, X_2, \cdots, X_{d-1})B$, then $I^3 = QI^2$ by Lemma 2.1 (5), since $S = BS_1$. However, because $\ell_A(I^2/QI) = \ell_A(\mathfrak{a}_1) = d-1$, we have $\operatorname{depth} G \geq 1$ by Proposition 2.5 (2), which is impossible. Therefore $\mathfrak{a}/(X_1, X_2, \cdots, X_{d-1})B \neq (0)$, so that we have

$$\mathfrak{a} = (X_1, X_2, \cdots, X_{d-1}, X_d^{\alpha})B$$

for some $\alpha \geq 1$. Notice that $\alpha = 1$ or $\alpha = 2$ by Lemma 2.1 (4). We must show that $\alpha = 1$.

Assume that $\alpha = 2$. Let us write, for each $1 \leq i \leq d$, $X_i = \overline{b_i t}$ with $b_i \in Q$, where $\overline{b_i t}$ denotes the image of $b_i t \in T$ in $B = T/\mathfrak{m}T$. Then $\mathfrak{a} = (\overline{b_1 t}, \overline{b_2 t}, \cdots, \overline{b_{d-1} t}, \overline{(b_d t)^2})$. Notice that

$$Q = (b_1, b_2, \cdots, b_d),$$

because $\{X_i\}_{1 \leq i \leq d}$ is a k-basis of B_1. We now choose elements $f_i \in S_1$ for $1 \leq i \leq d-1$ and $f_d \in S_2$ so that $\varphi(f_i) = X_i$ for $1 \leq i \leq d-1$ and $\varphi(f_d) = X_d^2$. Let $z_i \in I^2$ for $1 \leq i \leq d-1$ and $z_d \in I^3$ such that $\{f_i\}_{1 \leq i \leq d-1}$ and f_d are, respectively, the images of $\{z_i t\}_{1 \leq i \leq d-1}$ and $z_d t^2$ in S. We now consider the relations $X_i f_1 = X_1 f_i$ in S for $1 \leq i \leq d-1$ and $X_d^2 f_1 = X_1 f_d$, that is

$$b_i z_1 - b_1 z_i \in Q^2 I$$

for $1 \leq i \leq d-1$ and

$$b_d^2 z_1 - b_1 z_d \in Q^3 I.$$

Notice that

$$Q^3 = b_1 Q^2 + (b_2, b_3, \cdots, b_{d-1})^2 {\cdot} (b_2, b_3, \cdots, b_d) + b_d^2 Q$$

and write

$$b_d^2 z_1 - b_1 z_d = b_1\tau_1 + \tau_2 + b_d^2\tau_3$$

with $\tau_1 \in Q^2I$, $\tau_2 \in (b_2, b_3, \cdots, b_{d-1})^2{\cdot}(b_2, b_3, \cdots, b_d)I$, and $\tau_3 \in QI$. Then

$$b_d^2(z_1 - \tau_3) = b_1(\tau_1 + z_d) + \tau_2 \in (b_1) + (b_2, b_3, \cdots, b_{d-1})^2.$$

Hence $z_1 - \tau_3 \in (b_1) + (b_2, b_3, \cdots, b_{d-1})^2$, because the sequence $b_1, b_2, \cdots, b_d$ is A-regular. Let $z_1 - \tau_3 = b_1h + h'$ with $h \in A$ and $h' \in (b_2, b_3, \cdots, b_{d-1})^2$. Then since

$$b_1[b_d^2h - (\tau_1 + z_d)] = \tau_2 - b_d^2h' \in (b_2, b_3, \cdots, b_d)^3,$$

we have $b_d^2h - (\tau_1 + z_d) \in (b_2, b_3, \cdots, b_d)^3$, whence $b_d^2h \in I^3$.

We need the following.

Claim. *$h \notin I$ but $h \in \tilde{I}$. Hence $\tilde{I} \neq I$.*

Proof. If $h \in I$, then $b_1h \in QI$, so that $z_1 = b_1h + h' + \tau_3 \in QI$, whence $f_1 = 0$ in S (cf. Lemma 2.1 (2)), which is impossible. Let $1 \le i \le d-1$. Then

$$b_iz_1 - b_1z_i = b_i(b_1h + h' + \tau_3) - b_1z_i = b_1(b_ih - z_i) + b_i(h' + \tau_3) \in Q^2I.$$

Therefore, because $b_i(h' + \tau_3) \in Q^2I$, we get

$$b_1(b_ih - z_i) \in (b_1) \cap Q^2I.$$

Notice that

$$\begin{aligned}(b_1) \cap Q^2I &= (b_1) \cap [b_1QI + (b_2, b_3, \cdots, b_d)^2I]\\ &= b_1QI + [(b_1) \cap (b_2, b_3, \cdots, b_d)^2I]\\ &= b_1QI + b_1(b_2, b_3, \cdots, b_d)^2\\ &= b_1QI\end{aligned}$$

and we have $b_ih - z_i \in QI$, whence $b_ih \in I^2$ for $1 \le i \le d-1$. Consequently $b_i^2h \in I^3$ for all $1 \le i \le d$, so that $h \in \tilde{I}$, whence $\tilde{I} \neq I$. $\square$

Because $\ell_A(\tilde{I}/I) \ge 1$, we have

$$\begin{aligned}e_1 &= e_0 - \ell_A(A/I) + 1\\ &= \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}) + [1 - \ell_A(\tilde{I}/I)]\\ &\le \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I})\\ &\le \mathrm{e}_1(\tilde{I})\\ &= e_1,\end{aligned}$$

where $\mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}) \leq \mathrm{e}_1(\tilde{I})$ is the inequality of Northcott for the ideal $\tilde{I}$ (cf. Corollary 2.3). Hence $\ell_A(\tilde{I}/I) = 1$ and $\mathrm{e}_1(\tilde{I}) = \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I})$, so that

$$\tilde{I} = I + (h) \quad \text{and} \quad \tilde{I}^2 = Q\tilde{I}$$

by Corollary 2.3 (recall that Q is a reduction of $\tilde{I}$ also). We then have, thanks to [4, Corollary 3.1], that $I^3 = QI^2$, which is a required contradiction. This completes the proof of Theorem 1.1 and that of Theorem 3.1 as well. $\square$

4. An example

Lastly we construct one example which satisfies condition (3) in Theorem 1.1. Our goal is the following. See [2, Section 5] for the detailed proofs.

Theorem 4.1. *Let $0 < c \leq d$ be integers. Then there exists an $\mathfrak{m}$-primary ideal I in a Cohen-Macaulay local ring $(A, \mathfrak{m})$ such that*

$$d = \dim A, \quad \mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 1, \quad \text{and} \quad c = \ell_A(I^2/QI)$$

for some reduction $Q = (a_1, a_2, \cdots, a_d)$ of I.

To construct necessary examples we may assume that $c = d$.

Let m, $d > 0$ be integers. Let

$$U = k[\{X_j\}_{1 \leq j \leq m}, Y, \{V_i\}_{1 \leq i \leq d}, \{Z_i\}_{1 \leq i \leq d}]$$

be the polynomial ring with $m + 2d + 1$ indeterminates over an infinite field k and let

$$\begin{aligned} \mathfrak{a} \;=\; & [(X_j \mid 1 \leq j \leq m) + (Y)]\cdot[(X_j \mid 1 \leq j \leq m) + (Y) + (V_i \mid 1 \leq i \leq d)] \\ & + (V_iV_j \mid 1 \leq i, j \leq d,\ i \neq j) + (V_i^2 - Z_iY \mid 1 \leq i \leq d). \end{aligned}$$

We put $C = U/\mathfrak{a}$ and denote the images of X_j, Y, V_i, and Z_i in C by x_j, y, v_i, and a_i, respectively. Then $\dim C = d$, since $\sqrt{\mathfrak{a}} = (X_j \mid 1 \leq j \leq m) + (Y) + (V_i \mid 1 \leq i \leq d)$. Let $M = C_+ := (x_j \mid 1 \leq j \leq m) + (y) + (v_i \mid 1 \leq i \leq d) + (a_i \mid 1 \leq i \leq d)$ be the graded maximal ideal in C. Let Λ be a subset of $\{1, 2, \cdots, m\}$. We put

$$J = (a_i \mid 1 \leq i \leq d) + (x_\alpha \mid \alpha \in \Lambda) + (v_i \mid 1 \leq i \leq d) \quad \text{and} \quad \mathfrak{q} = (a_i \mid 1 \leq i \leq d).$$

Then $M^2 = \mathfrak{q}M$, $J^2 = \mathfrak{q}J + \mathfrak{q}y$, and $J^3 = \mathfrak{q}J^2$, whence $\mathfrak{q}$ is a reduction of both M and J, and $a_1, a_2, \cdots, a_d$ is a homogeneous system of parameters for the graded ring C.

Let $A = C_M$, $I = JA$, and $Q = \mathfrak{q}A$. We are now interested in the Hilbert coefficients $e_i's$ of the ideal I as well as the structure of the associated graded ring and the Sally module of I. We then have the following, which shows that the ideal I is a required example.

Theorem 4.2. *The following assertions hold true.*

(1) *A is a Cohen-Macaulay local ring with* $\dim A = d$.

(2) *$S \cong B_+$ as graded T-modules, whence $\ell_A(I^2/QI) = d$.*

(3) $\mathrm{e}_0(I) = m + d + 2$ *and* $\mathrm{e}_1(I) = \sharp\Lambda + d + 1$.

(4) $\mathrm{e}_i(I) = 0$ *for all* $2 \leq i \leq d$.

(5) *G is a Buchsbaum ring with* $\operatorname{depth} G = 0$ *and* $\mathbb{I}(G) = d$.

Proof. See [2, Theorem 5.2] □

References

[1] S. Goto, K. Nishida, and K. Ozeki, *Sally modules of rank one*, Michigan Math. J. (to appear).

[2] S. Goto, K. Nishida, and K. Ozeki, *The structure of Sally modules of rank one*, Math. Res. Lett. (to appear).

[3] C. Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J., Vol **34**, 1987, 293–318.

[4] Y. Kinoshita, K. Nishida, K. Sakata, and, R. Shinya, *An upper bound on the reduction number of an ideal*, Preprint (2007), arXiv: math.AC/0711.4880.

[5] M. Narita, *A note on the coefficients of Hilbert characteristic functions in semi-regular rings*, Proc. Cambridge Philos. Soc. **59**, 1963, 269–275.

[6] D. G. Northcott, *A note on the coefficients of the abstract Hilbert function*, J. London Math. Soc., Vol **35**, 1960, 209–214.

[7] L. J. Ratliff and D. Rush, *Two notes on reductions of ideals*, Indiana Univ. Math. J. **27**, 1978, 929-934.

[8] J. D. Sally, *Cohen-Macaulay local rings of maximal embedding dimension*, J. Algebra, **56**, 1979, 168–183.

[9] J. D. Sally, *Tangent cones at Gorenstein singularities*, Composito Math. **40**, 1980, 167–175.

[10] J. D. Sally, *Hilbert coefficients and reduction number 2*, J. Alg. Geo. and Sing. **1**, 1992, 325–333.

[11] W. V. Vasconcelos, *Hilbert Functions, Analytic Spread, and Koszul Homology*, Contemporary Mathematics, Vol **159**, 1994, 410–422.

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