The differential module of the polynomial ring
with
the action of the symmetric group

Nagoya, 20, November, 2007
H. Morita, A. Wachi, J. Watanabe

Let $R = K[x_1, \cdots, x_k]$ be the polynomial ring over K, a field of characteristic 0. Let Ω be the module of differentials:

$$\Omega := Rdx_1 \oplus \cdots \oplus Rdx_k$$

Let $G := S_k$ be the symmetric group in k letters. Let G act on R by permutation of the variables. Extend the action to

$$\wedge^j \Omega$$

for

$$j = 0, 1, \cdots, k.$$

We consider the following problems:

Problem 1. Decompose $\wedge^j \Omega$ into irreducible S_k-modules.

Problem 2. Determine the Hilbert series of the isotypic components of

$$\wedge^j \Omega.$$

Recall that the irreducible modules of S_k are parametrized by the partitions of k. Thus we write W^λ for the irreducible S_k-module corresponding to $\lambda \vdash k$.

Let $Y^\lambda(-)$ be the functor from the category of S_k-modules to itself

"to extract the isotypic component"

belonging to λ. Note that it is an exact functor.

For a graded vector space M,

$$h(M, q)$$

denotes the Hilbert series of M. (This is a power series in q with positive integers as coefficients.)

Example 1 : The case where $k = 2$.

Assume $k = 2$. Then $R = K[x, y]$. There are only two partitions: $\lambda = \begin{cases} 2, \\ 11. \end{cases}$ We want to determine

$$h(Y^\lambda(\wedge^j \Omega), q)$$

for $j = 0, 1, 2$.

For $j = 0$, it is easy to determine the Hilbert series since

$$\begin{cases} Y^{(2)}(\wedge^0 \Omega) = R^G = K[x + y, xy] \\ Y^{(11)}(\wedge^0 \Omega) = (x - y)R^G \end{cases}$$

and since R is the direct sum:

$$R = R^G \oplus (x - y)R^G.$$

For $j = 2$, we have

$$\wedge^2 \Omega \cong R(dx \wedge dy).$$

Thus we have

$$\begin{cases} h(Y^{(2)}(\wedge^2 \Omega), q) = h(Y^{(11)}(R), q), \\ h(Y^{(11)}(\wedge^2 \Omega), q) = h(Y^{(2)}(R), q). \end{cases}$$

For $j = 1$, we have to decompose the module

$$\Omega \cong Rdx \oplus Rdy.$$

As is easily seen, symmetric 1-forms are of the form either $sdx + sdy$ with $s \in R^G$ or $adx - ady$ with $a \in (x - y)R^G$, and alternating 1-forms are either $sdx - sdy$ or $adx + ady$.

Thus we have obtained the following table for $h(Y^\lambda(\wedge^j \Omega), q)$.

	$j = 0$	$j = 1$	$j = 2$
$\lambda = (2)$	$\frac{1}{(1-q)(1-q^2)}$	$\frac{1}{(1-q)^2}$	$\frac{q}{(1-q)(1-q^2)}$
$\lambda = (11)$	$\frac{q}{(1-q)(1-q^2)}$	$\frac{1}{(1-q)^2}$	$\frac{1}{(1-q)(1-q^2)}$

Example 2: The case where $j = 0$

Fix $j = 0$. So $\wedge^j \Omega = R$. If $\lambda = (k)$, the trivial partition, then $Y^\lambda(R)$ is R^G, the ring of invariants. So we have

$$h(R^G, q) = \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)},$$

since R^G is generated by the elementary symmetric functions.

Put $\overline{R} := R/(R^G_+)$. Then it is easy to see that

$$\begin{aligned} h(Y^\lambda(R), q) &= h(R^G, q) h(Y^\lambda(\overline{R}), q) \\ &= \frac{h(Y^\lambda(\overline{R}), q)}{(1-q)(1-q^2)\cdots(1-q^k)}. \end{aligned}$$

By a result of Terasoma-Yamada. the numerator (which is a polynomial)

$$h(Y^\lambda(\overline{R}), q)$$

is known to be the q-analog of the hook length formula multiplied by the dimension of W^λ with a certain shift of degree determined by λ.

Main result: The general case

Write

$$\Omega(-n) = R dx_1 \oplus \cdots \oplus R dx_n,$$

when we give dx_i degree n. We have been considering

$$\Omega = \Omega(0).$$

Similarly $R(-n)$ denotes the free module of rank 1 generated by a generator of degree n. Thus

$$\begin{aligned} \wedge^j (\Omega(-n)) &\cong R^{\binom{k}{j}}(-jn) \\ &\cong \left(\wedge^j \Omega\right)(-jn) \end{aligned}$$

Also put

$$A(n) = R/(x_1^n, \cdots, x_k^n).$$

For simplicity put $F = \Omega(-n)$. We would like to construct a minimal free resolution of $A(n)$ as an R-module such that the boundary maps are compatible with the action of S_k. For this the usual minimal free resolution suffices:

$$\to \wedge^3 F \to \quad \wedge^2 F \to \quad \wedge^1 F \to \quad \wedge^0 F \to \quad A(n) \to \quad 0 \tag{1}$$

We want to know

$$\xi_j := h(Y^\lambda(\wedge^j\Omega), q).$$

Assume that we know

$$h_n := h(Y^\lambda(A(n)), q),$$

for $n = 0, 1, 2, \cdots$. Fix $\lambda \vdash k$ and apply the functor $Y^\lambda(-)$ to the sequence (1) above. Then we have

$$\to \quad Y^\lambda(\wedge^3 F) \to \quad Y^\lambda(\wedge^2 F) \to \quad Y^\lambda(\wedge^1 F) \to \quad Y^\lambda(\wedge^0 F) \to \quad Y^\lambda(A(n)) \to \quad 0.$$

Since the sequence is exact it gives us:

$$h_n = \sum_{j=0}^{k} (-1)^j q^{nj} \xi_j \tag{2}$$

This means that we have an infinite set of linear equations relating $\{\xi_i\}$ and $\{h_i\}$. For example if $k = 3$, we have

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -q & q^2 & -q^3 \\ 1 & -q^2 & q^4 & -q^6 \\ 1 & -q^3 & q^6 & -q^9 \\ 1 & -q^4 & q^8 & -q^{12} \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ \vdots \end{pmatrix}$$

Note that any maximal minor of the matrix $\{(-q^n)^j\}$ is non-zero. Thus we have proved the following theorem.

Theorem With $\lambda \vdash k$ fixed, the set of Hilbert series

$$\xi_0, \xi_1, \xi_2, \cdots, \xi_k.$$

in consideration are determined by any $(k+1)$ terms of

$$h_0, h_1, h_2, \cdots.$$

Consequenctly the infinite sequece

$$h_0, h_1, h_2, \cdots.$$

is determined by any $k+1$ terms.

Actually h_0 and h_1 are known for all $\lambda \vdash k$. (This is trivial.) Hence the above theorem is rephrased as follows:

Theorem′ With $\lambda \vdash k$ fixed, any $k-1$ terms in the infinite series

$$h_2, h_3, \cdots$$

determine

$$\xi_0, \xi_1, \xi_2, \cdots, \xi_k$$

and they determine all

$$h_2, h_3, \cdots$$

As is easily conceived we have the duality

$$Y^{\lambda}(\wedge^j \Omega) \cong Y^{\overline{\lambda}}(\wedge^{k-j}\Omega)$$

Thus we have the following

Theorem″ Any $[(k+1)/2]$ terms in the infinite series

$$h_0, h_1, h_2, \cdots$$

for all $\lambda \vdash k$ determine

$$\xi_0, \xi_1, \xi_2, \cdots, \xi_k.$$

A result of Morita-Wachi-Watanabe says that

$$h_n$$

is the q-analog of the Weyl dimension formula. It means that we have determined

$$h(Y^{\lambda}(\wedge^j \Omega), q)$$

for all $\begin{cases} \lambda \vdash k, \\ j = 0, 1, \cdots, k. \end{cases}$

q-analog of the hook length formula

For $\lambda \vdash k$ let W^{λ} be the irreducibe S_k-module corresponding to λ. Then dim W^{λ} is given by

$$\dim\ W^{\lambda} = \frac{k!}{\prod h_{ij}} \tag{3}$$

where h_{ij} is the hook length at the (i, j)-th position. The following is an example which shows the matrix $\{h_{ij}\}$ for the Young diagram $\lambda = (5, 3, 1)$.

7	5	4	2	1
4	2	1		
1				

In (3) replace integer a by the polynomial

$$\begin{aligned} [a] : &= \frac{1-q^a}{1-q} \\ &= 1 + q + \cdots + q^{a-1} \end{aligned}$$

It is the "q-analog of the hook length formula."

Since there are same number of integers in the denominator and enumerator of (3), it is the same if we replace a by

$$1 - q^a.$$

q-analog of the Weyl dimension formula

Let $\lambda \vdash k$. Let $n > 0$ be any integer. Let V^λ be the irreducible $GL(n)$-module. Then dim V^λ is given by

$$\dim\ V^\lambda = \frac{\mu_1! \mu_2! \cdots \mu_n!}{(n-1)!(n-2)! \cdots 2!1! \prod h_{ij}}$$

where

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$$

and

$$\mu := (\mu_1, \mu_2, \cdots, \mu_n)$$

is defined by

$$\mu = \lambda + (n-1, n-2, \cdots, 1, 0).$$

If λ has more than n parts, we let dim $V^\lambda = 0$.

The Hilbert series h_n of the module

$$Y^\lambda(A(n))$$

is given by the q-analog of the Weyl dimension formula multiplied by $\dim W^\lambda$ with a certain shift of degrees.