Quotient categories of homotopy categories

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Abstract

We introduce the homotopy category of unbounded complexes with bounded homologies. We study a recollement of its a quotient by the homotopy category of bounded complexes. This leads to the existence of quotient categories which are equivalent to a homotopy category of acyclic comlpexes, that is a stable derived category. In the case of a coherent ring R of self-injective dimension both sides, we show that the above recollement are triangulated equivalent to a recollement of the stable module category of Cohen-Macaulay R-modules.

1 Introduction

We study two types of triangulated categories in this paper. One is the categories of homotopy classes of chain complexes, equipped with triangles induced by chain maps and mapping cones. The other is stable module categories that are module categories mod projective modules. A stable module category is not triangulated in general. If the module category is Frobenius, then it's projective stabilization is triangulated. This type of triangulates categories are called algebraic triangulated categories. The well-known example is a stable module category of Cohen-Macaulay modules over Gorenstein rings.

Let R be a two-sided noetherian ring. The catogories of right R-modules, of finitely generated right R-modules and of finitely generated projective right R-modules are denoted by Mod$R$ and mod$R$, and *proj* R respectively. Let $\mathsf{K} = \mathsf{K}(\mathrm{proj}R)$ be the category of homotopy classes of complexes of finitely generated R-projective complexes. The following triangulated subcategories of K are of our concern.

$$\mathsf{K}^{\infty,b} = \{C \in \mathsf{K} \mid \mathrm{H}^i(C) = 0 \text{ (except for finite } i\text{'s)}\}$$

$$\mathsf{K}^{-,b} = \{C \in \mathsf{K}^{\infty,b} \mid C^i = 0 \text{ (for sufficiently large } i)\}$$

$$\mathsf{K}^{\infty,\emptyset} = \{C \in \mathsf{K}^{\infty,b} \mid \mathrm{H}^i(C) = 0 \ (i \in \mathbf{Z})\}$$

$$\mathsf{K}^{b} = \{C \in \mathsf{K} \mid C^i = 0 \text{ (except for finite } i\text{'s)}\}$$

Those triangulated categories are all épaisse, so the quotient categories are again triangulated.

Definition 1.1 ([Iw]) *A two-sided noetherian ring is called Iwanaga-Gorenstein if* $\mathrm{id}_R R < \infty$ *and* $\mathrm{id}_{R^{op}}\ R < \infty$*.*

If R is an Iwanaga-Gorenstein ring, we define a subcategory $\mathrm{CM}(R)$ of $\mathrm{mod}R$ as $\mathrm{CM}(R) = \{X \in \mathrm{mod}R \mid \mathrm{Ext}^i_R(X, R) = 0 \quad (i > 0)\}$.

Theorem 1.2 (Buchweitz [Bu]) *Assume* R *is Iwanaga-Gorenstein. The quotient category* $\mathsf{K}^{-,b}/\mathsf{K}^b$ *is triangle equivalent to the stable module category* $\underline{\mathrm{CM}}(R)$*.*

On the other hand, we observe the following.

Theorem 1.3 *If* R *is Iwanaga-Gorenstein.The quotient category* $\mathsf{K}^{\infty,b}/\mathsf{K}^{-,b}$ *is equivalent to the stable module category* $\underline{\mathrm{CM}}(R)$*.*

Naturally, the question arises: What is $\mathsf{K}^{\infty,b}/\mathsf{K}^b$? Is it realizable as a stable module category?

2 Operations and functors on $\mathsf{K}^{\infty,b}$

For an object A of $\mathsf{K}^{\infty,b}$, define objects X_A and T_A of $\mathsf{K}^{\infty,\emptyset}$ as follows.

Let l be the smallest integer such that $\mathrm{H}_l(A^*) \neq 0$. Then $Cok\, d_A^{l-1}$ is a maximal Cohen-Macaulay module. Define $X_A \in \mathsf{K}^{\infty,\emptyset}$ as

$$\tau_{\leq l} X_A = \tau_{\leq l} A$$

and

$$\cdots \to X_A^{l+1*} \to X_A^{l+2*} \to (Cok\, d_A^{l-1})^* \to 0$$

is exact. Then X_A is totally acyclic and $\mathrm{id}_{Cok\, d_A^{l-1*}}$ induces a canonical chain map $\xi_A : X_A \to A$ as $\xi_A^i = \mathrm{id} \quad (i \leq l)$.

Similarly, let r be the largest integer such that $\mathrm{H}^r(A) \neq 0$. Then $Ker\, d_A^r$ is a maximal Cohen-Macaulay module. Define $T_A \in \mathsf{K}^{\infty,\emptyset}$ as

$$\tau_{\geq r} X_A = \tau_{\geq r} A$$

and

$$\cdots \to T_A^{r-1} \to T_A^r \to (Ker\, d_A^r) \to 0$$

is exact. Then T_A is totally acyclic and $\mathrm{id}_{Ker\, d_A^r}$ induces a canonical chain map $\zeta_A : A \to T_A$ as $\zeta_A^i = \mathrm{id} \quad (i \geq r)$.

Set a chain maps $l_A : L_A \to A$ and $r_{L_A} : L_A \to R_{L_A}$ as follows:

$$\tau_{\leq 0} L_A = \tau_{\leq 0} X_A, \tau_{\geq 1} L_A = \tau_{\geq 1} A,$$

$$\tau_{\leq 0} l_A = \tau_{\leq 0} \xi_A, \tau_{\geq 1} l_A = \tau_{\geq 1} \mathrm{id}_A,$$

$$\tau_{\leq 0} R_{L_A} = \tau_{\leq 0} L_A, \tau_{\geq 1} R_{L_A} = \tau_{\geq 1} T_{L_A},$$

$$\tau_{\leq 0} r_{L_A} = \tau_{\leq 0} \mathrm{id}_{L_A}, \tau_{\geq 1} r_{L_A} = \tau_{\geq 1} \zeta_A$$

Obviously $C(l_A)$ and $C(r_{L_A})$ belongs to K^b, hence as an object of $\mathsf{K}^{\infty,b}/\mathsf{K}^b$, A is isomorphic to the complex

$$R_{L_A} : \cdots \to {X_A}^{-1} \to {X_A}^{0} \to {T_A}^{1} \to {T_A}^{2} \to \cdots$$

We may assume $\lambda_A = \mathrm{H}^0(\tau_{\leq 0}\xi_A\zeta_A) : Cok\, d_{X_A}^{-1} \to Ker\, d_{T_A}^{1}$ to be surjective by adding some split exact sequence of projective modules if necessary.

3 The category of morphisms

We define category $\mathrm{Mor}(R)$ as follows: objects of $\mathrm{Mor}(R)$ are the morphisms $\alpha : X_\alpha \to T_\alpha$ of $\mathrm{Mod}(R)$. For $\alpha, \beta \in \mathrm{mor}(R)$, we define

$$\mathrm{Mor}(R)(\alpha,\beta) = \{(f_X, f_T) \in \mathrm{Hom}_R(X_\alpha, X_\beta) \times \mathrm{Hom}_R(T_\alpha, T_\beta) \mid f_T\alpha = \beta f_X\}.$$

And the subcategory $\mathrm{mor}_s^{CM}(R)$ of $\mathrm{Mor}(R)$ consists of the objects $\alpha : X_\alpha \to T_\alpha$ of $\mathrm{CM}(R)$ that are surjective. The structure of $\mathrm{mor}_s^{CM}(R)$ is obtained by the next lemma.

Lemma 3.1 *Let $T_2(R)$ be the category of 2×2 upper triangular matrices with entries in R. Then $\mathrm{Mod}(T_2(R))$ is equivalent to $\mathrm{Mor}(R)$. And $\mathrm{mor}_s^{CM}(R)$ is equivalent to the category $\mathrm{CM}(T_2(R))$.*

proof. An object $f : X_f \to T_f$ of $\mathrm{Mor}(R)$ corresponds to an $T_2(R)$-module $M_f = X_f \times T_f$ where $(x\ t)\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} = (xa\ f(x)b + tc)$. This correspondence gives an equivalence between $\mathrm{CM}(T_2(R))$ and $\mathrm{mor}_i^{CM}(R)$ consisting of injective maps $\alpha : X_\alpha \to T_\alpha$ with $X_\alpha, T_\alpha, \mathrm{Cok} f \in \mathrm{CM}(R)$. Obviously $\mathrm{mor}_i^{CM}(R)$ is equivalent to $\mathrm{mor}_s^{CM}(R)$. (q.e.d.)

Thus $\mathrm{mor}_s^{CM}(R)$ is a Frobenius category together with projective-injective objects consisting of $p \in \mathrm{mor}_s^{CM}(R)$ that X_p and T_p are projective modules. Hence the stable category $\underline{\mathrm{mor}}_s^{CM}(R)$ is triangulated. We shall construct a functor between $\mathsf{K}^{\infty,b}/\mathsf{K}^b$ and $\underline{\mathrm{mor}}_s^{CM}(R)$.

Let $\alpha : X_\alpha \to T_\alpha$ be an object of $\mathrm{mor}_s^{CM}(R)$ and let F_{X_α} and F_{T_α} be acyclic projective complexes such that $\mathrm{H}^0(\tau_{\leq 0}F_{X_\alpha}) = X_\alpha$ and $\mathrm{H}^0(\tau_{\leq 0}F_{T_\alpha}) = T_\alpha$. Set natural maps $\rho : F^0_{X_\alpha} \to X_\alpha$ and $\epsilon : T_\alpha \to F_{T_\alpha}$. Make a projective complex F_α as

$$\tau_{\leq 0}F_\alpha = \tau_{\leq 0}F_{X_\alpha},\ \tau_{\geq 1}F_\alpha = \tau_{\geq 1}F_{T_\alpha},\ d_{F_\alpha} = \epsilon\alpha\rho.$$

Lemma 3.2 *1) A morphism $f \in \mathrm{mor}_s^{CM}(R)(\alpha,\beta)$ induces a chain map $F_f : F_\alpha \to F_\beta$.*
2) For morphisms $f \in \mathrm{mor}_s^{CM}(R)(\alpha,\beta)$ and $g \in \mathrm{mor}_s^{CM}(R)(\beta,\gamma)$, $F_{gf} = F_gF_f$.
3) An exact sequence $0 \to \alpha \xrightarrow{f} \beta \xrightarrow{g} \gamma \to 0$ in $\mathrm{mor}_s^{CM}(R)$ induces an exact sequence $0 \to F_\alpha \xrightarrow{F_f} F_\beta \xrightarrow{F_g} F_\gamma \to 0$ in $\mathsf{C}^{\infty,b}$.

4) An object p of $\mathrm{mor}_s^{CM}(R)$ is projective if and only if F_p is a bounded complex.

Lemma 3.3 *The operation F gives a functor $\mathrm{mor}_s^{CM}(R) \to \mathsf{K}^{\infty,b}$. And F induces a functor $\underline{F} : \underline{\mathrm{mor}}_s^{CM}(R) \to \mathsf{K}^{\infty,b}/\mathsf{K}^b$.*

Proposition 3.4 *The functor $\underline{F} : \underline{\mathrm{mor}}_s^{CM}(R) \to \mathsf{K}^{\infty,b}/\mathsf{K}^b$ is triangulated.*

proof Let

$$\underline{\alpha} \xrightarrow{\underline{f}} \underline{\beta} \xrightarrow{\underline{g}} \underline{\gamma} \xrightarrow{\underline{h}} \Sigma\underline{\alpha}$$

be a triangle in $\underline{\mathrm{mor}}_s^{CM}(R)$. That is, the injective hull $\alpha \xrightarrow{\epsilon} q$ of α and f make a push-out diagram which implies a commutative diagram in $\mathrm{CM}(\Lambda^*)$ with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & \alpha & \xrightarrow{\epsilon} & q & \xrightarrow{\rho} & \Sigma\alpha & \to & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle w} & & \| & & \\
0 & \to & \beta & \xrightarrow{g} & \gamma & \xrightarrow{h} & \Sigma\alpha & \to & 0.
\end{array}$$

This induces a commutative diagram in $\mathsf{C}^{\infty,b}$ with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & F_\alpha & \xrightarrow{F_\epsilon} & F_q & \xrightarrow{F_\rho} & F_{\Sigma\alpha} & \to & 0 \\
 & & \downarrow{\scriptstyle F_f} & & \downarrow{\scriptstyle F_w} & & \| & & \\
0 & \to & F_\beta & \xrightarrow{F_g} & F_\gamma & \xrightarrow{F_h} & F_{\Sigma\alpha} & \to & 0
\end{array}$$

It remains to show that there is a functorial isomorphism $F_{\Sigma\alpha} \cong \Sigma F_\alpha$ in $\mathsf{K}^{\infty,b}/\mathsf{K}^b$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_\alpha & \xrightarrow{F_\epsilon} & F_q & \xrightarrow{F_\rho} & F_{\Sigma\alpha} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F_f} & & \downarrow{\scriptstyle F_w} & & \| & & \\
0 & \longrightarrow & F_\beta & \longrightarrow & F_\gamma & \longrightarrow & F_{\Sigma\alpha} & \longrightarrow & 0
\end{array}$$

induces a morphism between triangles in $\mathsf{K}^{\infty,b}$:

$$\begin{array}{ccccccc}
F_\alpha & \xrightarrow{F_\epsilon} & F_q & \xrightarrow{F_\rho} & F_{\Sigma\alpha} & \xrightarrow{\pi_\alpha} & \Sigma F_\alpha \\
\downarrow{\scriptstyle F_f} & & \downarrow{\scriptstyle F_w} & & \| & & \downarrow{\scriptstyle \Sigma F_f} \\
F_\beta & \longrightarrow & F_\gamma & \longrightarrow & F_{\Sigma\alpha} & \longrightarrow & \Sigma F_\alpha
\end{array}$$

Since $F_q \in \mathsf{K}^b$, it is easy to see that π_α is a functorial isomorphism in $\mathsf{K}^{\infty,\emptyset}/\mathsf{K}^b$, and we have a triangle in $\mathsf{K}^{\infty,\emptyset}/\mathsf{K}^b$:

$$F_\alpha \xrightarrow{F_f} F_\beta \xrightarrow{F_g} F_\gamma \xrightarrow{F_\alpha \pi_\alpha} \Sigma F_\alpha$$

(q.e.d.)

Theorem 3.5 *The category* $\mathsf{K}^{\infty,b}/\mathsf{K}^b$ *is triangle equivalent to* $\underline{\mathrm{mor}}_s^{CM}(R)$.

We shall show that $\underline{F}$ is a category equivalence. We have already seen that $\underline{F}$ is dense from the previous section. For proving $\underline{F}$ is fully faithful, we use the notion of t-strucrture.

4 Stable t-structures

Definition 4.1 ([Mi1]) *For full subcategories* $\mathcal{U}$ *and* $\mathcal{V}$ *of a triangulate category* $\mathcal{C}$*,* $(\mathcal{U},\mathcal{V})$ *is called a stable t-structure in* $\mathcal{C}$ *provided that*

- $\mathcal{U}$ *and* $\mathcal{V}$ *are stable for translations.*
- $\mathrm{Hom}_{\mathcal{C}}(\mathcal{U},\mathcal{V})=0$.
- *For every* $X\in\mathcal{C}$*, there exists a triangle* $U\to X\to V\to\Sigma U$ *with* $U\in\mathcal{U}$ *and* $V\in\mathcal{V}$.

Proposition 4.2 ([BBD], [Mi1]) *Let* $\mathcal{C}$ *be a triangulated category. The following hold.*

1 *Let* $(\mathcal{U},\mathcal{V})$ *be a stable t-structure in* $\mathcal{C}$*,* $i_*:\mathcal{U}\to\mathcal{C}$ *and* $j_*:\mathcal{V}\to\mathcal{C}$ *the canonical embeddings. Then there are a right adjoint* $i^!:\mathcal{C}\to\mathcal{U}$ *of* i_* *and a left adjoint* $j^*:\mathcal{C}\to\mathcal{V}$ *of* j_* *which satisfy the following.*

   (a) $j^*i_*=0,\ i^!j_*=0$.

   (b) The adjunction arrows $i_*i^!\to\mathbf{1}_{\mathcal{C}}$ *and* $\mathbf{1}_{\mathcal{C}}\to j_*j^*$ *imply a triangle* $i_*i^!X\to X\to j_*j^*X\to\Sigma i_*i^!X$ *for any* $X\in\mathcal{C}$.

   In this case, j^* *(resp.,* $i^!$*) implies the triangulated equivalence* $\mathcal{C}/\mathcal{U}\simeq\mathcal{V}$ *(resp.,* $\mathcal{C}/\mathcal{V}\simeq\mathcal{U}$*).*

2 *If* $\{\mathcal{C},\mathcal{C}'';j^*,j_*\}$ *(resp.,* $\{\mathcal{C},\mathcal{C}'';j_!,j^*\}$*) is a localization (resp., a colocalization) of* $\mathcal{C}$*, thai is,* j_* *(resp.,* i_**) is a fully faithful right (resp., left) adjoint of* $i^!$*, then* $(\mathrm{Ker}j^*,\mathrm{Im}j_*)$ *(resp.,* $(\mathrm{Im}j_!,\mathrm{Ker}j^*)$*) is a stable t-structure. In this case, the adjunction arrow* $\mathbf{1}_{\mathcal{C}}\to j_*j^*$ *(resp.,* $j_!j^*\to\mathbf{1}_{\mathcal{C}}$*) implies triangles*

$$U\to X\to j_*j^*X\to\Sigma U$$
$$(\text{resp., } j_!j^*X\to X\to V\to\Sigma j_!j^*X)$$

   with $U\in\mathrm{Ker}j^*$*,* $j_*j^*X\in\mathrm{Im}j_*$ *(resp.,* $j_!j^*X\in\mathrm{Im}j_!$*,* $V\in\mathrm{Ker}j^*$*) for all* $X\in\mathcal{C}$.

Proposition 4.3 *Let* R *be a coherent ring. Then we have the following.*

- $(\mathsf{K}^{-,b},\mathsf{K}^{\infty,\emptyset})$ *is a stable t-structure of* $\mathsf{K}^{\infty,b}$*. Hence* $(\mathsf{K}^{-,b}/\mathsf{K}^b,\mathsf{K}^{\infty,\emptyset})$ *is a stable t-structure of* $\mathsf{K}^{\infty,b}/\mathsf{K}^b$.
- $(\mathsf{K}^{+,b}/\mathsf{K}^b,\mathsf{K}^{-,b}/\mathsf{K}^b)$ *is a stable t-structure of* $\mathsf{K}^{\infty,b}/\mathsf{K}^b$.

- *If R is Iwanaga-Gorenstein, then $(\mathsf{K}^{\infty,\emptyset}/\mathsf{K}^b, \mathsf{K}^{+,b}/\mathsf{K}^b)$ is a stable t-structure of $\mathsf{K}^{\infty,b}/\mathsf{K}^b$.*

Let R be an Iwanaga-Gorenstein ring. Let $\underline{\mathcal{CM}_0}$ (resp., $\underline{\mathcal{CM}_1}$, $\underline{\mathcal{CM}_p}$) be the full subcategory of $\underline{\mathrm{mor}}_s^{CM}(R)$ consisting of objects of the form $X \to 0$ (resp., $S \xrightarrow{=} S$, $P \to T$, with P being projective).

Proposition 4.4 *The following are stable t-structures of $\underline{\mathrm{mor}}_s^{CM}(R)$.*

$$(\underline{\mathcal{CM}_0}, \underline{\mathcal{CM}_1}),\ (\underline{\mathcal{CM}_p}, \underline{\mathcal{CM}_0}),\ (\underline{\mathcal{CM}_1}, \underline{\mathcal{CM}_p}).$$

Proposition 4.5 *The triangulated functor F induces equivalences*

$$\begin{aligned} \underline{F}\mid_{\underline{\mathcal{CM}_0}} &: \underline{\mathcal{CM}_0} \to \mathsf{K}^{-,b}/\mathsf{K}^b, \\ \underline{F}\mid_{\underline{\mathcal{CM}_1}} &: \underline{\mathcal{CM}_1} \to \mathsf{K}^{\infty,\emptyset}, \\ \text{and } \underline{F}\mid_{\underline{\mathcal{CM}_p}} &: \underline{\mathcal{CM}_p} \to \mathsf{K}^{+,b}/\mathsf{K}^b. \end{aligned}$$

Now we focus on the stable t-structures $(\mathsf{K}^{-,b}/\mathsf{K}^b, \mathsf{K}^{\infty,\emptyset})$ of $\mathsf{K}^{\infty,b}/\mathsf{K}^b$, and $(\underline{\mathcal{CM}_0}, \underline{\mathcal{CM}_1})$ of $\underline{\mathrm{mor}}_s^{CM}(R)$. For a given object A of $\mathsf{K}^{\infty,b}/\mathsf{K}^b$, there uniquely exists a triangle

$$A_- \to A \to A_{ac} \to \Sigma A_-$$

with $A_- \in \mathsf{K}^{-,b}/\mathsf{K}^b$ and $A_{ac} \in \mathsf{K}^{\infty,\emptyset}/\mathsf{K}^b$. And for each object $\underline{\alpha}$ of $\underline{\mathrm{mor}}_s^{CM}(R)$, there uniquely exists a triangle

$$\underline{\alpha}_0 \to \underline{\alpha} \to \underline{\alpha}_1 \to \Sigma\underline{\alpha}_0$$

with $\underline{\alpha}_0 \in \underline{\mathcal{CM}_0}$ and $\underline{\alpha}_1 \in \underline{\mathcal{CM}_1}$. From Proposition 4.5, we have $(\underline{F}_{\underline{\alpha}})_- \cong \underline{F}_{\underline{\alpha}_0}$ and $(\underline{F}_{\underline{\alpha}})_{ac} \cong \underline{F}_{\underline{\alpha}_1}$.

Lemma 4.6 *For objects $\underline{\alpha}$ and $\underline{\beta}$ of $\underline{\mathrm{mor}}_s^{CM}(R)$, $\underline{F}$ induces an isomorphism*

$$\mathrm{Hom}_{\underline{\mathrm{mor}}_s^{CM}(R)}(\underline{\alpha}_1, \underline{\beta}_0) \cong \mathrm{Hom}_{\mathsf{K}^{\infty,b}/\mathsf{K}^b}((\underline{F}_{\underline{\alpha}})_{ac}, (\underline{F}_{\underline{\beta}})_-).$$

The proof of Theorem 3.5. We have only to show that $\underline{F}$ is fully faithful. Let $\underline{\alpha}$ and $\underline{\beta}$ be objects of $\underline{\mathrm{mor}}_s^{CM}(R)$. The triangles

$$\begin{gathered} \underline{\alpha}_0 \to \underline{\alpha} \to \underline{\alpha}_1 \to \Sigma\underline{\alpha}_0, \\ \underline{\beta}_0 \to \underline{\beta} \to \underline{\beta}_1 \to \Sigma\underline{\beta}_0 \end{gathered}$$

induce a diagram of abelian groups with exact rows and columns

$$\begin{array}{ccccc} \underline{\mathrm{mor}}_s^{CM}(R)(\alpha_1, \beta_0) & \longrightarrow & \underline{\mathrm{mor}}_s^{CM}(R)(\alpha_1, \beta) & \longrightarrow & \underline{\mathrm{mor}}_s^{CM}(R)(\alpha_1, \beta_1) \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\mathrm{mor}}_s^{CM}(R)(\alpha, \beta_0) & \longrightarrow & \underline{\mathrm{mor}}_s^{CM}(R)(\alpha, \beta) & \longrightarrow & \underline{\mathrm{mor}}_s^{CM}(R)(\alpha, \beta_1) \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\mathrm{mor}}_s^{CM}(R)(\alpha_0, \beta_0) & \longrightarrow & \underline{\mathrm{mor}}_s^{CM}(R)(\alpha_0, \beta) & \longrightarrow & \underline{\mathrm{mor}}_s^{CM}(R)(\alpha_0, \beta_1) \end{array}$$

From Proposition 4.5, $\underline{\mathrm{mor}}_s^{CM}(R)(\alpha_0,\beta_0) \cong \mathsf{K}^{\infty,b}/\mathsf{K}^b((\underline{F_\alpha})_-,(\underline{F_\beta})_-)$ and $\underline{\mathrm{mor}}_s^{CM}(R)(\alpha_1,\beta_1) \cong \mathsf{K}^{\infty,b}/\mathsf{K}^b((\underline{F_\alpha})_{ac},(\underline{F_\beta})_{ac})$. By Lemma 4.6, $\underline{\mathrm{mor}}_s^{CM}(R)(\alpha_1,\beta_0) \cong \mathsf{K}^{\infty,b}/\mathsf{K}^b((\underline{F_\alpha})_{ac},(\underline{F_\beta})_-)$. These together give us
$\underline{\mathrm{mor}}_s^{CM}(R)(\alpha_1,\beta) \cong \mathsf{K}^{\infty,b}/\mathsf{K}^b((\underline{F_\alpha})_{ac},\underline{F_\beta})$ and $\underline{\mathrm{mor}}_s^{CM}(R)(\alpha_0,\beta_0) \cong \mathsf{K}^{\infty,b}/\mathsf{K}^b(\underline{F_\alpha},(\underline{F_\beta})_0)$. Since $(\underline{\mathcal{CM}}_0,\underline{\mathcal{CM}}_1)$ and $(\mathsf{K}^{-,b}/\mathsf{K}^b,\mathsf{K}^{\infty,\emptyset})$ are stable t-structures of $\underline{\mathrm{mor}}_s^{CM}(R)$ and $\mathsf{K}^{\infty,b}/\mathsf{K}^b$ respectively, both $\underline{\mathrm{mor}}_s^{CM}(R)(\alpha_0,\beta_1)$ and $\mathsf{K}^{\infty,b}/\mathsf{K}^b((\underline{F_\alpha})_-,(\underline{F_\beta})_{ac})$ vanish. Therefore $\underline{\mathrm{mor}}_s^{CM}(R)(\alpha,\beta_1) \cong \underline{\mathrm{mor}}_s^{CM}(R)(\alpha_1,\beta_1) \cong \mathsf{K}^{\infty,b}/\mathsf{K}^b((\underline{F_\alpha})_{ac},(\underline{F_\beta})_{ac}) \cong \mathsf{K}^{\infty,b}/\mathsf{K}^b(\underline{F_\alpha},(\underline{F_\beta})_{ac})$. Similarly $\underline{\mathrm{mor}}_s^{CM}(R)(\alpha_0,\beta) \cong \mathsf{K}^{\infty,b}/\mathsf{K}^b((\underline{F_\alpha})_-,\underline{F_\beta})$. Now $\underline{\mathrm{mor}}_s^{CM}(R)(\alpha,\beta) \cong \mathsf{K}^{\infty,b}/\mathsf{K}^b((\underline{F_\alpha}),\underline{F_\beta})$ comes from Five lemma. (q.e.d.)

Together with Theorem 3.1, we obtain Buchweitz-type theorem:

Theorem 4.7 *If R is Iwanaga-Gorenstein, then $\mathsf{K}^{\infty,b}/\mathsf{K}^b$ is triangle equivalent to $\underline{\mathrm{CM}}(T_2(R))$.*

5 Recollements

Let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ be triangulated subcategories of a triangulated category $\mathcal{C}$. Suppose $(\mathcal{U},\mathcal{V})$ and $(\mathcal{V},\mathcal{W})$ are both stable t-structures of $\mathcal{C}$. From Prop 4.2, the canonical embedding $j_* : \mathcal{V} \to \mathcal{C}$ and the quotient $s^* : \mathcal{C} \to \mathcal{C}/\mathcal{V}$ have right adjoints $j^! : \mathcal{C} \to \mathcal{V}$ and $s^* : \mathcal{C}/\mathcal{V} \to \mathcal{C}$ since $(\mathcal{U},\mathcal{V})$ is a stable t-structure. And a stable t-structure $(\mathcal{V},\mathcal{W})$ produeces left adjoints $j^* : \mathcal{C} \to \mathcal{V}$ of j_* and $s_! : \mathcal{C}/\mathcal{V} \to \mathcal{C}$ of $s^* : \mathcal{C}/\mathcal{V} \to \mathcal{C}$ respectively.

Definition 5.1 ([BBD]) *A nine-tuple $\{\mathcal{C}',\mathcal{C},\mathcal{C}''; j^*, j_*, j^!, s_!, s^*, s_*\}$ consisting of triangulated categories and functors*

$$\mathcal{C}' \;\begin{matrix}\xleftarrow{j^*}\\ \xrightarrow{j_*}\\ \xleftarrow{j^!}\end{matrix}\; \mathcal{C} \;\begin{matrix}\xleftarrow{s_!}\\ \xrightarrow{s^*}\\ \xleftarrow{s_*}\end{matrix}\; \mathcal{C}''$$

is called a recollement if it satisfies the following:

- *j_*, $s_!$, and s_* are fully faithful.*
- *(j^*, j_*), $(j_*, j^!)$, $(s_!, s^*)$, and (s^*, s_*) are adjoint pairs.*
- *$j^* s_! = 0$, $s^* j_* = 0$, and $j^! s_* = 0$.*
- *For each object C of $\mathcal{C}$ has triangles*

$$j_*j^!C \to C \to s_!s^*C \to \Sigma j_*j^!C,$$
$$s_*s^*C \to C \to j_*j^*C \to \Sigma s_*s^*C.$$

Proposition 5.2 ([BBD], [Mi1]) *1) If $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{W})$ are stable t-structures of $\mathcal{C}$, then the canonical embedding $j_* : \mathcal{V} \to \mathcal{C}$ produces a recollement*

$$\mathcal{V} \;\underset{\xleftarrow{j^!}}{\overset{\xleftarrow{j^*}}{\xrightarrow{j_*}}}\; \mathcal{C} \;\underset{\xleftarrow{s_*}}{\overset{\xleftarrow{s_!}}{\xrightarrow{s^*}}}\; \mathcal{C}/\mathcal{V}$$

2) If $\{\mathcal{C}', \mathcal{C}, \mathcal{C}''; j^*, j_*, j^!, s_!, s^*, s_*\}$ is a recollement, then $(\mathrm{Im} j_*, \mathrm{Im} s_*)$ and $(\mathrm{Im} s_!, \mathrm{Im} j_*)$ are stable t-structures.

Remember that if R is Iwanaga-Gorenstein, three triangulated subcategories $\mathsf{K}^{-,b}/\mathsf{K}^b$, $\mathsf{K}^{\infty,\emptyset}$, and $\mathsf{K}^{+,b}/\mathsf{K}^b$ form three stable t-structures in $\mathsf{K}^{\infty,b}$: $(\mathsf{K}^{-,b}/\mathsf{K}^b, \mathsf{K}^{\infty,\emptyset})$, $(\mathsf{K}^{\infty,\emptyset}, \mathsf{K}^{+,b}/\mathsf{K}^b)$ and $(\mathsf{K}^{+,b}/\mathsf{K}^b, \mathsf{K}^{-,b}/\mathsf{K}^b)$. This implies there are three recollements with respect to the canonical embeddings of each subcategories to $\mathsf{K}^{\infty,b}$.

Definition 5.3 *Let $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$ be triangulated subcategories of a triangulated category $\mathcal{C}$. We call $(\mathcal{U}_1,\mathcal{U}_2,\mathcal{U}_3)$ a triangle of recollements in $\mathcal{C}$ if $(\mathcal{U}_1,\mathcal{U}_2$, $(\mathcal{U}_2,\mathcal{U}_3$, and $(\mathcal{U}_2,\mathcal{U}_3$ are stable t-structures of $\mathcal{C}$. In this case, there are recollements*

$$\mathcal{U}_n \;\underset{\xleftarrow{i_n^!}}{\overset{\xleftarrow{i_n^*}}{\xrightarrow{j_{n*}}}}\; \mathcal{C} \;\underset{\xleftarrow{j_{n*}}}{\overset{\xleftarrow{j_{n!}}}{\xrightarrow{j_n^*}}}\; \mathcal{C}/\mathcal{U}_n$$

for any $n \mod 3$ such that the essential image $\mathrm{Im} j_{n!}$ is $\mathcal{U}_{n-1}$, and that the essential image $\mathrm{Im} j_{n*}$ is $\mathcal{U}_{n+1}$. Therefore, $\mathcal{U}_1$,$\mathcal{U}_2$ and $\mathcal{U}_3$ are triangulated equivalent.

Theorem 5.4 *If R is Iwanaga-Gorenstein, then $(\mathsf{K}^{-,b}/\mathsf{K}^b, \mathsf{K}^{\infty,\emptyset}, \mathsf{K}^{+,b}/\mathsf{K}^b)$ is a triangle of recollements in $\mathsf{K}^{\infty,b}/\mathsf{K}^b$. There is a triangulated equivalence between $\underline{\mathrm{mor}}_s^{CM}(R) \cong \underline{\mathrm{CM}}(T_2(R))$ and $\mathsf{K}^{\infty,b}/\mathsf{K}^b$ that induces the correspondence between a triangle of recollements $(\underline{\mathcal{CM}}_0, \underline{\mathcal{CM}}_1, \underline{\mathcal{CM}}_p)$ and $(\mathsf{K}^{-,b}/\mathsf{K}^b, \mathsf{K}^{\infty,\emptyset}, \mathsf{K}^{+,b}/\mathsf{K}^b)$.*

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