QUASI-SOCLE IDEALS IN LOCAL RINGS WITH GORENSTEIN TANGENT CONES

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1. Introduction

This talk aims at a study of quasi-socle ideals in a local ring with the Gorenstein tangent cone. Our purpose is to answer Question 1.1 below, of when the graded rings associated to the ideals are Cohen-Macaulay and/or Gorenstein rings, estimating their reduction numbers with respect to minimal reductions.

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $Q = (x_1, x_2, \cdots, x_d)$ be a parameter ideal in A and let $q \geq 1$ be an integer. We put $I = Q : \mathfrak{m}^q$ and refer to those ideals as quasi-socle ideals in A. Then one can ask the following questions, which are the main subject of the present research.

Question 1.1.
(1) Find the conditions under which $I \subseteq \overline{Q}$, where $\overline{Q}$ stands for the integral closure of Q.
(2) When $I \subseteq \overline{Q}$, estimate or describe the reduction number $\mathrm{r}_Q(I) = \min\ \{0 \leq n \in \mathbb{Z} \mid I^{n+1} = QI^n\}$ of I with respect to Q in terms of some invariants of Q or A.
(3) Clarify what kind of ring-theoretic properties of the graded rings associated to the ideal I

$$\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n,\ \mathrm{G}(I) = \bigoplus_{n\geq 0} I^n/I^{n+1},\ \text{and}\ \mathrm{F}(I) = \bigoplus_{n\geq 0} I^n/\mathfrak{m}I^n$$

enjoy.

In this talk we shall focus our attention on a certain special kind of quasi-socle ideals. We now assume that the tangent cone, that is the associated graded ring $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ of $\mathfrak{m}$, is a Gorenstein ring and that the maximal ideal $\mathfrak{m}$ contains a system $x_1, x_2, \cdots, x_d$ of elements such that the ideal $(x_1, x_2, \cdots, x_d)$ is a reduction of $\mathfrak{m}$. Let $a_1, a_2, \cdots, a_d$, and q be positive integers and we put $Q = (x_1^{a_1}, x_2^{a_2}, \cdots, x_d^{a_d})$ and $I = Q : \mathfrak{m}^q$. Let $\overline{A} = A/Q$, $\overline{\mathfrak{m}} = \mathfrak{m}/Q$, and $\overline{I} = I/Q$. Let $\rho = \max\ \{n \in \mathbb{Z} \mid \overline{\mathfrak{m}}^n \neq (0)\}$, that is the index of nilpotency of the ideal $\overline{\mathfrak{m}}$ and put $\ell = \rho + 1 - q$. We then have the following, which are the answers to Question 1.1 in our specific setting.

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Theorem 1.2. *The following three conditions are equivalent to each other.*

(1) $I \subseteq \overline{Q}$.
(2) $\mathfrak{m}^q I = \mathfrak{m}^q Q$.
(3) $\ell \geq a_i$ *for all* $1 \leq i \leq d$.

When this is the case, the following assertions hold true.

(i) $\mathrm{r}_Q(I) = \lceil \frac{q}{\ell} \rceil := \min \{n \in \mathbb{Z} \mid \frac{q}{\ell} \leq n\}$.
(ii) *The graded rings* $\mathrm{G}(I)$ *and* $\mathrm{F}(I)$ *are Cohen-Macaulay.*

Theorem 1.3. *Suppose that* $\ell \geq a_i$ *for all* $1 \leq i \leq d$*. Then we have the following.*

(i) $\mathrm{G}(I)$ *is a Gorenstein ring if and only if* $\ell \mid q$.
(ii) $\mathcal{R}(I)$ *is a Gorenstein ring if and only if* $q = (d-2)\ell$.

Our setting naturally contains the case where A is a regular local ring with $x_1, x_2, \cdots, x_d$ a regular system of parameters, the case where A is an abstract hypersurface with the infinite residue class field, and the case where $A = R_M$ is the localization of the homogeneous Gorenstein ring $R = k[R_1]$ over an infinite field $k = R_0$ at the irrelevant maximal ideal $M = R_+$. In Section 3 we will explore a few examples, including these three cases, in order to see how Theorems 1.2 and 1.3 work for the analysis of concrete examples. The proofs of Theorems 1.2 and 1.3 themselves shall be given in Section 2.

2. Proof of Theorems 1.2 and 1.3

The purpose of this section is to prove Theorems 1.2 and 1.3. First of all, let us restate our setting, which we shall maintain throughout this talk.

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. We assume that the associated graded ring $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ of $\mathfrak{m}$ is Gorenstein and that the maximal ideal $\mathfrak{m}$ contains a system $x_1, x_2, \cdots, x_d$ of elements which generates a reduction of $\mathfrak{m}$ (the latter condition is satisfied if the filed $A/\mathfrak{m}$ is infinite). Hence A is a Gorenstein ring and the initial forms $\{X_i\}_{1 \leq i \leq d}$ of $\{x_i\}_{1 \leq i \leq d}$ with respect to $\mathfrak{m}$ constitute a regular sequence in $\mathrm{G}(\mathfrak{m})$ and we have a canonical isomorphism

$$\mathrm{G}(\mathfrak{m}/(x_1, x_2, \cdots, x_d)) \cong \mathrm{G}(\mathfrak{m})/(X_1, X_2, \cdots, X_d)$$

of graded A-algebras ([VV]). Let $a_1, a_2, \cdots, a_d$, and q be positive integers and we put

$$Q = (x_1^{a_1}, x_2^{a_2}, \cdots, x_d^{a_d}) \quad \text{and} \quad I = Q : \mathfrak{m}^q.$$

Let $\overline{A} = A/Q$, $\overline{\mathfrak{m}} = \mathfrak{m}/Q$, and $\overline{I} = I/Q$. Then

$$\mathrm{G}(\overline{\mathfrak{m}}) \cong \mathrm{G}(\mathfrak{m})/({X_1}^{a_1}, {X_2}^{a_2}, \cdots, {X_d}^{a_d}),$$

whence $\mathrm{G}(\overline{\mathfrak{m}})$ is a Gorenstein ring. Let $\rho = \max\ \{n \in \mathbb{Z} \mid \overline{\mathfrak{m}}^n \neq (0)\}$, that is the index of nilpotency of the ideal $\overline{\mathfrak{m}}$, and we have $\rho = \mathrm{a}(\mathrm{G}(\overline{\mathfrak{m}})) = \mathrm{a}(\mathrm{G}(\mathfrak{m})) + \sum_{i=1}^{d} a_i$, where $\mathrm{a}(*)$ denotes the a-invariant of the corresponding graded ring ([GW, (3.1.4)]).

Let $\ell = \rho + 1 - q$. By [Wat] (see [O, Theorem 1.6] also) we then have the following.

Proposition 2.1. $(0) : \overline{\mathfrak{m}}^i = \overline{\mathfrak{m}}^{\rho+1-i}$ *for all* $i \in \mathbb{Z}$. *In particular* $\overline{I} = (0) : \overline{\mathfrak{m}}^q = \overline{\mathfrak{m}}^{\ell}$ *whence* $I = Q + \mathfrak{m}^{\ell}$.

The key for our proof of Theorem 1.2 is the following.

Lemma 2.2. *Suppose that* $\ell \geq a_i$ *for all* $1 \leq i \leq d$. *Then*

$$Q \cap \mathfrak{m}^{n\ell+m} \subseteq \mathfrak{m}^m Q I^{n-1}$$

for all $m \geq 0$ *and* $n \geq 1$.

Proof. We have

$$Q \cap \mathfrak{m}^{n\ell+m} = \sum_{i=1}^{d} x_i^{a_i} \mathfrak{m}^{n\ell+m-a_i}$$

since $x_1, x_2, \cdots, x_d$ is a super regular sequence with respect to $\mathfrak{m}$. Because

$$n\ell + m - a_i = (n-1)\ell + m + (\ell - a_i) \geq (n-1)\ell + m$$

for each $1 \leq i \leq d$, we get

$$\mathfrak{m}^{n\ell+m-a_i} \subseteq \mathfrak{m}^{(n-1)\ell+m} = \mathfrak{m}^m \cdot (\mathfrak{m}^{\ell})^{n-1}.$$

Therefore, since $\mathfrak{m}^{\ell} \subseteq I$ by Proposition 2.1, we have

$$\begin{aligned} Q \cap \mathfrak{m}^{n\ell+m} &= \sum_{i=1}^{d} x_i^{a_i} \mathfrak{m}^{n\ell+m-a_i} \\ &\subseteq \sum_{i=1}^{d} x_i^{a_i} \mathfrak{m}^m (\mathfrak{m}^{\ell})^{n-1} \\ &\subseteq \mathfrak{m}^m Q I^{n-1} \end{aligned}$$

as is claimed. □

Let us now prove Theorem 1.2.

Proof of Theorem 1.2. (2) $\Rightarrow$ (1) This is well-known. See [NR].

(3) $\Rightarrow$ (2) By Proposition 2.1 we get $\mathfrak{m}^q I = \mathfrak{m}^q Q + \mathfrak{m}^{q+\ell}$, whence $\mathfrak{m}^{q+\ell} \subseteq Q$, so that $\mathfrak{m}^{q+\ell} = Q \cap \mathfrak{m}^{q+\ell} \subseteq \mathfrak{m}^q Q$ by Lemma 2.2, because $\ell \geq a_i$ for all $1 \leq i \leq d$. Thus $\mathfrak{m}^q I = \mathfrak{m}^q Q$.

(1) $\Rightarrow$ (3) Let $1 \leq i \leq d$ be an integer. Then $x_i^{\ell} \in \mathfrak{m}^{\ell} \subseteq I \subseteq \overline{Q}$. Consequently, x_i^{ℓ} is integral over $Q = (x_1^{a_1}, x_2^{a_2}, \cdots, x_d^{a_d})$ so that, thanks to the monomial property of the regular sequence $\underline{x} = x_1, x_2, \cdots, x_d$, we get $\frac{\ell}{a_i} \geq 1$. Hence $\ell \geq a_i$ for all $1 \leq i \leq d$.

Let us now consider assertions (i) and (ii). Let $n \geq 1$ be an integer. Then $I^n = QI^{n-1} + \mathfrak{m}^{n\ell}$ since $I = Q + \mathfrak{m}^\ell$ (Proposition 2.1), so that

$$Q \cap I^n = QI^{n-1} + [Q \cap \mathfrak{m}^{n\ell}] \subseteq QI^{n-1}$$

because $Q \cap \mathfrak{m}^{n\ell} \subseteq QI^{n-1}$ by Lemma 2.2. Therefore $Q \cap I^n = QI^{n-1}$ for all $n \geq 1$, whence $\mathrm{G}(I)$ is a Cohen-Macaulay ring ([VV, Corollary 2.7]).

We will show that $\mathrm{r}_Q(I) = \lceil \frac{q}{\ell} \rceil$. Notice that

$$\mathrm{r}_Q(I) = \min\{n \geq 0 \mid I^{n+1} \subseteq Q\},$$

because $I^{n+1} = QI^n$ if and only if $I^{n+1} \subseteq Q$. Firstly, suppose that $I^{n+1} \subseteq Q$. We then have $\overline{\mathfrak{m}}^{(n+1)\ell} = (0)$ (recall that $\overline{I} = \overline{\mathfrak{m}}^\ell$), whence $(n+1)\ell \geq \rho + 1$. Therefore

$$n + 1 \geq \frac{\rho+1}{\ell} = \frac{q+\ell}{\ell} = \frac{q}{\ell} + 1,$$

because $\ell = \rho + 1 - q$, so that we have $n \geq \frac{q}{\ell}$.

If $n \geq \frac{q}{\ell}$, then $(n+1)\ell \geq (\frac{q}{\ell} + 1)\ell = q + \ell = \rho + 1$ and so $\overline{I}^{n+1} = \overline{\mathfrak{m}}^{(n+1)\ell} = (0)$, whence $I^{n+1} \subseteq Q$. Thus $\mathrm{r}_Q(I) = \lceil \frac{q}{\ell} \rceil$.

To see that $\mathrm{F}(I)$ is a Cohen-Macaulay ring, it suffices to show that

$$Q \cap \mathfrak{m}I^n = \mathfrak{m}QI^{n-1}$$

for all $n \geq 1$. By Lemma 2.2 we have

$$\begin{aligned} Q \cap \mathfrak{m}I^n &= Q \cap [\mathfrak{m}QI^{n-1} + \mathfrak{m}^{n\ell+1}] \\ &= \mathfrak{m}QI^{n-1} + [Q \cap \mathfrak{m}^{n\ell+1}] \\ &\subseteq \mathfrak{m}QI^{n-1}, \end{aligned}$$

whence $Q \cap \mathfrak{m}I^n = \mathfrak{m}QI^{n-1}$. $\square$

Assume that $\ell \geq a_i$ for all $1 \leq i \leq d$ and let Y_i's be the initial forms of $x_i^{a_i}$'s with respect to I. Then $Y_1, Y_2, \cdots, Y_d$ is a homogeneous system of parameters of $\mathrm{G}(I)$, since Q is a reduction of I (Theorem 1.2). It therefore constitutes a regular sequence in $\mathrm{G}(I)$, because $\mathrm{G}(I)$ is a Cohen-Macaulay ring by Theorem 1.2 (ii), so that we have a canonical isomorphism

$$\mathrm{G}(\overline{I}) \cong \mathrm{G}(I)/(Y_1, Y_2, \cdots, Y_d)$$

of graded A-algebras ([VV]). Hence $\mathrm{a}(\mathrm{G}(\overline{I})) = \mathrm{a}(\mathrm{G}(I)) + d$. Let r be the index of nilpotency of $\overline{I}$, that is $r = \mathrm{a}(\mathrm{G}(\overline{I}))$. Then since $r = \mathrm{r}_Q(I)$ (recall that ${x_1}^{a_1}, {x_2}^{a_2}, \cdots, {x_d}^{a_d}$ is a super regular sequence with respect to I) and $\mathrm{a}(\mathrm{G}(I)) = \mathrm{a}(\mathrm{G}(\overline{I})) - d$ ([GW, (3.1.6)]), by Theorem 1.2 (i) we have the following.

Lemma 2.3. *Suppose that $\ell \geq a_i$ for all $1 \leq i \leq d$. Then $\mathrm{a}(\mathrm{G}(I)) = \lceil \frac{q}{\ell} \rceil - d$.*

Corollary 2.4. *Assume that $\ell \geq a_i$ for all $1 \leq i \leq d$. Then $\mathcal{R}(I)$ is a Cohen-Macaulay ring if and only if $\lceil \frac{q}{\ell} \rceil < d$. When this is the case, $d \geq 2$.*

Proof. Since $\mathrm{G}(I)$ is a Cohen-Macaulay ring by Theorem 1.2 (ii), $\mathcal{R}(I)$ is a Cohen-Macaulay ring if and only if $a(\mathrm{G}(I)) < 0$ ([TI]). By Lemma 2.3 the latter condition is equivalent to saying that $\lceil \frac{q}{\ell} \rceil < d$ (cf. [GSh, Remark (3.10)]). When this is case, $d \geq 2$ because $0 < \lceil \frac{q}{\ell} \rceil$. $\square$

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. (i) Notice that $\mathrm{G}(I)$ is a Gorenstein ring if and only if so is the graded ring

$$\mathrm{G}(\overline{I}) = \mathrm{G}(I)/(Y_1, Y_2, \cdots, Y_d),$$

where Y_i's stand for the initial forms of $x_i^{a_i}$'s with respect to I. Let r be the index of nilpotency of $\overline{I}$. Then $r = \mathrm{r}_Q(I) = \lceil \frac{q}{\ell} \rceil$, and $\mathrm{G}(\overline{I})$ is a Gorenstein ring if and only if the equality

$$(0) : \overline{I}^i = \overline{I}^{r+1-i}$$

holds true for all $i \in \mathbb{Z}$ ([O, Theorem 1.6]). Hence if $\mathrm{G}(I)$ is a Gorenstein ring, we have $(0) : \overline{I} = \overline{I}^r = \overline{\mathfrak{m}}^{r\ell}$. On the other hand, since $\overline{I} = \overline{\mathfrak{m}}^\ell$ and $q = \rho + 1 - \ell$, by Proposition 2.1 we get

$$(0) : \overline{I} = (0) : \overline{\mathfrak{m}}^\ell = \overline{\mathfrak{m}}^q.$$

Therefore $q = r\ell$, since $\overline{\mathfrak{m}}^{r\ell} = \overline{\mathfrak{m}}^q \neq (0)$. Thus $\ell \mid q$ and $r = \frac{q}{\ell}$.

Conversely, suppose that $\ell \mid q$. Hence $r = \frac{q}{\ell}$ by Theorem 1.2 (i). Let $i \in \mathbb{Z}$. Then since $\overline{I} = \overline{\mathfrak{m}}^\ell$, we get $\overline{I}^{r+1-i} = \overline{\mathfrak{m}}^{(r+1-i)\ell}$, while

$$(0) : \overline{I}^i = (0) : \overline{\mathfrak{m}}^{i\ell} = \overline{\mathfrak{m}}^{\rho+1-i\ell}$$

by Proposition 2.1. We then have $(0) : \overline{I}^i = \overline{I}^{r+1-i}$ for all $i \in \mathbb{Z}$, since

$$(r + 1 - i)\ell = q + \ell - i\ell = \rho + 1 - i\ell.$$

Thus $\mathrm{G}(\overline{I})$ is a Gorenstein ring, whence so is $\mathrm{G}(I)$.

(ii) The Rees algebra $\mathcal{R}(I)$ of I is a Gorenstein ring if and only if $\mathrm{G}(I)$ is a Gorenstein ring and $a(\mathrm{G}(I)) = -2$, provided $d \geq 2$ ([I, Corollary (3.7)]). Suppose that $\mathcal{R}(I)$ is a Gorenstein ring. Then $d \geq 2$ by Corollary 2.4. Since $a(\mathrm{G}(I)) = \mathrm{r}_Q(I) - d = -2$, by assertion (i) and Theorem 1.2 (i) we have $\frac{q}{\ell} = \mathrm{r}_Q(I) = d - 2$, whence $q = (d - 2)\ell$. Conversely, suppose that $q = (d - 2)\ell$. Then $d \geq 3$ since $q \geq 1$. By assertion (i) and Theorem 1.2 (i) $\mathrm{G}(I)$ is a Gorenstein ring with $\mathrm{r}_Q(I) = \frac{q}{\ell} = d - 2$, whence $a(\mathrm{G}(I)) = (d - 2) - d = -2$, so that $\mathcal{R}(I)$ is a Gorenstein ring. $\square$

Example 2.5. Suppose that $\rho \geq 5$ is an odd integer, say $\rho = 2\tau + 1$ with $\tau \geq 2$. Let $q = \rho - 1$. Then $\ell = \rho + 1 - q = 2$. Hence, choosing $a_i \leq 2$ for all $1 \leq i \leq d$, we have $I = Q + \mathfrak{m}^2 \subseteq \overline{Q}$ with $\mathrm{r}_Q(I) = \tau$ by Theorem 1.2. Since $\ell \mid q$, by Theorem 1.3 (i) the ring $\mathrm{G}(I)$ is Gorenstein. The ring $\mathcal{R}(I)$ is by Theorem 1.3 (ii) a Gorenstein ring, if $d = \tau + 2$.

3. Examples and applications

In this section we shall discuss some applications of Theorems 1.2 and 1.3. Let us begin with the case where A is a regular local ring.

3.1. **The case where A is a regular local ring.** Let A be a regular local ring with $x_1, x_2, \cdots, x_d$ a regular system of parameters. Similarly as in the previous sections, let

$$Q = (x_1^{a_1}, x_2^{a_2}, \cdots, x_d^{a_d}) \text{ and } I = Q : \mathfrak{m}^q$$

with positive integers $a_1, a_2, \cdots, a_d$, and q. Then $\mathrm{G}(\mathfrak{m}) = k[X_1, X_2, \cdots, X_d]$ is the polynomial ring, where $k = A/\mathfrak{m}$ and X_i's are the initial forms of x_i's, so that we have

$$\rho = \sum_{i=1}^{d} a_i - d \text{ and } \ell = \sum_{i=1}^{d} (a_i - 1) + 1 - q,$$

since $\mathrm{a}(\mathrm{G}(\mathfrak{m})) = -d$. Notice that the condition that

$$\ell \geq \max \{a_i \mid 1 \leq i \leq d\}$$

is equivalent to saying that

$$\sum_{j \neq i} a_j \geq q + d - 1$$

for all $1 \leq i \leq d$, because $\ell - a_i = \sum_{j \neq i} a_j - (q + d - 1)$. When this is the case, $d \geq 2$.

Example 3.1. The following assertions hold true.

(1) Let $d = 2$. Then $I \subseteq \overline{Q}$ if and only if $\min\{a_1, a_2\} \geq q + 1$.
(2) Let $d = 3$. Then $I \subseteq \overline{Q}$ if and only if $\min\{a_i + a_j \mid 1 \leq i < j \leq 3\} \geq q + 2$.
(3) Choose integers a and q so that $2 \leq a \leq d$ and $(d-1)(a-1) < q \leq d(a-1)$. Let $a_i = a$ for all $1 \leq i \leq d$. Then $I \subsetneq A$ but $I \not\subseteq \overline{Q}$. For example, let $d = 3, a = 2$, and $q = 3$. Then

$$(x_1^2, x_2^2, x_3^2) : \mathfrak{m}^3 = \mathfrak{m} \not\subseteq \overline{(x_1^2, x_2^2, x_3^2)}.$$

Example 3.2. The following assertions hold true.

(1) Let $d = 2$ and assume that $I \subseteq \overline{Q}$. Then $\mathrm{G}(I)$ is not a Gorenstein ring.
(2) Suppose that $d \geq 3$ and let $n \geq d - 1$ be an integer. Let $a_1 = d - 1$, $a_i = n$ for all $2 \leq i \leq d$, and $q = (d - 2)n$. Then $\mathcal{R}(I)$ is a Gorenstein ring.
(3) Suppose that $d = 5$ and let $a_i = 4$ for all $1 \leq i \leq 5$. Let $q = 8$. Then $I \subseteq \overline{Q}$ and $\mathrm{G}(I)$ is a Gorenstein ring with $\mathrm{r}_Q(I) = 1$, but $\mathcal{R}(I)$ is not a Gorenstein ring.

Since the base ring A is regular, the Cohen-Macaulayness in Rees algebras $\mathcal{R}(I)$ follows from that of associated graded rings $\mathrm{G}(I)$ ([L]). Let us note a brief proof in our context.

Proposition 3.3. *Suppose that $\ell \geq a_i$ for all $1 \leq i \leq d$. Then the Rees algebra $\mathcal{R}(I)$ is a Cohen-Macaulay ring.*

Proof. By Corollary 2.4 we have only to show $\lceil \frac{q}{\ell} \rceil < d$. Let $a_k = \max\{a_i \mid 1 \leq i \leq d\}$. Then because $\ell \geq a_k$, we have

$$\frac{q}{\ell} + 1 = \frac{\rho + 1}{\ell} \leq \frac{\sum_{j=1}^{d}(a_j - 1) + 1}{a_k} = \sum_{j \neq k} \frac{a_j - 1}{a_k} + 1 < d,$$

whence $\lceil \frac{q}{\ell} \rceil < d$ as is wanted. □

Let $L = \{(\alpha_1, \alpha_2, \cdots, \alpha_d) \in \mathbb{Z}^d \mid \alpha_i \geq 0 \text{ for all} 1 \leq i \leq d\}$. For each $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in L$ we put $x^\alpha = \prod_{i=1}^{d} x_i^{\alpha_i}$. Let $\mathfrak{a}$ be an ideal in A. Then we say that $\mathfrak{a}$ is a monomial ideal, if $\mathfrak{a}$ is generated by monomials in $\{x_i\}_{1 \leq i \leq d}$, that is $\mathfrak{a} = (x^\alpha \mid \alpha \in \Lambda)$ for some $\Lambda \subseteq L$. Monomial ideals behave very well as if they were monomial ideals in the polynomial ring $k[x_1, x_2, \cdots, x_d]$ over a field k (see [HS] for details). For instance, the integral closure $\overline{Q}$ of our monomial ideal Q is also a monomial ideal and we have the following.

Proposition 3.4 ([HS])**.** *Let $\Delta = \{\alpha \in L \mid \sum_{i=1}^{d} \frac{\alpha_i}{a_i} \geq 1\}$. Then $\overline{Q} = (x^\alpha \mid \alpha \in \Delta)$.*

Corollary 3.5. *Suppose that $d \geq 2$ and let $n \geq 2$ be an integer. We put $\mathfrak{q} = (x_1^{n-1}, x_2^n, \cdots, x_d^n)$. Then $\overline{\mathfrak{q}} = \mathfrak{q} + \mathfrak{m}^n = (x_1^{n-1}) + \mathfrak{m}^n$ and all the powers $\overline{\mathfrak{q}}^m$ $(m \geq 1)$ are integrally closed.*

Proof. Let $J = \mathfrak{q} + \mathfrak{m}^n$ and $\mathfrak{a} = (x_1^n, x_2^n, \cdots, x_d^n)$. Then $\mathfrak{a} \subseteq \mathfrak{q}$ and $\mathfrak{m}^n \subseteq \overline{\mathfrak{a}}$, so that $J \subseteq \overline{\mathfrak{q}}$. Let $m \geq 1$ be an integer and put $K = (x_1^{m(n-1)}, x_2^{mn}, \cdots, x_d^{mn})$. We will show that $\overline{K} \subseteq J^m$. Let $\alpha \in L$ and assume that $\frac{\alpha_1}{m(n-1)} + \sum_{i=2}^{d} \frac{\alpha_i}{mn} \geq 1$. We want to show that $x^\alpha \in J^m$. We may assume that $\alpha_1 < m(n-1)$. Let $\alpha_1 = (n-1)i + j$ with $i, j \in \mathbb{Z}$ such that $0 \leq j < (n-1)$. Then $0 \leq i < m$. Since $\frac{\alpha_1}{m(n-1)} + \sum_{i=2}^{d} \frac{\alpha_i}{mn} \geq 1$, we get

$$n\alpha_1 + (n-1) \cdot \sum_{i=2}^{d} \alpha_i \geq mn(n-1),$$

so that

$$(n-1) \cdot \sum_{i=2}^{d} \alpha_i \geq mn(n-1) - n\alpha_1 = n[(n-1)(m-i) - j],$$

whence

$$\sum_{i=2}^{d} \alpha_i \geq n(m-i) - \frac{nj}{n-1}.$$

Because $\frac{nj}{n-1} = j + \frac{j}{n-1}$ and $0 \leq j < n-1$, we have $\frac{nj}{n-1} = j + \frac{j}{n-1} < j + 1$ and so

$$\sum_{i=2}^{d} \alpha_i \geq n(m-i) - j.$$

Thus

$$x^\alpha = x_1^{(n-1)i} \cdot x_1^j x_2^{\alpha_2} \cdots x_d^{\alpha_d} \in x_1^{(n-1)i}\mathfrak{m}^{n(m-i)} \subseteq J^m,$$

whence $\overline{K} \subseteq J^m$ by Proposition 3.4.

Because $J^m \subseteq \overline{\mathfrak{q}}^m$ and $\mathfrak{q}^m \subseteq \overline{K}$, we have $J^m \subseteq \overline{\mathfrak{q}}^m \subseteq \overline{\mathfrak{q}^m} \subseteq \overline{K}$, whence $J^m = \overline{\mathfrak{q}}^m = \overline{\mathfrak{q}^m} = K$. Letting $m = 1$, we get $J = \overline{\mathfrak{q}}$. This completes the proof of Corollary 3.5. $\square$

Thanks to Corollary 3.5, we get the following characterization for quasi-socle ideals $I = Q : \mathfrak{m}^q$ to be integrally closed.

Theorem 3.6. *Suppose that $d \geq 2$ and $a_i \geq 2$ for all $1 \leq i \leq d$. Then the following two conditions are equivalent to each other.*

(1) *$I = \overline{Q}$.*

(2) *Either* (a) *$a_i = \ell$ for all $1 \leq i \leq d$, or* (b) *there exists $1 \leq j \leq d$ such that $a_i = \ell$ if $i \neq j$ and $a_j = \ell - 1$.*

When this is the case, $\overline{I^n} = I^n$ for all $n \geq 1$, whence $\mathcal{R}(I)$ is a Cohen-Macaulay normal domain.

Proof. (1) $\Rightarrow$ (2) Since $I = \overline{Q}$, we get $q \leq \rho$ and $I = Q + \mathfrak{m}^\ell$ (Proposition 2.1). Notice that

$$Q \subseteq I = Q : \mathfrak{m}^q \subsetneq (Q : \mathfrak{m}^q) : \mathfrak{m} = Q : \mathfrak{m}^{q+1},$$

because $I \subsetneq A$. Hence $Q : \mathfrak{m}^{q+1} \nsubseteq \overline{Q}$. Consequently $\ell - 1 = \rho + 1 - (q+1) < a_i$ for some $1 \leq i \leq d$ by Theorem 1.2, so that, thanks to Theorem 1.2 again, we have

$$\ell = a_i \geq a_j$$

for all $1 \leq j \leq d$. Let $\Delta = \{1 \leq j \leq d \mid a_j < \ell\}$. We then have the following.

Claim. (1) *$a_j = \ell - 1$, if $j \in \Delta$.*

(2) *$\sharp\Delta \leq 1$.*

Proof. Let $j \in \Delta$. Then $a_j < \ell = a_i$ whence $j \neq i$ and $\ell \geq 3$. Let $\alpha = (a_j - 1)\mathbf{e}_j + (a_i - a_j)\mathbf{e}_i$. Then $\alpha \in L$ but, thanks to the monomial property of ideals, $x^\alpha \notin Q + \mathfrak{m}^\ell = I = \overline{Q}$, because $\sum_{k=1}^d \alpha_k = a_i - 1 = \ell - 1$ and $x^\alpha \notin Q$. Consequently, $\sum_{k=1}^d \frac{\alpha_k}{a_k} < 1$ by Proposition 3.4, so that $1 < \frac{1}{a_j} + \frac{a_j}{a_i}$, because

$$\frac{a_j - 1}{a_j} + \frac{a_i - a_j}{a_i} < 1.$$

Let $n = a_i - a_j$. Then $a_j(a_i - a_j) < a_i$ as $1 < \frac{1}{a_j} + \frac{a_j}{a_i}$, whence $a_j n < a_i = a_j + n$ so that $0 \leq (a_j - 1)(n - 1) < 1$. Hence $n = 1$ (recall that $a_j \geq 2$) and $a_j = a_i - 1 = \ell - 1$.

Assume $\sharp\Delta \geq 2$ and choose $j, k \in \Delta$ so that $j \neq k$. We put $y = x_j x_k^{\ell-2}$. We then have $y^{\ell-1} = (x_j^{\ell-1})(x_k^{\ell-1})^{\ell-2} = (x_j^{a_j})(x_k^{a_k})^{\ell-2} \in Q^{\ell-1}$, because $a_j = a_k = \ell - 1$ by assertion (1). Hence $y \in \overline{Q} = Q + \mathfrak{m}^\ell$, which is impossible because $y \notin Q$ (recall that $\ell \geq 3$) and $y \notin \mathfrak{m}^\ell$, thanks to the monomial property of ideals. Hence $\sharp\Delta \leq 1$. $\square$

If $\Delta = \emptyset$, we then have $\ell = a_j$ for all $1 \le j \le d$. If $\Delta \ne \emptyset$, letting $\Delta = \{j\}$, we get $a_i = \ell$ if $i \ne j$ and $a_j = \ell - 1$. This proves the implication (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1) Suppose condition (b) is satisfied. Then $I = Q + \mathfrak{m}^\ell = (x_j^{\ell-1}) + \mathfrak{m}^\ell = \overline{Q}$ by Proposition 2.1 and Corollary 3.5. Suppose condition (a) is satisfied. Then $I \subseteq \overline{Q}$ by Theorem 1.2 and $I = Q + \mathfrak{m}^\ell = \mathfrak{m}^\ell$ by Proposition 2.1, whence $I = \overline{Q}$. In each case all the powers of I are integrally closed (see Corollary 3.5 for case (b)), whence the last assertion follows from Proposition 3.3. $\square$

Example 3.7. Suppose that $d \ge 3$ and let $n \ge d-1$ be an integer. We look at the ideal

$$Q = (x_1^{d-1}, x_2^n, x_3^n, \cdots, x_d^n)$$

and let $q = n(d-2)$. Then $\ell = n$, as $\rho = nd - (n+1)$, whence $I \subseteq \overline{Q}$ and $I = Q + \mathfrak{m}^n = (x_1^{d-1}) + \mathfrak{m}^n$. The ring $\mathcal{R}(I)$ is by Theorem 1.3 (ii) a Gorenstein ring, since $q = (d-2)\ell$. If $n = d$, then $I = (x_1^{d-1}) + \mathfrak{m}^d$ and $\overline{I^m} = I^m$ for all $m \ge 1$ by Corollary 3.5, so that $\mathcal{R}(I)$ is a Gorenstein normal ring.

3.2. **The case where $A = R_M$.** Our setting naturally contains the case where $A = R_M$ is the localization of the homogeneous Gorenstein ring $R = k[R_1]$ over an infinite field $k = R_0$ at the irrelevant maximal ideal $M = R_+$. Let us note one example.

Example 3.8. Let $S = k[X, Y, Z]$ be the polynomial ring over an infinite field k and let $R = S/fS$, where $0 \ne f \in S$ is a form with degree $n \ge 2$. Then R is a homogeneous Gorenstein ring with $\dim R = 2$. Let x_1, x_2 be a linear system of parameters in R and let $M = R_+$. We look at the local ring $A = R_M$. Let $a_1 = 2$, $a_2 = n$, and $q = n$. Let $Q = (x_1^2, x_2^n)A$ and $I = Q : \mathfrak{m}^q$, where $\mathfrak{m} = MA$. Then

$$\rho = \mathrm{a}(R) + (a_1 + a_2) = 2n - 1.$$

Hence $\ell = q = n$, so that $I \subseteq \overline{Q}$, $I = Q + \mathfrak{m}^n = (x_1^2) + \mathfrak{m}^n$, and $\mathrm{G}(I)$ is a Gorenstein ring with $\mathrm{r}_Q(I) = 1$ (Theorems 1.2 and 1.3). We have $Q \not\subseteq \mathfrak{m}^q$, if $n \ge 3$.

3.3. **The case where $A = k[[t^a, t^b]]$.** Let $1 < a < b$ be integers with $\mathrm{GCD}(a, b) = 1$. We look at the ring $A = k[[t^a, t^b]] \subseteq k[[t]]$, where $k[[t]]$ denotes the formal powers series ring over a field k. We put $x = t^a$ and $y = t^b$. Then A is a one-dimensional Gorenstein local ring and $\mathfrak{m} = (x, y)$. Because $A \cong k[[X, Y]]/(X^b - Y^a)$ where $k[[X, Y]]$ denotes the formal powers series ring over the field k, we get

$$\mathrm{G}(\mathfrak{m}) \cong k[X, Y]/(Y^a).$$

Let $n, q \ge 1$ be integers, and put $Q = (x^n)$ and $I = Q : \mathfrak{m}^q$. Then because $a(\mathrm{G}(\mathfrak{m})) = a - 2$, we have $\rho = a + n - 2$ and $\ell = (a + n) - (q + 1)$. Consequently $I \subseteq \overline{Q}$ if and only if $q < a$ (Theorem 1.2), whence the condition that $I \subseteq \overline{Q}$ is independent of the choice of the integer $n \geq 1$. When this is the case, by Theorems 1.2 and 1.3 we have the following.

Theorem 3.9. *The following assertions hold true.*
(1) $\mathrm{r}_Q(I) = \lceil \frac{q}{(a+n)-(q+1)} \rceil$.
(2) *The graded rings* $\mathrm{G}(I)$ *and* $\mathrm{F}(I)$ *are Cohen-Macaulay rings.*
(3) *The ring* $\mathrm{G}(I)$ *is a Gorenstein ring if and only if* $(a+n)-(q+1)$ *divides* q.

Hence, if $q = a-1$, we then have, for each integer $n \geq 1$ such that $n \mid q$, that $\mathrm{G}(I)$ is a Gorenstein ring.

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