Remarks on equivalences of additive subcategories

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Abstract

We study category equivalences between some additive subcategories of module categories. As its application, we show that the group of autofunctors of the category of reflexive modules over a normal domain is isomorphic to the divisor class group.

1 A necessary condition for equivalences of additive subcategories

Let $R$ be a commutative ring. We denote the category of all finitely generated $R$-modules by $R$-mod, and the full subcategory of $R$-mod consisting of all reflexive modules by $\text{ref}(R)$. If $R$ is a Cohen-Macaulay local ring, we denote the category of maximal Cohen-Macaulay modules by $\text{CM}(R)$ as a full subcategory of $R$-mod. By an additive subcategory we always mean a full subcategory which is closed under finite direct sums and direct summands.

Theorem 1. Let $A$ and $B$ be commutative rings. Let $\mathcal{C}$ (resp. $\mathcal{D}$) be an additive full subcategory of $A$-mod (resp. $B$-mod) which contains a nontrivial free module. If there is a category equivalence between $\mathcal{C}$ and $\mathcal{D}$, then $A \cong B$ as a ring.

Moreover, if $F$ and $G$ are the functors which give the equivalences above, then $F$ and $G$ are of the forms $F(X) \cong \text{Hom}_A(G(B), X)$ and $G(Y) \cong \text{Hom}_B(F(A), Y)$ for each $X \in \mathcal{C}$, $Y \in \mathcal{D}$.

proof. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ be functors satisfying $F \cdot G \cong 1_\mathcal{D}$ and $G \cdot F \cong 1_\mathcal{C}$. We denote the $B$-module $F(A)$ by $M$ and the $A$-module $G(B)$ by $N$. Since $F$ and $G$ are fully faithful functors, there exist isomorphisms as rings $\text{End}_B(M) \cong \text{End}_A(A) = A$ and $\text{End}_A(N) \cong \text{End}_B(B) = B$. Thus there are natural maps as follows:

\[
\begin{align*}
B & \xrightarrow{\beta} \text{End}_B(M) & \xrightarrow{\cong} & A & \xrightarrow{\alpha} & \text{End}_A(N) & \xrightarrow{\cong} & B \\
& b & \xrightarrow{b_M} & a & \xrightarrow{a_N} & b',
\end{align*}
\]

(1.1)

where $b_M$ (resp. $a_N$) denotes the multiplication map on $M$ (resp. $N$) by $b$ (resp. $a$).

The title of the talk has been changed.
First of all, we claim that \( b - b' \in \text{Ann}_B M \) for such \( b \) and \( b' \) as above. Since \( M \) is finitely generated \( B \)-module, we can take a finite free cover of \( M \) and get the following diagram.

\[
\begin{array}{c}
B^{\oplus n} \longrightarrow M \\
\downarrow b' \quad \downarrow b_M \\
B^{\oplus n} \longrightarrow M
\end{array}
\]

Applying the functor \( G \) to this diagram, we have a diagram

\[
\begin{array}{c}
N^{\oplus n} \longrightarrow A \\
\downarrow a_N \quad \downarrow a_A \\
N^{\oplus n} \longrightarrow A,
\end{array}
\]

which is commutative. Since \( G \) is an equivalence, this implies that the first diagram is also commutative. Hence we have \( b - b' \in \text{Ann}_B M \) as desired.

We denote \( \text{Ann}_B M \) by \( b \) and \( \text{Ann}_A N \) by \( a \). Note that the map \( A \xrightarrow{\alpha} \text{End}_A(N) \cong B \) induces an injective mapping \( q : A/a \to B \). We define the map \( B \xrightarrow{\beta} \text{End}_B(M) \cong A \) as the composition of \( B \xrightarrow{\beta} \text{End}_B(M) \cong A \) with the natural projection \( A \to A/a \).

Secondly, we claim that \( \ker p = b \). Since \( \beta(b) = 0 \), it is clear that \( \ker p \supseteq b \). To prove the converse let \( b \in \ker p \). Then, since \( \beta(b) \in a \), we have \( b' = 0 \) as in the notation as in (1.1). Since we have shown that \( b - b' \in b \), we have \( b \in b \) and the equality \( \ker p = b \) is proved. Therefore the mapping \( p \) induces an isomorphism \( A/a \cong B/b \).

Thirdly we note that any object \( Y \in \mathcal{D} \) has structure of a \((B,A)\)-bimodule. In fact, the category \( \mathcal{D} \) is a full subcategory of \( B\text{-mod} \), therefore \( Y \) is naturally equipped with left \( B \)-module structure. Since \( F \) is a dense functor, there exists an object \( X \in \mathcal{C} \) such that \( F(X) \cong Y \) and there is a natural ring homomorphism

\[
A \to \text{End}_A(X) \cong \text{End}_B(Y),
\]

which maps \( a \) to \( F(a_X) \). Now, for any \( a \in A \) and \( y \in Y \), we define \( y \circ a := F(a_X)(y) \). Since \( A \) is a commutative ring, it yields right \( A \)-module structure for \( Y \in \mathcal{D} \). Since the equality \( (b \circ y) \circ a = F(a_X)(by) = bF(a_X)(y) = b \circ (y \circ a) \) holds for \( a \in A, b \in B \) and \( y \in Y \), we see that \( Y \) has structure of a \((B,A)\)-bimodule. Similarly any object \( X \in \mathcal{C} \) has structure of an \((A,B)\)-bimodule.

Since \( F \) is an equivalence, there exists an isomorphism as \( A \)-modules for any object \( X \in \mathcal{C} \):

\[
_AX = \text{Hom}_A(AA, A_X) \cong \text{Hom}_B(BM_A, BFX).
\]

The second part of the theorem follows from this isomorphism.
To complete the proof, we need to show $a = b = (0)$. For this, we note from the definition of bimodule structure that $N$ is isomorphic to $\text{Hom}_B(BM_A, B)$ as an $(A, B)$-bimodule. In particular, there are isomorphisms of $B$-modules;

$$B \cong \text{End}_A N \cong \text{End}_A(\text{Hom}_B(BM_A, B)).$$

Since any element $b \in b$ acts as a zero map on $\text{Hom}_B(BM_A, B)$, it must be zero as an element of $\text{End}_A(\text{Hom}_B(BM_A, B))$. Consequently we have $b = 0$. Thus $b = 0$, and $a = 0$ as well.

**Corollary 2.** Let $A$ and $B$ be Cohen-Macaulay local rings. Then $\text{CM}(A)$ and $\text{CM}(B)$ are equivalent as additive categories if and only if $A$ is isomorphic to $B$ as a ring.

Our theorem is somehow a generalization of Morita equivalence theorem which deals with abelian categories over non-commutative rings. See [3]. The difference is that, assuming rings are commutative, we are concerned with additive subcategories which are not necessarily abelian and our functors are not necessarily exact.

## 2 Groups of autofunctors over additive subcategories

Let $R$ be a commutative ring and let $\mathcal{C}$ be an additive subcategory of $R$-mod. By an autofunctor $F$ on $\mathcal{C}$, we mean a covariant functor $F : \mathcal{C} \rightarrow \mathcal{C}$ which gives rise to an equivalence of categories. We denote by $\text{Aut}(\mathcal{C})$ the group of all isomorphism classes of autofunctors over $\mathcal{C}$. By an easy observation using Morita equivalence, it is known that $\text{Aut}(R$-mod) is isomorphic to the Picard group $\text{Pic}(R)$. As an application of Theorem 1 we can show the following theorem.

**Theorem 3.** Let $A$ be a Noetherian normal domain. Then there is an isomorphism of groups

$$\text{Aut}(\text{ref}(A)) \cong C\ell(A),$$

where $C\ell(A)$ denotes the divisor class group of $A$.

**proof.** It follows from Theorem 1 that any $F \in \text{Aut}(\text{ref}(A))$ has a description $F \cong \text{Hom}_A(M, \_)$ for some reflexive $A$-module $M$. Since $F$ is an autofunctor, there exists a functor $G$ of the form $\text{Hom}_B(N, \_)$ for some $N \in \text{ref}(A)$ satisfying $F \cdot G \cong G \cdot F \cong 1_{\text{ref}(A)}$. Hence we have a sequence of isomorphisms of $A$-modules

$$A \cong G \cdot F(A) \cong \text{Hom}_A(N, \text{Hom}_A(M, A)) \cong \text{Hom}_A(M \otimes_A N, A),$$

which forces $\text{rank} M = 1$. Thus $M$ defines the divisor class $[M]$ in $C\ell(A)$. We define a homomorphism $\alpha : \text{Aut}(\text{ref}(A)) \rightarrow C\ell(A)$ by mapping an autofunctor $F \cong \text{Hom}_R(M, \_)$ to $[M]$. 
We should remark that \( \alpha \) is a well-defined mapping. But it is clear from Yoneda’s lemma which claims that if \( \text{Hom}_R(M, \ ) \cong \text{Hom}_R(M', \ ) \) as functors on \( \text{ref}(A) \) for \( M, M' \in \text{ref}(A) \), then \( M \cong M' \) as \( A \)-modules. It is also not difficult to verify that \( \alpha \) is a homomorphism of groups. In fact this follows from the isomorphism of functors on \( \text{ref}(A) \):

\[
\text{Hom}_A(M, \ ) \cdot \text{Hom}_A(N, \ ) \cong \text{Hom}_A((M \otimes_A N)^*, \ ).
\]

We only have to show that \( \alpha \) is an isomorphism. It is obvious from the definition that \( \alpha \) is injective. In the rest we shall show that \( \alpha \) is surjective. For this let \( [I] \in C\ell(A) \) be an arbitrary element, where \( I \) is a divisorial fractional ideal of \( A \). It is enough to see that \( \text{Hom}_A(I, \ ) \) is a well-defined autofunctor on \( \text{ref}(A) \).

First we remark from Bourbaki [2, Chapter VII, §2] that an \( A \)-lattice \( M \) is reflexive if and only if the equality \( M = \bigcap_{p \in H(R)} M_p \) holds, where \( H(A) \) is the set of all prime ideal of height one. Secondly we note that that the equality \( \text{Hom}_A(X,Y) = \bigcap_{p \in H(R)} \text{Hom}_A(X_p,Y_p) \) holds for \( X, Y \in \text{ref}(A) \). In fact, any \( f \in \bigcap_{p \in H(R)} \text{Hom}_A(X_p,Y_p) \) maps \( X \) to \( Y_p \) for all \( p \in H(A) \), hence \( f(X) \subseteq \bigcap_{p \in H(R)} Y_p = Y \), and thus \( f \in \text{Hom}_A(X,Y) \).

Combining the above two claims we see that \( \text{Hom}_A(I, X) \) is a reflexive lattice for any \( X \in \text{ref}(A) \). Hence \( \text{Hom}_A(I, \ ) \) yields a functor from \( \text{ref}(A) \) to itself.

Since \( I \) is a divisorial ideal, there exists an ideal \( J \) with \( [J] = [-I] \) in \( C\ell(A) \), i.e. \( (I \otimes_A J)^* \cong A \) where \( (\ )^* \) denotes \( \text{Hom}_A(\ , A) \).

Therefore there are isomorphisms of functors on \( \text{ref}(A) \):

\[
\text{Hom}_A(J, \text{Hom}_A(I, \ )) \cong \text{Hom}_A(I \otimes_A J, \ ) \cong \text{Hom}_A((I \otimes_A J)^*, \ ) \cong \text{Hom}_A(A, \ ) = 1_{\text{ref}(A)}.
\]

This shows that \( \text{Hom}(I, \ ) \) is an autofunctor over \( \text{ref}(A) \) as desired, and the proof is completed.

**Corollary 4.** Let \( A \) be a normal domain of dimension at most two. Then \( \text{Aut}(CM(A)) \cong C\ell(A) \).

**proof.** In fact, the equality \( CM(A) = \text{ref}(A) \) holds in this case.

Compared with the corollary, the groups of autofunctors of \( CM(A) \) are expected to be rather small for higher dimensional rings \( A \). In fact we can prove the following theorem.
**Theorem 5.** Let $A$ be a Cohen-Macaulay local ring. Suppose that $A$ has only an isolated singularity with $\dim A \geq 3$. Then $\text{Aut}(\text{CM}(A))$ is a trivial group.

**proof.** Let $F$ be an autofunctor over $\text{CM}(A)$. By virtue of Theorem 1, there exists a maximal Cohen-Macaulay module $M$ with $F \cong \text{Hom}_A(M, \ )$. Assume that $M$ is not free, and we shall show a contradiction. For this, take a free cover $F$ of $M$ and we obtain an exact sequence

$$0 \to \Omega(M) \to F \to M \to 0.$$  

Recall that $\Omega(M)$ is also a maximal Cohen-Macaulay module. Apply $\text{Hom}_A(M, \ )$ to the sequence, and we get an exact sequence

$$0 \to \text{Hom}(M, \Omega(M)) \to \text{Hom}(M, F) \to \text{Hom}(M, M) \to \text{Ext}^1(M, \Omega(M)).$$

Note that $f \neq 0$ holds, since $M$ is not free. Because $A$ is an isolated singularity, we see that $M_\mathfrak{p}$ is free for any $\mathfrak{p} \in \text{Spec}(R)$ except the maximal ideal of $A$. This implies that the image $\text{Im}(f)$ is a nontrivial $A$-module of finite length. On the other hand, we notice that the modules $\text{Hom}(M, M)$ and $\text{Hom}(M, F)$ have depth at least two. (Actually this follows from a general fact that if $\text{depth} Y \geq 2$ and if $\text{Hom}_A(X,Y) \neq 0$, then $\text{depth} \text{Hom}_A(X,Y) \geq 2$ for $X, Y \in A$-mod.) Hence we conclude from the depth argument [1, Proposition 1.2.9] that $\text{depth}(\text{Hom}(M, \Omega(M))) = 2$. This is a contradiction, because $\text{Hom}(M, \Omega(M)) \cong F'(\Omega(M))$ is a maximal Cohen-Macaulay over $A$ and $\text{depth}(A) \geq 3$. 

**Example 6.** Let $k$ be a field and set $A = k[[x, y, z]]/(x^2 - yz)$. Let $\mathfrak{p}$ be a prime ideal of $A$ generated by $\{x, y\}$. It is known that $A$ is a normal Gorenstein domain of dimension two and $\mathfrak{p}$ is a unique indecomposable non-free maximal Cohen-Macaulay module over $A$. The class group $\text{Cl}(A)$ is generated by the class of $\mathfrak{p}$ and it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Hence we have $\text{Aut}(\text{ref}(A)) \cong \mathbb{Z}/2\mathbb{Z}$. In fact, the functor $F = \text{Hom}_A(\mathfrak{p}, \ )$ is a unique nontrivial autofunctor over $\text{ref}(A)$.

**References**


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