On Wilking's criterion for the Ricci flow

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Let (M, g) be a Riemannian manifold. We have the following "classical" notions of positive curvature: Positivity of

- scalar curvature
- Ricci curvature
- sectional curvature
- Pinched sectional curvatures: $\delta < K < 1$ for some $\delta > 0$.

For each $p \in M$, the curvature tensor R of M gives rise to a symmetric operator

$$\mathcal{R} : \wedge^2 T_pM \to \wedge^2 T_pM$$

defined by

$$\langle \mathcal{R}(X \wedge Y), U \wedge V \rangle := R(X, Y, U, V)$$

Here the inner product on $\wedge^2 T_pM$ is induced from g. One can then consider the following notions of positivity:

- Positivity of the symmetric operator $\mathcal{R}$.
- k-positivity of $\mathcal{R}$, i.e., positivity of the sum of the the k smallest eigenvalues of $\mathcal{R}$.

One can complexify $\wedge^2 T_pM$ and consider the complex-linear extension

$$\mathcal{R} : \wedge^2 T_pM \otimes \mathbb{C} \to \wedge^2 T_pM \otimes \mathbb{C}$$

and the Hermitian extension of $\langle \, , \, \rangle$.

- The positivity or k-positivity of $\mathcal{R}$ on $\wedge^2 T_pM \otimes \mathbb{C}$ is equivalent to the corresponding property of $\mathcal{R}$ on $\wedge^2 T_pM$.
- Positive complex sectional curvature:

$$\langle \mathcal{R}(X \wedge Y), X \wedge Y \rangle = R(X, Y, \overline{X}, \overline{Y}) > 0$$

for all $X, Y \in T_pM \otimes \mathbb{C}$ with $X \wedge Y \neq 0$.

- Positive isotropic curvature.

  The notion of positive isotropic curvature was introduced by Micallef and Moore in 1989. They showed that the second variation formula for the energy of a harmonic map from a surface into a Riemannian manifold can be written in a form where the curvature part is in terms of isotropic curvature.

  Using this they showed that *if* (M, g) *is a compact Riemannian n-manifold with positive isotropic curvature then* $\pi_i(M) = \{0\}$ *for* $2 \leq i \leq [n/2]$. It follows by classical results in geometric topology that if M is, in addition, simply-connected then M is homeomorphic to S^n.

Since strict quarter-pinching of sectional curvatures or positivity of curvature operator implies positive isotropic curvature it follows that a compact simply-connected Riemannian n manifold M with either of these two properties must be homeomorphic to S^n.

A curvature tensor R has *positive isotropic curvature* if

$$\langle \mathcal{R}(X \wedge Y), X \wedge Y \rangle > 0$$

for all $X, Y \in T_pM \otimes \mathbb{C}$ such that X, Y generate an isotropic 2-plane.

Let $(\ ,\ )$ denote the $\mathbb{C}$-bilinear extension of g to $T_pM \otimes \mathbb{C}$. A vector X is called *isotropic* if $(X, X) = 0$. An *isotropic subspace* is a subspace all of whose elements are isotropic.

A simple calculation shows that X, Y generate an isotropic 2-plane if and only if $(X, X) = (Y, Y) = (X, Y) = 0$.

This condition is equivalent to the following: For every orthonormal 4-frame $\{e_1, e_2, e_3, e_4\}$ of real vectors $e_i \in T_pM$ we have

$$K_{13} + K_{14} + K_{23} + K_{24} - 2R_{1234} > 0,$$

where K denotes sectional curvature.

There is an associated positivity condition, that of *positive isotropic curvature on $M \times \mathbb{R}$*: It is equivalent to the condition

$$K_{13} + \mu^2 K_{14} + K_{23} + \mu^2 K_{24} - 2\mu R_{1234} > 0$$

for any orthonormal 4-frame $\{e_1, e_2, e_3, e_4\} \subset T_pM$ and any $\mu \in [-1, 1]$.

For $k \geq 2$, $M \times \mathbb{R}^k$ can never have positive isotropic curvature.

Among all these conditions only the following are preserved by Ricci flow:

Positivity of

- scalar curvature (Hamilton 1982)
- curvature operator (Hamilton 1986)
- 2-positivity of curvature operator (Hamilton 1986)
- complex sectional curvature (Ni-Wolfson 2008)
- isotropic curvature (Nguyen 2008, Brendle-Schoen 2008)
- isotropic curvature on $M \times \mathbb{R}$ (Brendle 2008)

We note that *all the above conditions except positive scalar curvature imply positive isotropic curvature.*

We begin by identifying $\wedge^2\mathbb{R}^n$ with $\mathbf{so}(n,\mathbb{R})$:

$$\phi : \wedge^2\mathbb{R}^n \to \mathbf{so}(n,\mathbb{R})$$

where

$$\phi(u \wedge v)(x) = \langle u, x\rangle v - \langle v, x\rangle u,$$

for $u, v \in \mathbb{R}^n$.

By choosing a linear isometry

$$L : T_pM \to \mathbb{R}^n$$

we then have an identification

$$\Phi := \phi \circ L \wedge L : \wedge^2 T_pM \to \mathbf{so}(n,\mathbb{R}).$$

Complexifying we get

$$\Phi : \wedge^2 T_pM \otimes \mathbb{C} \to \mathbf{so}(n, \mathbb{C}).$$

With this identification we regard the curvature operator as an operator on $\mathbf{so}(n, \mathbb{C})$ and denote it again by $\mathcal{R}$. In this formalism

(i) $\mathcal{R}$ is a positive operator if and only if

$$\langle \mathcal{R}(X), X \rangle > 0 \quad \forall X \in \mathbf{so}(n, \mathbb{C}).$$

(ii) Every nonzero simple element in $\wedge^2 T_pM \otimes \mathbb{C}$ corresponds to a $X \in \mathbf{so}(n, \mathbb{C})$ with $Rank(X) = 2$.

Hence $\mathcal{R}$ has positive complex sectional curvature if and only if

$$\langle \mathcal{R}(X), X \rangle > 0 \quad \forall X \in S := \{X \in \mathbf{so}(n, \mathbb{C}) \mid Rank(X) = 2\}.$$

(iii) A simple element in $\wedge^2 T_pM \otimes \mathbb{C}$ represents an isotropic 2-plane if and only if it corresponds to a $X \in \mathbf{so}(n, \mathbb{C})$ satisfying $X^2 = 0,\ Rank(X) = 2$.

$\mathcal{R}$ has positive isotropic curvature if and only if

$$\langle \mathcal{R}(X), X \rangle > 0 \quad \forall X \in S := \{X \in \mathbf{so}(n, \mathbb{C}) \mid X^2 = 0,\ Rank(X) = 2\}.$$

(iv) $M \times \mathbb{R}$ has positive isotropic curvature if and only if

$$\langle \mathcal{R}(X), X \rangle > 0 \quad \forall X \in S := \{X \in \mathbf{so}(n, \mathbb{C}) \mid X^3 = 0,\ Rank(X) = 2\}.$$

Note that in all these cases the set S is $Ad_{SO(n,\mathbb{C})}$-invariant, where

$$SO(n,\mathbb{C}) = \{P \in M(n,\mathbb{C}) \mid PP^t = I\}$$

is the complex Lie group corresponding to the Lie algebra $\mathbf{so}(n,\mathbb{C})$.

Next we consider the evolution of the curvature operator under the Ricci flow: This is given by

$$\frac{d\mathcal{R}}{dt} = \Delta\mathcal{R} + Q(\mathcal{R})$$

where $Q(\mathcal{R})$ is a certain quadratic expression in the components of $\mathcal{R}$.

In many situations one can apply the maximum principle for systems and reduce the study of the above PDE to the ODE

$$\frac{d\mathcal{R}}{dt} = Q(\mathcal{R}).$$

The following theorem of B. Wilking recovers most of the known Ricci flow invariant positive curvature conditions and generates many new conditions:

Theorem (B. Wilking, 2010): *Let S be an $Ad_{SO(n,\mathbb{C})}$-invariant subset of $\mathbf{so}(n,\mathbb{C})$. Then the cone*

$$C(S) := \{\mathcal{R} \in S^2(\mathbf{so}(n,\mathbb{R})) \mid \langle \mathcal{R}(X), X\rangle > 0 \;\; \forall X \in S\}$$

is preserved by the ODE

$$\frac{d\mathcal{R}}{dt} = Q(\mathcal{R})$$

We say that a Riemannian manifold M has *positive S-curvature* if the curvature tensor belongs to $C(S)$ at every point of M.

By the maximum principle for systems we have

Corollary: *Let (M, g_0) be a compact Riemannian manifold with positive S-curvature. If $g(t)$, $t \in [0, T]$ is the solution to the Ricci flow with $g(0) = g_0$ then $(M, g(t))$ has positive S-curvature for all $t \in [0, T]$.*

To motivate our main result we note that the normalized Ricci flow converges for certain positive curvatures : In fact, by the work of S. Brendle positive isotropic curvature on $M \times \mathbb{R}$ guarantees convergence to a constant positive sectional curvature metric. This recovers all of the previously known Sphere Theorems.

On the other hand, the normalized Ricci flow does not converge, in general, for other notions of positive curvature. For instance, the product metric on $S^1 \times S^{n-1}$ has positive isotropic curvature and it is easy to see that the normalized Ricci flow starting at the product metric does not converge to a metric on $S^1 \times S^{n-1}$.

In the presence of positive curvature, if the Ricci flow does not converge one expects the formation of “necks", i.e., the regions of large curvature along the flow should resemble the product $I \times \mathbb{R}$ where $I \subset S^{n-1}$ is an interval. Since positive S-curvature is preserved by the flow, it would follow that $\mathbb{R} \times S^{n-1}$ has positive S-curvature.

Hence if one starts with a manifold M with positive S-curvature and $p \in M$ then one should be deform the metric on $M \setminus \{p\}$ so that it looks like a cylinder $\mathbb{R} \times S^{n-1}$ in a deleted neighborhood of p. It would then follow that the connected sum of two manifolds with positive S-curvature should admit a metric with positive S-curvature.

Motivated by these considerations we proceed as follows:

Let $\mathcal{C}$ be the class of all closed $Ad_{SO(n,\mathbb{C})}$-invariant subsets of $\mathbf{so}(n, \mathbb{C})$ and $\mathcal{C}_0$ be the subcollection consisting of those invariant sets which contain a nonzero simple element of the form $\phi(e \wedge u)$ with $e \in \mathbb{R}^n$, $u \in \mathbb{C}^n$ and $(e, u) = 0$. It can be checked that the condition $S \in \mathcal{C} \setminus \mathcal{C}_0$ can be restated as follows:

An $Ad_{SO(n,\mathbb{C})}$-invariant subset S belongs to $\mathcal{C} \setminus \mathcal{C}_0$ if and only if the product manifold $\mathbb{R} \times S^{n-1}$ has positive S-curvature.

We then have

Theorem (H. Gururaja, S. Maity, H.S.): Let S be an $Ad_{SO(n,\mathbb{C})}$-invariant subset of $\mathbf{so}(n,\mathbb{C})$ such that $S\cup\{0\}$ is a closed subset of $\mathbf{so}(n,\mathbb{C})$.

(i) If $S\in\mathcal{C}\setminus\mathcal{C}_0$ then the connected sum of any two Riemannian manifolds with positive S-curvature also admits a metric with positive S-curvature.

(ii) If $S\in\mathcal{C}_0$ and M is any compact Riemannian manifold with positive S-curvature then the normalized Ricci flow on M converges to a metric of constant positive sectional curvature.

This result is sharp in the sense that for any $S \in \mathcal{C} \setminus \mathcal{C}_0$ there is a Riemannian manifold (M, g) with positive S-curvature such that the normalized Ricci of (M, g) does not converge to a metric on M.

Indeed, as noted earlier, the standard product manifold $S^{n-1} \times S^1$ has positive S-curvature for any $S \in \mathcal{C} \setminus \mathcal{C}_0$.

Moreover for any $S \in \mathcal{C}_0$ there are compact Riemannian manifolds M_1 and M_2 with positive S-curvature whose connected sum does not admit a metric with positive S-curvature.

We can just take $M_1 = M_2$ to be a non-simply connected quotient of S^n with the canonical metric. It follows from the Theorem that $M_1 \# M_2$ cannot admit a metric with positive S-curvature.

Alternatively one can check that positive S-curvature implies positive Ricci curvature when $S \in \mathcal{C}_0$. Hence the Myers-Bonnet theorem on the finiteness of fundamental group rules out the existence of a metric with positive S-curvature on $M_1 \# M_2$.

If we drop the assumption of $S \cup \{0\}$ being closed, we have the following result:

Proposition (H. Gururaja, S. Maity, H.S.): Let $S \in \mathcal{C}_0$ be an $Ad_{SO(n,\mathbb{C})}$-invariant subset of $\mathbf{so}(n, \mathbb{C})$ and let M be a compact Riemannian manifold with positive S-curvature. Then one of the following holds:

(1) The normalized Ricci flow on M converges to a metric of constant positive sectional curvature

(2) M is Kähler and the normalized Ricci flow on M converges to a metric of constant positive holomorphic sectional curvature

(3) M is isometric to a rank-1 symmetric space.

A corollary of the proof of the Theorem and Proposition is that positive isotropic curvature on $M \times \mathbb{R}$ is the “weakest" curvature condition for which we have a Sphere Theorem.

Corollary: Let S be an $Ad_{SO(n,\mathbb{C})}$-invariant subset of $\mathbf{so}(n, \mathbb{C})$. The normalized Ricci flow $g(t)$ of any compact Riemannian manifold (M, g) with positive S-curvature converges to a metric of constant positive sectional curvature if and only if the product metric $g(t) + ds^2$ on $M \times \mathbb{R}$ has positive isotropic curvature for any $t > 0$.

The fact that a Sphere Theorem holds for positive isotropic curvature metrics on $M \times \mathbb{R}$ is a result of S. Brendle.

Our second result is the following:

Theorem(H. Gururaja, S. Maity, H.S.) Let S be an $Ad_{SO(n,\mathbb{C})}$-invariant subset of $\mathbf{so}(n,\mathbb{C})$. Then every element of $C(S)$ has nonnegative isotropic curvature.

Wilking's criterion actually is a general statement about Lie algebras and, in particular, a version for Kähler curvature tensors. In this case one considers $Ad_{GL(n,\mathbb{C})}$-invariant subsets of $\mathbf{gl}(n,\mathbb{C})$ and our result says that *any Kähler curvature tensor in $C(S)$ has nonnegative orthogonal bisectional curvature.*

This illustrates a fundamental difference between the Riemannian and Kähler cases: One knows (by the work of Chen and Chen-Sun-Tian) that the normalized Ricci flow on a compact Kahler manifold with positive orthogonal bisectional curvature converges to a metric with constant holomorphic sectional curvature. Hence the same happens for a compact Kähler manifold with positive S-curvature.

On the other hand, we know that the normalized Ricci flow in the Riemannian case does not converge, in general, when $S \in \mathcal{C}_0$.

The construction of a positive S-curvature metric on connected sums proceeds along the same lines as the Micallef-Wang construction of positive isotropic curvature on connected sums.

Suppose S is an $Ad_{SO(n,\mathbb{C})}$ invariant subset of $\mathbf{so}(n, \mathbb{C})$ such that $S \cup \{0\}$ is closed and $C(S)$ contains the curvature tensor of $\mathbb{R} \times S^{n-1}$.

Let M be a manifold with positive S-curvature. Let $p \in M$ and B be a geodesic ball centered at p and radius less than the injectivity radius at p.

The idea is to find a smooth positive function $u = u(r)$, where $r(x) = d(p, x)$, on $M \setminus \{p\}$ such that $u|_{M \setminus B} \equiv 1$ and near $r = 0$, $u^2 g$ is close to the product metric on $\mathbb{R}^+ \times S^{n-1}(\rho)$ in the C^2 topology, for a suitable choice of ρ. This can be done so that $u^2 g$ has positive S-curvature.

One can then deform this to the product metric again maintaining positive S-curvature.

If, on the other hand, $C(S)$ does not contain the curvature tensor of $\mathbb{R} \times S^{n-1}$ then one can easily show that $M \times \mathbb{R}$ has positive isotropic curvature.

The result of Brendle them implies that the normalized Ricci flow on M converges.

The main ingredient here is Wilking's version of the Brendle-Schoen strong maximum principle:

Let S be an $Ad_{SO(n,\mathbb{C})}$ invariant subset of $\mathbf{so}(n,\mathbb{C})$ and (M,g) be a compact n-manifold with nonnegative S-curvature.

Let $g(t)$ be the solution to Ricci flow starting at g. For $p \in M$ and $t > 0$, let $S_t(p) \subset \wedge^2 T_p(M)$ be the subset corresponding to S at time t i.e., $S_t(p) = \rho^{-1}_{g(t)}(S))$ and let

$$T_t(p) := \{X \in S_t(p) \ : \ \langle R(t)(X), X\rangle_t = 0\}.$$

Then the set $\bigcup_{p\in M} T_t(p)$ is invariant under parallel transport.

Combining this with the Berger-Simons holonomy theorem and previous work of the speaker one gets the required result.

Let S be an $Ad_{SO(n,\mathbb{C})}$ invariant subset of $\mathbf{so}(n,\mathbb{C})$. One first observes that *for any* $X \in S$ *such that* $X^2 \neq 0$ *there exists a sequence* $\{P_1, P_2, ..\} \subset SO(n,\mathbb{C})$ *such that* $lim_{k\to\infty}\|P_k X P_k^{-1}\| = \infty$.

Let $\lambda_k := \|P_k X P_k^{-1}\|^{-1}$. Then a subsequence of $T_k = \lambda_k P_k X P_k^{-1}$ converges to a nonzero $T \in \mathbf{so}(n,\mathbb{C})$. It can be readily checked that if p is the degree of the minimal polynomial of X, then $T^p = 0$.

Note that $\langle R(T), T\rangle = |\lambda_k|^2 lim_{k\to\infty}\langle R(P_k X P_k^{-1}),\ P_k X P_k^{-1}\rangle \geq 0$. In fact, if $T_1 \in O(T)$, where $O(T)$ denotes the adjoint orbit of T, then $\langle R(T_1), T_1\rangle \geq 0$.

The proof is completed by the following classical facts: Let **g** be a simple Lie algebra. Then there exists a nonzero nilpotent orbit of minimal dimension, denoted by O_{min}, which is contained in the closure of any nonzero nilpotent orbit. For $\mathbf{so}(n, \mathbb{C})$, $O_{min} = O(A)$ for any A satisfying $A^2 = 0$ and $rank(A) = 2$ i.e. $O_{min} = S_0$.

Hence $S_0 \subset \overline{O(T)}$ and $\langle R(v), v\rangle \geq 0$ for all $v \in S_0$ by continuity.

- Understand Ricci flow on compact manifolds with positive S-curvature where $S \in \mathcal{C} \setminus \mathcal{C}_0$. In particular, on a manifold with positive isotropic curvature.

  In dimension 4, Hamilton showed that one can perform Ricci flow with surgery on such manifolds to conclude that *every such manifold is finitely covered (up to diffeomorphism) by a connected sum of copies of $S^1 \times S^3$.*

  In dimensions ≥ 5, even the possible rescale blow-up limits are not known. In particular one has the following basic question: *Does any rescale blow-up limit have nonnegative curvature operator ?*

Independent of Ricci flow, one can try to understand the topology of manifolds M with positive isotropic curvature.

- In even dimensions Micallef and Wang proved that $b_2(M) = 0$. This is the only result about Betti numbers of such M.
- It is a conjecture of Gromov and Fraser that $\pi_1(M)$ contains a finite index free subgroup. According to a result of Fraser that $\pi_1(M)$ does not contain $\mathbb{Z} \oplus \mathbb{Z}$.
- It is not known if an exotic sphere can admit a metric with positive isotropic curvature.
- An answer to the following question would result in a complete understanding of the topology of M: Is a finite cover of M diffeomorphic to a connected sum of copies of $S^1 \times S^{n-1}$