# 3-Manifolds with Positive Flat Conformal Structure

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## Aim of this talk

**Setting** $M^n$ : a compact $n$-manifold $(n \geqq 3)$

$C$ : a flat conformal structure

**Thm (Schoen-Yau)**

$n \geqq 4$, $C > 0 \rightsquigarrow (M^n, C)$ : Kleinian

**Main Thm** $n = 3$, $C > 0 \rightsquigarrow (M^3, C)$ : Kleinian

$\Uparrow$

**folklore** by using Witten's $L^2$-spinor approch to Positive Mass Thm

**Note** $\exists$ many examples $(M^n, C)$ : **not** Kleinian

$( \rightsquigarrow C \leqq 0 )$

**Aim** Give another explicit proof of Main Thm by using

a **PMT** for **AF**-manifolds with **singularities**

## ▨1 Positivity / Flatness for Conf. Str.s ($n \geqq 3$)

- $C = \{e^{2f} \cdot g \mid f \in C^{\infty}_{+}(M^n)\}$ : a **conf. str.** on $M^n$

**Def.** $C$ : **flat** $\Leftrightarrow$ ${}^{\forall}p \in M$, ${}^{\exists}U_p$, ${}^{\exists}g \in C$

**s.t.** $g|_{U_p} = (dx^1)^2 + \cdots + (dx^n)^2$ : **flat metric**

( $\rightsquigarrow$ $(M, C)$ : **confomally flat** )

**Fact** ${}^{\exists}g \in C$ **s.t.**

(1) $R_g > 0$ on $M$ ( $\rightsquigarrow$ $C > 0$ : **positive** )

(2) $R_g \equiv 0$ on $M$ ( $\rightsquigarrow$ $C = 0$ : **zero** )

(3) $R_g < 0$ on $M$ ( $\rightsquigarrow$ $C < 0$ : **negative** )

**Note** (1), (2), (3) : **mutually exclusive**

## ∅2 Kleinian Manifold, Developing Map

- $C_0 = [g_0]$ : the **standard** flat conf. str. on $S^n$

  $\Omega \subset S^n$ : an open set

  $\Gamma$ : a discrete subgr. of $\mathrm{Conf}(S^n, C_0)$

  $\Gamma \curvearrowright \Omega$ : $\Gamma(\Omega) = \Omega$,
  **freely** and **properly discontinuously** on $\Omega$

  $\rightsquigarrow$ $(\Omega/\Gamma, C_0)$ : loc. conf. flat $n$-mfd

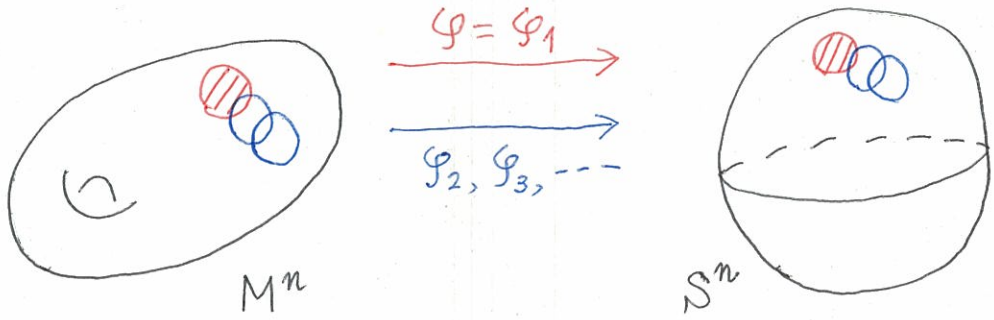
**Def.** $(M^n, C)$ : **Kleinian** $\Leftrightarrow$ $^\exists(\Omega, \Gamma)$ as above

**s.t.** $(M^n, C) \cong (\Omega/\Gamma, C_0) =: \Omega/\Gamma$ : **conf. diffeo.**

- $(M^n, C)$ : **conf. flat.** $\rightsquigarrow$ $^\forall p \in M$, $^\exists (U_p, \varphi)$

**s.t.** $\varphi : (U_p, C) \to (\varphi(U_p), C_0)$ : conf. diffeo.

$\rightsquigarrow$




$\Downarrow$ by **Liouville's Thm for** $\mathrm{Conf}(S^n, C_0)$

& **monodromy arg.**

$^\exists$dev : $(\widetilde{M}, \widetilde{C}) \to (S^n, C_0)$ : conf. imm.

$\nwarrow$ **developing map** unique up to $^\exists \gamma \in \mathrm{Conf}(S^n, C_0)$

$^\exists \rho : \pi_1(M) \to \mathrm{Conf}(S^n, C_0)$ : **holonomy rep.**

**s.t.**

$$\begin{array}{ccccc} & & \text{dev} & & \\ & \widetilde{M} & \longrightarrow & S^n & \\ \gamma & \downarrow & \circlearrowright & \downarrow & \rho(\gamma) \\ & \widetilde{M} & \longrightarrow & S^n & \\ & & \text{dev} & & \end{array}$$

for $^\forall \gamma \in \pi_1(M)$

- $(M^n, C)$ : conf. flat, compact $n$-mfd

$\boxed{\textbf{Criterion}}$ dev : $(\widetilde{M}, \widetilde{C}) \to (S^n, C_0)$ : **injective**

$\rightsquigarrow$ $\rho : \pi_1(M) \to \mathrm{Conf}(S^n, C_0)$ : <u>faithful</u> <u>discrete</u> rep.

& $\rho(\pi_1(M)) \curvearrowright \mathrm{dev}(\widetilde{M})$ : <u>freely</u>, <u>properly disconti.</u>

$\rightsquigarrow$ $(M^n, C) \cong (\mathrm{dev}(\widetilde{M})/\rho(\pi_1(M)),\ C_0)$ : **Kleinian**

**<u>Schoen-Yau</u>** $(M^n, C)$ : <u>conf. flat</u>, compact $n$-mfd

<u>$n \geqq 4$</u>, <u>$C > 0$</u> $\rightsquigarrow$ dev : $(\widetilde{M}^n, \widetilde{C}) \to (S^n, C_0)$ : **injective**

$\left( \rightsquigarrow \quad (M^n, C) : \textbf{Kleinian} \right)$

**<u>Note 1</u>** <u>$n = 3$</u> $\rightsquigarrow$ $^{\exists}$ **technical difficulty !**

**<u>Note 2</u>** <u>$n \geqq 3$</u>, <u>$|\pi_1(M)| < \infty$</u> $\rightsquigarrow$ $(\widetilde{M}^n, \widetilde{C}) \cong (S^n, C_0)$ by **Kuiper's Thm**

$\rightsquigarrow$ $(M^n, C) \cong (S^n/\mathrm{dev}(\pi_1(M)),\ C_0)$ : **Kleinian**

$\rightsquigarrow$ Assume $\boxed{|\pi_1(M)| = \infty}$

## ③ Another Schoen-Yau's Criterion

**Setting** $(M^n, C)$ : compact $n$-mfd with

$C > 0$, $|\pi_1(M)| = \infty$

Take $g \in C$ with $R_g > 0$

$L_g := -\alpha_n \Delta_g + R_g$ : **conf. Laplacian**, $\alpha_n := \frac{4(n-1)}{n-2}$

$G$ : **normalized Green's fct** for $L_g$ with pole $p \in M$

**i.e.**, $L_g \circ G = c_n \cdot \delta_p$ & $\lim_{q \to p} d_g(q,p)^{n-2} \cdot G(q) = 1$

$\mathcal{P} : (\widetilde{M}, \widetilde{C}, \tilde{p}) \to (M, C, p)$ : infinite unversal covering

$\tilde{g}$ : lift of $g$ to $\widetilde{M}$ ( $\rightsquigarrow$ $R_{\tilde{g}} > 0$ )

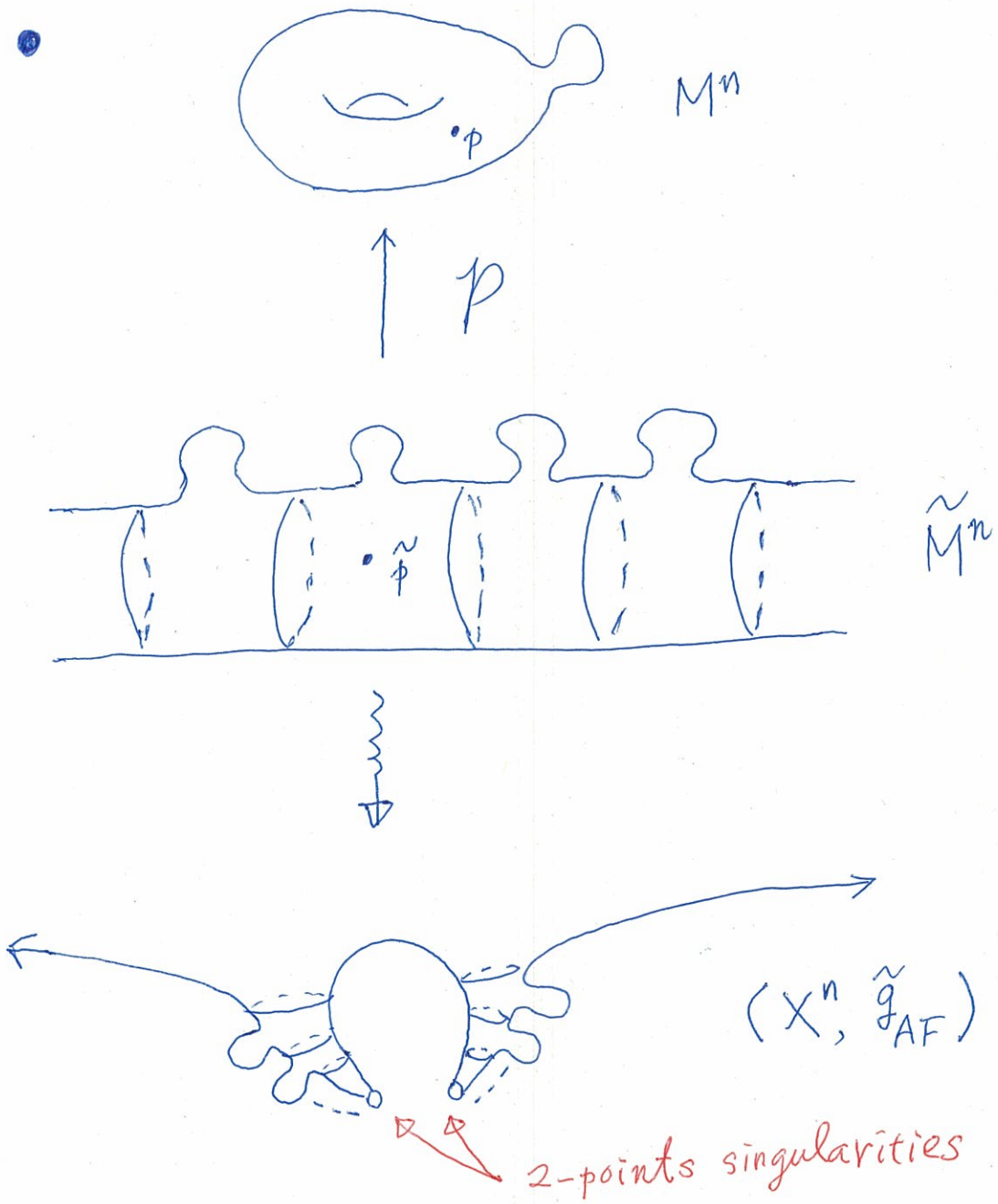
$\widetilde{G}$ : **n. G. fct** with pole $\tilde{p} \in \widetilde{M}$ for $L_{\tilde{g}} = -\alpha_n \Delta_{\tilde{g}} + R_{\tilde{g}}$

$\rightsquigarrow$ $\boxed{\mathcal{P}^* G = \Sigma_{\gamma \in \pi_1(M)} \widetilde{G} \circ \gamma \quad \text{on} \quad \widetilde{M}}$

$\rightsquigarrow$ $\tilde{g}_{AF} := \widetilde{G}^{4/n-2} \cdot \tilde{g}$ on $X := \widetilde{M} - \{\tilde{p}\}$

$\nwarrow$ **scalar-flat**, **asymptotically flat** $n$-mfd

**Note** $\mathrm{Sing}(X, \tilde{g}_{AF}) \neq \emptyset$




- $M^n := \#\ell(S^{n-1} \times S^1)$, $\ell \geqq 2$

$\rightsquigarrow \left|\mathrm{Sing}(X^n, \tilde{g}_{AF})\right| = \infty$

## Schoen-Yau's Criterion

$(M^n, g)$ : conf. flat, compact $n$-mfd with $R_g > 0$

Assume $\boxed{\text{mass}(X^n,\ \tilde{g}_{AF}) \geqq 0}$

$\rightsquigarrow$ dev : $(\widetilde{M}^n, [\tilde{g}]) \to (S^n, C_0)$ : **injective**

## Naive Question PMT for AF-mfd with singularities ?

- If true for $(M^3, C)$ : conf. flat $\rightsquigarrow$ Main Thm holds !

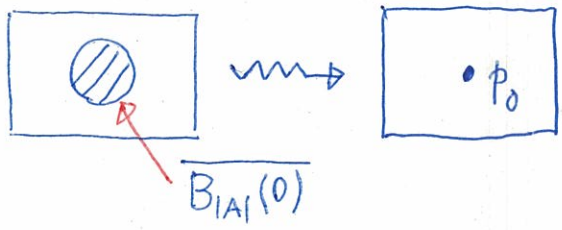
## PMT for AF-mfd with sing. **does not hold** generally !

$$\boxed{X_A^3 := (\ \mathbb{R}^3 - B_{|A|}(0),\ h_A := (1 + \frac{A}{|x|})^4 \delta_{ij}\ ) \quad \text{with} \quad A < 0}$$

$\Rightarrow$ $h_A$ : **scalar-flat AF-metric**, mass$(h_A) = A < 0$

- $\boxed{\text{2-sphere } \{|x| = |A|\}}$ **collapses** $p_0$ **!**

under $\boxed{\mathbb{R}^3 - B_{|A|}(0) \approx \mathbb{R}^3}$ $\rightsquigarrow$ $p_0$ : **singularity** !



$M_\infty$
$\downarrow$
$\vdots$
$\downarrow$
$M_k$
$\downarrow$
$\vdots$
$\downarrow$
$M_2$
$\downarrow$
$M$

## ⊘4 PMT for AF-mfd with singularities

**Setting** $(M^n, [g])$ : compact, $[g] > 0$

$(M^n_\infty, g_\infty) \to (M^n, g)$ : **normal infinite** Riem. covering with $M_\infty \ni p_\infty \mapsto p \in M$

$(M^n_k, g_k) \to (M^n, g)$ : **finite** Riem. covering with $(M^n_1, g_1) = (M^n, g)$

**s.t.**

(i) $\pi_1(M^n) \supsetneq \cdots \supsetneq \pi_1(M^n_k) \supsetneq \pi_1(M^n_{k+1}) \supsetneq \cdots \supsetneq \pi_1(M_\infty)$

(ii) $\bigcap_{k=1}^\infty \pi_1(M^n_k) = \pi_1(M^n_\infty)$

$\boxed{\text{P M T}}$ under the **above** setting

- $n = 3, 4, 5$ **or** $g$ : **conf. flat** near $p$ for $n = 6, 7$

$g_{\infty,AF} := G_\infty^{4/n-2} \cdot g_\infty$ : sacalar-flat, AF-metric on $M^n_\infty - \{p_\infty\}$

$$= \boxed{\left(1 + \frac{A}{|y|^{n-2}} + O(\frac{1}{|y|^{n-1}})\right)^{4/(n-2)} \cdot \delta_{ij} + O(\frac{1}{|y|^{n-1}})} \text{ near } \infty$$

$\rightsquigarrow$ (1) $\text{mass}(g_{\infty,AF}) = A \left( = \lim_{k\to\infty} \text{mass}(g_{k,AF}) \right) \geq 0$

(2) $\text{mass}(g_{\infty,AF}) = 0 \Rightarrow (M^n_\infty, [g_\infty]) \subset (S^n, C_0)$

**simp. conn. domain**

## ⊘5 Proof of Main Thm

**Recall** $(M^3, C)$ : conf. flat, compact 3-mfd with $C > 0$, $|\pi_1(M^3)| = \infty$

$\Downarrow$ by Results of Gromov-Lawson, Izeki

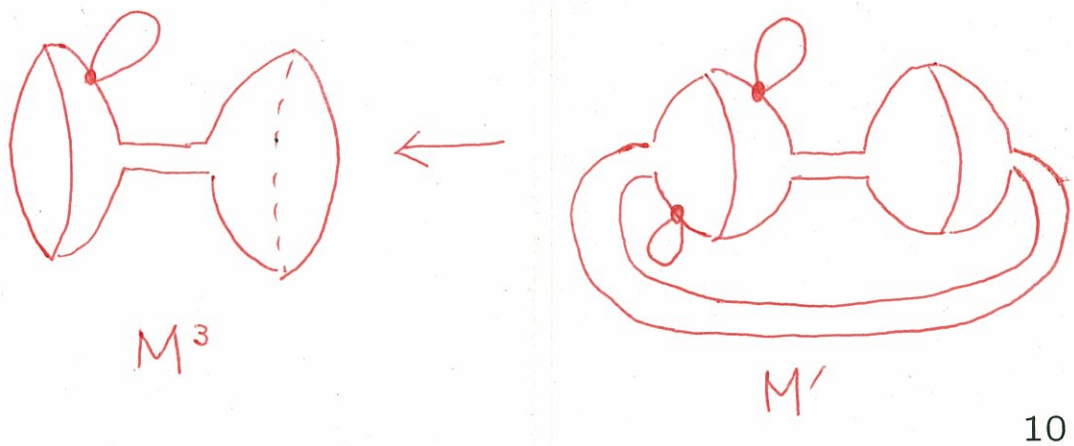
$$M^3 = (S^3/\Gamma_1) \sharp \cdots \sharp (S^3/\Gamma_k) \sharp \ell(S^2 \times S^1)$$

$\Downarrow$ taking approp. finite covering

$$M' = \sharp\, m(S^2 \times S^1)$$

**Ex.** $M^3 = (S^3/\mathbb{Z}_2) \sharp (S^3/\mathbb{Z}_2) \sharp \ell(S^2 \times S^1)$

$\rightsquigarrow$ $M' = \sharp\, (2\ell + 1)(S^2 \times S^1)$



- $\widetilde{M}^3$ : universal covering of $M^3$

$\rightsquigarrow$ $\widetilde{M}^3 \to M' = \sharp\, m(S^2 \times S^1)$ : universal covering

& $\pi_1(M') = \pi_1(\underbrace{S^1 \vee \cdots \vee S^1}_{m}) = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{m}$ : **free group**

$\rightsquigarrow$ ${}^{\exists}M_k^3 \to M'$ : non-trivial finite covering satisfying

(i) $\pi_1(M') \supsetneqq \cdots \supsetneqq \pi_1(M_k^3) \supsetneqq \cdots \supsetneqq \pi_1(\widetilde{M}^3) = \{e\}$

(ii) $\bigcap_{k=1}^{\infty} \pi_1(M_k^3) = \pi_1(\widetilde{M}^3) = \{e\}$

$\rightsquigarrow$ $\mathrm{mass}(\widetilde{M}^3 - \{\tilde{p}\}, \tilde{g}_{AF}) \geqq 0$ by **PMT**

$\rightsquigarrow$ dev : $(\widetilde{M}^3, \widetilde{C}) \to (S^3, C_0)$ : **injective**
by **Schoen-Yau's Criterion**

$\rightsquigarrow$ $(M^3, C)$ : **Kleinian** $\boxed{\text{QED}}$

## Counterexamples

(1) $\underline{M_a^n = \Sigma^{n-1}(-1) \times S_a^1}$, $S_a^1 := \mathbb{R}/\{t \mapsto t + a\}, a \in \mathbb{R}$

$\rightsquigarrow$ dev : $\mathbb{H}^{n-1} \times \mathbb{R} \to \mathbb{R}^n - \mathbb{R}^{n-2} \subset \widehat{\mathbb{R}}^n = S^n$

$a/\pi \notin \mathbb{Q}$ $\rightsquigarrow$ $\rho(\pi_(M_a)) \subset \mathrm{Conf}(S^n, C_0)$ : **not** discrete

$\rightsquigarrow$ $M_a$ : **not** Kleinian

(2) $N_a^n := (S^{n-1} \times S^1) \sharp M_a^n$ for $a/\pi \notin \mathbb{Q}$

$\rightsquigarrow$ dev : $\widetilde{N}_a^n \to (S^n, C_0)$ : **onto**

$\rightsquigarrow$ $N_a^n$ : **not** Kleinian