Anisotropic Variational Problems for Surfaces

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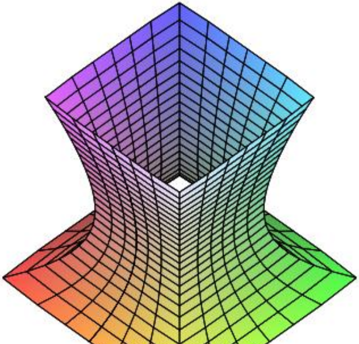

An anisotropic surface energy assigns to a surface a value which depends on the direction of the surface at each point.
e.g.

$$\mathcal{F} = \int_\Sigma \gamma(\nu)\, d\Sigma\,, \qquad \gamma : S^2 \to \mathbf{R}^+$$

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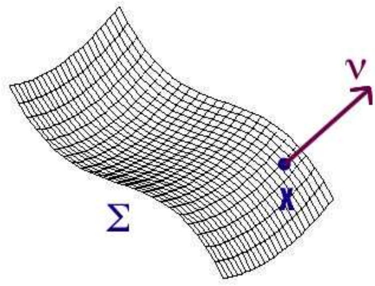


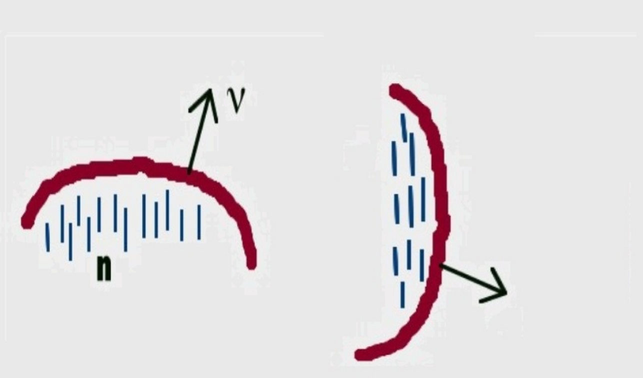


Figure: Nematic liquid crystal drops with homogeneously aligned director field

$$\gamma = 1 + a(\nu \cdot n)^2 \, , a > 0$$

$$\mathcal{F} = \int_{\Sigma} \gamma(\nu) \, d\Sigma \, .$$

First variation

$$\text{Free energy} = \mathcal{F} = \int_\Sigma \gamma(\nu)\, d\Sigma$$

Variation: $X_\epsilon = X + \epsilon(\delta X) + \dots$

First variation

$$\delta\mathcal{F} = \partial_\epsilon \mathcal{F}[X_\epsilon]_{\epsilon=0} = -\int_\Sigma \Lambda\, \delta X \cdot \nu\, d\Sigma + \text{boundary terms}$$

$\Lambda =$ anisotropic mean curvature.

$$\delta\ \text{Volume} = \int_\Sigma \delta X \cdot \nu\, d\Sigma\,,$$

therefore $\Lambda \equiv$ constant characterizes critical points of $\mathcal{F}$ subject to a volume constraint.

Calculation of the anisotropic mean curvature :

Homogeneously extend the energy density:

$$\tilde{\gamma}(Y) := |Y|\gamma(\frac{Y}{|Y|})$$

Cahn-Hoffman field:

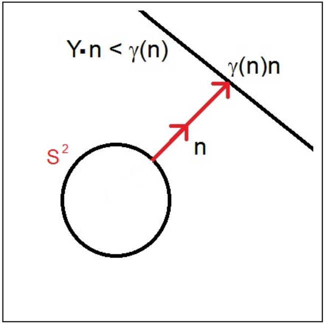
$$\xi(\nu(p)) := \nabla\tilde{\gamma}(\nu(p)) : \ .$$

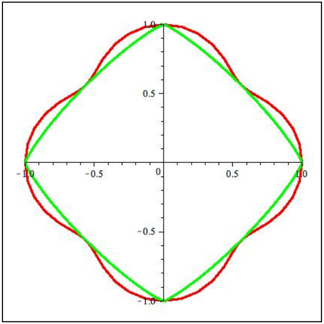
Anisotropic mean curvature:

$$\begin{aligned} \Lambda_p &= -\nabla \cdot \xi_{\nu(p)} \\ &= -\mathrm{trace}\big((D^2\gamma + \gamma I)\, d\nu\big) \end{aligned}$$

Wulff Construction

$$\text{Wulff shape} = W = \partial \bigcap_{n \in S^2} \{Y \cdot n \leq \gamma(n)\}$$





Convexity condition: W is a smooth, convex surface.

Wulff's Theorem

(Dinghas, Liebmann, Herring, Chandrasekhar, Taylor, Brothers-Morgan, Fonseca..)

The Wulff shape W is, up to rescaling, the unique minimizer of the free energy $\mathcal{F}$ among all closed surfaces enclosing the same volume as W.

In particular, W has constant anisotropic mean curvature $\Lambda\,(=-2)$.

Isoperimetric inequality for $\mathcal{F}$

$$\frac{(\mathcal{F}[W])^3}{(V[W])^2} \leq \frac{(\mathcal{F}[\Sigma])^3}{(V[\Sigma])^2}\,.$$

Variation of the anisotropic mean curvature

$$X_\epsilon = X + \epsilon(\delta X) + ...$$

$$\delta X = \psi\nu + T$$

$\delta\Lambda = J[\psi] + \nabla\Lambda \cdot T$ where

$$J[\psi] = \mathrm{Div}_\Sigma[(D^2\gamma + \gamma I)\nabla\psi] + \langle (D^2\gamma + \gamma I) \cdot d\nu, d\nu\rangle\psi \ ,$$

Because of the convexity condition, the operator J is elliptic for any sufficiently smooth surface. This implies that the equation for constant, or more generally prescribed, anisotropic mean curvature has a Maximum Principle.

In the smooth case:

Generalized Barbosa-doCarmo Theorem (P. 1997). *The only closed, stable hypersurfaces with constant anisotropic mean curvature are rescalings of the Wulff shape*.

Generalized Alexandroff Theorem (Ge, He, Li, Ma 2009.) *The only embedded closed hypersurfaces with $\Lambda \equiv$constant are rescalings of the Wulff shape.*

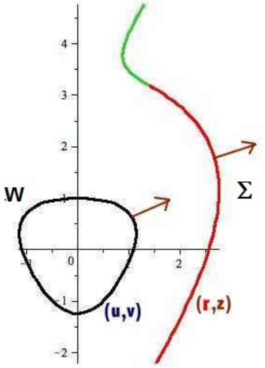
Generalized Hopf Theorem: Ando, (M. Koiso, P, 2009). *The only immersed topological spheres with $\Lambda \equiv$constant are rescalings of the Wulff shape.*

Examples of surfaces with constant anisotropic mean curvature.

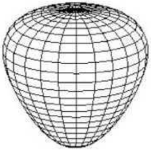
Axially symmetric surfaces with constant anisotropic mean curvature will be called *anisotropic Delaunay surfaces.* From Noether's Theorem:

$$(*)\, 2ur + \Lambda r^2 = c\,, \quad z = \int_0^v r_u\, dv\,. \quad c = \text{'flux parameter'}.$$

$(r, z) =$ generating curve of Σ, $(u, v) =$ generating curve of W

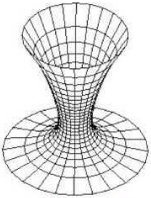


Wulff shape



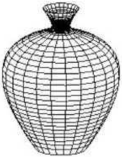
$\Lambda=-2,\ c=0$

anisotropic catenoid



$\Lambda=0,\ c<0$

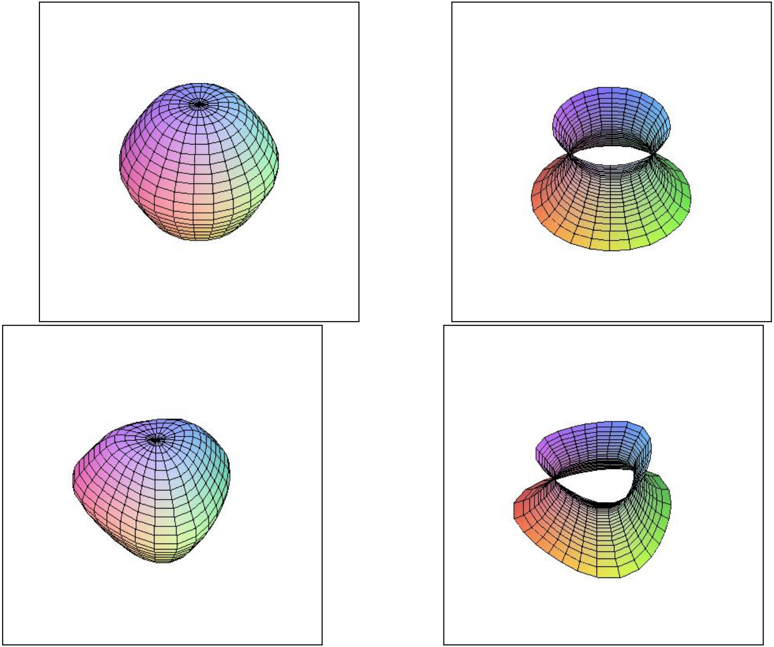
anisotropic unduloid

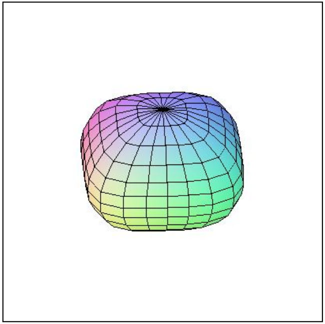
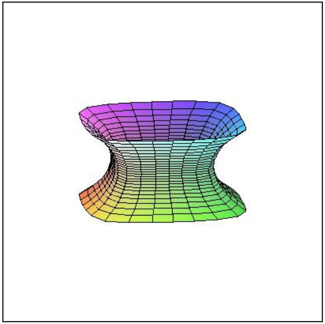
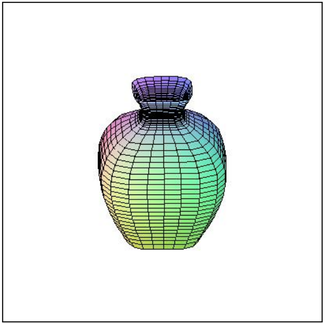
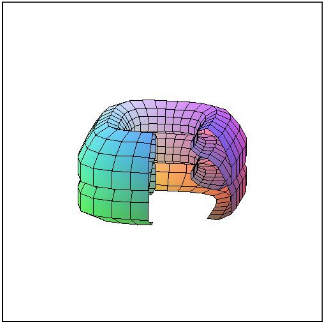


$\Lambda<0, c>0$

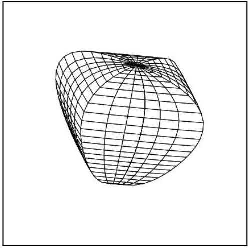
anisotropic nodoid

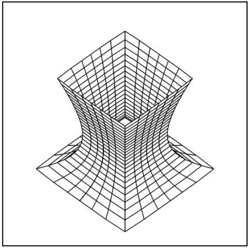
$\Lambda<0,\ c<0$



Equilibrium surfaces with edges.



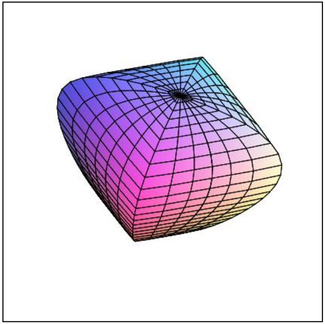
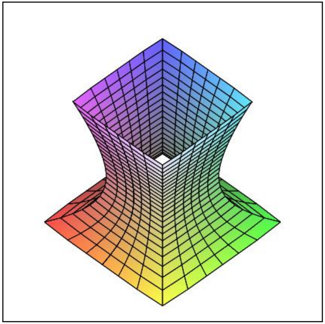
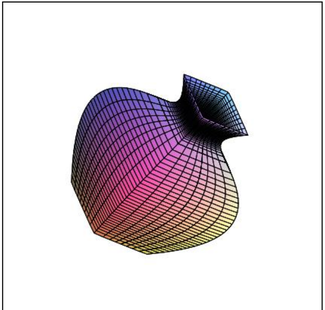
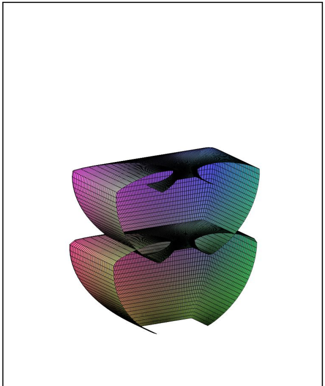


$$\delta\mathcal{F} = -\sum_j \left(\int_{\Sigma_j} \Lambda \delta X \cdot \nu \, d\Sigma - \oint_{\partial\Sigma_j} (\xi \times dX) \cdot \delta X \right). \qquad (1)$$

Equilibrium condition along an edge:

$(\xi_1 - \xi_2) \times T \equiv 0$ Force balancing.

This will clearly hold if ξ extends continuously to all of Σ.

Theorem

Let Σ be a closed embedded piecewise smooth surface which is in equilibrium for an anisotropic surface energy having piecewise smooth convex Wulff shape W. Assume that the Cahn-Hofmann map extends to a continuous map $\xi : \Sigma \to W$. Then if Σ is stable, $\Sigma = rW$ for some real number r.

Basic idea: Consider the variation $X_\epsilon = s(\epsilon)\big(X + \epsilon\xi\big)$, where $s(\epsilon)$ is defined by

$$V(X_\epsilon) \equiv \text{ constant.}$$

The last equation allows us to compute the coefficients of $\mathcal{F}[X_\epsilon]$ and we want to show

$$\mathcal{F}[X_\epsilon] = ... + \frac{\epsilon^2}{2}\int_\Sigma \gamma(\frac{K_\Sigma}{K_W} - \frac{\Lambda^2}{4})\, d\Sigma + ... \, .$$

Integrand is ≤ 0.

Unintegrated Minkowski type formulas:

$$\nabla \cdot J(X \times \xi)^T = 2\gamma + \Lambda q \,, \tag{2}$$

$$\nabla \cdot (d\xi + \Lambda\, I) J(X \times \xi)^T = \frac{2qK_\Sigma}{K_W} + \Lambda\gamma \,, \tag{3}$$

The force balancing condition $\Longrightarrow$

$$\int_\Sigma 2\gamma + \Lambda q \, d\Sigma = 0 = \int_\Sigma \frac{2qK_\Sigma}{K_W} + \Lambda\gamma \, d\Sigma \,.$$

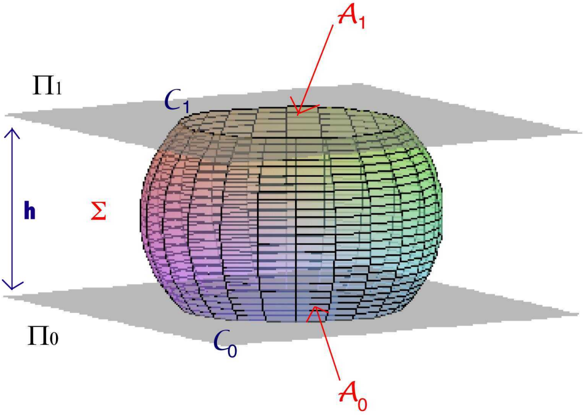
$$\text{stability} \Longrightarrow \frac{K_\Sigma}{K_W} - \frac{\Lambda^2}{4} \equiv 0$$

$$\Longrightarrow \quad \frac{(\mathcal{F}[W])^3}{(V[W])^2} \geq \frac{(\mathcal{F}[\Sigma])^3}{(V[\Sigma])^2} \,.$$

Reverse isoperimetric inequality!

Free boundary problem

Fixed volume of material trapped between two horizontal planes.



Consider the volume as the body of a physical drop. We will assign an energy to each part of the boundary of the drop.

- $\Sigma \longrightarrow \mathcal{F}[\Sigma] = \int_\Sigma \gamma(\nu)\, d\Sigma$.

- $\mathcal{A}_i \longrightarrow \omega_i \cdot \text{Area}\,(\mathcal{A}_i)$ where ω_i are constants. This is called the *wetting energy*. $\omega_i > 0$ is called lyophobic wetting, $\omega_i < 0$ is called lyophilic wetting.

- $C_i \longrightarrow \tau_i \mathcal{L}[C_i]$ where τ_i are constants and $\mathcal{L}$ is a one dimensional anisotropic energy. This term is called the *line tension*.

The total energy is:

$$\mathcal{E} := \mathcal{F} + \omega \cdot \mathcal{A} + \tau \cdot \mathcal{L}\ , \ \ \omega, \tau \in \mathbf{R}^2\ .$$

Here: $\omega \cdot \mathcal{A} = \omega_0 \mathcal{A}_0 + \omega_1 \mathcal{A}_1$, $\tau \cdot \mathcal{L} = \tau_0 \mathcal{L}_0 + \tau_1 \mathcal{L}_1$.

First assume $\gamma = \gamma(\nu_3)$, (or equivalently W is axially symmetric), and $\mathcal{L}[C_i]$ =Length$[C_i]$. The equilibrium conditions for the free boundary problem are:

$$\Lambda = \text{constant} \ \ \text{on} \ \ \Sigma\ ,$$

$$\xi \cdot E_3 = (-1)^{i+1}(\omega_i + \tau_i \kappa) \ \ \text{on} \ \ C_i\ , \ i = 0, 1\ .$$

Using the Maximum Principle, one obtains:(Koiso-P, 2009)

If $\tau_i \geq 0$ holds, all equilibrium surfaces are parts of anisotropic Delaunay surfaces.

Conversely, any part of an anisotropic Delaunay surface between two horizontal planes is in equilibrium for a one parameter continuum of energy functionals $\mathcal{E}$.

Basic problem: If the volume V, the plane separation h and the constants $\omega = (\omega_0, \omega_1)$, $\tau = (\tau_0, \tau_1)$ are specified, can we determine the equilibrium surface?

We will see that of we require that the equilibrium surfaces are stable, then in some cases we can predict the surface that will form.

Let W be the Wulff shape for the functional. It will be assumed that

(W1) W is a uniformly convex surface of revolution with vertical rotation axis.

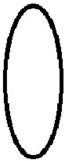
(W2) W is symmetric with respect to reflection through the horizontal plane $z = 0$.

(W3) The generating curve of W has non-decreasing curvature (with respect to the inward pointing normal) as a function of arc length on $\{z \geq 0\}$ as one moves in an upward direction.

e.g.

$$\gamma = 1 + e\nu_3^2, \quad 0 \leq e < 1$$

.



W3

No W3

$\omega = \tau = (0, 0)$

Theorem (Koiso-P.)
If Σ is an equilibrium surface with free boundary on two horizontal planes for the functional $\mathcal{F}$ and with the Wulff shape for the functional satisfying the conditions (W1) through (W3) stated above, then Σ is stable if and only if the surface is either homothetic to a half of the Wulff shape or a cylinder which is perpendicular to the boundary planes which satisfies

$$\frac{\mu_1(0)}{\mu_2(0)}\, h^2 \leq (\pi R)^2 \,.$$

μ_i are the principal curvatures of W along its equator.

The isotropic (CMC) case is due to Athanassenas and Vogel.

Without the curvature condition (W3), stable anisotropic unduloids form for some small values of the volume. The stability of some of these unduloids is proved via bifurcation theory. (Joint work with Koiso, Piccione)

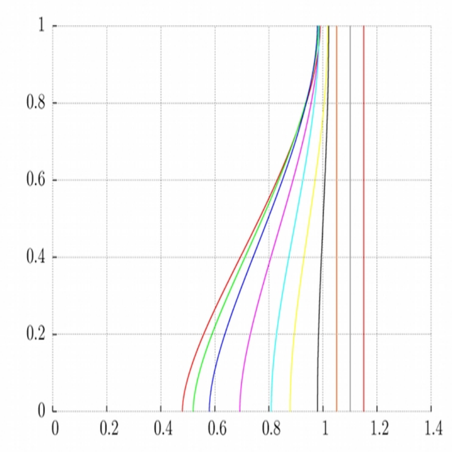
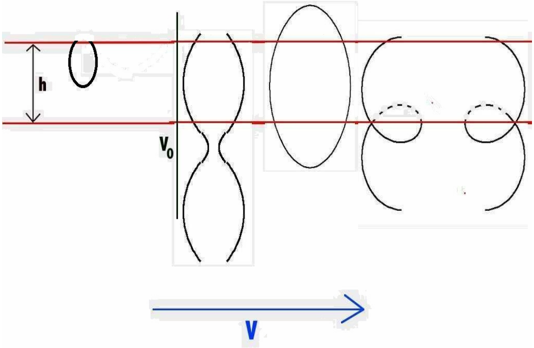


Figure: Results of simulation using $\gamma = 1 - 0.45\nu_3^2$.

Lyophobic wetting

In the case $\omega_0 = \omega_1 > 0$ and the curvature condition(W3) holds, we have the following:



Volume constrained axially symmetric anisotropic mean curvature flow (Joint work with Wenxiang Zhu)

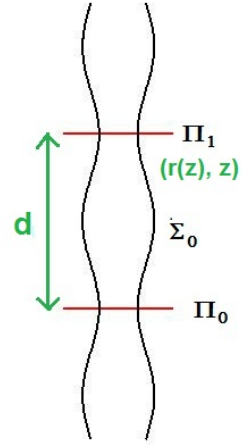
For a sufficiently smooth surface $X : \Sigma \to \mathbf{R}^3$, define

$$\bar{\Lambda} := \frac{\int_\Sigma \Lambda \, d\Sigma}{\int_\Sigma \, d\Sigma} \, ,$$

Volume constrained anisotropic mean curvature flow:

$$\frac{\partial X(p,t)}{\partial t} = (\Lambda - \bar{\Lambda}) \, \nu \, .$$

Volume preserving MCF has been studied extensively by Huisken,... Volume preserving anisotropic MCF has been treated by B. Andrews, McCoy...



$$\frac{\partial X}{\partial t} = (\Lambda - \bar{\Lambda})\,\nu\,.$$

$$\nu \cdot E_3 \equiv 0\,,\ \text{on}\ \partial\Sigma_t$$

Theorem

(P., Zhu) Assume that the Wulff shape of the functional is axially symmetric and is symmetric w.r.t. a horizontal plane. Also assume

- *The initial surface is axially symmetric*
- *The generating curve can be represented as a graph with vertical axis.*
- $$(*) \quad \mathcal{F}[\Sigma_0] < \gamma(E_3)\frac{V_0}{d} \, ,$$

Then the flow exists for all time and

$$\lim_{t\to\infty} \int_{\Sigma_t} (\Lambda - \bar{\Lambda})^2 \, d\Sigma = 0$$

Remark: Our proof uses many ideas of Athanessenas for the isotropic case.

- The flow preserves volume and decreases energy.

$$\begin{aligned}\frac{d\mathcal{F}}{dt} &= -\int_\Sigma \Lambda \partial_t X \cdot \nu \, d\Sigma + \oint_{\partial\Sigma} (\xi \times \partial_t X \cdot dX) \\ &= -\int_\Sigma \Lambda(\Lambda - \bar{\Lambda}) \, d\Sigma + \oint_{\partial\Sigma} (\xi \times (\Lambda - \bar{\Lambda})\nu \cdot dX) \\ &= -\int_\Sigma (\Lambda - \bar{\Lambda})^2 \, d\Sigma \, .\end{aligned}$$

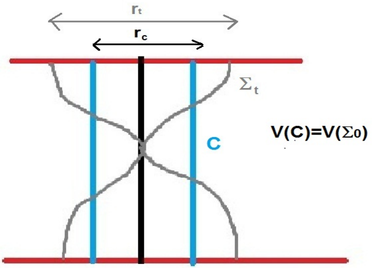
Since $\xi \,||\, \nu$ on $\partial\Sigma$

- If the generating curve is written $(r(z), z)$, then

$$\frac{dr}{dt} = (\Lambda - \bar{\Lambda})\sqrt{1 + r_z^2}$$

$$r_z = 0 \text{ , at } z = 0, d$$

- The condition $(*)$ $\quad \mathcal{F}[\Sigma_0] < \gamma(E_3)\frac{V_0}{d}$, is used to show that there is no pinching: .



$$\mathcal{F}[\Sigma_t] \leq \mathcal{F}[\Sigma_0] < \gamma(E_3)\frac{V_0}{d} = \gamma(E_3)\pi r_C^2 \leq \gamma(E_3)\pi r_t^2 = \mathcal{F}[D_{r_t}].$$

Calibration argument shows that D_{r_t} is, in fact, the minimzer.
With a bit more work: $r \geq c(\Sigma_0) > 0$, $t \in [0, T)$

$$\Lambda = \lambda_1 + \lambda_2,$$

$\lambda_k =$ anisotropic principal curvatures. $|\lambda_2| = |\frac{-z'(s)}{r\mu_2}| \leq C$

$$|\bar{\Lambda}| \leq C\,.$$

- The Maximum Principle for the parabolic operator

  $$u \mapsto P[u] := \partial_t u - \nabla \cdot A \nabla u\,, \qquad A := D^2\gamma + \gamma I$$

  applied to the function $\omega := \sqrt{1 + r_z^2}$ is used to show $\omega \leq c_1(\Sigma_0)e^{at}$, hence the generating curve is a graph over the vertical axis.

- $P[\Lambda] = \langle A d\nu, d\nu \rangle (\Lambda - \bar{\Lambda})$. The Max. Principle is again used to bound $|\Lambda| \leq c(\Sigma_0)e^{at}$. Since one (anisotropic) principle curvature is already bounded, we obtain a bound on the other.

The inequalities:

$$P[|\nabla^m \Lambda|^2] \le -\sigma_m |\nabla^{m+1}\Lambda|^2 + \tau_m(1 + |\nabla^m \Lambda|^2)\,,$$

$$0 \le \sigma_m, \tau_m = \tau_m(|\nabla^j \Lambda|^2)\,, j < m$$

lead to bounds:

$$(*) \quad |\nabla^m \Lambda|^2 \le C_m(T)\,,\ \ t \in [0, T)$$

By using the Codazzi equations $|\nabla^m \lambda_2|$ can be bounded by $|\nabla^j \lambda_k|$, $k = 1, 2$ and $j < m$. By induction, the bound (*) leads to bounds on $|\nabla^m \lambda_1|$, $m = 1, 2, ...$

Standard techniques then imply the long time existence.

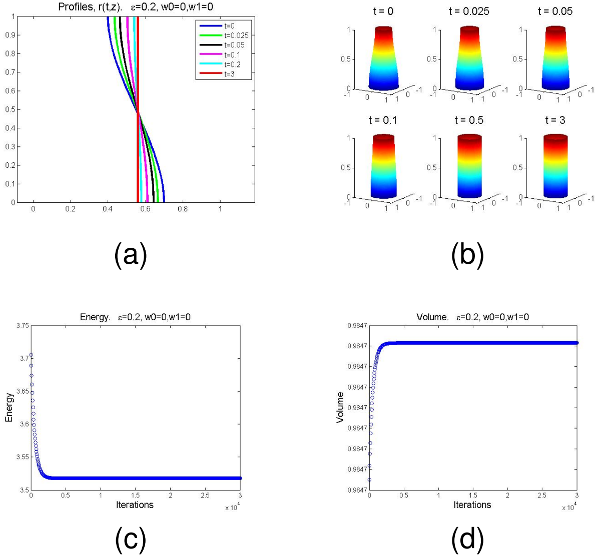


Figure: Example 1. Snapshots of surface profiles, energy functional, and volumes. $\gamma = 1 + 0.2\nu_3^2$, $r(0,0) = 0.7$, $r(0,1) = 0.4$.

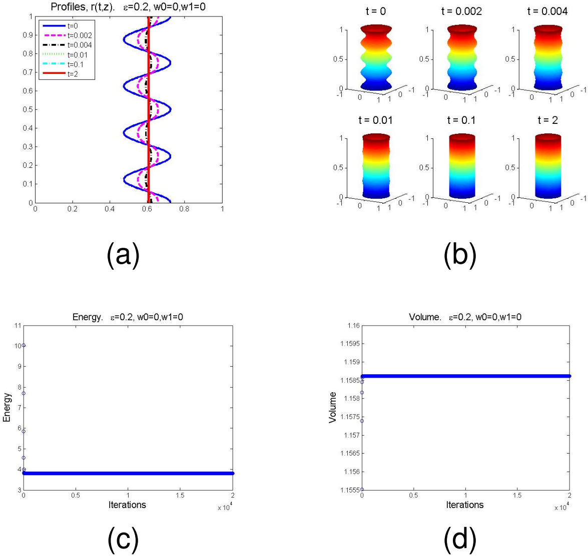


Figure: Snapshots of surface profiles, energy functional, and volumes. $\gamma = 1 + 0.2\nu_3^2$, $r(0, z) = 1 + \cos(8\pi z)/4$.

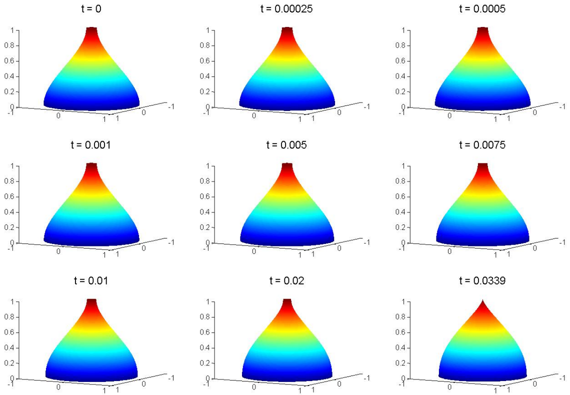


Figure: Snapshots of the 3D surfaces. $\gamma = 1 + 0.2\nu_3^2$, $r(0,0) = 0.9$, $r(0,1) = 0.1$.

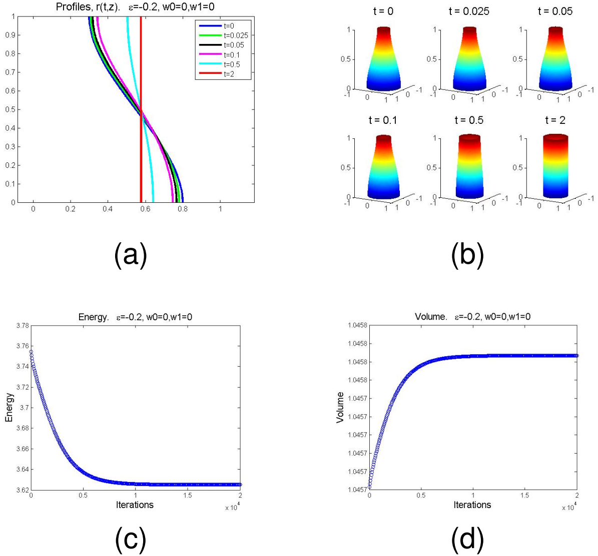


Figure: Snapshots of surface profiles, energy functional, and volumes. $\gamma = 1 - 0.2\nu_3^2$, $r(0,0) = 0.8$, $r(0,1) = 0.3$.

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Thank You!