Remarks on equivalences of additive subcategories

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Abstract

We study category equivalences between some additive subcategories of module categories. As its application, we show that the group of autofunctors of the category of reflexive modules over a normal domain is isomorphic to the divisor class group.

1 A necessary condition for equivalences of additive subcategories

Let R be a commutative ring. We denote the category of all finitely generated R-modules by R-mod, and the full subcategory of R-mod consisting of all reflexive modules by $\mathrm{ref}(R)$. If R is a Cohen-Macaulay local ring, we denote the category of maximal Cohen-Macaulay modules by $\mathrm{CM}(R)$ as a full subcategory of R-mod. By an additive subcategory we always mean a full subcategory which is closed under finite direct sums and direct summands.

Theorem 1. Let A and B be commutative rings. Let $\mathfrak{C}$ (resp. $\mathfrak{D}$) be an additive full subcategory of A-mod (resp. B-mod) which contains a nontrivial free module. If there is a category equivalence between $\mathfrak{C}$ and $\mathfrak{D}$, then $A \cong B$ as a ring.

Moreover, if F and G are the functors which give the equivalences above, then F and G are of the forms $F(X) \cong \mathrm{Hom}_A(G(B),\ X)$ and $G(Y) \cong \mathrm{Hom}_B(F(A),\ Y)$ for each $X \in \mathfrak{C},\ Y \in \mathfrak{D}$.

proof. Let $F\colon \mathfrak{C} \to \mathfrak{D}$ and $G\colon \mathfrak{D} \to \mathfrak{C}$ be functors satisfying $F \cdot G \cong 1_{\mathfrak{D}}$ and $G \cdot F \cong 1_{\mathfrak{C}}$. We denote the B-module $F(A)$ by M and the A-module $G(B)$ by N. Since F and G are fully faithful functors, there exist isomorphisms as rings $\mathrm{End}_B(M) \cong \mathrm{End}_A(A) = A$ and $\mathrm{End}_A(N) \cong \mathrm{End}_B(B) = B$. Thus there are natural maps as follows:

$$\begin{array}{ccccccccc} B & \xrightarrow{\beta} & \mathrm{End}_B(M) & \xrightarrow{\cong} & A & \xrightarrow{\alpha} & \mathrm{End}_A(N) & \xrightarrow{\cong} & B \\ b & \longrightarrow & b_M & \longrightarrow & a & \longrightarrow & a_N & \longrightarrow & b', \end{array} \qquad (1.1)$$

where b_M (resp. a_N) denotes the multiplication map on M (resp. N) by b (resp. a).

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First of all, we claim that $b - b' \in \mathrm{Ann}_B M$ for such b and b' as above. Since M is finitely generated B-module, we can take a finite free cover of M and get the following diagram.

$$\begin{array}{ccc} B^{\oplus n} & \longrightarrow & M \\ \Big\downarrow b'_B & & \Big\downarrow b_M \\ B^{\oplus n} & \longrightarrow & M \end{array}$$

Applying the functor G to this diagram, we have a diagram

$$\begin{array}{ccc} N^{\oplus n} & \longrightarrow & A \\ \Big\downarrow a_N & & \Big\downarrow a_A \\ N^{\oplus n} & \longrightarrow & A, \end{array}$$

which is commutative. Since G is an equivalence, this implies that the first diagram is also commutative. Hence we have $b - b' \in \mathrm{Ann}_B M$ as desired.

We denote $\mathrm{Ann}_B M$ by $\mathfrak{b}$ and $\mathrm{Ann}_A N$ by $\mathfrak{a}$. Note that the map $A \xrightarrow{\alpha} \mathrm{End}_A(N) \cong B$ induces an injective mapping $q : A/\mathfrak{a} \to B$. We define the map $p : B \to A/\mathfrak{a}$ as the composition of $B \xrightarrow{\beta} \mathrm{End}_B(M) \cong A$ with the natural projection $A \to A/\mathfrak{a}$.

Secondly, we claim that $\mathrm{Ker}\ p = \mathfrak{b}$. Since $\beta(\mathfrak{b}) = 0$, it is clear that $\mathrm{Ker}\ p \supseteq \mathfrak{b}$. To prove the converse let $b \in \mathrm{Ker}\ p$. Then, since $\beta(b) \in \mathfrak{a}$, we have $b' = 0$ as in the notation as in (1.1). Since we have shown that $b - b' \in \mathfrak{b}$, we have $b \in \mathfrak{b}$ and the equality $\mathrm{Ker}\ p = \mathfrak{b}$ is proved. Therefore the mapping p induces an isomorphism $A/\mathfrak{a} \cong B/\mathfrak{b}$.

Thirdly we note that any object $Y \in \mathfrak{D}$ has structure of a (B, A)-bimodule. In fact, the category $\mathfrak{D}$ is a full subcategory of B-mod, therefore Y is naturally equipped with left B-module structure. Since F is a dense functor, there exists an object $X \in \mathfrak{C}$ such that $F(X) \cong Y$ and there is a natural ring homomorphism

$$A \to \mathrm{End}_A(X) \cong \mathrm{End}_B(Y),$$

which maps a to $F(a_X)$. Now, for any $a \in A$ and $y \in Y$, we define $y \circ a := F(a_X)(y)$. Since A is a commutative ring, it yields right A-module structure for $Y \in \mathfrak{D}$. Since the equality $(b \circ y) \circ a = F(a_X)(by) = bF(a_X)(y) = b \circ (y \circ a)$ holds for $a \in A$, $b \in B$ and $y \in Y$, we see that Y has structure of a (B, A)-bimodule. Similarly any object $X \in \mathfrak{C}$ has structure of an (A, B)-bimodule.

Since F is an equivalence, there exists an isomorphism as A-modules for any object $X \in \mathfrak{C}$:

$${}_A X = \mathrm{Hom}_A({}_A A, {}_A X) \cong \mathrm{Hom}_B({}_B M_A, {}_B F(X)).$$

The second part of the theorem follows from this isomorphism.

To complete the proof, we need to show $\mathfrak{a} = \mathfrak{b} = (0)$. For this, we note from the definition of bimodule structure that N is isomorphic to $\mathrm{Hom}_B({}_BM_A, B)$ as an (A, B)-bimodule. In particular, there are isomorphisms of B-modules;

$$B \cong \mathrm{End}_A N \cong \mathrm{End}_A(\mathrm{Hom}_B({}_BM_A, B)).$$

Since any element $b \in \mathfrak{b}$ acts as a zero map on $\mathrm{Hom}_B({}_BM_A, B)$, it must be zero as an element of $\mathrm{End}_A(\mathrm{Hom}_B({}_BM_A, B))$. Consequently we have $b = 0$. Thus $\mathfrak{b} = 0$, and $\mathfrak{a} = 0$ as well. □

Corollary 2. Let A and B be Cohen-Macaulay local rings. Then $\mathrm{CM}(A)$ and $\mathrm{CM}(B)$ are equivalent as additive categories if and only if A is isomorphic to B as a ring.

Our theorem is somehow a generalization of Morita equivalence theorem which deals with abelian categories over non-commutative rings. See [3]. The difference is that, assuming rings are commutative, we are concerned with additive subcategories which are not necessarily abelian and our functors are not necessarily exact.

2 Groups of autofunctors over additive subcategories

Let R be a commutative ring and let $\mathfrak{C}$ be an additive subcategory of R-mod. By an autofunctor F on $\mathfrak{C}$, we mean a covariant functor $F : \mathfrak{C} \to \mathfrak{C}$ which gives rise to an equivalence of categories. We denote by $\mathrm{Aut}(\mathfrak{C})$ the group of all isomorphism classes of autofunctors over $\mathfrak{C}$. By an easy observation using Morita equivalence, it is known that $\mathrm{Aut}(R\text{-mod})$ is isomorphic to the Picard group $\mathrm{Pic}(R)$. As an application of Theorem 1 we can show the following theorem.

Theorem 3. Let A be a Noetherian normal domain. Then there is an isomorphism of groups

$$\mathrm{Aut}(\mathrm{ref}(A)) \cong C\ell(A),$$

where $C\ell(A)$ denotes the divisor class group of A.

proof. It follows from Theorem 1 that any $F \in \mathrm{Aut}(\mathrm{ref}(A))$ has a description $F \cong \mathrm{Hom}_A(M, \ )$ for some reflexive A-module M. Since F is an autofunctor, there exists a functor G of the form $\mathrm{Hom}_B(N, \ )$ for some $N \in \mathrm{ref}(A)$ satisfying $F \cdot G \cong G \cdot F \cong 1_{\mathrm{ref}(A)}$. Hence we have a sequence of isomorphisms of A-modules

$$A \cong G \cdot F(A) \cong \mathrm{Hom}_A(N, \mathrm{Hom}_A(M, A)) \cong \mathrm{Hom}_A(M \otimes_A N, A),$$

which forces $\mathrm{rank} M = 1$. Thus M defines the divisor class $[M]$ in $C\ell(A)$. We define a homomorphism $\alpha : \mathrm{Aut}(\mathrm{ref}(A)) \to C\ell(A)$ by mapping an autofunctor $F \cong \mathrm{Hom}_R(M, \ )$ to $[M]$.

We should remark that α is a well-defined mapping. But it is clear from Yoneda's lemma which claims that if $\mathrm{Hom}_R(M, \ ) \cong \mathrm{Hom}_R(M', \ )$ as functors on $\mathrm{ref}(A)$ for $M, M' \in \mathrm{ref}(A)$, then $M \cong M'$ as A-modules. It is also not difficult to verify that α is a homomorphism of groups. In fact this follows from the isomorphism of functors on $\mathrm{ref}(A)$;

$$\mathrm{Hom}_A(M, \ \ ) \cdot \mathrm{Hom}_A(N, \ \ ) \cong \mathrm{Hom}_A((M \otimes_A N)^{**}, \ \ ).$$

We only have to show that α is an isomorphism. It is obvious from the definition that α is injective. In the rest we shall show that α is surjective. For this let $[I] \in C\ell(A)$ be an arbitrary element, where I is a divisorial fractional ideal of A. It is enough to see that $\mathrm{Hom}_A(I, \ \ )$ is a well-defined autofunctor on $\mathrm{ref}(A)$.

First we remark from Bourbaki [2, Chapter VII, §2] that an A-lattice M is reflexive if and only if the equality $M = \bigcap_{\mathfrak{p} \in H(R)} M_{\mathfrak{p}}$ holds, where $H(A)$ is the set of all prime ideal of height one. Secondly we note that that the equality

$$\mathrm{Hom}_A(X, Y) = \bigcap_{\mathfrak{p} \in H(R)} \mathrm{Hom}_A(X_{\mathfrak{p}}, Y_{\mathfrak{p}})$$

holds for $X, Y \in \mathrm{ref}(A)$. In fact, any $f \in \bigcap_{\mathfrak{p} \in H(R)} \mathrm{Hom}_A(X_{\mathfrak{p}}, Y_{\mathfrak{p}})$ maps X to $Y_{\mathfrak{p}}$ for all $\mathfrak{p} \in H(A)$, hence $f(X) \subseteq \bigcap_{\mathfrak{p} \in H(R)} Y_{\mathfrak{p}} = Y$, and thus $f \in \mathrm{Hom}_A(X, Y)$.

Combining the above two claims we see that $\mathrm{Hom}_A(I, X)$ is a reflexive lattice for any $X \in \mathrm{ref}(A)$. Hence $\mathrm{Hom}_A(I, \ \ )$ yields a functor from $\mathrm{ref}(A)$ to itself.

Since I is a divisorial ideal, there exists an ideal J with $[J] = -[I]$ in $C\ell(A)$, i.e. $(I \otimes_A J)^{**} \cong A$ where $(\ )^*$ denotes $\mathrm{Hom}_A(\ , A)$.

Therefore there are isomorphisms of functors on $\mathrm{ref}(A)$;

$$\begin{aligned} \mathrm{Hom}_A(J, \mathrm{Hom}_A(I, \ \ )) &\cong \mathrm{Hom}_A(I \otimes_A J, \ \ ) \cong \mathrm{Hom}_A((I \otimes_A J)^{**}, \ \ ) \\ &\cong \mathrm{Hom}_A(A, \ \ ) = 1_{\mathrm{ref}(A)}. \end{aligned}$$

This shows that $\mathrm{Hom}(I, \ )$ is an autofunctor over $\mathrm{ref}(A)$ as desired, and the proof is completed. □

Corollary 4. Let A be a normal domain of dimension at most two. Then $\mathrm{Aut}(\mathrm{CM}(A)) \cong C\ell(A)$.

proof. In fact, the equality $\mathrm{CM}(A) = \mathrm{ref}(A)$ holds in this case. □

Compared with the corollary, the groups of autofunctors of $\mathrm{CM}(A)$ are expected to be rather small for higher dimensional rings A. In fact we can prove the following theorem.

Theorem 5. Let A be a Cohen-Macaulay local ring. Suppose that A has only an isolated singularity with $\dim A \geq 3$. Then $\mathrm{Aut}(\mathrm{CM}(A))$ is a trivial group.

proof. Let F be an autofunctor over $\mathrm{CM}(A)$. By virtue of Theorem 1, there exists a maximal Cohen-Macaulay module M with $F \cong \mathrm{Hom}_A(M,\ \ )$. Assume that M is not free, and we shall show a contradiction. For this, take a free cover F of M and we obtain an exact sequence

$$0 \longrightarrow \Omega(M) \longrightarrow F \longrightarrow M \longrightarrow 0.$$

Recall that $\Omega(M)$ is also a maximal Cohen-Macaulay module. Apply $\mathrm{Hom}_A(M,\ \ )$ to the sequence, and we get an exact sequence

$$0 \to \mathrm{Hom}(M, \Omega(M)) \to \mathrm{Hom}(M, F) \to \mathrm{Hom}(M, M) \xrightarrow{f} \mathrm{Ext}^1(M, \Omega(M)) \ .$$

Note that $f \neq 0$ holds, since M is not free. Because A is an isolated singularity, we see that $M_{\mathfrak{p}}$ is free for any $\mathfrak{p} \in \mathrm{Spec}(R)$ except the maximal ideal of A. This implies that the image $\mathrm{Im}(f)$ is a nontrivial A-module of finite length. On the other hand, we notice that the modules $\mathrm{Hom}(M, M)$ and $\mathrm{Hom}(M, F)$ have depth at least two. (Actually this follows from a general fact that if $\mathrm{depth} Y \geq 2$ and if $\mathrm{Hom}_A(X, Y) \neq 0$, then depth $\mathrm{Hom}_A(X, Y) \geq 2$ for $X, Y \in A$-mod.) Hence we conclude from the depth argument [1, Proposition 1.2.9] that $\mathrm{depth}(\mathrm{Hom}(M, \Omega(M))) = 2$. This is a contradiction, because $\mathrm{Hom}(M, \Omega(M)) \cong F(\Omega(M))$ is a maximal Cohen-Macaulay over A and $\mathrm{depth}(A) \geq 3$. □

Example 6. Let k be a field and set $A = k[[x, y, z]]/(x^2 - yz)$. Let $\mathfrak{p}$ be a prime ideal of A generated by $\{x, y\}$. It is known that A is a normal Gorenstein domain of dimension two and $\mathfrak{p}$ is a unique indecomposable non-free maximal Cohen-Macaulay module over A. The class group $C\ell(A)$ is generated by the class of $\mathfrak{p}$ and it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Hence we have $\mathrm{Aut}(\mathrm{ref}(A)) \cong \mathbb{Z}/2\mathbb{Z}$. In fact, the functor $F = \mathrm{Hom}_A(\mathfrak{p},\ \ )$ is a unique nontrivial autofunctor over $\mathrm{ref}(A)$.

References

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