Willmore two-spheres in $S^{n+2}$ via Loop group theory

Peng Wang (with Josef Dorfmeister)

Tongji University
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Background

- \( x : M \rightarrow S^{n+2} \) Willmore surface: critical surface of the Willmore functional

\[
W(M) = \int_M (H^2 - K + 1) dM
\]

- Bryant, R. (1984), \( x : M \rightarrow S^3 \) Willmore,
  1. harmonicity of conformal Gauss map

\[
Gr : M \rightarrow Gr_{3,1}(\mathbb{R}^5_1) = S^4_1,
\]

  2. Duality theorems.
  3. Willmore \( S^2 \) \( \implies \) conformal to minimal surface in \( \mathbb{R}^3 \).
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Peng Wang (with Josef Dorfmeister) Willmore two-spheres in $S^{n+2}$ via Loop group theory
Ejiri (1988) $x : M \rightarrow S^{n+2}$ Willmore:

- harmonicity of conformal Gauss map

\[ Gr : M \rightarrow Gr_{3,1}(\mathbb{R}^{n+4}_1) \]

- S-Willmore surface: Willmore surface with a dual surface,
- Classification of S-Willmore $S^2$ in $S^{n+2}$.
- All Willmore $S^2$ in $S^4$ are S-Willmore.
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$$S^2 \setminus \{p_1, \ldots, p_n\} \xrightarrow{\text{Willmore}} S^4$$

$\pi$ $\downarrow$

$S^4 \rightarrow R^4$

$(\text{anti-)holo}$ $\downarrow$

Twistor map

$S^2 \xrightarrow{\text{Willmore}} S^4$

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$S^2 \setminus \{p_1, \cdots, p_n\} \xrightarrow{\text{Willmore}} S^4$

$\xrightarrow{\text{minimal}}$

$\pi$

$\downarrow$

$R^4$

$CP^3 \xrightarrow{(\text{anti-})\text{holo}} S^4$

$\xrightarrow{\text{Twistor map}}$

$S^2 \xrightarrow{\text{Willmore}} S^4$
Questions:

- Are there Willmore two spheres in $S^5$ or $S^6$ which are non-S-Willmore?
- Classification of all Willmore $S^2$ in $S^{n+2}$.
- How to do with Willmore surfaces by use of the theory on harmonic maps into symmetric spaces?
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Uhlenbeck, K. (1989): All harmonic $S^2$ in $U(n)$. finite uniton.


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Our main strategy

- Study Willmore surface by considering the conformal harmonic Gauss map via loop group methods.
- Harmonic map from $S^2$ into compact Lie group is of finite uniton. For the non-compact case, this property holds too.
- One can describe the conformal harmonic maps of finite uniton explicitly, and as an application, giving all Willmore two-spheres (may have branch points).
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Basic methods of our work

- Moving frame of Willmore surface in $S^{n+2}$ by Burstall-Pedit-Pinkall.
- DPW methods for harmonic maps in symmetric space, i.e., using Lie-algebra-valued meromorphic 1-form (Normalized potential) to describe harmonic maps.
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Main results

- Harmonic maps into compact symmetric space v.s. non compact symmetric spaces.
- From Willmore surface to the conformal Gauss map, and how to go back.
- The finite uniton case: classification of nilpotent normalized potential, and going back to the corresponding Willmore surfaces.
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G K compact. \( f : M^2 \rightarrow G/K \) harmonic

\[ \rightarrow F(z, \bar{z}, \lambda) : M^2 \rightarrow \Lambda G_{\sigma}, \lambda \in S^1. \quad \rightarrow \]

\[ F(z, \bar{z}, \lambda) = F_-(z, \bar{z}, \lambda) F_+(z, \bar{z}, \lambda) \quad \text{(Birkhoff decomposition)} \]

\[ F_- dF_- = \eta = \lambda^{-1} \eta_{-1} dz. \quad \text{(meromorphic) Normalized potential} \]

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\[ \rightarrow F_- = F(z, \bar{z}, \lambda) F_+(z, \bar{z}, \lambda) \quad \text{Iwasawa decomposition} \]

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Burstall-Guest: \( f \) finite uniton \( \iff \eta_{-1} \) locates in some nilpotent Lie subalgebra.
Strategy of DPW

- \( G/K \) compact. \( f : M^2 \to G/K \) harmonic

\[ \implies F(z, \bar{z}, \lambda) : M^2 \to \Lambda G_\sigma, \lambda \in S^1. \implies \]
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Non-compact case vs compact case

- $G$ non-compact Lie group, $G/K$ inner symmetric. $\implies$
- $U \subset G^C$, $U$ compact, and $U^C = G^C$, $(U \cap K^C)^C = K^C$.

- $f : M^2 \rightarrow G/K$, $f$ harmonic, $\implies$ (Iwasawa)
  $f_U : M^2 \rightarrow U/(U \cap K^C)$
  $f$ has the same normalized potential as $f_U$.
  Especially, $f$ is of finite uniton if and only if $f_U$ is of finite uniton.
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Let $C^{n+3}$ be the light cone of Lorentz-Minkowski space $\mathbb{R}^{n+4}_1$, then $S^{n+2} = Q^{n+2} = \{ [x] \in \mathbb{R}P^{n+3} | x \in C^{n+3} \setminus \{0\} \}$.

The conformal group of $S^{n+2}: = SO(1, n + 3)$.

$y : M \to S^{n+2}$ immersion, the conformal Gauss map

$$Gr : M \to Gr_{3,1}(\mathbb{R}^{n+4}_1) = SO(1, n + 3)/SO(1, 3) \times SO(n).$$

$Gr$ corresponds to the mean curvature sphere congruence. $y$ is a conformal enveloping surface of $Gr$.

$y$ Willmore $\iff$ $Gr$ harmonic
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From Willmore surfaces to harmonic maps

- $y : M \to S^{n+2}$ Willmore $\implies$

  $Gr : M \to SO(1, n+3)/SO(1, 3) \times SO(n)$ harmonic, the Maurer-Cartan form is of the form

  $$\alpha' = \begin{pmatrix} A_1 & B_1 \\ -B_1^t I_{1,3} & A_2 \end{pmatrix} dz,$$

  with $B_1^t I_{1,3} B_1 = 0. (\implies \text{Rank}(B_1) \leq 2).$

- $y$ S-Willmore $\iff B_1$ is of rank one.
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Let \( f : M \to SO(1, n + 3)/SO(1, 3) \times SO(n) \) be a harmonic map with its Maurer-Cartan form of \( f \) satisfying \( B_1^t I_{1,3} B_1 = 0 \).

- \( f \) envelops a pair of dual Willmore surfaces (hence S-Willmore) \( \iff \text{Rank}(B_1) = 1 \). (One of them may degenerate to a point).

- \( f \) envelops a unique surface \( y \iff \text{Rank}(B_1) = 2 \). (\( y \) may degenerate to a point).
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- $f$ envelopes a unique surface $y$ \iff $\text{Rank}(B_1) = 2$. ($y$ may degenerate to a point).
Let \( f : M \to SO(1, n + 3)/SO(1, 3) \times SO(n) \) be a harmonic map with \( B_1^t I_{1,3} B_1 = 0 \). Then there exists an enveloping surface of \( f \) degenerating to a point, if and only if the normalized potential is of the form

\[
\eta = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{pmatrix} dz, \quad \hat{B}_1 = (v_1, \cdots, v_n),
\]

with

\[
v_j \hookrightarrow \text{Span}_\mathbb{C} \left\{ (1, 1, 0, 0)^t, (0, 0, 1, i)^t \right\}, \quad j = 1, \cdots, n.
\]
Examples of Willmore surfaces of finite uniton in $S^n$

- Minimal surfaces in $\mathbb{R}^n$.
- Surfaces in $S^4$ coming from (anti-)holomorphic curves of the twistor bundle $\mathbb{C}P^3$. 

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For a harmonic map $f : M \to SO(1, 7)/SO(1, 3) \times SO(4)$ of finite uniton, with $B_1^t I_{1,3} B_1 = 0$. Suppose that the normalized potential

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Then up to a conjugation of $SO(1, 3) \times SO(4)$, $\hat{B}_1$ must be one of the three cases:

1. $v_j \hookrightarrow \text{Span}_\mathbb{C} \left\{ (1, 1, 0, 0)^t, (0, 0, 1, i)^t \right\}, j = 1, \cdots, 4.$
2. $v_2 = iv_1, v_3 = iv_4.$
3. $v_2 = iv_1, v_3, v_4 \hookrightarrow \text{Span}_\mathbb{C} \left\{ (1, 1, 0, 0)^t, (0, 0, 1, i)^t \right\}.$
Willmore surfaces of finite uniton in $S^6$

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For a harmonic map $f : M \to SO(1, 7)/SO(1, 3) \times SO(4)$ of finite uniton, with $B^t_1 I_{1,3} B_1 = 0$. Suppose that the normalized potential

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3. $v_2 = iv_1, v_3, v_4 \leftrightarrow \text{Span}_\mathbb{C} \{(1, 1, 0, 0)^t, (0, 0, 1, i)^t\}.$
Going back to Willmore surfaces of finite uniton in $S^6$

- Case (1).

\[ \text{Rank}(\hat{B}_1) = 1 \iff y \text{ conformal to minimal surface in } \mathbb{R}^6, \]
\[ \text{Rank}(\hat{B}_1) = 2 \iff y \text{ degenerates to a point.} \]

- Case (2) $\Rightarrow y$ totally isotropic. For Case (2)$\setminus$Case (1).

\[ \text{Rank}(\hat{B}_1) = 1 \iff y \text{ S-Willmore,} \]
\[ \text{Rank}(\hat{B}_1) = 2 \iff y \text{ not S-Willmore.} \]

- Case (3)$\setminus$Case (2) and Case (1): $\Rightarrow \text{Rank}(\hat{B}_1) = 2$

$y$ having non isotropic Hopf differential, not S-Willmore.

- Case (1) and Case (2) are all the cases such that $f$ is $S^1$-invariant.
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y \text{ having non isotropic Hopf differential, not S-Willmore.}

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Examples of Case (2)

The normalized potential

\[ \eta = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}^t_1 I_{1,3} & 0 \end{pmatrix} \, dz, \]

with

\[ \hat{B}_1 = \frac{1}{2} \begin{pmatrix} 2iz & -2z & -i & 1 \\ -2iz & 2z & -i & 1 \\ -2 & -2i & -z & -iz \\ 2i & -2 & -iz & z \end{pmatrix}. \]
\[
Y = \begin{pmatrix}
(1 + r^2 + \frac{5r^4}{4} + \frac{4r^6}{9} + \frac{r^8}{36}) \\
(1 - r^2 - \frac{3r^4}{4} + \frac{4r^6}{9} - \frac{r^8}{36}) \\
- i \left( z - \bar{z} \right) \left( 1 + \frac{r^6}{9} \right) \\
\left( z + \bar{z} \right) \left( 1 + \frac{r^6}{9} \right) \\
- i \left( \lambda^{-1} z^2 - \lambda \bar{z}^2 \right) \left( 1 - \frac{r^4}{12} \right) \\
\left( \lambda^{-1} z^2 + \lambda \bar{z}^2 \right) \left( 1 - \frac{r^4}{12} \right) \\
- \frac{i}{2} r^2 \left( \lambda^{-1} z - \lambda \bar{z} \right) \left( 1 + \frac{4r^2}{3} \right) \\
\frac{r^2}{2} \left( \lambda^{-1} z + \lambda \bar{z} \right) \left( 1 + \frac{4r^2}{3} \right)
\end{pmatrix}, \quad r = |z|.
\]

\[
y = [Y] : S^2 \to S^6 \text{ is a totally isotropic immersed Willmore sphere which is not S-Willmore.}
\]
$y = \begin{bmatrix}
(1 + r^2 + \frac{5r^4}{4} + \frac{4r^6}{9} + \frac{r^8}{36}) \\
(1 - r^2 - \frac{3r^4}{4} + \frac{4r^6}{9} - \frac{r^8}{36}) \\
- \frac{1}{2} \left( \lambda^{-1}z^2 - \lambda \bar{z}^2 \right) (1 - \frac{r^4}{12}) \\
\left( \lambda^{-1}z^2 + \lambda \bar{z}^2 \right) (1 - \frac{r^4}{12}) \\
- \frac{1}{2} \frac{r^2}{2} \left( \lambda^{-1}z - \lambda \bar{z} \right) (1 + \frac{4r^2}{3}) \\
\frac{r^2}{2} \left( \lambda^{-1}z + \lambda \bar{z} \right) (1 + \frac{4r^2}{3})
\end{bmatrix}$

, $r = |z|$. 

$y = [Y] : S^2 \rightarrow S^6$ is a totally isotropic immersed Willmore sphere which is not S-Willmore.
The $S^4$ case

- Case (3) can not happen.

- Case (1) $\implies$ minimal surfaces in $R^4$.

- For case (2), $rank(B_1) = 1$. The corresponding Willmore surfaces are always S-Willmore having isotropic Hopf differential. $\Rightarrow$ holomorphic or anti-holomorphic curves in $CP^3$. 

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Willmore two-spheres in $S^{n+2}$ via Loop group theory
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Suppose that $\hat{B}_1 = (v_1, \cdots, v_{2m})$. Then up to a conjugation of $SO(1, 3) \times SO(2m)$, $\hat{B}_1$ must be one of the $(m + 1)$ cases:

1. 
   
   $v_j \mapsto \text{Span}_\mathbb{C} \left\{ (1, 1, 0, 0)^t, (0, 0, 1, i)^t \right\}, j = 1, \cdots, 2m.$

2. 
   
   $v_2 = iv_1, v_j \mapsto \text{Span}_\mathbb{C} \left\{ (1, 1, 0, 0)^t, (0, 0, 1, i)^t \right\}, j = 3, \cdots, 2m.$

3. 
   
   $v_2 = iv_1, v_4 = iv_3, \cdots, v_{2m} = iv_{2m-1}.$


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Thank you!