Shestakov-Umirbaev reductions and Nagata's conjecture on a polynomial automorphism

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Abstract

In 2003, Shestakov-Umirbaev solved Nagata's conjecture on an automorphism of a polynomial ring. In the present paper, we reconstruct their theory by using the "generalized Shestakov-Umirbaev inequality", which was recently given by the author. As a consequence, we obtain a more precise tameness criterion for polynomial automorphisms. In particular, we show that no tame automorphism of a polynomial ring admits a reduction of type IV.

1 Introduction

Let k be a field, n a natural number, and $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ the polynomial ring in n variables over k. In the present paper, we discuss the structure of the automorphism group $\mathrm{Aut}_k\, k[\mathbf{x}]$ of $k[\mathbf{x}]$ over k. Let $F : k[\mathbf{x}] \to k[\mathbf{x}]$ be an endomorphism over k. We identify F with the n-tuple $(f_1, \ldots, f_n)$ of elements of $k[\mathbf{x}]$, where $f_i = F(x_i)$ for each i. Then, F is an automorphism if and only if the k-algebra $k[\mathbf{x}]$ is generated by $f_1, \ldots, f_n$. Note that the sum $\deg F := \sum_{i=1}^{n} \deg f_i$ of the total degrees of $f_1, \ldots, f_n$ is at least n whenever F is an automorphism. An automorphism F is said to be *affine* if $\deg F = n$. If this is the case, then there exist $(a_{i,j})_{i,j} \in GL_n(k)$ and $(b_i)_i \in k^n$ such that $f_i = \sum_{j=1}^{n} a_{i,j} x_j + b_i$ for each i. We say that F is *elementary* if there exist

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$l \in \{1, \ldots, n\}$ and $\phi \in k[x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_n]$ such that $f_l = x_l + \phi$ and $f_i = x_i$ for each $i \neq l$. The subgroup $\mathrm{T}_k\, k[\mathbf{x}]$ of $\mathrm{Aut}_k\, k[\mathbf{x}]$ generated by affine automorphisms and elementary automorphisms is called the *tame subgroup*. An automorphism is said to be *tame* if it belongs to $\mathrm{T}_k\, k[\mathbf{x}]$.

It is a fundamental question in polynomial ring theory whether $\mathrm{T}_k\, k[\mathbf{x}] = \mathrm{Aut}_k\, k[\mathbf{x}]$ holds for each n, which is called the *tame generators problem*. The equality is obvious if $n = 1$. This also holds true if $n = 2$. It was shown by Jung [4] in 1942 when k is of characteristic zero, and by van der Kulk [5] in 1953 when k is an arbitrary field. These results are consequences of the fact that each automorphism of $k[\mathbf{x}]$ but an affine automorphism admits an elementary reduction if $n = 2$. Here, we say that F *admits an elementary reduction* if $\deg(F \circ E) < \deg F$ for some elementary automorphism E, that is, there exist $l \in \{1, \ldots, n\}$ and $\phi \in k[f_1, \ldots, f_{l-1}, f_{l+1}, \ldots, f_n]$ such that $\deg(f_l - \phi) < \deg f_l$. By the Jung-van der Kulk theorem, in case $n = 2$, we may find elementary automorphisms $E_1, \ldots, E_r$ for some $r \in \mathbf{N}$ such that

$$\deg F > \deg(F \circ E_1) > \cdots > \deg(F \circ E_1 \circ \cdots \circ E_r) = 2$$

for each $F \in \mathrm{Aut}_k\, k[\mathbf{x}]$ with $\deg F > 2$. This implies that F is tame.

When $n = 3$, the structure of $\mathrm{Aut}_k\, k[\mathbf{x}]$ becomes far more difficult. In 1972, Nagata [9] conjectured that the automorphism

$$F = (x_1 - 2(x_1x_3 + x_2^2)x_2 - (x_1x_3 + x_2^2)^2x_3, x_2 + (x_1x_3 + x_2^2)x_3, x_3) \quad (1.1)$$

is not tame. This famous conjecture was finally solved in the affirmative by Shestakov-Umirbaev [11] in 2003 for a field k of characteristic zero. Therefore, $\mathrm{T}_k\, k[\mathbf{x}] \neq \mathrm{Aut}_k\, k[\mathbf{x}]$ if $n = 3$. However, the question remains open for $n \geq 4$.

Shestakov-Umirbaev [11] showed that, if F does not admit an elementary reduction for $F \in \mathrm{T}_k\, k[\mathbf{x}]$ with $\deg F > 3$, then there exists a sequence of elementary automorphisms $E_1, \ldots, E_r$, where $r \in \{2, 3, 4\}$, with certain conditions such that $\deg(F \circ E_1 \circ \cdots \circ E_r) < \deg F$. If this is the case, then F is said to *admit a reduction of type I, II, III or IV* according to the conditions on F and $E_1, \ldots, E_r$. Nagata's automorphism is not affine, and does not admit neither an elementary reduction nor reductions of these four types. Therefore, Nagata's automorphism is not tame. We note that there exist tame automorphisms which admit reductions of type I (see [1], [7] and [11]), but it is not known whether there exist automorphisms admitting reductions of the other types.

Shestakov-Umirbaev [11] used an inequality [10, Theorem 3] concerning the total degrees of polynomials as a crucial tool. This result was recently generalized by the author in [6]. The purpose of this paper is to reconstruct the Shestakov-Umirbaev theory using the generalized inequality. As a consequence, we obtain a more precise tameness criterion for polynomial automorphisms. In particular, we show that no tame automorphism of $k[\mathbf{x}]$ admits a reduction of type IV.

This report consists of the first two sections of [8], which is available at

`http://arxiv.org/PS_cache/arxiv/pdf/0801/0801.0117v1.pdf`

Although the full version of [8] is 48 pages long, the details are carefully explained. It is said that the theory of Shestakov and Umirbaev is difficult and still not widely understood. I hope that our article will be helpful in understanding how the tame generators problem was solved.

2 Main result

In what follows, we assume that the field k is of characteristic zero. Let Γ be a totally ordered $\mathbf{Z}$-module, and $\omega = (\omega_1, \ldots, \omega_n)$ an n-tuple of elements of Γ with $\omega_i > 0$ for $i = 1, \ldots, n$. We define the *ω-weighted grading* $k[\mathbf{x}] = \bigoplus_{\gamma \in \Gamma} k[\mathbf{x}]_\gamma$ by setting $k[\mathbf{x}]_\gamma$ to be the k-vector subspace generated by the monomials $x_1^{a_1} \cdots x_n^{a_n}$ of $k[\mathbf{x}]$ with $\sum_{i=1}^n a_i \omega_i = \gamma$ for each $\gamma \in \Gamma$. For $f \in k[\mathbf{x}] \setminus \{0\}$, we define the *$\omega$-weighted degree* $\deg_\omega f$ of f to be the maximum among $\gamma \in \Gamma$ with $f_\gamma \neq 0$, where $f_\gamma \in k[\mathbf{x}]_\gamma$ for each γ such that $f = \sum_{\gamma \in \Gamma} f_\gamma$. We define $f^\omega = f_\delta$, where $\delta = \deg_\omega f$. In case $f = 0$, we set $\deg_\omega f = -\infty$, i.e., a symbol which is less than any element of Γ. For example, if $\Gamma = \mathbf{Z}$ and $\omega_i = 1$ for $i = 1, \ldots, n$, then the ω-weighted degree is the same as the total degree. For each k-vector subspace V of $k[\mathbf{x}]$, we define V^ω to be the k-vector subspace of $k[\mathbf{x}]$ generated by $\{f^\omega \mid f \in V \setminus \{0\}\}$. For each l-tuple $F = (f_1, \ldots, f_l)$ of elements of $k[\mathbf{x}]$ for $l \in \mathbf{N}$, we define $\deg_\omega F = \sum_{i=1}^l \deg_\omega f_i$.

For each $\sigma \in \mathfrak{S}_l$, we define $F_\sigma = (f_{\sigma(1)}, \ldots, f_{\sigma(l)})$, where $\mathfrak{S}_l$ is the symmetric group of $\{1, \ldots, l\}$ for each $l \in \mathbf{N}$.

The degree of a differential form defined in [6] is important in our theory. Let $\Omega_{k[\mathbf{x}]/k}$ be the module of differentials of $k[\mathbf{x}]$ over k, and $\bigwedge^l \Omega_{k[\mathbf{x}]/k}$ the l-th exterior power of the $k[\mathbf{x}]$-module $\Omega_{k[\mathbf{x}]/k}$ for $l \in \mathbf{N}$. Then, we may uniquely express each $\theta \in \bigwedge^l \Omega_{k[\mathbf{x}]/k}$ as

$$\theta = \sum_{1 \le i_1 < \cdots < i_l \le n} f_{i_1, \ldots, i_l} dx_{i_1} \wedge \cdots \wedge dx_{i_l},$$

where $f_{i_1, \ldots, i_l} \in k[\mathbf{x}]$ for each $i_1, \ldots, i_l$. Here, df denotes the differential of f for each $f \in k[\mathbf{x}]$. We define

$$\deg_\omega \theta = \max\{\deg_\omega(f_{i_1, \ldots, i_l} x_{i_1} \cdots x_{i_l}) \mid 1 \le i_1 < \cdots < i_l \le n\}.$$

If $\theta \ne 0$, then it follows that

$$\deg_\omega \theta \ge \min\{\omega_{i_1} + \cdots + \omega_{i_l} \mid 1 \le i_1 < \cdots < i_l \le n\} > 0. \tag{2.1}$$

We remark that $f_1, \ldots, f_l$ are algebraically independent over k if and only if $df_1 \wedge \cdots \wedge df_l \ne 0$ for $f_1, \ldots, f_l \in k[\mathbf{x}]$. Actually, this condition is equivalent to the condition that the rank of the l by n matrix $((f_i)_{x_j})_{i,j}$ is equal to l (cf. [3, Proposition 1.2.9]). Here, f_{x_i} denotes the partial derivative of f in x_i for each $f \in k[\mathbf{x}]$ and $i \in \{1, \ldots, n\}$. By definition, it follows that

$$\sum_{i=1}^{l} \deg_\omega df_i \ge \deg_\omega(df_1 \wedge \cdots \wedge df_l). \tag{2.2}$$

In (2.2), the equality holds if and only if $f_1^\omega, \ldots, f_l^\omega$ are algebraically independent over k. Actually, we may write $df_1 \wedge \cdots \wedge df_l = df_1^\omega \wedge \cdots \wedge df_l^\omega + \eta$, where $\eta \in \bigwedge^l \Omega_{k[\mathbf{x}]/k}$ with $\deg_\omega \eta < \sum_{i=1}^{l} \deg_\omega f_i$. For each $f \in k[\mathbf{x}] \setminus k$, we have

$$\deg_\omega df = \max\{\deg_\omega(f_{x_i} x_i) \mid i = 1, \ldots, n\} = \deg_\omega f, \tag{2.3}$$

since $df = \sum_{i=1}^{n} f_{x_i} dx_i$. If $f_1, \ldots, f_n \in k[\mathbf{x}]$ are algebraically independent over k, then

$$\sum_{i=1}^{n} \deg_\omega f_i = \sum_{i=1}^{n} \deg_\omega df_i \ge \deg_\omega(df_1 \wedge \cdots \wedge df_n) \ge \sum_{i=1}^{n} \omega_i =: |\omega| \tag{2.4}$$

by (2.1), (2.3) and (2.4). As will be shown in Lemma 6.1(i), if $\deg_\omega F = |\omega|$ for $F \in \mathrm{Aut}_k k[\mathbf{x}]$, then F is tame.

Now, consider the set $\mathcal{T}$ of triples $F = (f_1, f_2, f_3)$ of elements of $k[\mathbf{x}]$ such that f_1, f_2 and f_3 are algebraically independent over k. We identify each $F \in \mathcal{T}$ with the injective homomorphism $F : k[\mathbf{y}] \to k[\mathbf{x}]$ defined by $F(y_i) = f_i$ for $i = 1, 2, 3$, where $k[\mathbf{y}] = k[y_1, y_2, y_3]$ is the polynomial ring in three variables over k. Let $\mathcal{E}_i$ denote the set of elementary automorphisms E of $k[\mathbf{y}]$ such that $E(y_j) = y_j$ for each $j \neq i$ for $i \in \{1, 2, 3\}$, and $\mathcal{E} = \bigcup_{i=1}^{3} \mathcal{E}_i$. We say that $F = (f_1, f_2, f_3)$ *admits an elementary reduction* for the weight ω if $\deg_\omega(F \circ E) < \deg_\omega F$ for some $E \in \mathcal{E}$, and call $F \circ E$ an *elementary reduction* of F for the weight ω.

Let $F = (f_1, f_2, f_3)$ and $G = (g_1, g_2, g_3)$ be elements of $\mathcal{T}$. We say that the pair (F, G) satisfies the *Shestakov-Umirbaev condition* for the weight ω if the following conditions hold:

(SU1) $g_1 = f_1 + af_3^2 + cf_3$ and $g_2 = f_2 + bf_3$ for some $a, b, c \in k$, and $g_3 - f_3$ belongs to $k[g_1, g_2]$;

(SU2) $\deg_\omega f_1 \leq \deg_\omega g_1$ and $\deg_\omega f_2 = \deg_\omega g_2$;

(SU3) $(g_1^\omega)^2 \approx (g_2^\omega)^s$ for some odd number $s \geq 3$;

(SU4) $\deg_\omega f_3 \leq \deg_\omega g_1$, and f_3^ω does not belong to $k[g_1^\omega, g_2^\omega]$;

(SU5) $\deg_\omega g_3 < \deg_\omega f_3$;

(SU6) $\deg_\omega g_3 < \deg_\omega g_1 - \deg_\omega g_2 + \deg_\omega(dg_1 \wedge dg_2)$.

Here, $h_1 \approx h_2$ (resp. $h_1 \not\approx h_2$) denotes that h_1 and h_2 are linearly dependent (resp. linearly independent) over k for each $h_1, h_2 \in k[\mathbf{x}] \setminus \{0\}$. We say that $F \in \mathcal{T}$ *admits a Shestakov-Umirbaev reduction* for the weight ω if there exist $G \in \mathcal{T}$ and $\sigma \in \mathfrak{S}_3$ such that (F_σ, G_σ) satisfies the Shestakov-Umirbaev condition, and call this G a *Shestakov-Umirbaev reduction* of F for the weight ω. As will be shown in Theorem 4.1(P6), $\deg_\omega G < \deg_\omega F$ if G is a Shestakov-Umirbaev reduction of F.

Note that (SU1) implies that there exist $E_i \in \mathcal{E}_i$ for $i = 1, 2, 3$ such that $F \circ E_1 = (f_1, g_2, f_3)$, $F \circ E_1 \circ E_2 = (g_1, g_2, f_3)$ and $F \circ E_1 \circ E_2 \circ E_3 = G$. Furthermore, $\delta := (1/2) \deg_\omega g_2$ belongs to Γ by (SU3).

Here is our main result.

Theorem 2.1 *Assume that* $n = 3$*, and* $\omega = (\omega_1, \omega_2, \omega_3)$ *is an element of* Γ^3 *such that* $\omega_i > 0$ *for each* i*. Then, each* $F \in \mathrm{T}_k\, k[\mathbf{x}]$ *with* $\deg_\omega F > |\omega|$ *admits an elementary reduction or a Shestakov-Umirbaev reduction for the weight* ω*.*

Note that F admits an elementary reduction for the weight ω if and only if f_i^ω belongs to $k[f_j, f_l]^\omega$ for some $i \in \{1, 2, 3\}$, where $j, l \in \mathbf{N} \setminus \{i\}$ with $1 \le j < l \le 3$. In case $\deg_\omega f_1$, $\deg_\omega f_2$ and $\deg_\omega f_3$ are pairwise linearly independent, this condition is equivalent to the condition that $\deg_\omega f_i$ belongs to the subsemigroup of Γ generated by $\deg_\omega f_j$ and $\deg_\omega f_l$ for some $i \in \{1, 2, 3\}$. Indeed, for each $\phi \in k[f_j, f_l] \setminus \{0\}$, there exist $p, q \in \mathbf{Z}_{\ge 0}$ such that $\deg_\omega \phi = \deg_\omega f_j^p f_l^q$, since ϕ is a linear combination of $f_j^p f_l^q$ for $(p, q) \in (\mathbf{Z}_{\ge 0})^2$ over k, in which $\deg_\omega f_j^p f_l^q \ne \deg_\omega f_j^{p'} f_l^{q'}$ whenever $(p, q) \ne (p', q')$. Here, $\mathbf{Z}_{\ge 0}$ denotes the set of nonnegative integers.

Using Theorem 2.1, we can verify that Nagata's automorphism is not tame. Let $\Gamma = \mathbf{Z}^3$ equipped with the lexicographic order, i.e., $a \le b$ if the first nonzero component of $b - a$ is positive for $a, b \in \mathbf{Z}^3$, and let $\omega = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, where $\mathbf{e}_i$ is the i-th standard unit vector of $\mathbf{R}^3$ for each i. Then, we have

$$\deg_\omega f_1 = (2, 0, 3),\ \deg_\omega f_2 = (1, 0, 2),\ \deg_\omega f_3 = (0, 0, 1).$$

Hence, $\deg_\omega F = (3, 0, 6) > (1, 1, 1) = |\omega|$. On the other hand, the three vectors above are pairwise linearly independent, while any one of them is not contained in the subsemigroup of $\mathbf{Z}^3$ generated by the other two vectors. Hence, F does not admit an elementary reduction for the weight ω. Since $(1/2) \deg_\omega f_i$ does not belong to $\Gamma = \mathbf{Z}^3$ for each $i \in \{1, 2, 3\}$, we know that F does not admit a Shestakov-Umirbaev reduction for the weight ω.

Therefore, we have the following corollary to Theorem 2.1.

Corollary 2.2 *Nagata's automorphism is not tame.*

We may also check that Nagata's automorphism does not admit a Shestakov-Umirbaev reduction in a different way as follows. By Theorem 4.1(P7), we know that $0 < \delta < \deg_\omega f_i \le s\delta$ holds each $i \in \{1, 2, 3\}$ if F admits a Shestakov-Umirbaev reduction for the weight ω. Hence, $s \deg_\omega f_i > \deg_\omega f_j$

for each $i, j \in \{1, 2, 3\}$. On the other hand, in the case of Nagata's automorphism, $l \deg_\omega f_3 = (0, 0, l)$ is less than $\deg_\omega f_i$ for $i = 1, 2$ for any $l \in \mathbf{N}$ by the definition of the lexicographic order. Therefore, F does not admit a Shestakov-Umirbaev reduction for the weight ω.

We define the *rank* of ω as the rank of the $\mathbf{Z}$-submodule of Γ generated by $\omega_1, \ldots, \omega_n$. If ω has maximal rank n, then the k-vector space $k[\mathbf{x}]_\gamma$ is of dimension at most one for each γ. Consequently, it follows that $\deg_\omega f = \deg_\omega g$ if and only if $f^\omega \approx g^\omega$ for each $f, g \in k[\mathbf{x}] \setminus \{0\}$. In such a case, the assertion of Theorem 2.1 can be proved more easily than the general case. Actually, we may omit a few lemmas and propositions needed to prove Theorem 2.1. We note that $\omega = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ has maximal rank three, and so it suffices to show the assertion of Theorem 2.1 in this special case to verify that Nagata's automorphism is not tame.

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