GOTZMANN IDEALS OF THE POLYNOMIAL RING

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Abstract. Let A denote the polynomial ring in n variables over a field. All the Gotzmann ideals of A with at most n generators will be classified. This is a joint work with Takayuki Hibi.

1. Introduction

Let $A = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K with each $\deg x_i = 1$. Let $<_{\mathrm{lex}}$ be the lexicographic order on A induced by the ordering $x_1 > x_2 > \cdots > x_n$. Recall that a *lexsegment* ideal of A is a monomial ideal I of A such that, for monomials u and v of A with $u \in I$, $\deg u = \deg v$ and $u <_{\mathrm{lex}} v$, one has $v \in I$. Let I be a homogeneous ideal of A and I^{lex} the (unique) lexsegment ideal ([2] and [11]) with the same Hilbert function as I. A homogeneous ideal I of A is said to be *Gotzmann* if the number of minimal generators of I is equal to that of I^{lex}. Gotzmann ideals were introduced by Herzog and Hibi [9] in the study of maximal Betti numbers of ideals for a given Hilbert function. Indeed, Herzgo and Hibi proved that a homogeneous ideal I is Gotzmann if and only if the graded Betti numbers of I are equal to those of I^{lex}. Our goal is to classify all the Gotzmann ideals of A generated by at most n homogeneous polynomials.

A homogeneous ideal I of A is said to have a *critical function* if I^{lex} is generated by at most n monomials. Let $1 \leq s \leq n$ and $f_1, \ldots, f_s$ homogeneous polynomials with

$$f_i \in K[x_i, x_{i+1}, \ldots, x_n]$$

for each $1 \leq i \leq s$ and with $\deg f_s > 0$. In [7] the ideal $I_{(f_1,\ldots,f_s)}$ of A defined by

$$I_{(f_1,\ldots,f_s)} = (f_1x_1, f_1f_2x_2, \ldots, f_1f_2\cdots f_{s-1}x_{s-1}, f_1f_2\cdots f_s) \tag{1}$$

was introduced. A homogeneous ideal I of A is called *canonical critical* if $I = I_{(f_1,\ldots,f_s)}$ for some homogeneous polynomials $f_1, \ldots, f_s$ with $f_i \in K[x_i, x_{i+1}, \ldots, x_n]$ for each $1 \leq i \leq s$ and with $\deg f_s > 0$, where $1 \leq s \leq n$.

Theorem 1.1. *Given a homogeneous ideal I of $A = K[x_1, \ldots, x_n]$, the following conditions are equivalent:*

(i) *I has a critical Hilbert function;*
(ii) *there exists a linear transformation φ on A such that $\varphi(I)$ is a canonical critical ideal;*
(iii) *I is a Gotzmann ideal generated by at most n homogeneous polynomials.*

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2. Universal lexsegment ideals and canonical critical ideals

In this section we study universal lexsegment ideals and canonical critical ideals. A monomial ideal I of $A = K[x_1, \dots, x_n]$ is said to be *universal lexsegment* if for all integers $m \geq 0$ the ideal $I \cdot K[x_1, \dots, x_{n+m}]$ is a lexsegment ideal of $K[x_1, \dots, x_{n+m}]$. Universal lexsegment ideals were introduced by Babson, Novik and Thomas [1]. We recall the following easy fact.

Lemma 2.1. *Let I be a monomial ideal of A. The following conditions are equivalent:*

(i) *I is universal lexsegment;*
(ii) *I is a lexsegment ideal generated by at most n monomials;*
(iii) *there exist integers $b_1, \dots, b_s \in \mathbb{Z}_{\geq 0}$ with $1 \leq s \leq n$ such that*

$$I = (x_1^{b_1+1}, x_1^{b_1}x_2^{b_2+1}, \dots, x_1^{b_1}x_2^{b_2}\cdots x_{s-1}^{b_{s-1}}x_s^{b_s+1}). \tag{2}$$

Proof. First, we will show that (ii) implies (iii). Let $I = (u_1, \dots, u_s)$ be a lexsegment ideal with $s \leq n$. Suppose $\deg u_1 \leq \cdots \leq u_s$ and $u_i >_{\mathrm{lex}} u_{i+1}$ if $\deg u_i = \deg u_{i+1}$. Since $u_1 = x_1^{\deg u_1}$, one has $u_1 = x_1^{b_1+1}$. Let $1 < k \leq \min\{n, \delta\}$ and suppose that $u_{k-1} = x_1^{b_1}x_2^{b_2}\cdots x_{k-1}^{b_{k-1}+1}$. Since the monomial ideal $(u_1, \dots, u_{k-1})$ is lexsegment, it follows that the smallest monomial with respect to $<_{\mathrm{lex}}$ of degree $\deg u_k$ belonging to $(u_1, \dots, u_{k-1})$ is $u_{k-1}x_n^{b_k}$. Since u_k is the biggest monomial with respect to $<_{\mathrm{lex}}$ which satisfies $\deg u_k = \deg(u_{k-1}x_n^{b_k})$ and $u_k <_{\mathrm{lex}} u_{k-1}x_n^{b_k}$, we have $u_k = (u_{k-1}/x_{k-1})x_k^{b_k+1}$. Thus $u_k = x_1^{b_1}x_2^{b_2}\cdots x_{k-1}^{b_{k-1}}x_k^{b_k+1}$, as desired.

The implication (iii) $\Rightarrow$ (i) is easy. We will show that (i) implies (ii). Suppose that I is universal lexsegment with $|G(I)| \geq n+1$. What we must prove is I is not universal lexsegment. Since $I' = I \cdot K[x_1, \dots, x_{n+1}]$ is a lexsegment ideal with $|G(I')| \geq n+1$, there exists a lexsegment ideal J of $K[x_1, \dots, x_{n+1}]$ such that $G(J) \subset G(I')$ and $|G(J)| = n+1$. Then, by the implication (ii) $\Rightarrow$ (iii), J must contains a generator which is divisible by x_{n+1}. Since $G(J) \subset G(I') = G(I) \subset A$, this is a contradiction. $\square$

Example 2.2. (a) The lexsegment ideal $(x_1^2, x_1x_2^2)$ of $K[x_1, x_2]$ is universal lexsegment. In fact, the ideal $(x_1^2, x_1x_2^2)$ of $K[x_1, \dots, x_m]$ is lexsegment for all $m \geq 2$.

(b) The lexsegment ideal $(x_1^3, x_1^2x_2, x_1x_2^2)$ of $K[x_1, x_2]$ is not universal lexsegment. Indeed, since $x_1x_2^2 <_{\mathrm{lex}} x_1^2x_3$ in $K[x_1, x_2, x_3]$, the ideal $(x_1^3, x_1^2x_2, x_1x_2^2)$ of $K[x_1, x_2, x_3]$ is not lexsegment.

By using Lemma 2.1, it is easy to characterize critical Hilbert functions. Indeed, it was shown in [12] that the Hilbert function $H(I, t)$ of the universal lexsegment ideal (2) is given by

$$H(I, t) = \binom{t - a_1 + n - 1}{n - 1} + \cdots + \binom{t - a_s + n - s}{n - s}, \tag{3}$$

where the sequence $(a_1, a_2, \dots, a_s)$ is defined by setting

$$a_i = \deg x_1^{b_1}\cdots x_{i-1}^{b_{i-1}}x_i^{b_i+1}, \qquad 1 \leq i \leq s.$$

Since a lexsegment ideal with a given Hilbert function is uniquely determined, it follows that a homogeneous ideal I of A is critical if and only if there exists a sequence $(a_1, \ldots, a_s)$ of integers with $0 < a_1 \leq a_2 \leq \cdots \leq a_s$, where $1 \leq s \leq n$, such that the Hilbert function of I is of the form (3).

Definition 2.3. A homogeneous ideal I of A with the Hilbert function (3) will be called a *critical ideal of type* $(a_1, a_2, \ldots, a_s)$.

Next, we study the property of canonical critical ideals. We require the following obvious facts.

Lemma 2.4. *Let $1 < s \leq n$. Fix homogeneous polynomials $f_1, \ldots, f_{s-1}$ with each $f_i \in K[x_i, \ldots, x_n]$. Let $g \in K[x_s, \ldots, x_n]$ be a homogeneous polynomial with $\deg g > 0$. Then*

$$f_1 f_2 \cdots f_{s-1} g \notin (f_1 x_1, f_1 f_2 x_2, \ldots, f_1 f_2 \cdots f_{s-1} x_{s-1}).$$

Corollary 2.5. *As a vector space over K the ideal* (1) *is the direct sum*

$$I_{(f_1,\ldots,f_s)} = \left(\bigoplus_{j=1}^{s-1} (f_1 \cdots f_j x_j) K[x_j, \ldots, x_n] \right) \bigoplus (f_1 \cdots f_s) K[x_s, \ldots, x_n]. \tag{4}$$

The above facts implies that canonical critical ideals are critical and Gotzmann.

Proposition 2.6. *Let $I_{(f_1,\ldots,f_s)}$ denote the ideal* (1).

(a) *$I_{(f_1,\ldots,f_s)}$ is a critical ideal of type $(a_1, \ldots, a_s)$, where $a_i = \deg f_1 f_2 \cdots f_i x_i$, $i = 1, \ldots, s-1$, and $a_s = \deg f_1 f_2 \cdots f_s$.*

(b) *$I_{(f_1,\ldots,f_s)}$ is minimally generated by*

$$\{f_1 x_1, \ldots, f_1 f_2 \cdots f_{s-1} x_{s-1}, f_1 f_2 \cdots f_s\}. \tag{5}$$

(c) *$I_{(f_1,\ldots,f_s)}$ is Gotzmann.*

Proof. The direct sum decomposition (4) says that the Hilbert function of $I_{(f_1,\ldots,f_s)}$ is of the form (3) and, in addition, that $I_{(f_1,\ldots,f_s)}$ is minimally generated by (5). Thus (a) and (b) follow. Since the lexsegment ideal with the Hilbert function (3) is the universal lexsegment ideal (2), one has $|G((I_{(f_1,\ldots,f_s)})^{\text{lex}})| = s$. Hence $I_{(f_1,\ldots,f_s)}$ is Gotzmann, as required. $\square$

3. Proof of Theorem 1.1

In the previous section, we already see that canonical critical ideals are Gotzmann ideals having at most n homogeneous generators. On the other hand, it is clear from the definition that Gotzmann ideals generated by at most n homogeneous generators have a critical Hilbert function. Thus, to complete the proof of Theorem 1.1, what we must prove is any critical ideal can be transformed into canonical critical ideals by a linear transformation of A.

For a monomial u of A, we write $m(u)$ for the largest integer j for which x_j divides u. A monomial ideal I of A is called *stable* if, for each monomial u belonging to $G(I)$ and for each $1 \leq i < m(u)$, one has $(x_i u)/x_{m(u)} \in I$.

Lemma 3.1. *A monomial ideal I of A which is both critical and stable is universal lexsegment.*

Proof. (Sketch.) Suppose $|G(I^{\text{lex}})| = s$. It follows from [12] that the projective dimension of S/I is equal to s. Thus, by the Eliahou–Kervaire resolution [6], it follows that there exists a monomial $u_s \in G(I)$ such that $m(u_s) = s$. Then, by using the definition of stable ideals, a straightforward computation implies that there are monomials $u_1, \ldots, u_{s-1} \in G(I)$ such that $m(u_k) = k$ for $k = 1, 2, \ldots, s-1$ and $\deg u_1 \leq \cdots \leq \deg u_s$ (e.g., [10, Lemma 1.3]). Since $|G(I)| \leq |G(I^{\text{lex}})| = s$, we have $G(I) = \{u_1, \ldots, u_s\}$.

Clearly, $u_1 = x_1^{b_1+1}$ for some $b_1 \geq 0$. Then, by arguing inductively, a routine computation implies that $I = (u_1, \ldots, u_s)$ is an ideal of the form (2). $\square$

Let I be an ideal of A. When K is infinite, given a monomial order σ on A, we write $\text{gin}_\sigma(I)$ for the generic initial ideal ([5] and [8]) of I with respect to σ.

Lemma 3.2. *Let I be a critical ideal of A. Then, for an arbitrary monomial order σ on A induced by the ordering $x_1 > \cdots > x_n$, the generic initial ideal $\text{gin}_\sigma(I)$ is stable. Thus in particular $\text{gin}_\sigma(I)$ is universal lexsegment.*

Proof. Since $\text{gin}_\sigma(I)$ is a critical monomial ideal, it follows from [12, Corollary 1.8] that $\text{gin}_\sigma(I)$ is Gotzmann. Thus in particular $\text{gin}_\sigma(I)$ is componentwise linear [9]. Hence [3, Lemma 1.4] says that $\text{gin}_{<_{\text{rev}}}(\text{gin}_\sigma(I)) = \text{gin}_\sigma(I)$ is stable. Here $<_{\text{rev}}$ is the reverse lexicographic order on A induced by the ordering $x_1 > \cdots > x_n$. Since $\text{gin}_\sigma(I)$ is both critical and stable, it follows from Lemma 3.1 that $\text{gin}_\sigma(I)$ is universal lexsegment. $\square$

Note that the above lemma is obvious in characteristic 0, since generic initial ideals are stable in characteristic 0.

Lemma 3.3. *Suppose that a homogeneous ideal I of A is a critical ideal of type $(a_1, \ldots, a_s)$, where $2 \leq s \leq n$. Then there exists a homogeneous polynomial f of A with $\deg f = a_1 - 1$ together with a homogeneous ideal J of A such that*

$$I = f \cdot J.$$

Proof. (Sketch.) By considering an extension field, we may assume that K is infinite. Then there is a linear transformation φ with $\text{in}_{<_{\text{lex}}}(\varphi(I)) = \text{gin}_{<_{\text{lex}}}(I)$. Considering $\varphi(I)$ instead of I, one may assume that $\text{in}_{<_{\text{lex}}}(I) = \text{gin}_{<_{\text{lex}}}(I)$. Lemma 3.2 says that $\text{in}_{<_{\text{lex}}}(I)$ is universal lexsegment. Hence

$$\text{in}_{<_{\text{lex}}}(I) = (x_1^{b_1+1}, x_1^{b_1}x_2^{b_2+1}, \ldots, x_1^{b_1} \cdots x_{s-1}^{b_{s-1}} x_s^{b_s+1}),$$

where $b_i = a_i - a_{i-1}$, $1 \leq i \leq s$, with $a_0 = 1$.

To simplify the notation, let $u_i = x_1^{b_1} \cdots x_i^{b_i}$ for $i = 1, \ldots, s$. Let $\mathcal{G} = \{g_1, \ldots, g_s\}$ be a Gröbner basis of I, where g_i is a homogeneous polynomial of A with $\text{in}_{<_{\text{lex}}}(g_i) = u_i x_i$ for each $1 \leq i \leq s$, and $\mathcal{G}' = \{g_2, \ldots, g_s\}$. We show that $\mathcal{G}'$ is a Gröbner basis with respect to $<_{\text{lex}}$. For $2 \leq i < j \leq s$, consider the S-polynomial

$$S(g_i, g_j) = (u_j/u_i)x_j g_i - x_i g_j.$$

Then, since $\mathrm{in}_{<_{\mathrm{lex}}}(g_1) >_{\mathrm{lex}} \mathrm{in}_{<_{\mathrm{lex}}}(S(g_i, g_j))$ and since $\mathcal{G}$ is a Gröbner basis, a remainder of the S-polynomial of g_i and g_j with respect to $\mathcal{G}'$ can be 0. Hence $\mathcal{G}'$ is a Gröbner basis with respect to $<_{\mathrm{lex}}$, as desired.

Now, we prove Lemma 3.3 by using induction on s. Suppose $s > 2$ (the proof for $s = 2$ is similar). Let J be the ideal of A generated by $\mathcal{G}'$. Since

$$\mathrm{in}_{<_{\mathrm{lex}}}(J) = (u_2x_2, \ldots, u_sx_s) = x_1^{b_1}(x_2^{b_2+1}, x_2^{b_2}x_3^{b_3+1}, \ldots, x_2^{b_2}\cdots x_{s-1}^{b_{s-1}}x_s^{b_s+1})$$

and since

$$(x_2^{b_2+1}, x_2^{b_2}x_3^{b_3+1}, \ldots, x_2^{b_2}\cdots x_{s-1}^{b_{s-1}}x_s^{b_s+1})$$

is universal lexsegment in $K[x_2, \ldots, x_n, x_1]$, the ideal J is a critical ideal of type $(a_2, \ldots, a_s)$. The induction hypothesis guarantees the existence of a homogeneous polynomial f_0 of A with $\deg(f_0) = a_2 - 1$ which divides each of $g_2, \ldots, g_s$. Since $\mathrm{in}_{<_{\mathrm{lex}}}(f_0)$ divides $\mathrm{in}_{<_{\mathrm{lex}}}(g_i) = u_ix_i$ for each $1 < i \leq s$, one has $\mathrm{in}_{<_{\mathrm{lex}}}(f_0) = u_2$. Let $g_i' = g_i/f_0$ for $i = 2, \ldots, s$. Thus in particular $\mathrm{in}_{<_{\mathrm{lex}}}(g_2') = u_2x_2/u_2 = x_2$.

Now, divide the S-polynomial of g_1 and g_2 by $\mathcal{G}$, say,

$$x_2^{b_2+1}g_1 - x_1(f_0g_2') = q_1g_1 + q_2(f_0g_2') + \cdots + q_s(f_0g_s'),$$

where $q_1, \ldots, q_s$ are homogeneous polynomials of A with

$$\mathrm{in}_{<_{\mathrm{lex}}}(q_1g_1) \leq_{\mathrm{lex}} \mathrm{in}_{<_{\mathrm{lex}}}(x_2^{b_2+1}g_1 - x_1(f_0g_2'))$$

and with

$$\mathrm{in}_{<_{\mathrm{lex}}}(q_k(f_0g_k')) \leq_{\mathrm{lex}} \mathrm{in}_{<_{\mathrm{lex}}}(x_2^{b_2+1}g_1 - x_1(f_0g_2'))$$

for each $2 \leq k \leq s$. Let

$$f_0h = q_2(f_0g_2') + \cdots + q_s(f_0g_s').$$

Thus

$$(x_2^{b_2+1} - q_1)g_1 = f_0(x_1g_2' + h).$$

Since $\mathrm{in}_{<_{\mathrm{lex}}}(x_2^{b_2+1} - q_1) = x_2^{b_2+1}$, $\mathrm{in}_{<_{\mathrm{lex}}}(g_1) = x_1^{b_1+1}$ and $\mathrm{in}_{<_{\mathrm{lex}}}(x_1g_2' + h) = x_1x_2$, it follows that $x_1g_2' + h$ can divide neither $x_2^{b_2+1} - q_1$ nor g_1. Thus $x_1g_2' + h$ is a product $(x_1 + h_1)(x_2 + h_2)$, where h_1 and h_2 are homogeneous polynomials of A with $\deg h_1 = \deg h_2 = 1$, such that $x_1 + h_1$ divides g_1 and $x_2 + h_2$ divides $x_2^{b_2+1} - q_1$. Let $f = g_1/(x_1 + h_1)$. Then $\deg f = a_1 - 1$ and f divides both g_1 and f_0. $\square$

We are now in the position to give a proof of Theorem 1.1.

Proof of Theorem 1.1. What we must prove is (i) implies (ii). This will be achieved by induction on s. Let $I \subset A$ be a critical ideal of type $(a_1, \ldots, a_s)$. If $s = 1$, then the statement is obvious.

Let $s > 1$. Lemma 3.3 guarantees that $I = f \cdot J$, where f is a homogeneous polynomial of A with $\deg f = a_1 - 1$ and where J is a homogeneous ideal of A. The Hilbert function of J is $H(J, t) = H(I, t + a_1 - 1)$. Hence J is a critical ideal of type$(1, a_2 - a_1 + 1, \ldots, a_s - a_1 + 1)$. Since $H(J, 1) \neq 0$, there exists a linear transformation φ on A with $x_1 \in \varphi(J)$. Let J' be the ideal

$$J' = \varphi(J) \cap K[x_2, \ldots, x_n]$$

of $K[x_2, \ldots, x_n]$. Then

$$\varphi(J) = x_1 K[x_1, \ldots, x_n] \bigoplus J'.$$

A straightforward computation implies that the ideal J' of $K[x_2, \ldots, x_n]$ is a critical ideal of type $(a_2 - a_1 + 1, \ldots, a_s - a_1 + 1)$. The induction hypothesis then guarantees the existence of a linear transformation ψ on $K[x_2, \ldots, x_n]$ such that $\psi(J')$ is a canonical critical ideal of $K[x_2, \ldots, x_n]$, say

$$\psi(J') = (f_2 x_2, \ldots, f_2 \cdots f_{s-1} x_{s-1}, f_2 \cdots f_s),$$

where $f_i \in K[x_i, x_{i+1}, \ldots, x_n]$ for each $2 \leq i \leq s$ and where $\deg f_s > 0$. Now, regarding ψ to be a linear transformation on A by setting $\psi(x_1) = x_1$, one has

$$\begin{aligned}(\psi \circ \varphi)(I) &= ((\psi \circ \varphi)(f)) \cdot ((\psi \circ \varphi)(J)) \\ &= ((\psi \circ \varphi)(f)) \cdot (\psi(x_1 A \bigoplus J')) \\ &= (\psi \circ \varphi)(f) \cdot (x_1 A \bigoplus \psi(J')).\end{aligned}$$

Let $f_1 = (\psi \circ \varphi)(f)$. Then it follows that

$$(\psi \circ \varphi)(I) = (f_1 x_1, f_1 f_2 x_2, \ldots, f_1 f_2 \cdots f_{s-1} x_{s-1}, f_1 f_2 \cdots f_s)$$

as desired. $\square$

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