

CONTRAVARIANTLY FINITE RESOLVING SUBCATEGORIES OVER A GORENSTEIN LOCAL RING

RYO TAKAHASHI

INTRODUCTION

The notion of a contravariantly finite subcategory (of the category of finitely generated modules) was first introduced over artin algebras by Auslander and Smalø [5] in connection with studying the problem of which subcategories admit almost split sequences. The notion of a resolving subcategory was introduced by Auslander and Bridger [2] in the study of modules of Gorenstein dimension zero, which are now also called totally reflexive modules. There is an application of contravariantly finite resolving subcategories to the study of the finitistic dimension conjecture [4].

This paper deals with contravariantly finite resolving subcategories over commutative rings. Let R be a commutative noetherian henselian local ring. We denote by $\text{mod } R$ the category of finitely generated R -modules, by $\mathcal{F}(R)$ the full subcategory of free R -modules, and by $\mathcal{C}(R)$ the full subcategory of maximal Cohen-Macaulay R -modules. The subcategory $\mathcal{F}(R)$ is always contravariantly finite, and so is $\mathcal{C}(R)$ provided that R is Cohen-Macaulay. The latter fact is known as the Cohen-Macaulay approximation theorem, which was shown by Auslander and Buchweitz [3].

In this paper, we shall prove the following theorem; the category of finitely generated modules over a henselian Gorenstein local ring possesses only three contravariantly finite resolving subcategories.

Theorem A. *If R is Gorenstein, then all the contravariantly finite resolving subcategories of $\text{mod } R$ are $\mathcal{F}(R)$, $\mathcal{C}(R)$ and $\text{mod } R$.*

The main theorem of this paper asserts the following: let \mathcal{X} be a resolving subcategory of $\text{mod } R$ such that the residue field of R has a right \mathcal{X} -approximation. Assume that there exists an R -module $G \in \mathcal{X}$ of infinite projective dimension with $\text{Ext}_R^i(G, R) = 0$ for $i \gg 0$. Let M be an R -module such that each $X \in \mathcal{X}$ satisfies $\text{Ext}_R^i(X, M) = 0$ for $i \gg 0$. Then M has finite injective dimension. From this result, we will prove the following two theorems. Theorem A will be obtained from Theorem B. The assertion of Theorem C is a main result of [12], which has been a motivation for this paper. (Our way of obtaining Theorem C is quite different from the original proof given in [12].)

Theorem B. *Let $\mathcal{X} \neq \text{mod } R$ be a contravariantly finite resolving subcategory of $\text{mod } R$. Suppose that there is an R -module $G \in \mathcal{X}$ of infinite projective dimension such that $\text{Ext}_R^i(G, R) = 0$ for $i \gg 0$. Then R is Cohen-Macaulay and $\mathcal{X} = \mathcal{C}(R)$.*

Theorem C (Christensen-Piepmeyer-Striuli-Takahashi). *Suppose that there is a nonfree R -module in $\mathcal{G}(R)$. If $\mathcal{G}(R)$ is contravariantly finite in $\text{mod } R$, then R is Gorenstein.*

Here, $\mathcal{G}(R)$ denotes the full subcategory of totally reflexive R -modules. A totally reflexive module, which is also called a module of Gorenstein dimension (G-dimension) zero, was defined by Auslander [1] as a common generalization of a free module and a maximal Cohen-Macaulay module over a Gorenstein local ring. Auslander and Bridger [2] proved that the full subcategory of totally reflexive modules over a left and right noetherian ring is resolving. The other details of totally reflexive modules are stated in [2] and [11].

If R is Gorenstein, then $\mathcal{G}(R)$ coincides with $\mathcal{C}(R)$, and so $\mathcal{G}(R)$ is contravariantly finite by virtue of the Cohen-Macaulay approximation theorem. Thus, Theorem C can be viewed as the converse of this fact. Theorem C implies the following: let R be a homomorphic image of a regular local ring. Suppose that there is a nonfree totally reflexive R -module and are only finitely many nonisomorphic indecomposable totally reflexive R -modules. Then R is an isolated simple hypersurface singularity. For the details, see [12].

CONVENTIONS

In the rest of this paper, we assume that all rings are commutative and noetherian, and that all modules are finitely generated. Unless otherwise specified, let R be a henselian local ring. The unique maximal ideal of R and the residue field of R are denoted by \mathfrak{m} and k , respectively. We denote by $\text{mod } R$ the category of finitely generated R -modules. By a *subcategory* of $\text{mod } R$, we always mean a full subcategory of $\text{mod } R$ which is closed under isomorphisms. Namely, in this paper, a subcategory \mathcal{X} of $\text{mod } R$ means a full subcategory such that every R -module which is isomorphic to some R -module in \mathcal{X} is also in \mathcal{X} .

1. CONTRAVARIANT FINITENESS OF TOTALLY REFLEXIVE MODULES

In this section, we will state background materials which motivate the main results of this paper. We start by recalling the definition of a totally reflexive module.

Definition 1.1. We denote by $(-)^*$ the R -dual functor $\text{Hom}_R(-, R)$. An R -module M is called *totally reflexive* (or *of Gorenstein dimension zero*) if

- (1) the natural homomorphism $M \rightarrow M^{**}$ is an isomorphism, and
- (2) $\text{Ext}_R^i(M, R) = \text{Ext}_R^i(M^*, R) = 0$ for any $i > 0$.

We introduce three subcategories of $\text{mod } R$ which will often appear throughout this paper.

We denote by $\mathcal{F}(R)$ the subcategory of $\text{mod } R$ consisting of all free R -modules, by $\mathcal{G}(R)$ the subcategory of $\text{mod } R$ consisting of all totally reflexive R -modules, and by $\mathcal{C}(R)$ the subcategory of $\text{mod } R$ consisting of all maximal Cohen-Macaulay R -modules. By definition, $\mathcal{F}(R)$ is contained in $\mathcal{G}(R)$. If R is Cohen-Macaulay, then $\mathcal{G}(R)$ is contained in $\mathcal{C}(R)$. If R is Gorenstein, then $\mathcal{G}(R)$ coincides with $\mathcal{C}(R)$.

Next, we recall the notion of a right approximation over a subcategory of $\text{mod } R$.

Definition 1.2. Let \mathcal{X} be a subcategory of $\text{mod } R$.

- (1) Let $\phi : X \rightarrow M$ be a homomorphism of R -modules with $X \in \mathcal{X}$. We say that ϕ is a *right \mathcal{X} -approximation* (of M) if the induced homomorphism $\text{Hom}_R(X', \phi) : \text{Hom}_R(X', X) \rightarrow \text{Hom}_R(X', M)$ is surjective for any $X' \in \mathcal{X}$.

- (2) We say that \mathcal{X} is *contravariantly finite* (in $\text{mod } R$) if every R -module has a right \mathcal{X} -approximation.

The following result is well-known.

Theorem 1.3 (Auslander-Buchweitz). *Let R be a Cohen-Macaulay local ring. Then $\mathcal{C}(R)$ is contravariantly finite.*

Corollary 1.4. *If R is Gorenstein, then $\mathcal{G}(R)$ is contravariantly finite.*

The converse of this corollary essentially holds:

Theorem 1.5. [12] *Suppose that there is a nonfree totally reflexive R -module. If $\mathcal{G}(R)$ is contravariantly finite in $\text{mod } R$, then R is Gorenstein.*

This theorem yields the following corollary, which is a generalization of [21, Theorem 1.3].

Corollary 1.6. *Let R be a non-Gorenstein local ring. If there is a nonfree totally reflexive R -module, then there are infinitely many nonisomorphic indecomposable totally reflexive R -modules.*

Combining this with [25, Theorems (8.15) and (8.10)] (cf. [14, Satz 1.2] and [10, Theorem B]), we obtain the following result.

Corollary 1.7. *Let R be a homomorphic image of a regular local ring. Suppose that there is a nonfree totally reflexive R -module but there are only finitely many nonisomorphic indecomposable totally reflexive R -modules. Then R is a simple hypersurface singularity.*

2. CONTRAVARIANTLY FINITE RESOLVING SUBCATEGORIES

In this section, we will give the main theorem of this paper and several results it yields. One of them implies Theorem 1.5, which is the motive fact of this paper.

First of all, we recall the definition of the syzygies of a given module. Let M be an R -module and n a positive integer. Let

$$F_{\bullet} = (\cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \rightarrow 0)$$

be a minimal free resolution of M . We define the n th syzygy $\Omega^n M$ of M as the image of the homomorphism d_n . We set $\Omega^0 M = M$.

We recall the definition of a resolving subcategory.

Definition 2.1. A subcategory \mathcal{X} of $\text{mod } R$ is called *resolving* if it satisfies the following four conditions.

- (1) \mathcal{X} contains R .
- (2) \mathcal{X} is closed under direct summands: if M is an R -module in \mathcal{X} and N is a direct summand of M , then N is also in \mathcal{X} .
- (3) \mathcal{X} is closed under extensions: for an exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ of R -modules, if L and N are in \mathcal{X} , then M is also in \mathcal{X} .
- (4) \mathcal{X} is closed under kernels of epimorphisms: for an exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ of R -modules, if M and N are in \mathcal{X} , then L is also in \mathcal{X} .

Now we state the main theorem in this paper.

Theorem 2.2. *Let \mathcal{X} be a resolving subcategory of $\text{mod } R$ such that the residue field k has a right \mathcal{X} -approximation. Assume that there exists an R -module $G \in \mathcal{X}$ of infinite projective dimension such that $\text{Ext}_R^i(G, R) = 0$ for $i \gg 0$. Let M be an R -module such that each $X \in \mathcal{X}$ satisfies $\text{Ext}_R^i(X, M) = 0$ for $i \gg 0$. Then M has finite injective dimension.*

We shall prove Theorem 2.2 in the next section. In the rest of this section, we will state and prove several results by using Theorem 2.2. We begin with two corollaries which are immediately obtained.

Corollary 2.3. *Let \mathcal{X} be a resolving subcategory of $\text{mod } R$ which is contained in the subcategory $\{M \mid \text{Ext}_R^i(M, R) = 0 \text{ for } i \gg 0\}$ of $\text{mod } R$. Suppose that in \mathcal{X} there is an R -module of infinite projective dimension. If k has a right \mathcal{X} -approximation, then R is Gorenstein.*

Proof. Each module X in \mathcal{X} satisfies $\text{Ext}_R^i(X, R) = 0$ for $i \gg 0$. Hence Theorem 2.2 implies that R has finite injective dimension as an R -module. \square

Corollary 2.4. *Let \mathcal{X} be one of the following.*

- (1) $\mathcal{G}(R)$.
- (2) The subcategory $\{M \mid \text{Ext}_R^i(M, R) = 0 \text{ for } i > n\}$ of $\text{mod } R$, where n is a non-negative integer.
- (3) The subcategory $\{M \mid \text{Ext}_R^i(M, R) = 0 \text{ for } i \gg 0\}$ of $\text{mod } R$.

Suppose that in \mathcal{X} there is an R -module of infinite projective dimension. If k has a right \mathcal{X} -approximation, then R is Gorenstein.

Proof. The subcategory \mathcal{X} of $\text{mod } R$ is resolving. Since \mathcal{X} is contained in the subcategory $\{M \mid \text{Ext}_R^i(M, R) = 0 \text{ for } i \gg 0\}$, the assertion follows from Corollary 2.3. \square

Remark 2.5. Corollary 2.4 implies Theorem 1.5. Indeed, any nonfree totally reflexive module has infinite projective dimension by [11, (1.2.10)].

For a subcategory \mathcal{X} of $\text{mod } R$, let \mathcal{X}^\perp (respectively, ${}^\perp\mathcal{X}$) denote the subcategory of $\text{mod } R$ consisting of all R -modules M such that $\text{Ext}_R^i(X, M) = 0$ (respectively, $\text{Ext}_R^i(M, X) = 0$) for all $X \in \mathcal{X}$ and $i > 0$. Applying Wakamatsu's lemma to a resolving subcategory, we obtain the following lemma.

Lemma 2.6. *Let \mathcal{X} be a resolving subcategory of $\text{mod } R$. If an R -module M has a right \mathcal{X} -approximation, then there is an exact sequence $0 \rightarrow Y \rightarrow X \rightarrow M \rightarrow 0$ of R -modules with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^\perp$.*

By using this lemma and the theorem which was formerly called "Bass' conjecture", we obtain another corollary of Theorem 2.2.

Corollary 2.7. *Let \mathcal{X} be a resolving subcategory of $\text{mod } R$ such that k has a right \mathcal{X} -approximation and that k is not in \mathcal{X} . Assume that there is an R -module $G \in \mathcal{X}$ with $\text{pd}_R G = \infty$ and $\text{Ext}_R^i(G, R) = 0$ for $i \gg 0$. Then R is Cohen-Macaulay and $\dim R > 0$.*

Before giving the next corollary of Theorem 2.2, we establish an easy lemma without proof.

Lemma 2.8. (1) Let \mathcal{X} be a contravariantly finite resolving subcategory of $\text{mod } R$. Then, $k \in \mathcal{X}$ if and only if $\mathcal{X} = \text{mod } R$.

(2) Let \mathcal{X} be a resolving subcategory of $\text{mod } R$. Suppose that every R -module in ${}^\perp(\mathcal{X}^\perp)$ admits a right \mathcal{X} -approximation. Then $\mathcal{X} = {}^\perp(\mathcal{X}^\perp)$.

(3) Let M and N be nonzero R -modules. Assume either that M has finite projective dimension or that N has finite injective dimension. Then one has an equality

$$\sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\} = \text{depth } R - \text{depth}_R M.$$

Now we can show the following corollary. There are only two contravariantly finite resolving subcategories possessing such G as in the corollary.

Corollary 2.9. Let \mathcal{X} be a contravariantly finite resolving subcategory of $\text{mod } R$. Assume that there is an R -module $G \in \mathcal{X}$ with $\text{pd}_R G = \infty$ and $\text{Ext}_R^i(G, R) = 0$ for $i \gg 0$. Then either of the following holds.

(1) $\mathcal{X} = \text{mod } R$,

(2) R is Cohen-Macaulay and $\mathcal{X} = \mathcal{C}(R)$.

Proof. Suppose that $\mathcal{X} \neq \text{mod } R$. Then k is not in \mathcal{X} . By Corollary 2.7, R is Cohen-Macaulay.

First, we show that $\mathcal{C}(R)$ is contained in \mathcal{X} . For this, let M be a maximal Cohen-Macaulay R -module. We have only to prove that M is in ${}^\perp(\mathcal{X}^\perp)$. Let N be a nonzero R -module in \mathcal{X}^\perp . Theorem 2.2 implies that N is of finite injective dimension. Since M is maximal Cohen-Macaulay, we have $\sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\} = 0$. Therefore $\text{Ext}_R^i(M, N) = 0$ for all $N \in \mathcal{X}^\perp$ and $i > 0$. It follows that M is in ${}^\perp(\mathcal{X}^\perp)$, as desired.

Next, we show that \mathcal{X} is contained in $\mathcal{C}(R)$. We have an exact sequence $0 \rightarrow Y \rightarrow X \rightarrow k \rightarrow 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^\perp$ by Lemma 2.6. Since k is not in \mathcal{X} , the module Y is nonzero. By Theorem 2.2, Y has finite injective dimension. For a nonzero R -module X' in \mathcal{X} , we have equalities $0 \geq \sup\{i \mid \text{Ext}_R^i(X', Y) \neq 0\} = \text{depth } R - \text{depth}_R X' = \dim R - \text{depth}_R X'$. Therefore X' is a maximal Cohen-Macaulay R -module, as desired. \square

Next, we study contravariantly finite resolving subcategories all of whose objects X satisfy $\text{Ext}_R^{\geq 0}(X, R) = 0$. We start by considering special ones among such subcategories.

Proposition 2.10. Let \mathcal{X} be a contravariantly finite resolving subcategory of $\text{mod } R$. Suppose that every R -module in \mathcal{X} has finite projective dimension. Then either of the following holds.

(1) $\mathcal{X} = \mathcal{F}(R)$,

(2) R is regular and $\mathcal{X} = \text{mod } R$.

Proof. If $\mathcal{X} = \text{mod } R$, then our assumption says that all R -modules have finite projective dimension. Hence R is regular. Assume that $\mathcal{X} \neq \text{mod } R$. Then there is an R -module M which is not in \mathcal{X} . There is an exact sequence $0 \rightarrow Y \rightarrow X \rightarrow M \rightarrow 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^\perp$ by Lemma 2.6. Note that $Y \neq 0$ as $M \notin \mathcal{X}$. Fix a nonzero R -module $X' \in \mathcal{X}$. We have $\text{Ext}_R^i(X', Y) = 0$ for all $i > 0$, and hence $\text{pd}_R X' = \sup\{i \mid \text{Ext}_R^i(X', Y) \neq 0\}$.

$0\} = 0$ by the Auslander-Buchsbaum formula. Hence X' is free. This means that \mathcal{X} is contained in $\mathcal{F}(R)$. On the other hand, \mathcal{X} contains $\mathcal{F}(R)$ since \mathcal{X} is resolving. Therefore $\mathcal{X} = \mathcal{F}(R)$. \square

Combining Proposition 2.10 with Corollary 2.9, we can get the following.

Corollary 2.11. *Let \mathcal{X} be a contravariantly finite resolving subcategory of $\text{mod } R$. Suppose that every module $X \in \mathcal{X}$ is such that $\text{Ext}_R^i(X, R) = 0$ for $i \gg 0$. Then one of the following holds.*

- (1) $\mathcal{X} = \mathcal{F}(R)$,
- (2) R is Gorenstein and $\mathcal{X} = \mathcal{C}(R)$,
- (3) R is Gorenstein and $\mathcal{X} = \text{mod } R$.

Proof. The corollary follows from Proposition 2.10 in the case where all R -modules in \mathcal{X} are of finite projective dimension. So suppose that in \mathcal{X} there exists an R -module of infinite projective dimension. Then Corollary 2.9 shows that either of the following holds.

- (1) $\mathcal{X} = \text{mod } R$,
- (2) R is Cohen-Macaulay and $\mathcal{X} = \mathcal{C}(R)$.

By the assumption that every $X \in \mathcal{X}$ satisfies $\text{Ext}_R^i(X, R) = 0$ for $i \gg 0$, we have $\text{Ext}_R^i(k, R) = 0$ for $i \gg 0$ in the case (i). In the case (ii), since $\Omega^d k$ is in \mathcal{X} where $d = \dim R$, we have $\text{Ext}_R^{i+d}(k, R) \cong \text{Ext}_R^i(\Omega^d k, R) = 0$ for $i \gg 0$. Thus, in both cases, the ring R is Gorenstein. \square

Finally, we obtain the following result from Corollary 2.11 and Theorem 1.3. It says that the category of finitely generated modules over a Gorenstein local ring possesses only three contravariantly finite resolving subcategories.

Corollary 2.12. *Let R be a Gorenstein local ring. Then all the contravariantly finite resolving subcategories of $\text{mod } R$ are $\mathcal{F}(R)$, $\mathcal{C}(R)$ and $\text{mod } R$.*

3. PROOF OF THE MAIN THEOREM

Let M be an R -module. Take a minimal free resolution $F_\bullet = (\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \rightarrow 0)$ of M . We define the *transpose* $\text{Tr } M$ of M as the cokernel of the R -dual homomorphism $d_1^* : F_0^* \rightarrow F_1^*$ of d_1 . The transpose $\text{Tr } M$ has no nonzero free summand.

For an R -module M , let M^*M be the ideal of R generated by the subset

$$\{ f(x) \mid f \in M^*, x \in M \}$$

of R . Note that M has a nonzero free summand if and only if $M^*M = R$.

Proposition 3.1. *Let \mathcal{X} be a subcategory of $\text{mod } R$ and $0 \rightarrow Y \xrightarrow{f} X \rightarrow M \rightarrow 0$ an exact sequence of R -modules with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^\perp$. Let $G \in \mathcal{X}$, set $H = \text{Tr } \Omega G$, and suppose that $(H^*H)M = 0$. Let $0 \rightarrow K \xrightarrow{g} F \xrightarrow{h} H \rightarrow 0$ be an exact sequence of R -modules with F free. Then the induced sequence*

$$0 \longrightarrow K \otimes_R Y \xrightarrow{g \otimes_R Y} F \otimes_R Y \xrightarrow{h \otimes_R Y} H \otimes_R Y \longrightarrow 0$$

is exact, and the map $h \otimes_R Y$ factors through the map $F \otimes_R f : F \otimes_R Y \rightarrow F \otimes_R X$.

Proof. We can show that there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H \otimes_R Y & \xrightarrow{\delta} & H \otimes_R X & \xrightarrow{\varepsilon} & H \otimes_R M & \longrightarrow & 0 \\
& & \alpha \downarrow \cong & & \beta \downarrow & & \gamma \downarrow 0 & & \\
0 & \longrightarrow & \text{Hom}_R(H^*, Y) & \xrightarrow{\zeta} & \text{Hom}_R(H^*, X) & \xrightarrow{\eta} & \text{Hom}_R(H^*, M) & \longrightarrow & 0
\end{array}$$

with exact rows, and see that δ is a split monomorphism. Thus, the homomorphism $h \otimes_R Y$ factors through the homomorphism $F \otimes_R f$. We have isomorphisms $\text{Tor}_1^R(H, Y) = \text{Tor}_1^R(\text{Tr } \Omega G, Y) \cong \underline{\text{Hom}}_R(\Omega G, Y) = 0$, which completes the proof of the proposition. \square

Now we can prove the following, which will play a key role in the proof of Theorem 2.2.

Proposition 3.2. *Let \mathcal{X} be a subcategory of $\text{mod } R$ which is closed under syzygies. Let $0 \rightarrow Y \rightarrow X \rightarrow M \rightarrow 0$ be an exact sequence of R -modules with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^\perp$. Suppose that there is an R -module $G \in \mathcal{X}$ with $\text{pd}_R G = \infty$ and $\text{Ext}_R^i(G, R) = 0$ for $i \gg 0$. Put $H_i = \text{Tr } \Omega(\Omega^i G)$ and assume that $((H_i)^* H_i)M = 0$ for $i \gg 0$. Let $D = (D^j)_{j \geq 0} : \text{mod } R \rightarrow \text{mod } R$ be a contravariant cohomological δ -functor. If $D^j(X) = 0$ for $j \gg 0$, then $D^j(Y) = D^j(M) = 0$ for $j \gg 0$.*

Proof. Replacing G with $\Omega^i G$ for $i \gg 0$, we may assume that $\text{Ext}_R^i(G, R) = 0$ for all $i > 0$ and that $((H_i)^* H_i)M = 0$ for all $i \geq 0$. Let $F_\bullet = (\cdots \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \rightarrow 0)$ be a minimal free resolution of G . Dualizing this by R , we easily see that $H_i \cong (\Omega^{i+3} G)^*$ and $\Omega H_i \cong (\Omega^{i+2} G)^*$ for $i \geq 0$. By Proposition 3.1, for each integer $i \geq 0$ we have an exact sequence

$$0 \rightarrow (\Omega^{i+2} G)^* \otimes_R Y \rightarrow (F_{i+2})^* \otimes_R Y \xrightarrow{f_i} (\Omega^{i+3} G)^* \otimes_R Y \rightarrow 0$$

such that f_i factors through $(F_{i+2})^* \otimes_R X$. The homomorphism $D^j(f_i)$ factors through $D^j((F_{i+2})^* \otimes_R X)$, which vanishes for $j \gg 0$. Hence $D^j(f_i) = 0$ for $j \gg 0$, and we obtain an exact sequence

$$0 \rightarrow D^j((F_{i+2})^* \otimes_R Y) \rightarrow D^j((\Omega^{i+2} G)^* \otimes_R Y) \xrightarrow{\varepsilon_{i,j}} D^{j+1}((\Omega^{i+3} G)^* \otimes_R Y) \rightarrow 0$$

for $i \geq 0$ and $j \gg 0$. Thus, there is a sequence

$$D^j((\Omega^{i+2} G)^* \otimes_R Y) \xrightarrow{\varepsilon_{i,j}} D^{j+1}((\Omega^{i+3} G)^* \otimes_R Y) \xrightarrow{\varepsilon_{i+1,j+1}} D^{j+2}((\Omega^{i+4} G)^* \otimes_R Y) \xrightarrow{\varepsilon_{i+2,j+2}} \cdots$$

of surjective homomorphisms of R -modules, and $\varepsilon_{i,j}$ is an isomorphism. It follows that $D^j((F_{i+2})^* \otimes_R Y) = 0$ for $i \geq 0$ and $j \gg 0$. Thus we have $D^j(Y) = 0$ for $j \gg 0$, and $D^j(M) = 0$ for $j \gg 0$. \square

Now we can prove our main theorem.

Proof of Theorem 2.2. Since k admits a right \mathcal{X} -approximation, there exists an exact sequence $0 \rightarrow Y \rightarrow X \rightarrow k \rightarrow 0$ of R -modules with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^\perp$ by Lemma 2.6. For an integer $i \geq 0$, put $H_i = \text{Tr } \Omega(\Omega^i G)$. The module H_i has no nonzero free summand. We have $(H_i)^* H_i \neq R$. Hence $((H_i)^* H_i)k = 0$ for $i \geq 0$. Applying Proposition 3.2 to the contravariant cohomological δ -functor $D = (\text{Ext}_R^j(_, M))_{j \geq 0}$, we obtain $D^j(k) = 0$ for $j \gg 0$. Namely, we have $\text{Ext}_R^j(k, M) = 0$ for $j \gg 0$, which implies that M has finite injective dimension. \square

REFERENCES

- [1] M. AUSLANDER, Anneaux de Gorenstein, et torsion en algèbre commutative. Séminaire d'Algèbre Commutative dirigé par Pierre Samuel, 1966/67, Texte rédigé, d'après des exposés de Maurice Auslander, Marquerite Mangeney, Christian Peskine et Lucien Szpiro. École Normale Supérieure de Jeunes Filles, *Secrétariat mathématique, Paris*, 1967.
- [2] M. AUSLANDER; M. BRIDGER, Stable module theory. Memoirs of the American Mathematical Society, No. 94, *American Mathematical Society, Providence, R.I.*, 1969.
- [3] M. AUSLANDER; R.-O. BUCHWEITZ, The homological theory of maximal Cohen-Macaulay approximations. Colloque en l'honneur de Pierre Samuel (Orsay, 1987), *Mém. Soc. Math. France (N.S.)* No. 38, (1989), 5–37.
- [4] M. AUSLANDER; I. REITEN, Applications of contravariantly finite subcategories. *Adv. Math.* **86** (1991), no. 1, 111–152.
- [5] M. AUSLANDER; S. O. SMALØ, Preprojective modules over Artin algebras. *J. Algebra* **66** (1980), no. 1, 61–122.
- [6] M. AUSLANDER; S. O. SMALØ, Almost split sequences in subcategories. *J. Algebra* **69** (1981), no. 2, 426–454.
- [7] L. L. AVRAMOV; A. MARTSINKOVSKY, Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension. *Proc. London Math. Soc. (3)* **85** (2002), no. 2, 393–440.
- [8] A. BELIGIANNIS, Cohen-Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras. *J. Algebra* **288** (2005), no. 1, 137–211.
- [9] W. BRUNS; J. HERZOG, Cohen-Macaulay rings. revised edition. Cambridge Studies in Advanced Mathematics, 39, *Cambridge University Press, Cambridge*, 1998.
- [10] R.-O. BUCHWEITZ; G.-M. GREUEL; F.-O. SCHREYER, Cohen-Macaulay modules on hypersurface singularities. II. *Invent. Math.* **88** (1987), no. 1, 165–182.
- [11] L. W. CHRISTENSEN, Gorenstein dimensions. Lecture Notes in Mathematics, 1747. *Springer-Verlag, Berlin*, 2000.
- [12] L. W. CHRISTENSEN; G. PIEPMEYER; J. STRIULI; R. TAKAHASHI, Finite Gorenstein representation type implies simple singularity. Preprint (2007), <http://arxiv.org/abs/0704.3421>.
- [13] E. E. ENOCHS; O. M. G. JENDA, Relative homological algebra. de Gruyter Expositions in Mathematics, 30. *Walter de Gruyter & Co., Berlin*, 2000.
- [14] J. HERZOG, Ringe mit nur endlich vielen Isomorphieklassen von maximalen, unzerlegbaren Cohen-Macaulay-Moduln. (German) *Math. Ann.* **233** (1978), no. 1, 21–34.
- [15] F. ISCHEBECK, Eine Dualität zwischen den Funktoren Ext und Tor. (German) *J. Algebra* **11** (1969), 510–531.
- [16] R. TAKAHASHI, On the category of modules of Gorenstein dimension zero. II. *J. Algebra* **278** (2004), no. 1, 402–410.
- [17] R. TAKAHASHI, Modules of G-dimension zero over local rings of depth two. *Illinois J. Math.* **48** (2004), no. 3, 945–952.
- [18] R. TAKAHASHI, On the category of modules of Gorenstein dimension zero. *Math. Z.* **251** (2005), no. 2, 249–256.
- [19] R. TAKAHASHI, Remarks on modules approximated by G-projective modules. *J. Algebra* **301** (2006), no. 2, 748–780.
- [20] R. TAKAHASHI, A new approximation theory which unifies spherical and Cohen-Macaulay approximations. *J. Pure Appl. Algebra* **208** (2007), no. 2, 617–634.
- [21] R. TAKAHASHI, On the number of indecomposable totally reflexive modules. *Bull. London Math. Soc.* **39** (2007), no. 3, 487–492.
- [22] R. TAKAHASHI, Contravariantly finite resolving subcategories over commutative rings, preprint (2007).
- [23] T. WAKAMATSU, Stable equivalence for self-injective algebras and a generalization of tilting modules. *J. Algebra* **134** (1990), no. 2, 298–325.
- [24] J. XU, Flat covers of modules. Lecture Notes in Mathematics, 1634, *Springer-Verlag, Berlin*, 1996.
- [25] Y. YOSHINO, Cohen-Macaulay modules over Cohen-Macaulay rings. London Mathematical Society Lecture Note Series, 146, *Cambridge University Press, Cambridge*, 1990.

- [26] Y. YOSHINO, Modules of G-dimension zero over local rings with the cube of maximal ideal being zero. *Commutative algebra, singularities and computer algebra (Sinaia, 2002)*, 255–273, NATO Sci. Ser. II Math. Phys. Chem., 115, *Kluwer Acad. Publ., Dordrecht*, 2003.
- [27] Y. YOSHINO, Approximations by modules of G-dimension zero. *Algebraic structures and their representations*, 119–125, *Contemp. Math.*, 376, *Amer. Math. Soc., Providence, RI*, 2005.

DEPARTMENT OF MATHEMATICAL SCIENCES, FACULTY OF SCIENCE, SHINSHU UNIVERSITY, 3-1-1
ASAHI, MATSUMOTO, NAGANO 390-8621, JAPAN

E-mail address: `takahasi@math.shinshu-u.ac.jp`