Surgery, concordance and isotopy of metrics of positive scalar curvature

Boris Botvinnik
University of Oregon, Eugene, USA

December 9th, 2011
The 10th Pacific Rim Geometry Conference
Osaka-Fukuoka, Japan

Notations:

- M is a closed manifold,
- $\mathcal{R}\mathrm{iem}(M)$ is the space of all Riemannian metrics,
- R_g is the scalar curvature for a metric g,
- $\mathcal{R}\mathrm{iem}^+(M)$ is the subspace of metrics with $R_g > 0$,
- "psc-metric" = "metric with positive scalar curvature".

Definition 1. Psc-metrics g_0 and g_1 are **psc-isotopic** if there is a smooth path of psc-metrics $g(t)$, $t \in [0,1]$, with $g(0) = g_0$ and $g(1) = g_1$.

Remark: In fact, g_0 and g_1 are psc-isotopic if and only if they belong to the same path-component in $\mathcal{R}\mathrm{iem}^+(M)$.

Remark: There are many examples of manifolds with infinite $\pi_0\mathcal{R}\mathrm{iem}^+(M)$. In particular, $\mathbf{Z} \subset \pi_0\mathcal{R}\mathrm{iem}^+(M)$ if M is spin and $\dim M = 4k+3,\ k \geq 1$.

Definition 2: Psc-metrics g_0 and g_1 are **psc-concordant** if there is a psc-metric $\bar{g}$ on $M \times I$ such that

$$\bar{g}|_{M\times\{i\}} = g_i, \quad i = 0, 1$$

with $\bar{g} = g_i + dt^2$ near $M \times \{i\}$, $i = 0, 1$.

Definition 2′: Psc-metrics g_0 and g_1 are **psc-concordant** if there is a psc-metric $\bar{g}$ on $M \times I$ such that

$$\bar{g}|_{M\times\{i\}} = g_i, \quad i = 0, 1.$$

with **minimal boundary condition** i.e. the mean curvature is zero along the boundary $M \times \{i\}$, $i = 0, 1$.

Remark: Definitions 2 and Definition $2'$ are equivalent.

[Akutagawa-Botvinnik, 2002]

Remark: Any psc-isotopic metrics are psc-concordant.

Question: Does psc-concordance imply psc-isotopy?

Remark: Definitions 2 and Definition $2'$ are equivalent.

[Akutagawa-Botvinnik, 2002]

Remark: Any psc-isotopic metrics are psc-concordant.

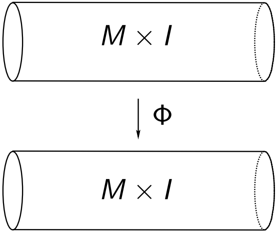
Question: Does psc-concordance imply psc-isotopy?

My goal today: To give some answers to this **Question**.

Topology:

A diffeomorphism $\Phi : M \times I \to M \times I$ is a **pseudo-isotopy** if

$$\Phi|_{M\times\{0\}} = Id_{M\times\{0\}}$$



Let $\mathrm{Diff}(M \times I, M \times \{0\}) \subset \mathrm{Diff}(M \times I)$ be the group of pseudo-isotopies.

A smooth function $\bar{\alpha} : M \times I \to I$ without critical points is called a **slicing function** if

$$\bar{\alpha}^{-1}(0) = M \times \{0\}, \qquad \bar{\alpha}^{-1}(1) = M \times \{1\}.$$

Let $\mathcal{E}(M \times I)$ be the space of slicing functions.

There is a natural map

$$\sigma : \mathrm{Diff}(M \times I, M \times \{0\}) \longrightarrow \mathcal{E}(M \times I)$$

which sends $\Phi : M \times I \longrightarrow M \times I$ to the function

$$\sigma(\Phi) = \pi_I \circ \Phi : M \times I \xrightarrow{\Phi} M \times I \xrightarrow{\pi_I} I.$$

Theorem.(J. Cerf) The map

$$\sigma : \mathrm{Diff}(M \times I, M \times \{0\}) \longrightarrow \mathcal{E}(M \times I)$$

is a homotopy equivalence.

Theorem. (J. Cerf) Let M be a closed simply connected manifold of dimension $\dim M \geq 5$. Then

$$\pi_0(\mathrm{Diff}(M \times I, M \times \{0\}) = 0.$$

Remark: In particular, for simply connected manifolds of dimension at least five any two diffeomorphisms which are **pseudo-isotopic**, are **isotopic**.

Remark: The group $\pi_0(\mathrm{Diff}(M \times I, M \times \{0\})$ is non-trivial for most non-simply connected manifolds.

Example: (D. Ruberman, '02) There exists a simply connected 4-manifold M^4 and psc-concordant psc-metrics g_0 and g_1 which are not psc-isotopic.

The obstruction comes from Seiberg-Witten invariant: in fact, it detects a gap between isotopy and pseudo-isotopy of diffeomorphisms for 4-manifolds.

In particular, the above psc-metrics g_0 and g_1 are isotopic in the moduli space $\mathcal{R}\mathrm{iem}^+(M)/\mathrm{Diff}(M)$.

Conclusion: It is reasonable to expect that psc-concordant metrics g_0 and g_1 are homotopic in the moduli space

$$\mathcal{R}\mathrm{iem}^+(M)/\mathrm{Diff}(M).$$

Theorem A. Let M be a closed compact manifold with $\dim M \geq 4$. Assume that $g_0, g_1 \in \mathcal{R}\mathrm{iem}^+(M)$ are two psc-concordant metrics. Then there exists a pseudo-isotopy

$$\Phi \in \mathrm{Diff}(M \times I, M \times \{0\}),$$

such that the psc-metrics g_0 and $(\Phi|_{M\times\{1\}})^* g_1$ are psc-isotopic.

According to J. Cerf, there is no obstruction for two pseudo-isotopic diffeomorphisms to be isotopic for simply connected manifolds of dimension at least five.
Thus **Theorem A** implies

Theorem B. Let M be a closed simply connected manifold with $\dim M \geq 5$. Then two psc-metrics g_0 and g_1 on M are psc-isotopic if and only if the metrics g_0, g_1 are psc-concordant.

We use the abbreviation "**(C$\Longleftrightarrow$I)**(M)" for the following statement:

"**Let** $g_0, g_1 \in \mathcal{R}\mathrm{iem}^+(M)$ **be any psc-concordant metrics. Then there exists a pseudo-isotopy**

$$\Phi \in \mathrm{Diff}(M \times I, M \times \{0\})$$

such that the psc-metrics

$$g_0 \quad \textbf{and} \quad (\Phi|_{M\times\{1\}})^* g_1$$

are psc-isotopic."

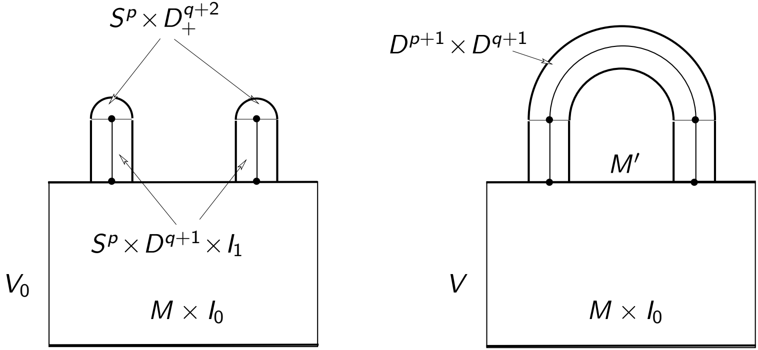
The strategy to prove Theorem A.

1. Surgery. Let M be a closed manifold, and $S^p \times D^{q+1} \subset M$.

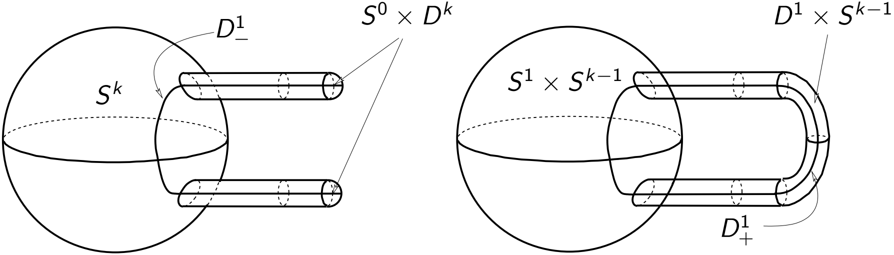
We denote by M' the manifold which is the result of the surgery along the sphere S^p:

$$M' = (M \setminus (S^p \times D^{q+1})) \cup_{S^p \times S^q} (D^{p+1} \times S^q).$$

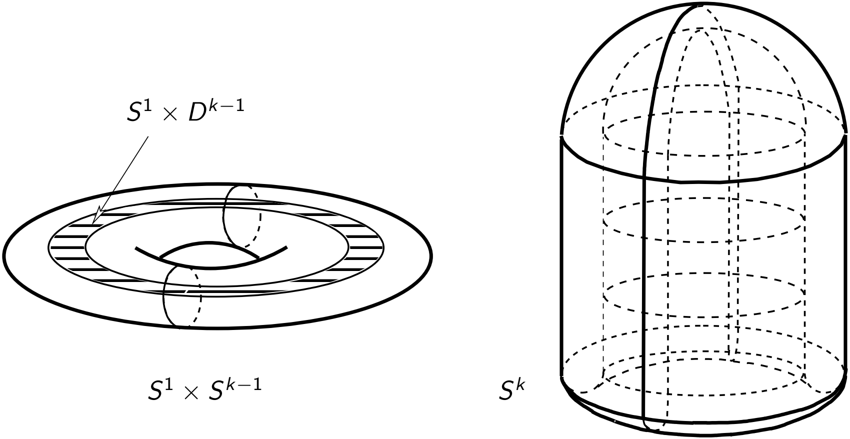
Codimension of this surgery is $q+1$.



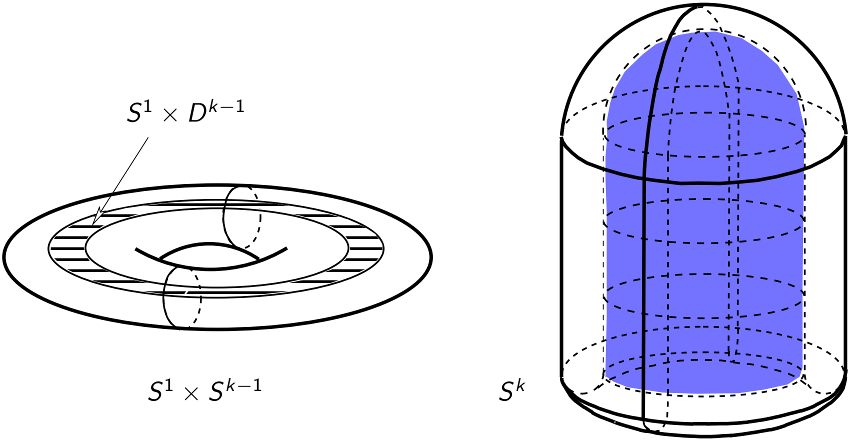
Example: surgeries $S^k \Longleftrightarrow S^1 \times S^{k-1}$.



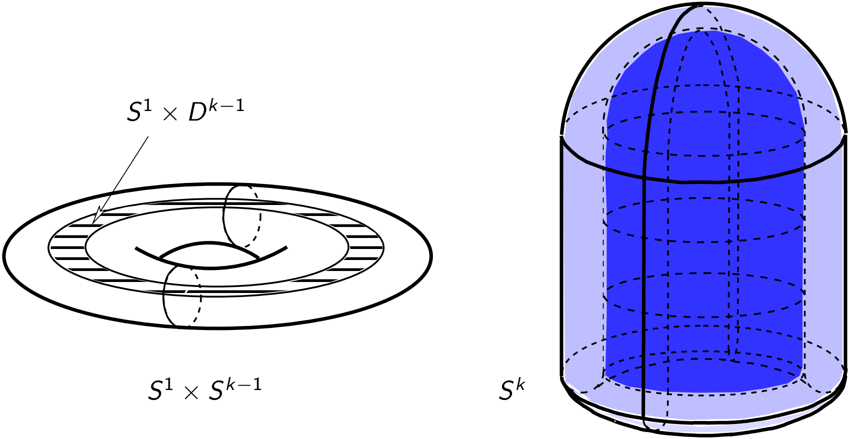
The first surgery on S^k to obtain $S^1 \times S^{k-1}$



The second surgery on $S^1 \times S^{k-1}$ to obtain S^k



The second surgery on $S^1 \times S^{k-1}$ to obtain S^k



The second surgery on $S^1 \times S^{k-1}$ to obtain S^k

Definition. Let M and M' be manifolds such that:

- M' can be constructed out of M by a finite sequence of surgeries of codimension at least three;
- M can be constructed out of M' by a finite sequence of surgeries of codimension at least three.

Then **M and M' are related by admissible surgeries**.

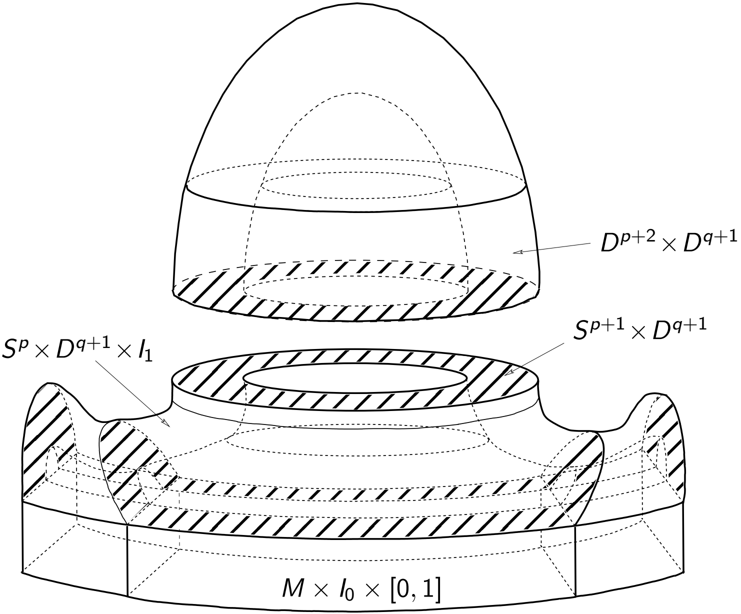
Examples: $M = S^k$ and $M' = S^3 \times T^{k-3}$;

$M \cong M \# S^k$ and $M' = M \# (S^3 \times T^{k-3})$, where $k \geq 4$.

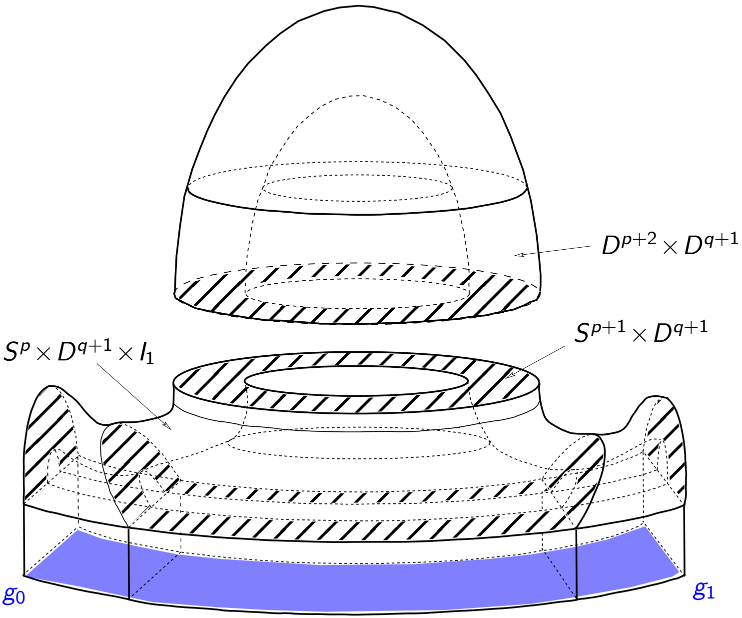
PSC-Concordance-Isotopy Surgery Lemma. Let M and M' be two closed manifolds related by admissible surgeries. Then the statements

$$(\mathbf{C} \Longleftrightarrow \mathbf{I})(M) \quad \text{and} \quad (\mathbf{C} \Longleftrightarrow \mathbf{I})(M')$$

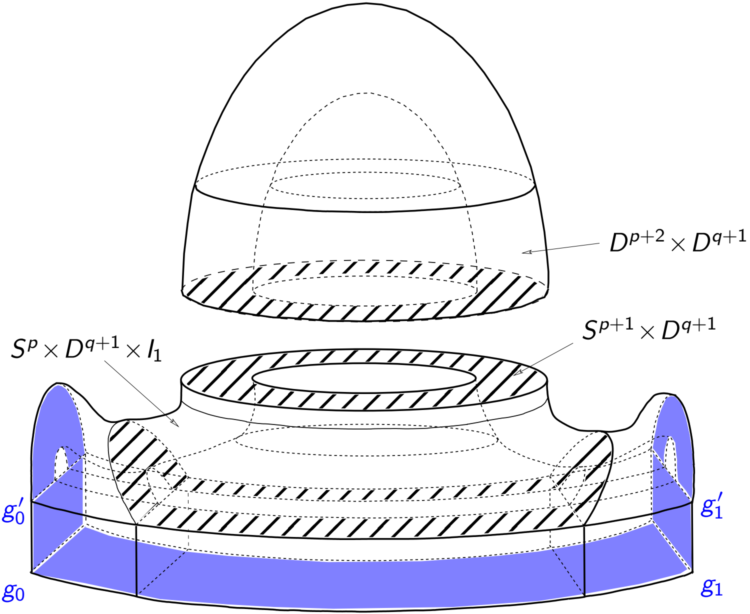
are equivalent.



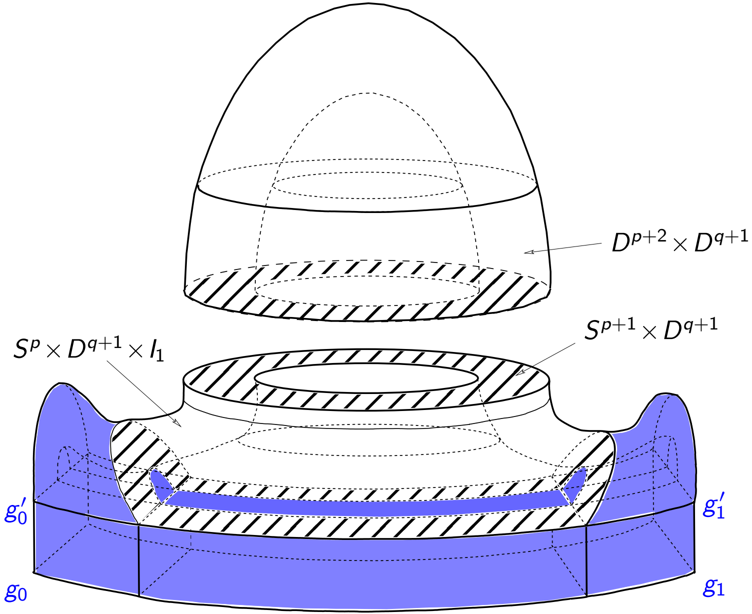
Proof of Surgery Lemma



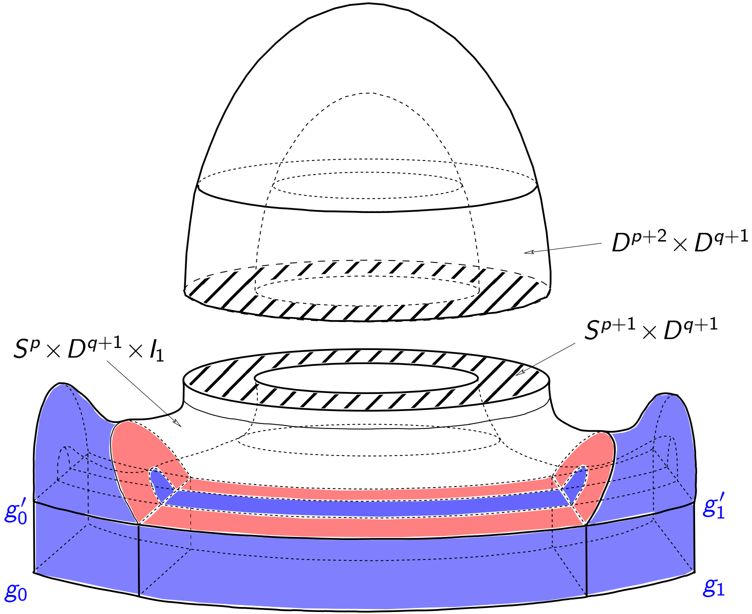
Proof of Surgery Lemma



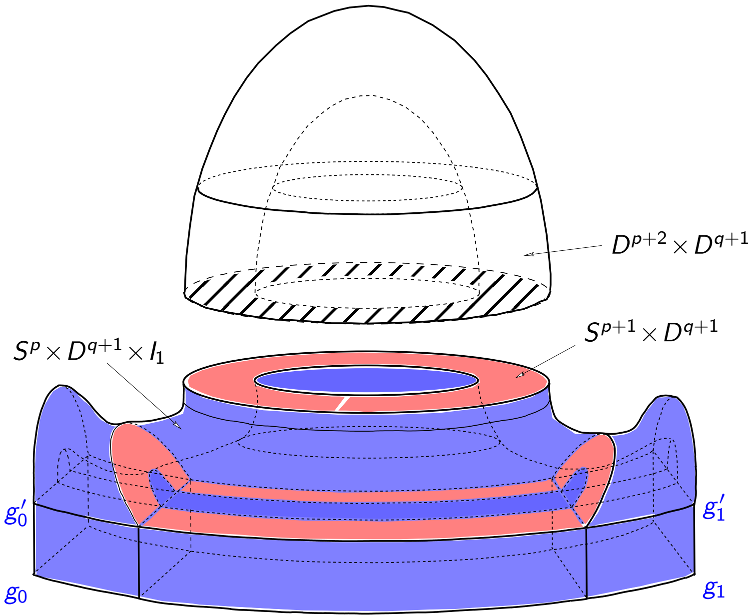
Proof of Surgery Lemma



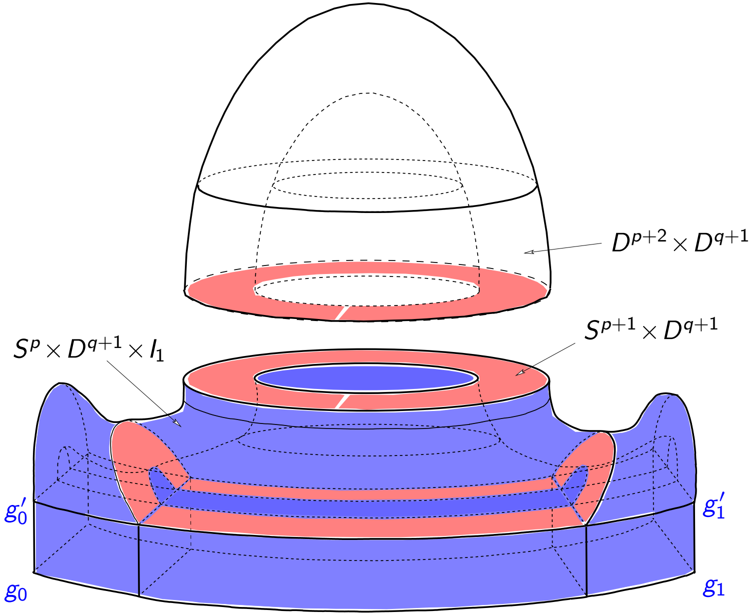
Proof of Surgery Lemma



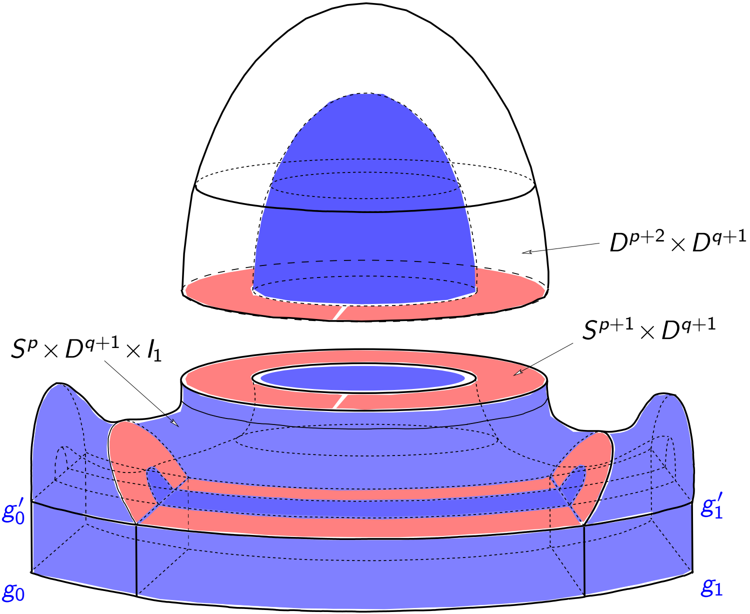
Proof of Surgery Lemma



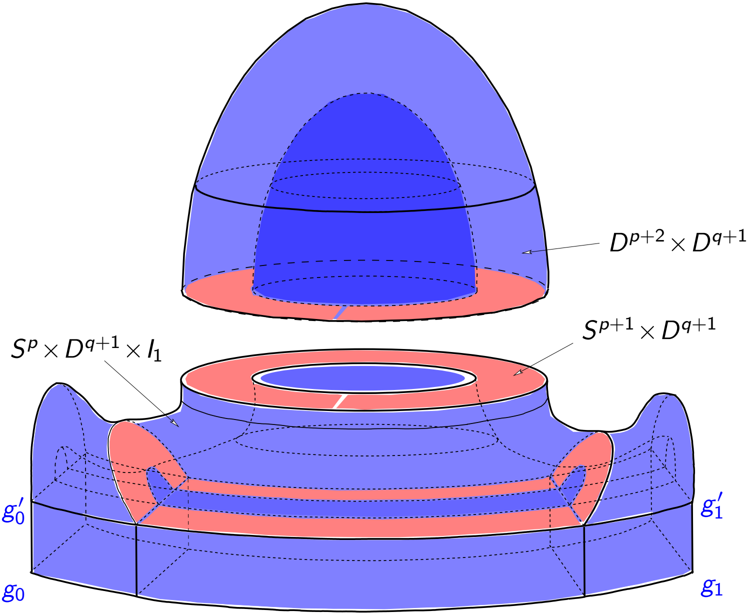
Proof of Surgery Lemma



Proof of Surgery Lemma



Proof of Surgery Lemma



Proof of Surgery Lemma

2. Surgery and Ricci-flatness.

Examples of manifolds which **do not admit any Ricci-flat metric:**

$$S^3, \qquad S^3 \times T^{k-3}.$$

Observation. Let M be a closed connected manifold with $\dim M = k \geq 4$. Then the manifold

$$M' = M\#(S^3 \times T^{k-3})$$

does not admit a Ricci-flat metric [Cheeger-Gromoll, 1971].

The manifolds M and M' are related by admissible surgeries.

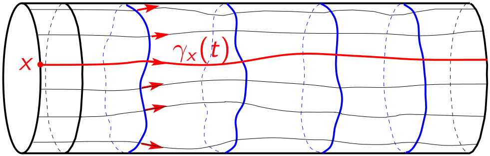
Surgery Lemma implies that it is enough to prove **Theorem A** for those manifolds which **do not admit any Ricci-flat metric.**

3. Pseudo-isotopy and psc-concordance.

Let $(M \times I, \bar{g})$ be a psc-concordance and $\bar{\alpha} : M \times I \to I$ be a slicing function. Let $\bar{C} = [\bar{g}]$ the conformal class. We use the vector field:

$$X_{\bar{\alpha}} = \frac{\nabla \bar{\alpha}}{|\nabla \bar{\alpha}|^2_{\bar{g}}} \in \mathfrak{X}(M \times I).$$

Let $\gamma_x(t)$ be the integral curve of the vector field $X_{\bar{\alpha}}$ such that $\gamma_x(0) = (x, 0)$.



Then $\gamma_x(1) \in M \times \{1\}$, and $d\bar{\alpha}(X_{\bar{\alpha}}) = \bar{g} \langle \nabla \bar{\alpha}, X_{\bar{\alpha}} \rangle = 1$.

We obtain a pseudo-isotopy: $\Phi : M \times I \to M \times I$ defined by the formula

$$\Phi : (x,t) \mapsto (\pi_M(\gamma_x(t)), \pi_I(\gamma_x(t))).$$

Lemma. (K. Akutagawa) Let $\bar{C} \in \mathcal{C}(M \times I)$ be a conformal class, and $\bar{\alpha} \in \mathcal{E}(M \times I)$ be a slicing function. Then there exists a unique metric $\bar{g} \in (\Phi^{-1})^* \bar{C}$ such that

$$\begin{cases} \bar{g} & = & \bar{g}|_{M_t} + dt^2 \quad \text{on} \quad M \times I \\ \mathrm{Vol}_{g_t}(M_t) & = & \mathrm{Vol}_{g_0}(M_0) \quad \text{for all} \quad t \in I \end{cases}$$

up to pseudo-isotopy Φ arising from $\bar{\alpha}$.

In particular, the function $(\Phi^{-1})^*\bar{\alpha}$ is just a standard projection $M \times I \to M$.

Conformal Laplacian and minimal boundary condition:

Let $(W, \bar{g})$ be a manifold with boundary ∂W, $\dim W = n$.

- $A_{\bar{g}}$ is the second fundamental form along ∂W;
- $H_{\bar{g}} = \operatorname{tr} A_{\bar{g}}$ is the mean curvature along ∂W;
- $h_{\bar{g}} = \frac{1}{n-1} H_{\bar{g}}$ is the “normalized” mean curvature.

Let $\tilde{g} = u^{\frac{4}{n-2}} \bar{g}$. Then

$$
\begin{aligned}
R_{\tilde{g}} &= u^{-\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2} \Delta_{\bar{g}} u + R_{\bar{g}} u \right) &= u^{-\frac{n+2}{n-2}} L_{\bar{g}} u \\
h_{\tilde{g}} &= \frac{2}{n-2} u^{-\frac{n}{n-2}} \left(\partial_\nu u + \frac{n-2}{2} h_{\bar{g}} u \right) &= u^{-\frac{n}{n-2}} B_{\bar{g}} u
\end{aligned}
$$

- Here ∂_ν is the derivative with respect to outward unit normal vector field.

The minimal boundary problem:

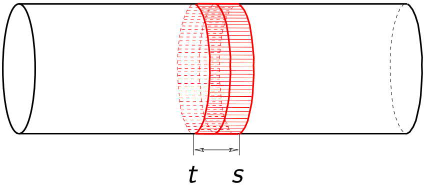
$$\begin{cases} L_{\bar{g}} u &= \frac{4(n-1)}{n-2} \Delta_{\bar{g}} u + R_{\bar{g}} u = \lambda_1 u \quad \text{on } W \\ B_{\bar{g}} u &= \partial_\nu u + \frac{n-2}{2} h_{\bar{g}} u = 0 \qquad \text{on } \partial W. \end{cases}$$

If u is the eigenfunction corresponding to the first eigenvalue, i.e. $L_{\bar{g}} u = \lambda_1 u$, and $\tilde{g} = u^{\frac{4}{n-2}} \bar{g}$, then

$$\begin{cases} R_{\tilde{g}} &= u^{-\frac{n+2}{n-2}} L_{\bar{g}} u = \lambda_1 u^{-\frac{4}{n-2}} \quad \text{on } W \\ h_{\tilde{g}} &= u^{-\frac{n}{n-2}} B_{\bar{g}} u = 0 \qquad \text{on } \partial W. \end{cases}$$

4. Sufficient condition. Let $(M \times I, \bar{g})$ be a Riemannian manifold with the minimal boundary condition, and let $\bar{\alpha} : M \times I \to I$ be a slicing function. For each $t < s$, we define:

$$W_{t,s} = \bar{\alpha}^{-1}([t,s]), \quad \bar{g}_{t,s} = \bar{g}|_{W_{t,s}}$$



Consider the conformal Laplacian $L_{\bar{g}_{t,s}}$ on $(W_{t,s}, \bar{g}_{t,s})$. Let $\lambda_1(L_{\bar{g}_{t,s}})$ be the first eigenvalue of $L_{\bar{g}_{t,s}}$ on $(W_{t,s}, \bar{g}_{t,s})$ with the minimal boundary condition.

We obtain a function $\Lambda_{(M\times I, \bar{g}, \bar{\alpha})} : (t,s) \mapsto \lambda_1(L_{\bar{g}_{t,s}})$.

Theorem 1. Let M be a closed manifold with $\dim M \geq 3$ which does not admit a Ricci-flat metric. Let $g_0, g_1 \in \mathcal{R}\mathrm{iem}^+(M)$ and $\bar{g}$ be a Riemannian metric on $M \times I$ with minimal boundary condition such that

$$\bar{g}|_{M\times\{0\}} = g_0, \qquad \bar{g}|_{M\times\{1\}} = g_1.$$

Assume $\bar{\alpha} : M \times I \to I$ is a slicing function such that $\Lambda_{(M\times I,\bar{g},\bar{\alpha})} \geq 0$. Then there exists a pseudo-isotopy

$$\Phi : M \times I \longrightarrow M \times I$$

such that the metrics g_0 and $(\Phi|_{M\times\{1\}})^* g_1$ are psc-isotopic.

Theorem 1. Let M be a closed manifold with $\dim M \geq 3$ which does not admit a Ricci-flat metric. Let $g_0, g_1 \in \mathcal{R}\mathrm{iem}^+(M)$ and $\bar{g}$ be a Riemannian metric on $M \times I$ with minimal boundary condition such that

$$\bar{g}|_{M\times\{0\}} = g_0, \quad \bar{g}|_{M\times\{1\}} = g_1.$$

Assume $\bar{\alpha} : M \times I \to I$ is a slicing function such that $\Lambda_{(M\times I, \bar{g}, \bar{\alpha})} \geq 0$. Then there exists a pseudo-isotopy

$$\Phi : M \times I \longrightarrow M \times I$$

such that the metrics g_0 and $(\Phi|_{M\times\{1\}})^* g_1$ are psc-isotopic.

Question: Why do we need the condition that M does not admit a Ricci-flat metric?

Assume the slicing function $\bar{\alpha}$ coincides with the projection

$$\pi_I : M \times I \to I.$$

Moreover, we assume that $\bar{g} = g_t + dt^2$ with respect to the coordinate system given by the projections

$$M \times I \xrightarrow{\pi_I} I, \quad M \times I \xrightarrow{\pi_M} M.$$

Let $L_{\bar{g}_{t,s}}$ be the conformal Laplacian on the cylinder $(W_{t,s}, \bar{g}_{t,s})$ with the minimal boundary condition, and $\lambda_1(L_{\bar{g}_{t,s}})$ be the first eigenvalue of the minimal boundary problem.

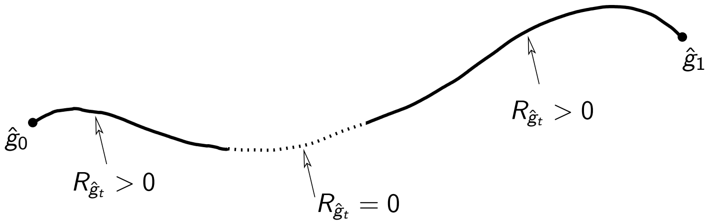
For given t we denote L_{g_t} the conformal Laplacian on the slice (M_t, g_t).

Lemma. The assumption $\lambda_1(L_{\bar{g}_{t,s}}) \geq 0$ for all $t < s$ implies that $\lambda_1(L_{g_t}) \geq 0$ for all t.

We find positive eigenfunctions $u(t)$ corresponding to the eigenvalues $\lambda_1(L_{g_t})$ and let $\hat{g}_t = u(t)^{\frac{4}{k-2}} g_t$. Then

$$R_{\hat{g}_t} = u(t)^{-\frac{4}{k-2}} \lambda_1(L_{g_t}) = \begin{cases} > 0 & \text{if } \lambda_1(L_{g_t}) > 0, \\ \equiv 0 & \text{if } \lambda_1(L_{g_t}) = 0. \end{cases}$$

Then we apply the Ricci flow:

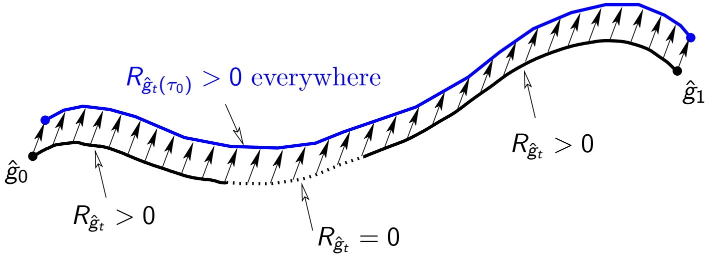


Ricci flow applied to the path $\hat{g}_t$.

We find positive eigenfunctions $u(t)$ corresponding to the eigenvalues $\lambda_1(L_{g_t})$ and let $\hat{g}_t = u(t)^{\frac{4}{k-2}} g_t$. Then

$$R_{\hat{g}_t} = u(t)^{-\frac{4}{k-2}} \lambda_1(L_{g_t}) = \begin{cases} > 0 & \text{if } \lambda_1(L_{g_t}) > 0, \\ \equiv 0 & \text{if } \lambda_1(L_{g_t}) = 0. \end{cases}$$

Then we apply the Ricci flow:



Ricci flow applied to the path $\hat{g}_t$.

We recall:

$$\frac{\partial R_{\hat{g}_t(\tau)}}{\partial \tau} = \Delta R_{\hat{g}_t(\tau)} + 2|\operatorname{Ric}_{\hat{g}_t(\tau)}|^2, \quad \hat{g}_t(0) = \hat{g}_t.$$

Remark: If $\lambda_1(L_{g_t}) = 0$, we really need the condition that M does not have a Ricci flat metric.

Then if the metric $\hat{g}_t$ is scalar flat, it cannot be Ricci-flat.

In the general case, there exists a pseudo-isotopy

$$\Phi : M \times I \longrightarrow M \times I$$

(given by the slicing function $\bar{\alpha}$) such that the metric $\Phi^* \bar{g}$ satisfies the above conditions.

5. Necessary Condition.

Theorem 2. Let M be a closed manifold with $\dim M \geq 3$, and $g_0, g_1 \in \mathcal{R}\mathrm{iem}(M)$ be two psc-concordant metrics. Then there exist

- a psc-concordance $(M \times I, \bar{g})$ between g_0 and g_1 and
- a slicing function $\bar{\alpha} : M \times I \to I$

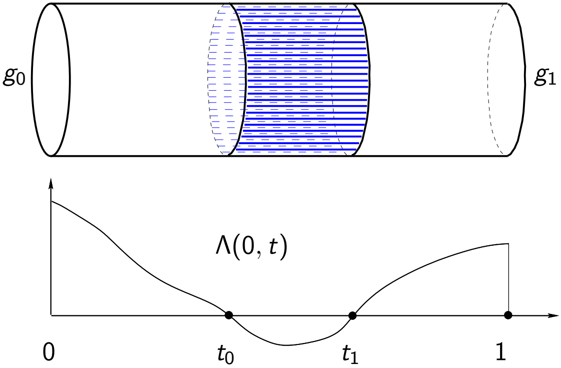
such that $\Lambda_{(M \times I, \bar{g}, \bar{\alpha})} \geq 0$.

Sketch of the proof. Let $g_0, g_1 \in \mathcal{R}\mathrm{iem}^+(M)$ be psc-concordant. We choose a psc-concordance $(M \times I, \bar{g})$ between g_0 and g_1 and a slicing function $\bar{\alpha} : M \times I \to I$.

The notations: $W_{t,s} = \bar{\alpha}^{-1}([t,s]), \quad \bar{g}_{t,s} = \bar{g}|_{W_{t,s}}$.

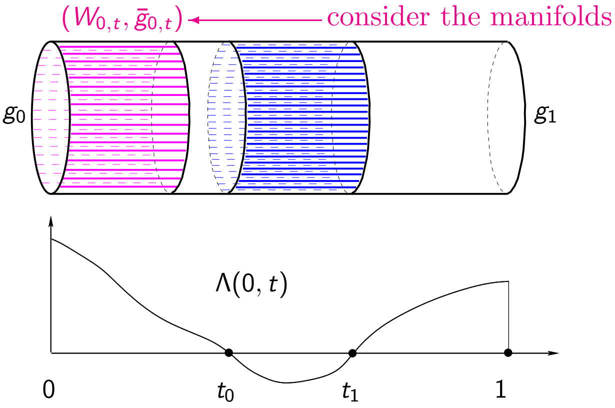
Key construction: a bypass surgery.

Example. We assume:



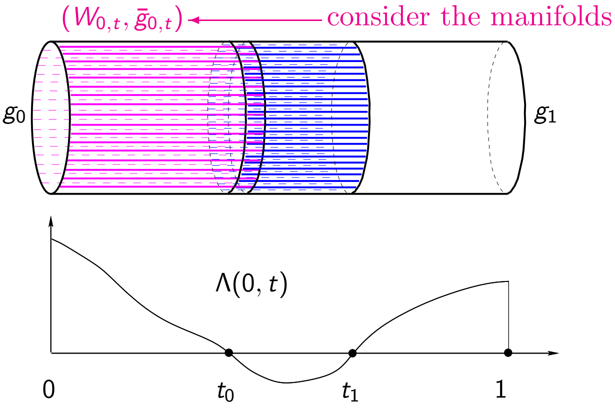
Key construction: a bypass surgery.

Example. We assume:



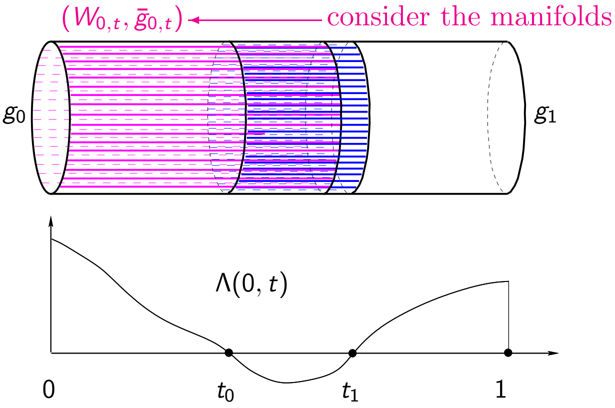
Key construction: a bypass surgery.

Example. We assume:



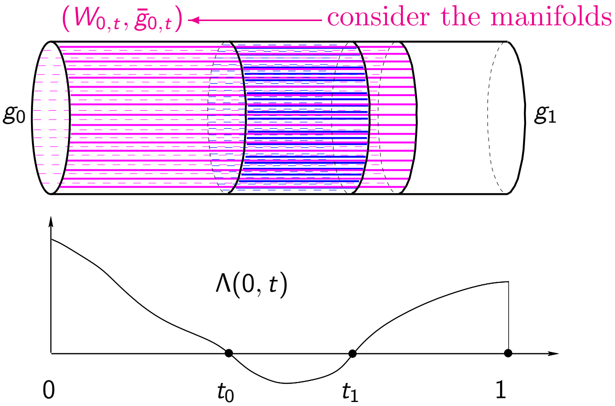
Key construction: a bypass surgery.

Example. We assume:



Key construction: a bypass surgery.

Example. We assume:



Recall the minimal boundary problem:

$$\begin{cases} L_{\bar{g}_{0,t}} u &= \frac{4(n-1)}{n-2} \Delta_{\bar{g}_{0,t}} u + R_{\bar{g}_{0,t}} u = \lambda_1 u \quad \text{on } W_{0,t} \\ B_{\bar{g}} u &= \partial_\nu u + \frac{n-2}{2} h_{\bar{g}_{0,t}} u = 0 \quad \text{on } \partial W_{0,t}. \end{cases}$$

where $\Lambda(0,t) = \lambda_1$ is the first eigenvalue of $L_{\bar{g}_{0,t}}$ with minimal boundary conditions.

If u is the eigenfunction corresponding to the first eigenvalue, and $\tilde{g}_{0,t} = u^{\frac{4}{n-2}} \bar{g}_{0,t}$, then

$$\begin{cases} R_{\tilde{g}_{0,t}} &= u^{-\frac{n+2}{n-2}} L_{\bar{g}_{0,t}} u = \lambda_1 u^{-\frac{4}{n-2}} \quad \text{on } W_{0,t} \\ h_{\tilde{g}_{0,t}} &= u^{-\frac{n}{n-2}} B_{\bar{g}_{0,t}} u = 0 \quad \text{on } \partial W_{0,t}. \end{cases}$$

There is the second boundary problem:

$$\begin{cases} L_{\bar{g}_{0,t}} u &= \frac{4(n-1)}{n-2}\Delta_{\bar{g}_{0,t}} u + R_{\bar{g}_{0,t}} u = 0 \quad \text{on } W_{0,t} \\ B_{\bar{g}} u &= \partial_\nu u + \frac{n-2}{2} h_{\bar{g}_{0,t}} u = \mu_1 u \quad \text{on } \partial W_{0,t}. \end{cases}$$

where μ_1 is the corresponding first eigenvalue.

If u is the eigenfunction corresponding to the first eigenvalue, and $\tilde{g}_{0,t} = u^{\frac{4}{n-2}} \bar{g}_{0,t}$, then

$$\begin{cases} R_{\tilde{g}_{0,t}} &= u^{-\frac{n+2}{n-2}} L_{\bar{g}_{0,t}} u = 0 \quad \text{on } W_{0,t} \\ h_{\tilde{g}_{0,t}} &= u^{-\frac{n}{n-2}} B_{\bar{g}_{0,t}} u = \mu_1 u^{-\frac{2}{n-2}} \quad \text{on } \partial W_{0,t}. \end{cases}$$

It is well-known that λ_1 and μ_1 have the same sign.
In particular, $\lambda_1 = 0$ if and only if $\mu_1 = 0$.

Concerning the manifolds $(W_{0,t}, \bar{g}_{0,t})$, there exist metrics $\hat{g}_{0,t} \in [\bar{g}_{0,t}]$ such that

(1) $R_{\hat{g}_{0,t}} \equiv 0, \; t_0 \leq t \leq t_1,$

$$\textbf{(2)} \quad H_{\hat{g}_{0,t}} \equiv \left\{ \begin{array}{cl} \xi_t > 0 & \text{if } 0 < t < t_0 \\ 0 & \text{if } t = t_0, \\ \xi_t < 0 & \text{if } t_0 \leq t \leq t_1 \\ 0 & \text{if } t = t_1, \\ \xi_t > 0 & \text{if } t_1 < t \leq 1. \end{array} \right\} \text{ along } \partial W_{0,t}.$$

Here the functions ξ_t depend continuously on t and

$$\mathrm{sign}(\xi_t) = \mathrm{sign}(\mu_1) = \mathrm{sign}(\lambda_1)$$

and $\lambda_1 = \Lambda(0, t)$.

Observation. Let $(V, \tilde{g})$ be a manifold with boundary ∂V and with $\lambda_1 = \mu_1 = 0$ (zero conformal class), and

$$\begin{cases} R_{\tilde{g}} \equiv 0 & \text{on } V \\ H_{\tilde{g}} = f & \text{on } \partial V \text{ (where } f \not\equiv 0\text{)} \end{cases}$$

Then $\displaystyle\int_{\partial V} f \, d\sigma < 0.$

Indeed, let $\bar{g}$ be such that $R_{\bar{g}} \equiv 0$ and $H_{\bar{g}} \equiv 0$. Then $\tilde{g} = u^{\frac{4}{n-2}} \bar{g}$, and

$$\begin{cases} \Delta_{\bar{g}} u \equiv 0 & \text{on } V \\ \partial_\nu u = b_n u^{\frac{n}{n-2}} f & \text{on } \partial V, \; b_n = \frac{2(n-1)}{n-2} \end{cases}$$

Integration by parts gives

$$\int_{\partial V} f \, d\sigma = b_n^{-1} \int_{\partial V} u^{-\frac{n}{n-2}} \partial_\nu u \, d\sigma < 0.$$

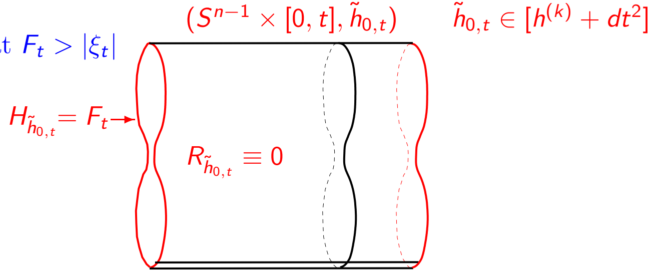
Theorem. (O. Kobayashi) Let $k >> 0$. There exists a metric $h^{(k)}$ on S^{n-1} (Osamu Kobayashi metric) such that

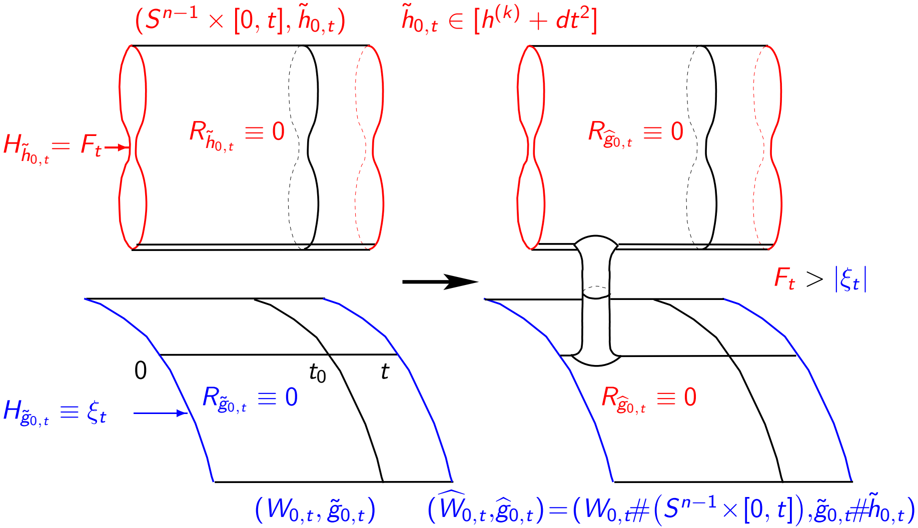
(a) $R_{h^{(k)}} > k$,

(b) $\mathrm{Vol}_{h^{(k)}}(S^{n-1}) = 1$.

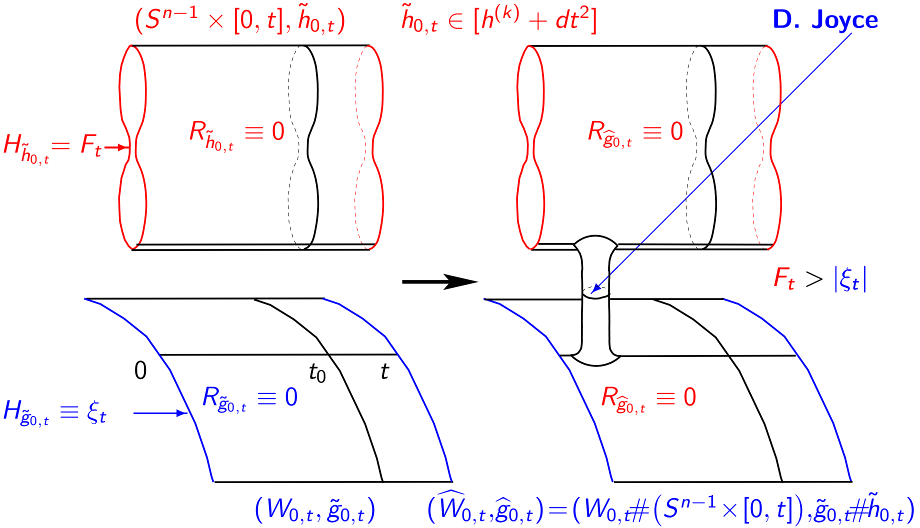
For $t > 0$, we construct the tube $(S^{n-1} \times [0,t], h^{(k)} + dt^2)$.

Choose k such that $F_t > |\xi_t|$

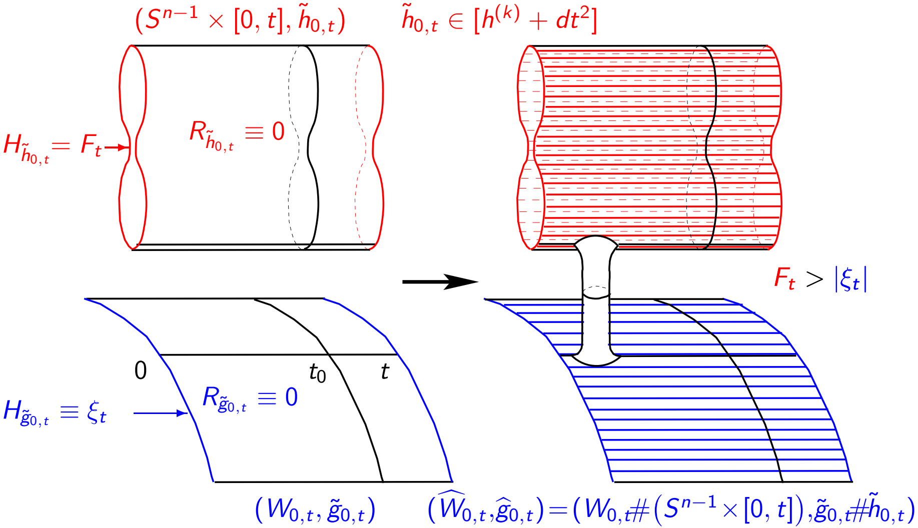




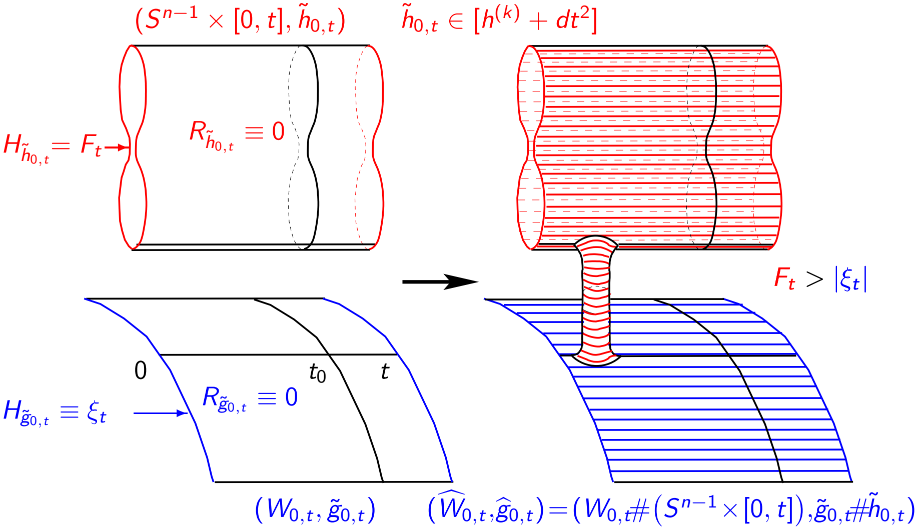
Assume that $(\widehat{W}_{0,t},\widehat{g}_{0,t})$ has zero conformal class. Then $\int_{\partial\widehat{W}_{0,t}}\widehat{H}_{0,t}d\sigma_{0,t}<0$; this fails since $F_t>|\xi_t|$. Thus $(\widehat{W}_{0,t},\widehat{g}_{0,t})$ cannot be of zero conformal class.



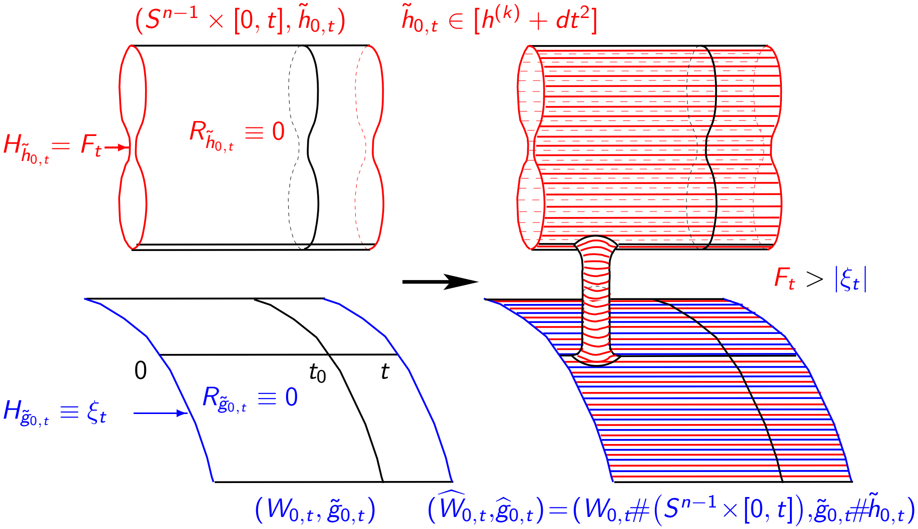
Assume that $(\widehat{W}_{0,t},\widehat{g}_{0,t})$ has zero conformal class. Then $\int_{\partial\widehat{W}_{0,t}}\widehat{H}_{0,t}d\sigma_{0,t}<0$; this fails since $F_t>|\xi_t|$. Thus $(\widehat{W}_{0,t},\widehat{g}_{0,t})$ cannot be of zero conformal class.



Assume that $(\widehat{W}_{0,t},\widehat{g}_{0,t})$ has zero conformal class. Then $\int_{\partial\widehat{W}_{0,t}}\widehat{H}_{0,t}d\sigma_{0,t}<0$; this fails since $F_t>|\xi_t|$. Thus $(\widehat{W}_{0,t},\widehat{g}_{0,t})$ cannot be of zero conformal class.

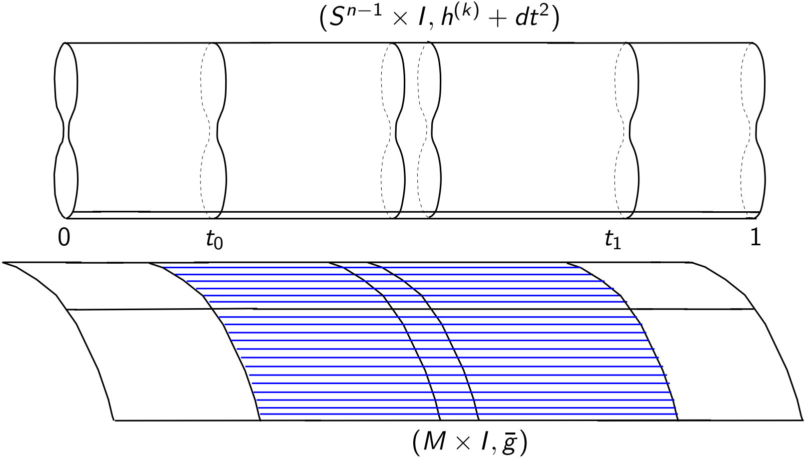


Assume that $(\widehat{W}_{0,t}, \widehat{g}_{0,t})$ has zero conformal class. Then $\int_{\partial \widehat{W}_{0,t}} \widehat{H}_{0,t} d\sigma_{0,t} < 0$; this fails since $F_t > |\xi_t|$. Thus $(\widehat{W}_{0,t}, \widehat{g}_{0,t})$ cannot be of zero conformal class.

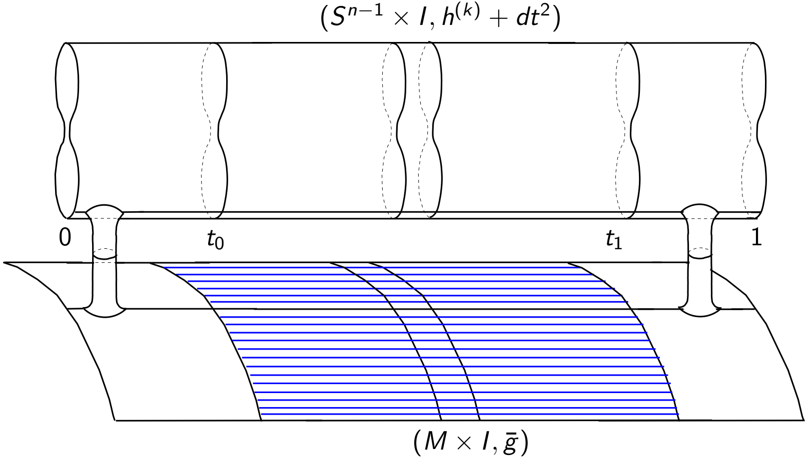


Assume that $(\widehat{W}_{0,t}, \widehat{g}_{0,t})$ has zero conformal class. Then $\int_{\partial \widehat{W}_{0,t}} \widehat{H}_{0,t} d\sigma_{0,t} < 0$; this fails since $F_t > |\xi_t|$. Thus $(\widehat{W}_{0,t}, \widehat{g}_{0,t})$ cannot be of zero conformal class.

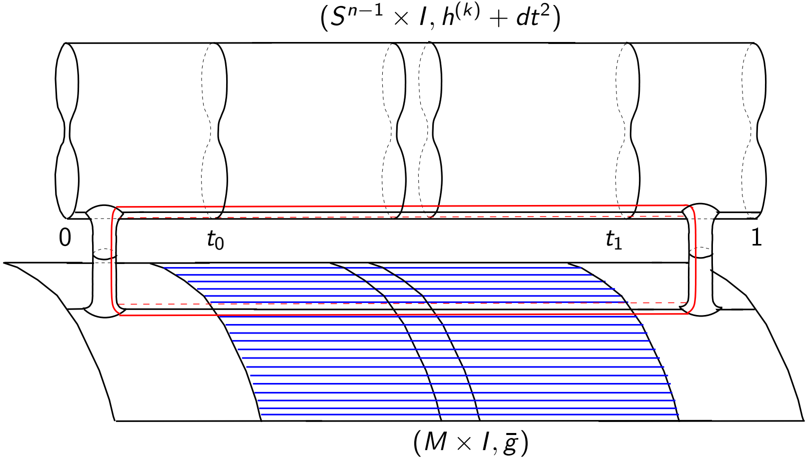
A bypass surgery:



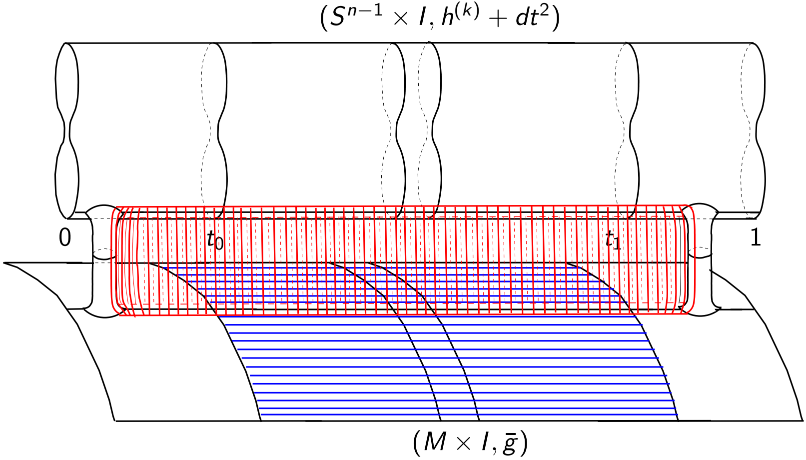
A bypass surgery:



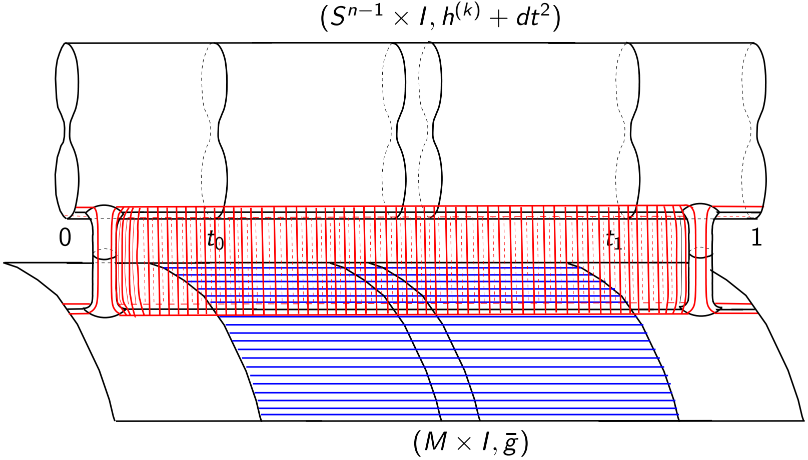
A bypass surgery:



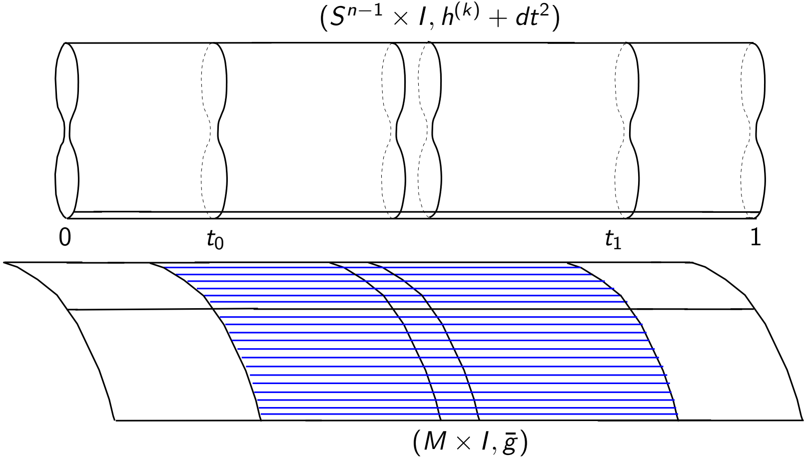
A bypass surgery:



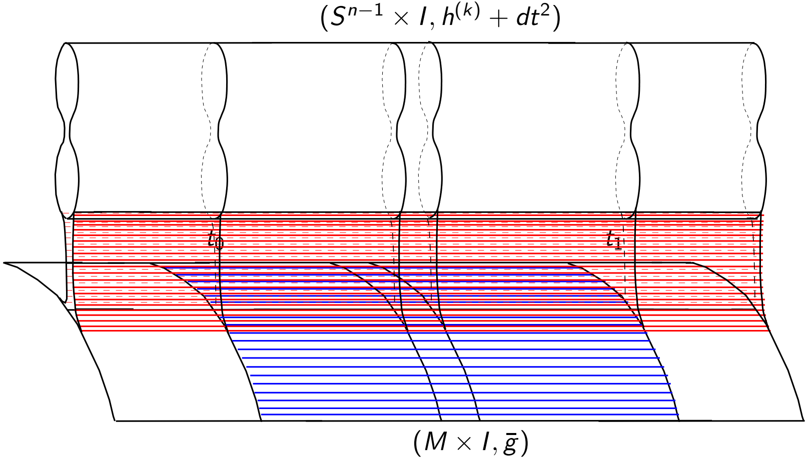
A bypass surgery:



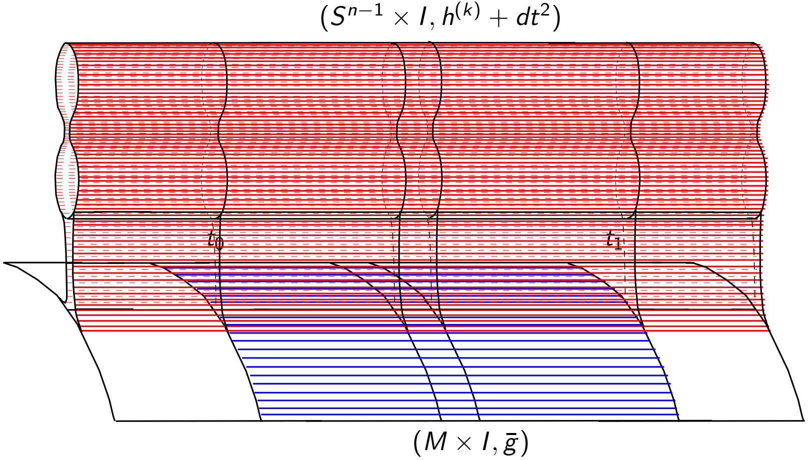
There is another bypass surgery:



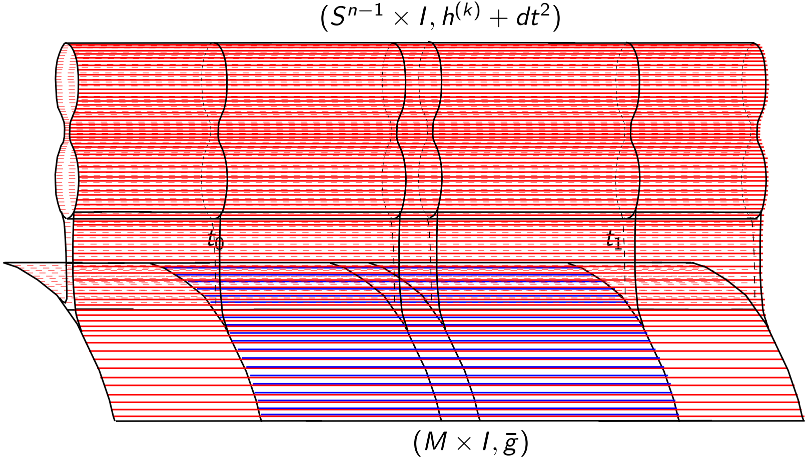
There is another bypass surgery:



There is another bypass surgery:



There is another bypass surgery:



THANK YOU!