RING EXTENSIONS OF AB RINGS

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1. Introduction

This is a report of our recent work. For any detail of this article, see the preprint [6]. Throughout the article, R denotes a commutative Noetherian ring with unity and $\dim(R) < \infty$.

Definition 1.1. We define

$$\text{Ext-index}(R) := \sup \{ \, n \mid \operatorname{Ext}^i_R(M,N) = 0 \text{ for } i > n \text{ and } \operatorname{Ext}^n_R(M,N) \neq 0 \text{ for some finitely generated } R\text{-modules } M \text{ and } N \, \}.$$

And the ring R is said to be an AB ring if it satisfies Ext-index$(R) < \infty$.

In this paper we are interested in the AB property for some ring extensions. Note that any rings of the following types are known to be AB rings.

(1) Complete intersections [3].
(2) Cohen-Macaulay local rings with minimal multiplicity [3] [5].
(3) Gorenstein local rings with codimension at most 4 [7].
(4) Golod rings [5].
(5) Artinian local rings $(R, \mathfrak{m})$ with any of the following conditions:
  (a) $\mathfrak{m}^3 = 0$ and $\mu(\mathfrak{m}) = 3$ [5].
  (b) $\mathfrak{m}^3 = 0$ and $2\mu(\mathfrak{m}) \geq \ell_R(R) + 1$ [4].

2. Trivial extension of a local ring by its residue class field

Let M be an R-module. Then the direct sum $R \oplus M$ is equipped with the product:

$$(r, m).(r', m') = (rr', rm' + r'm).$$

This makes $R \oplus M$ a ring which is called the trivial extension of R by M and denoted by $R(M)$. There is a ring homomorphism $\pi : R(M) \longrightarrow R$ with $\pi(r,m) = r$ and any R-module can be regarded as an $R(M)$-module through π.

This work was done during the visit of the first author to Department of Mathematics of Okayama University. The first author would like to thank Okayama University for their hospitality and facilities.

Lemma 2.1. *Let $(R, \mathfrak{m}, k)$ be an arbitrary local ring. Then for R-modules M and N and for $n \geq 1$, we have an isomorphism*

$$\mathrm{Tor}_n^{R(k)}(M,N) \cong \mathrm{Tor}_n^R(M,N) \oplus \coprod_{i+j=n-1} \mathrm{Tor}_i^{R(k)}(M,k) \otimes_k \mathrm{Tor}_j^R(k,N).$$

Remark 2.2. Let S be a local ring with residue class field ℓ and let M, N be S-modules such that $\ell_S(\mathrm{Tor}_n^S(M,N)) < \infty$ for all n. Then we can consider the generating function $P_{M,N}^S(t)$ defined by the equality

$$P_{M,N}^S(t) = \sum_{n \geq 0} \ell_S(\mathrm{Tor}_n^S(M,N))t^n.$$

Recall that the Poincaré series $P_M^S(t)$ of M is defined to be $P_{\ell,M}^S(t)$ and the Poincaré series $P_\ell^S(t)$ of S is denoted simply by $P_S(t)$. Remark that the following is known.

<u>A theorem of Gulliksen:</u> *Let $(R, \mathfrak{m})$ be a local ring and let M be a finitely generated R-module. Then the equality $P_{R(M)}(t) = P_R(t)(1 - P_M^R(t)\ t)^{-1}$ holds.*

Note that by the previous lemma we can show the equality

$$P_{M,N}^{R(k)}(t) = P_{M,N}^R(t) + P_M^{R(k)}(t)\ P_N^R(t)\ t,$$

for finitely generated modules M and N over an Artinian local ring R. Applying this to $M = N = k$, we have

$$P_{R(k)}(t) = P_R(t)(1 - P_R(t)\ t)^{-1},$$

which is a special case of the above mentioned theorem of Gulliksen.

Theorem 2.3. *Let $(R, \mathfrak{m}, k)$ be an arbitrary local ring and let M and N be nonzero non-free finitely generated $R(k)$-modules. Then $\mathrm{Tor}_n^{R(k)}(M,N) \neq 0$ for all $n \geq 3$.*

Proof. Set $A = R(k)$ and suppose $\mathrm{Tor}_n^A(M,N) = 0$ for some $n \geq 3$. Let $\mathfrak{n}$ be the maximal ideal of A and $x = (0,1) \in A$. Notice that $\mathfrak{n} = (0 :_A x)$ holds and $R \cong A/Ax$ as a ring.

Replacing M and N with their first syzygies, we may assume that $\mathrm{Tor}_n^A(M,N) = 0$ for some $n \geq 1$ and that $xM = 0$ and $xN = 0$. Thus we may assume M and N are modules over R through the identification $R \cong A/Ax$. Then by the previous lemma, the equality $\mathrm{Tor}_{n-1}^A(M,k) \otimes_k (N \otimes_R k) = 0$ holds. Since $N \otimes_R k \neq 0$, we see $\mathrm{Tor}_{n-1}^A(M,k) = 0$. This implies that M has finite projective dimension as an A-module. But $\mathrm{depth}(A) = 0$ and by Auslander-Buchsbaum formula, M is a free A-module. This is a contradiction. □

As applications of the theorem we can prove the following corollaries.

Corollary 2.4. *For an Artinian local ring $(R, \mathfrak{m}, k)$, the trivial extension $R(k)$ is an AB ring with* $\text{Ext-index}(R(k)) = 0$.

Corollary 2.5. *Let $(R, \mathfrak{m}, k)$ be an Artinian local ring. Suppose that M is a finitely generated $R(k)$-module such that $\text{Ext}^i_{R(k)}(M, M) = 0$ for all $i > 0$. Then M is either a free or an injective $R(k)$-module.*

Corollary 2.6. *Let $(R, \mathfrak{m}, k)$ be an Artinian local ring. And let $E = E_{R(k)}(k)$ be the injective envelope of the $R(k)$-module k. Then $\text{Ext}^i_{R(k)}(E, R(k)) \neq 0$ for some $i > 0$.*

Corollary 2.7 (Auslander-Reiten conjecture). *Let $(R, \mathfrak{m}, k)$ be an Artinian local ring and let M be a finitely generated $R(k)$-module such that $\text{Ext}^i_{R(k)}(M, M \oplus R(k)) = 0$ for all $i > 0$. Then M is a free $R(k)$-module.*

3. More ring extensions

Let R be an algebra over a field k. And let M be a module over the polynomial ring $R[x]$. The *specialization of M to an element $\alpha \in k$* is defined by

$$M_\alpha := M \otimes_{k[x]} (k[x]/(x - \alpha)k[x]).$$

Remark that if M is a finitely generated $R[x]$-module, then M_α is a finitely generated R-module.

Lemma 3.1. *Let R be a k-algebra and let $\alpha \in k$. Assume that $x - \alpha$ is a nonzero divisor on $R[x]$-modules M and N. Then we have the exact sequence*

$$0 \longrightarrow \text{Ext}^i_{R[x]}(M, N)_\alpha \longrightarrow \text{Ext}^i_R(M_\alpha, N_\alpha) \longrightarrow \text{Tor}_1^{k[x]}(\text{Ext}^{i+1}_{R[x]}(M, N),\ k[x]/(x - \alpha)) \longrightarrow 0,$$

for each $i \geq 0$.

Theorem 3.2. *Suppose that k is an uncountable field and R is a finite dimensional k-algebra which is AB. Then $R \otimes_k k(x)$ is AB with* $\text{Ext-index}(R \otimes_k k(x)) \leq \text{Ext-index}(R)$.

Proof. Set $b := \text{Ext-index}(R)$ and let M' and N' be finitely generated $R \otimes_k k(x)$-modules with $\text{Ext}^i_{R \otimes_k k(x)}(M', N') = 0$ for $i \gg 0$. We have to show that $\text{Ext}^i_{R \otimes_k k(x)}(M', N') = 0$ for $i > b$.

Note that $R \otimes_k k(x)$ is just a localization of $R[x]$ by a multiplicatively closed subset $k[x] \backslash \{0\}$. Hence we can choose a finitely generated $R[x]$-submodule M of M' (resp. N of N') so that $M \otimes_{k[x]} k(x) \cong M'$ (resp. $N \otimes_{k[x]} k(x) \cong N'$). Notice that $x - \alpha$ acts on M and N as a non-zero divisor.

Since we have an isomorphism $\mathrm{Ext}^i_{R\otimes_k k(x)}(M', N') \cong \mathrm{Ext}^i_{R[x]}(M, N) \otimes_{k[x]} k(x)$, we see that $\mathrm{Ext}^i_{R[x]}(M, N) \otimes_{k[x]} k(x) = 0$ for $i \gg 0$.

On the other hand, since R is a finite dimensional k-algebra, each module $\mathrm{Ext}^i_{R[x]}(M, N)$ ($i \geq 0$) is a finitely generated $k[x]$-module. Hence it has a decomposition as a $k[x]$-module as follows:

$$\mathrm{Ext}^i_{R[x]}(M, N) \cong \bigoplus_{j=1}^{s_i} k[x]/(f_{ij}(x)) \oplus k[x]^{r_i},$$

where $f_{ij}(x) \neq 0 \in k[x]$.

Since $\mathrm{Ext}^i_{R[x]}(M, N) \otimes_{k[x]} k(x)$ are vanishing for $i \gg 0$, we have $r_i = 0$ for $i \gg 0$. Since there are only countably many equations $f_{ij}(x)$, we can find an element $\alpha \in k$ with the property $f_{ij}(\alpha) \neq 0$ for all i, j. Then, since $x - \alpha$ acts bijectively on $k[x]/(f_{ij}(x))$, we see that $\mathrm{Tor}_1^{k[x]}(\mathrm{Ext}^{i+1}_{R[x]}(M, N),\ k[x]/(x - \alpha)) = 0$ for all i. And we see as well that $\mathrm{Ext}^i_{R[x]}(M, N)_\alpha = 0$ for $i \gg 0$. Therefore the previous lemma implies that $\mathrm{Ext}^i_R(M_\alpha, N_\alpha) = 0$ for $i \gg 0$. Thus, by the definition of Ext-index, we have $\mathrm{Ext}^i_R(M_\alpha, N_\alpha) = 0$ for all $i > b$. Since $\mathrm{Ext}^i_{R[x]}(M, N)_\alpha$ is a submodule of $\mathrm{Ext}^i_R(M_\alpha, N_\alpha)$, we have $\mathrm{Ext}^i_{R[x]}(M, N)_\alpha = 0$ for all $i > b$. This implies that $r_i = 0$ for $i > b$, which is equivalent to the vanishing $\mathrm{Ext}^i_{R[x]}(M, N) \otimes_{k[x]} k(x) = 0$ for $i > b$. $\square$

Remark 3.3. Let R be a Gorenstein local ring. Suppose there is an integer $n \geq 0$ such that $\mathrm{Ext}^i_R(M, N) = 0$ for $n + 1 \leq i \leq n + t$ and $\mathrm{Ext}^j_R(M, N) \neq 0$ for $j = n, n + t + 1$. In such a case we say that $\mathrm{Ext}_R(M, N)$ has a gap of length t. Set

$$\text{Ext-gap}(R) := \sup\{t \in \mathbb{N} \mid \mathrm{Ext}_R(M, N) \text{ has a gap of length } t\},$$

where "sup" is taken over all pairs (M, N) of finitely generated R-modules. R is called Ext-bounded if it has finite Ext-gap. Furthermore we should remark from [3] that it is known that $\text{Ext-gap}(R) < \infty \implies R$ is AB.

Keeping in mind this remark, we can prove the following statement completely in a same way as in the proof of Theorem 3.2:

Let R be a finite dimensional k-algebra where k is an infinite field. If R is Ext-bounded, then so is $R \otimes_k k(x)$.

We can also prove the following theorem in a similar way to the proof of Theorem 3.2.

Theorem 3.4. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay AB local ring with dualizing module. Suppose that R contains an uncountable coefficient field k. Then $R[x]_{\mathfrak{m}R[x]}$ is also a Cohen-Macaulay AB local ring.*

We finish this report by adding the following result.

Theorem 3.5. *Let $(R, \mathfrak{m}, k)$ be an Artinian Gorenstein AB local ring. Assume that the residue class field k is algebraically closed. Then the polynomial ring $R[x_1, ..., x_n]$ is also AB.*

Proof. It is enough to prove that $R[x_1, ..., x_n]_{\mathfrak{M}}$ is AB for every maximal ideal $\mathfrak{M}$ of $R[x_1, ..., x_n]$. Since R is Artinian, we see that $\mathfrak{M} \cap R = \mathfrak{m}$. Therefore, by Hilbert's Nullstellensatz, there are elements $r_1, \ldots, r_n \in R$ with $\mathfrak{M} = (\mathfrak{m}, x_1 - r_1, \ldots, x_n - r_n)R[x_1, \ldots, x_n]$. Since $R \cong R[x_1, ..., x_n]_{\mathfrak{M}}/(x_1 - r_1, ..., x_n - r_n)R[x_1, ..., x_n]_{\mathfrak{M}}$ is AB and since $\{x_1 - r_1, ..., x_n - r_n\}$ is a regular sequence contained in the radical of $R[x_1, ..., x_n]_{\mathfrak{M}}$, it is easy to see that $R[x_1, ..., x_n]_{\mathfrak{M}}$ is also AB. □

References

[1] L. L. Avramov and R.-O. Buchweitz, *Support varieties and cohomology over complete intersections*, Invent. Math. **142** (2000), 285–318.

[2] T. H. Gulliksen, *Massey operations and the Poincaré series of certain local rings*, J. Alg. **22** (1972), 223–232.

[3] C. Huneke and D. A. Jorgensen, *Symmetry in the vanishing of Ext over Gorenstein Rings*, Math. Scand. **93** (2003), 161–184.

[4] C. Huneke, L. M. Şega and A. N. Vraciu, *Vanishing of Ext and Tor over some Cohen-Macaulay local rings*, Illinois J. Math. **48** (2004), 295–317.

[5] D. A. Jorgensen and L. M. Şega, *Nonvanishing cohomology and classes of Gorenstein rings*, Adv. Math. **188** (2004), 470–490.

[6] S. Nasseh and Y. Yoshino, *On Ext-indices of ring extensions*, in preparation.

[7] L. M. Şega, *Vanishing of cohomology over Gorenstein rings of small codimension*, Proc. Amer. Math. Soc. **131** (2003), no. 8, 2313–2323 .

[8] C. A. Weibel, *An introduction to homological algebra*, Cambridge Stud. Adv. Math . 38 . (1994).

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