Hopf's theorem for surfaces with constant mean curvature and its generalizations

Naoya Ando

(Kumamoto University, JAPAN)

Hopf's theorem

Hopf's theorem

S: a surface in E^3.

Hopf's theorem

S: a surface in E^3.

Suppose that

- S is homeomorphic to S^2,

Hopf's theorem

S: a surface in E^3.

Suppose that

- S is homeomorphic to S^2,
- S has constant mean curvature.

Hopf's theorem

S: a surface in E^3.

Suppose that

・ S is homeomorphic to S^2,

・ S has constant mean curvature.

$\Rightarrow$ S is a round sphere.

Key points of the proof

Key points of the proof

- The index $i(a_0)$ of an isolated umbilical point a_0 on a surface with constant mean curvature is negative;

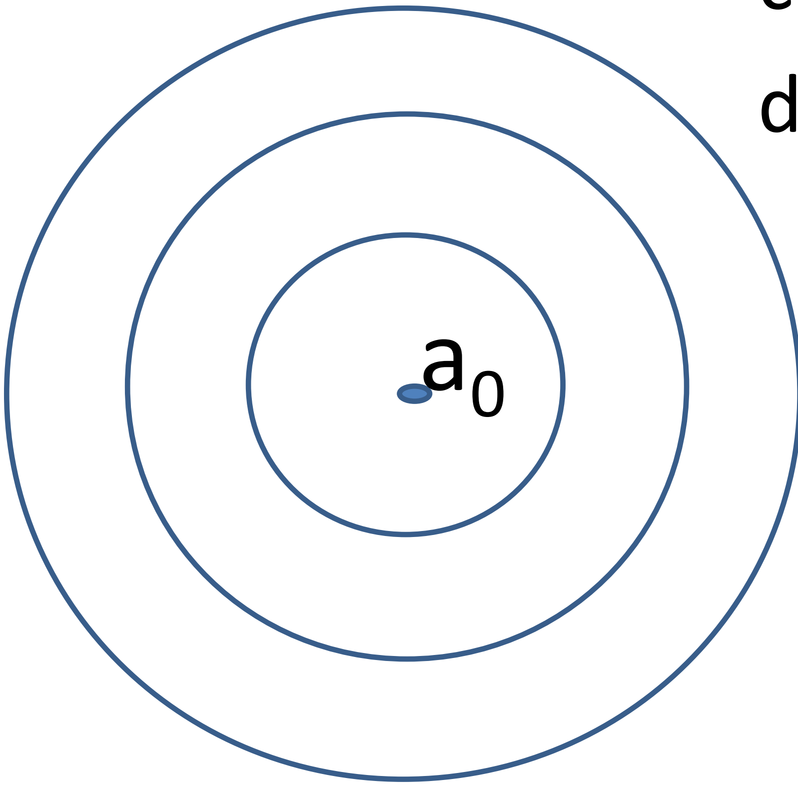
Key points of the proof

- The index $i(a_0)$ of an isolated umbilical point a_0 on a surface with constant mean curvature is negative;
- Hopf-Poincare's theorem:

$$\sum i(a_0)$$

$$= \text{the Euler number of } S.$$

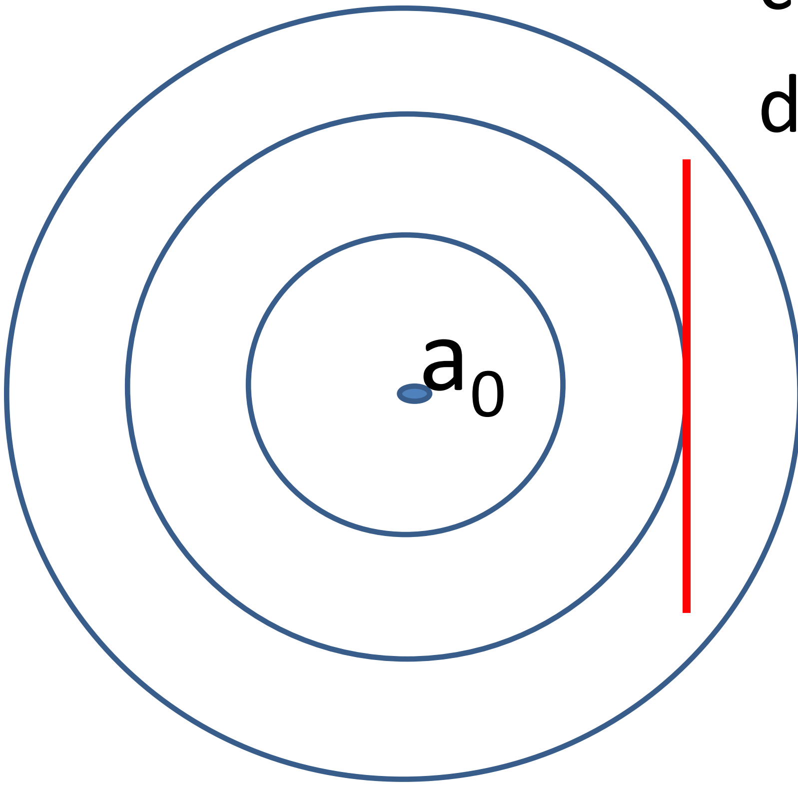
Example



If curvature lines (integral curves of a principal distribution) are as in the left figure, then

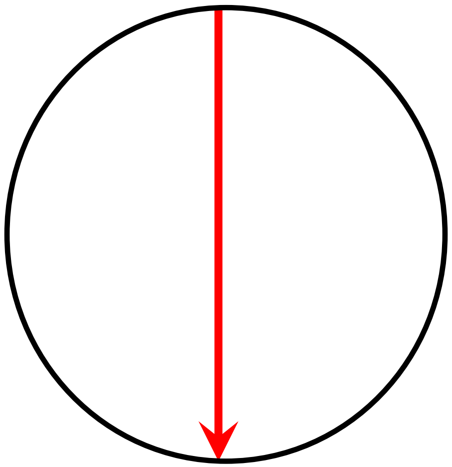
$i(a_0) =$

Example



If curvature lines (integral curves of a principal distribution) are as in the left figure, then

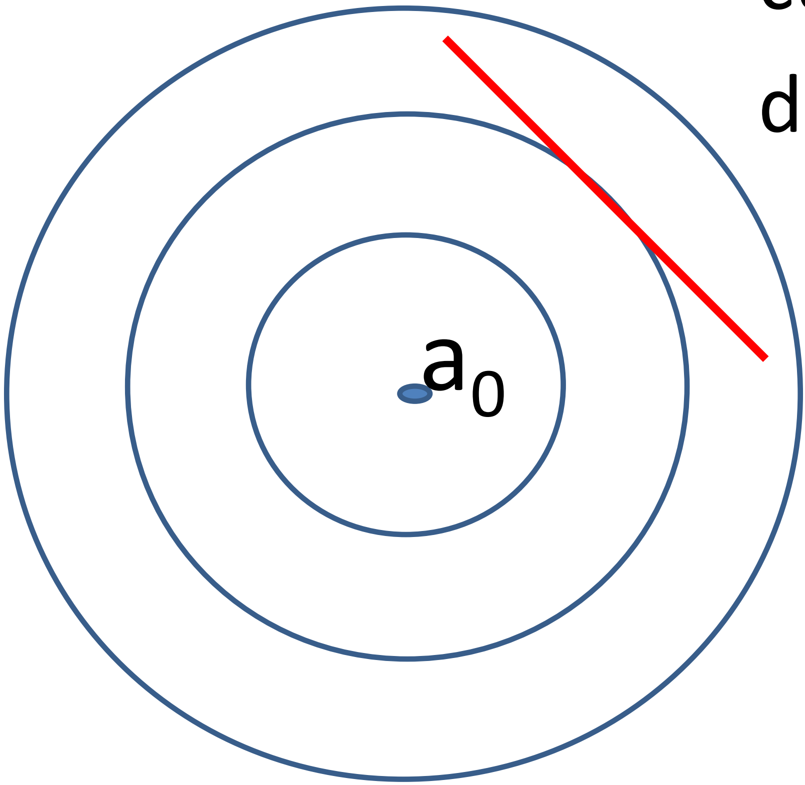
$i(a_0) =$

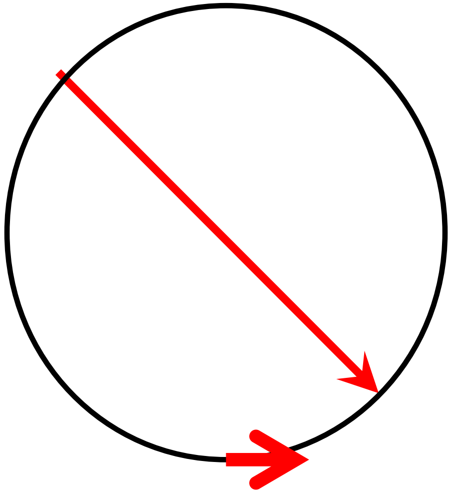


Example

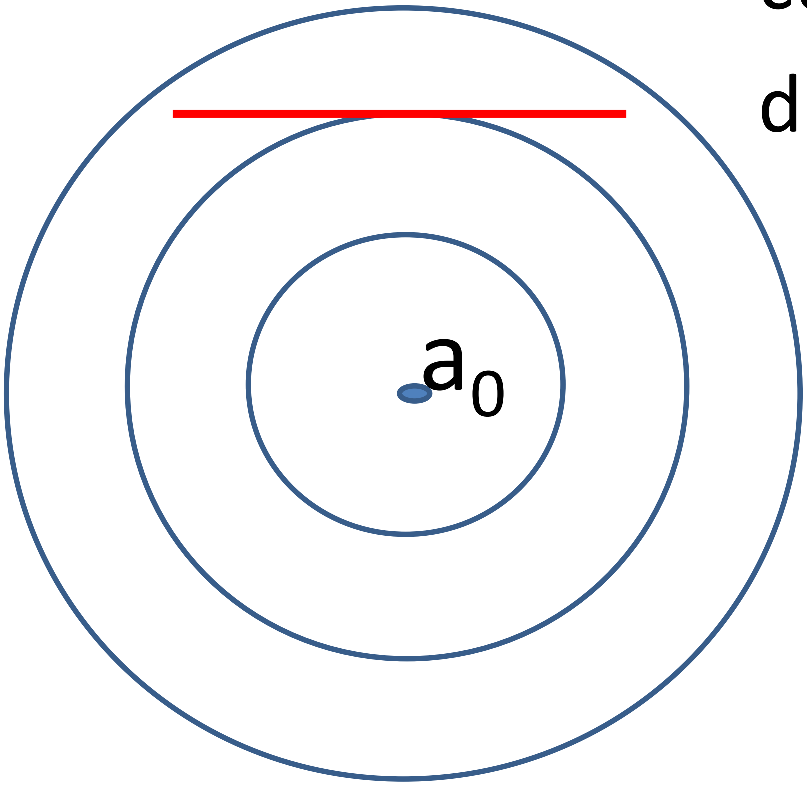
If curvature lines (integral curves of a principal distribution) are as in the left figure, then

$i(a_0) =$



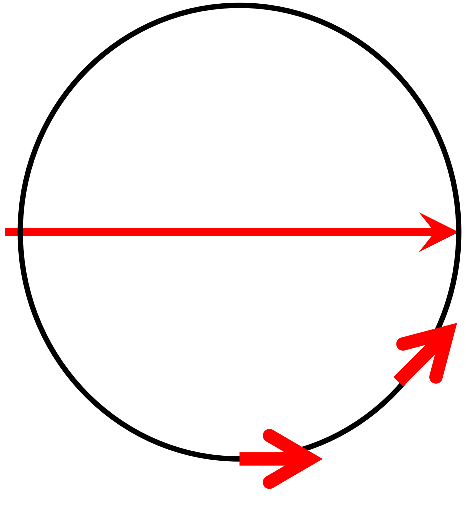


Example

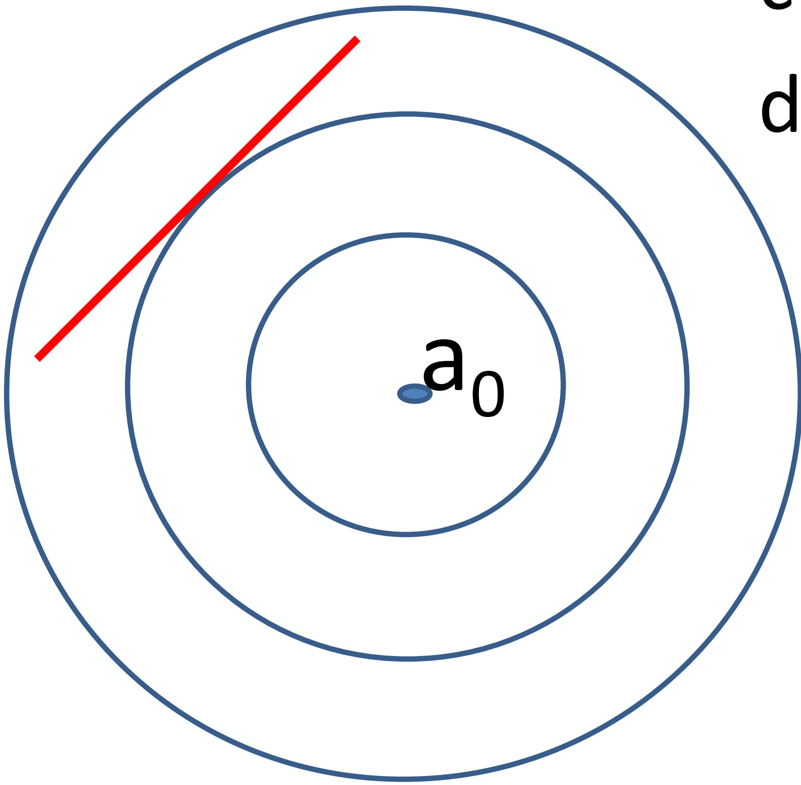


If curvature lines (integral curves of a principal distribution) are as in the left figure, then

$i(a_0) =$

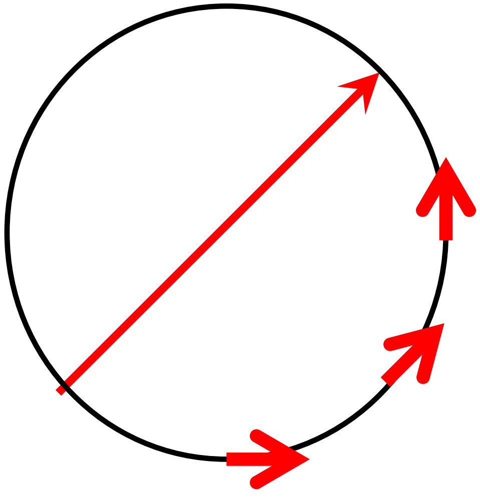


Example



If curvature lines (integral curves of a principal distribution) are as in the left figure, then

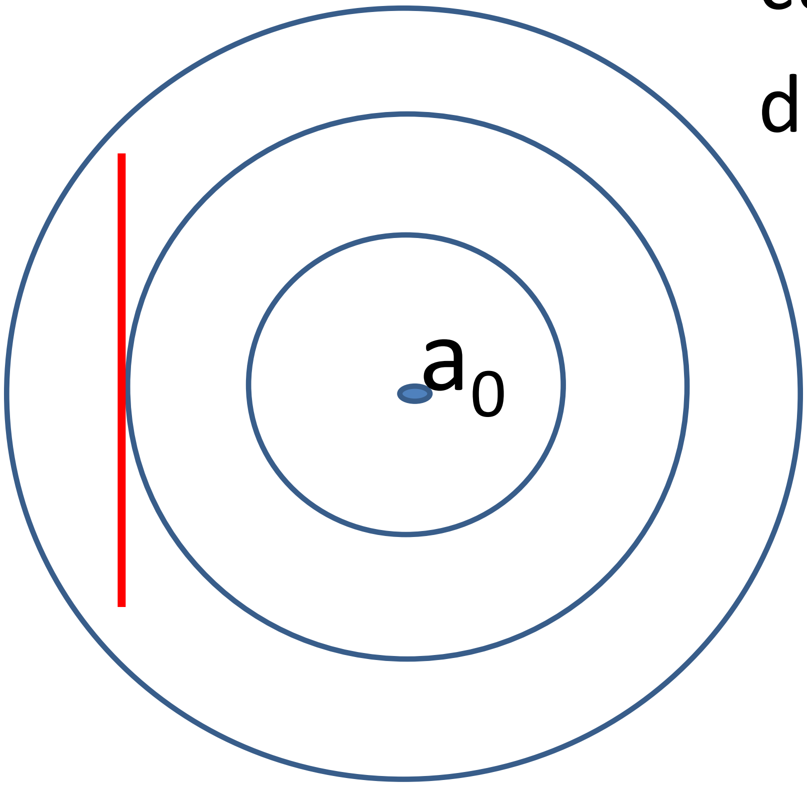
$i(a_0) =$

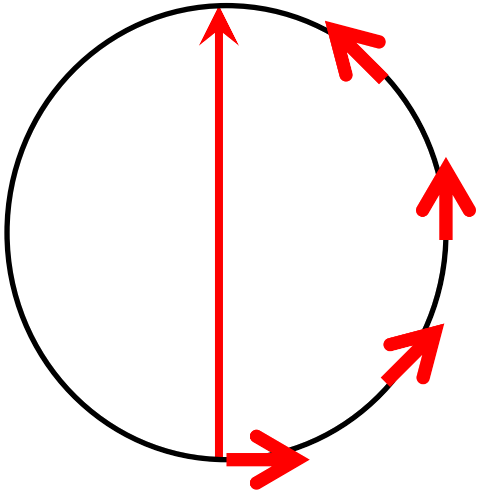


Example

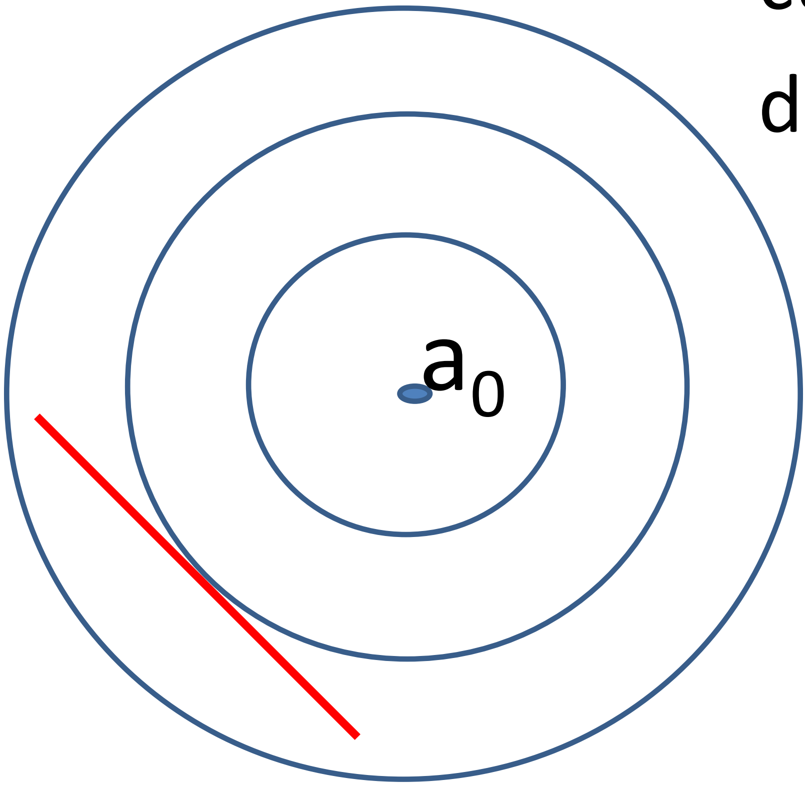
If curvature lines (integral curves of a principal distribution) are as in the left figure, then

$i(a_0) =$



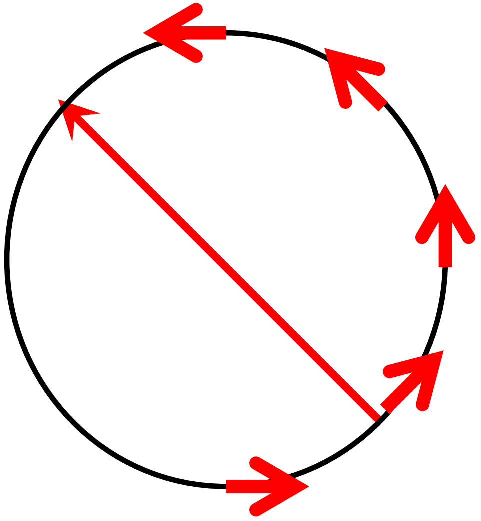


Example

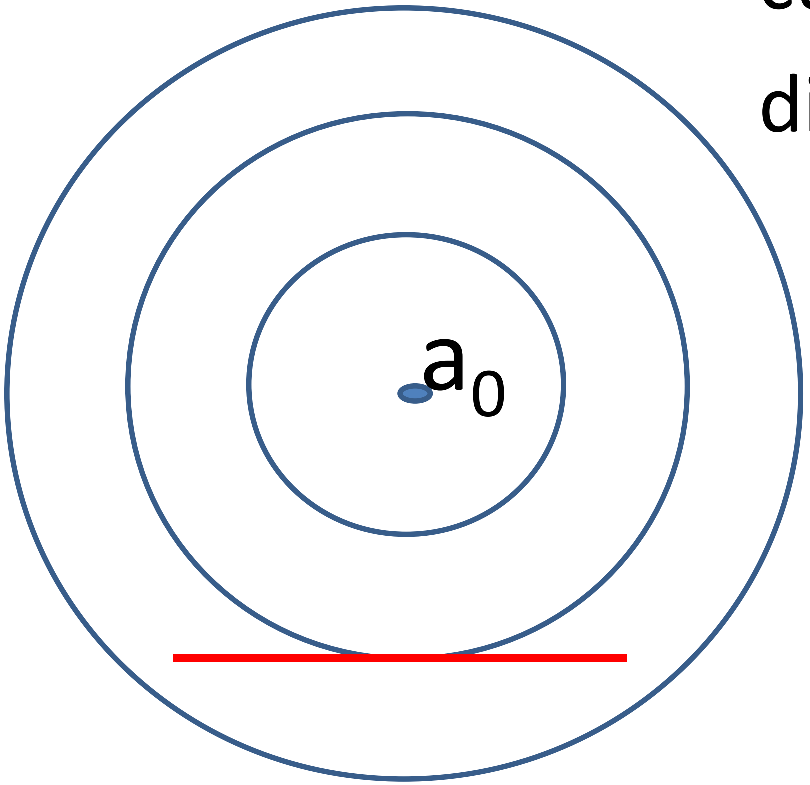


If curvature lines (integral curves of a principal distribution) are as in the left figure, then

$i(a_0) =$

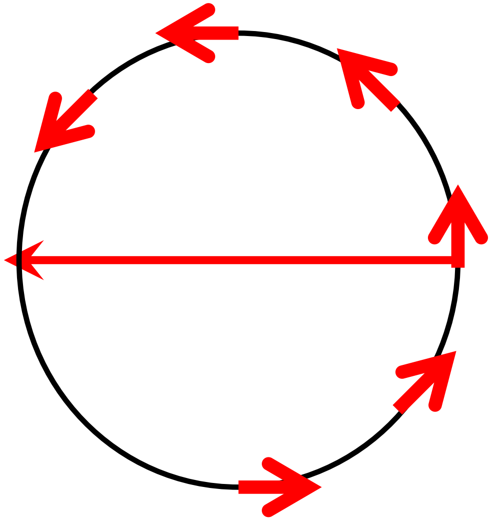


Example

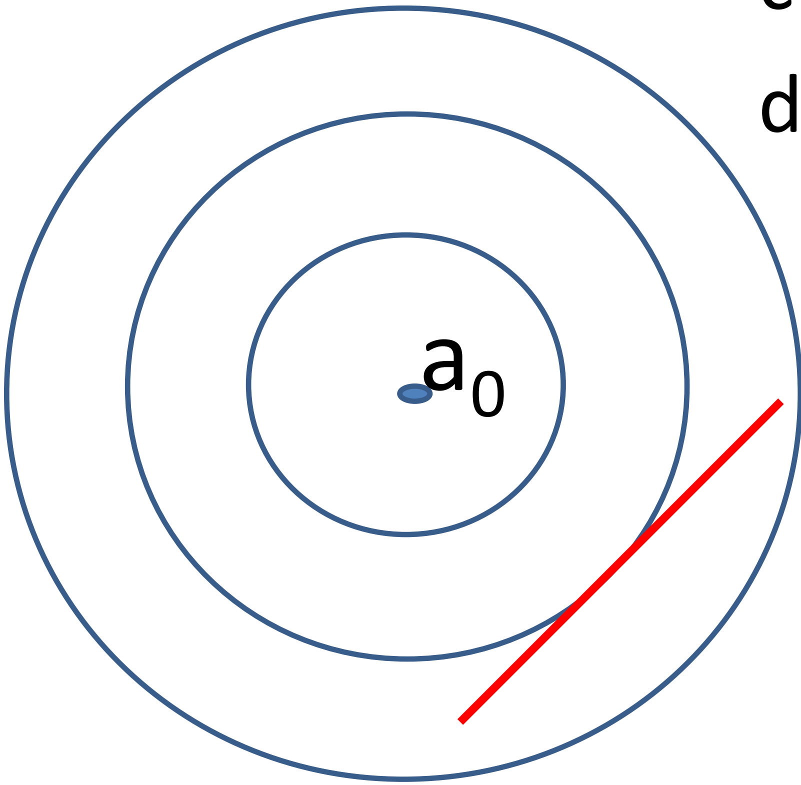


If curvature lines (integral curves of a principal distribution) are as in the left figure, then

$i(a_0) =$

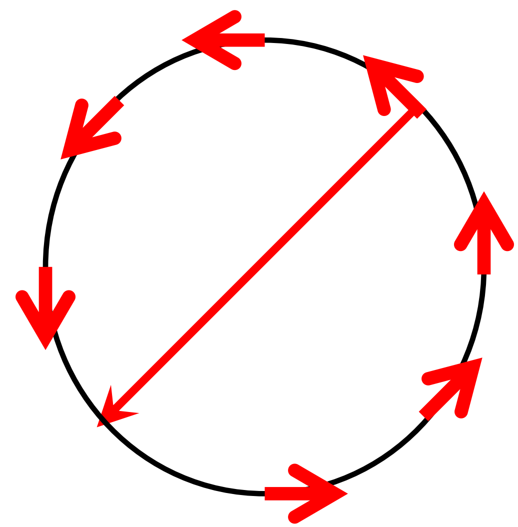


Example



If curvature lines (integral curves of a principal distribution) are as in the left figure, then

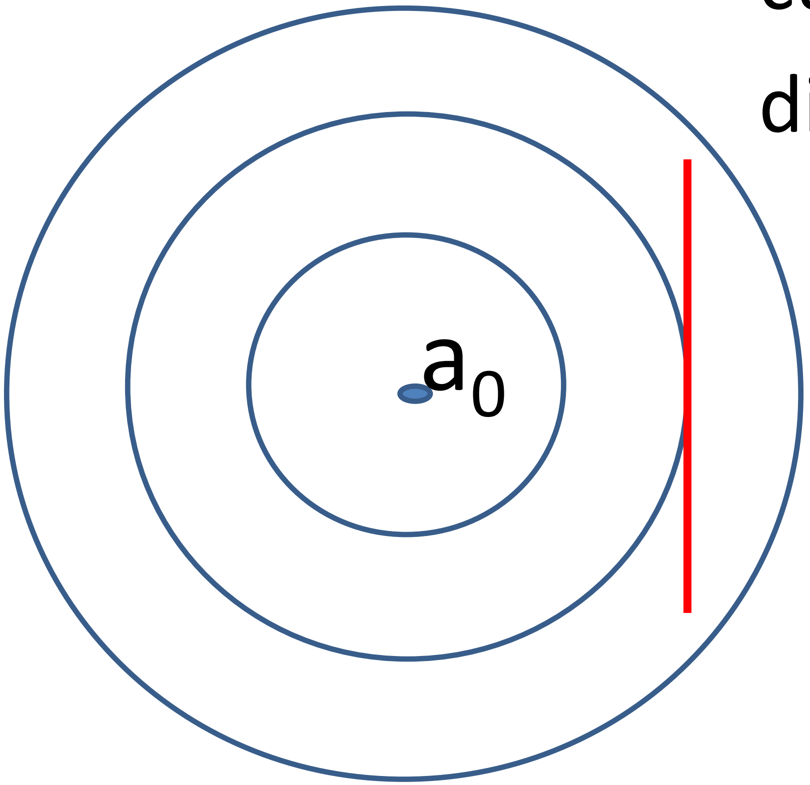
$i(a_0) =$

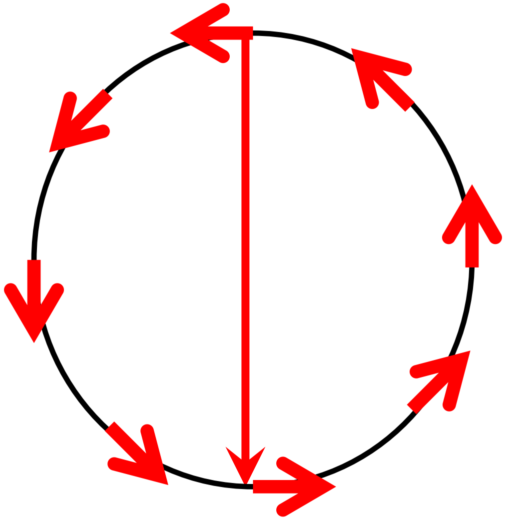


Example

If curvature lines (integral curves of a principal distribution) are as in the left figure, then

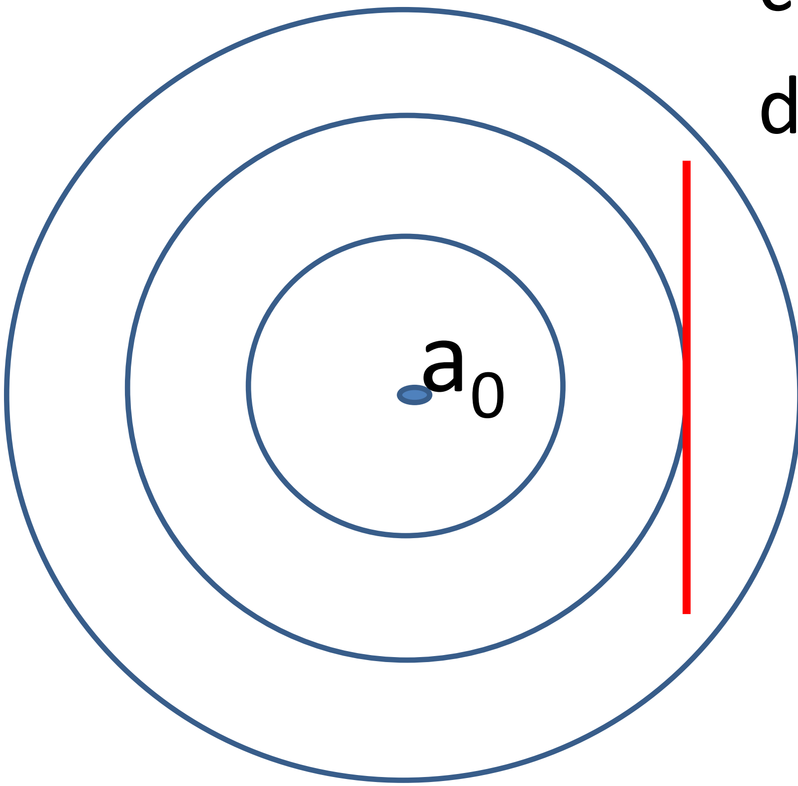
$i(a_0) =$

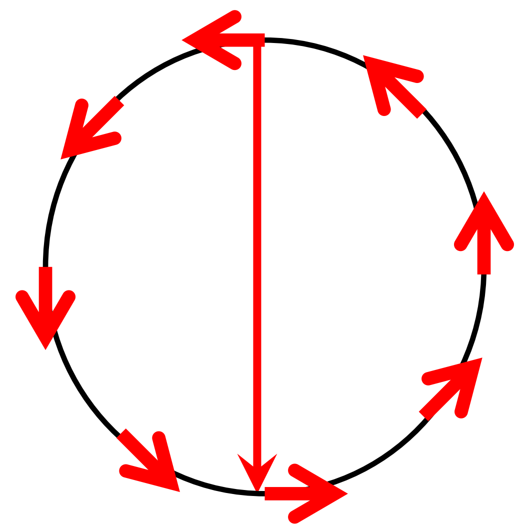




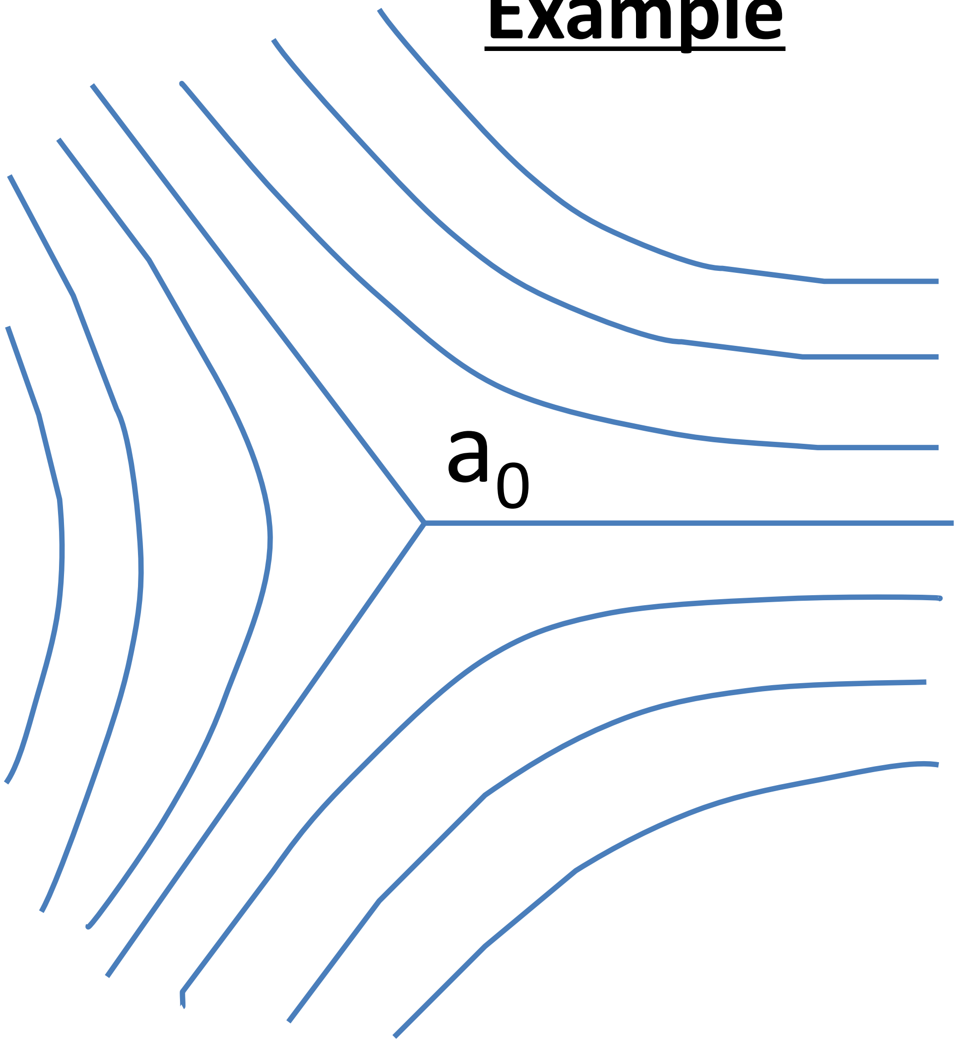
Example

If curvature lines (integral curves of a principal distribution) are as in the left figure, then $i(a_0) = 1$.





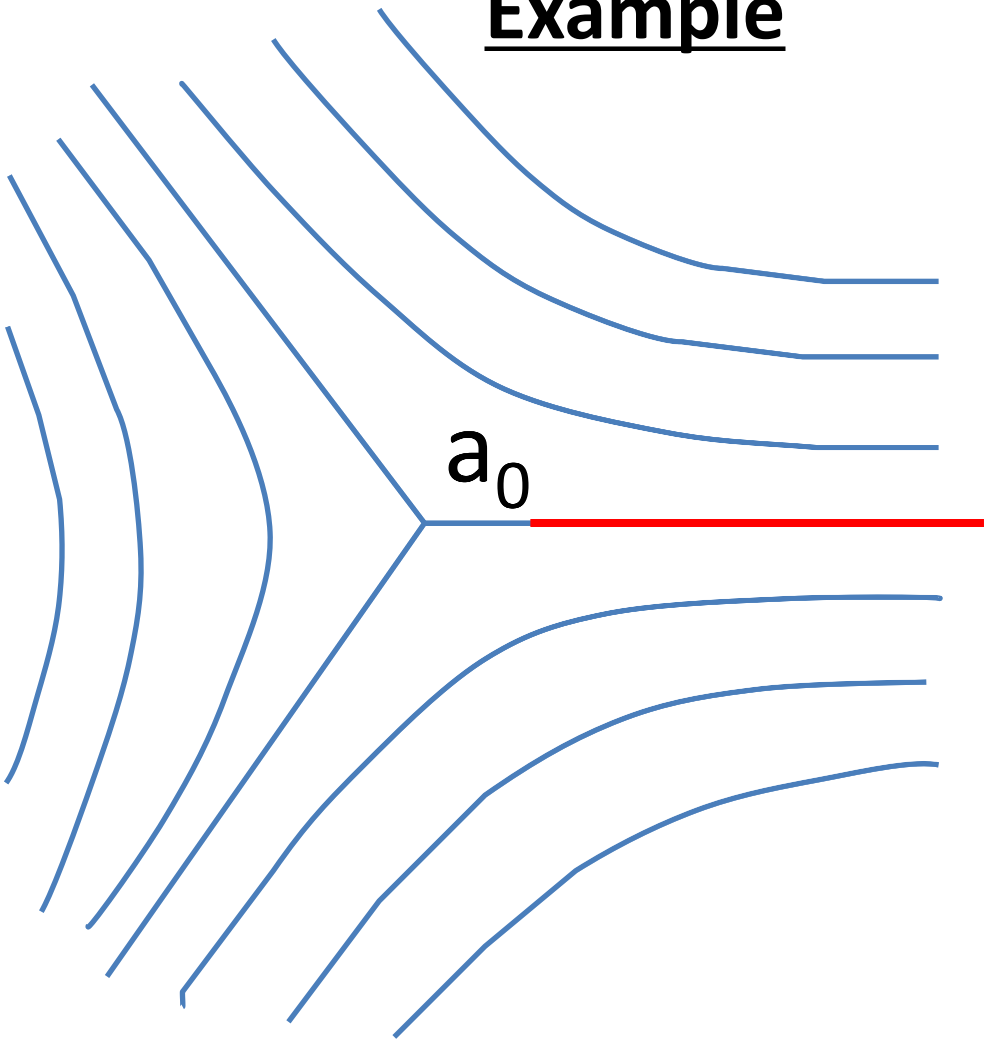
Example



If curvature lines are as in the left figure, then

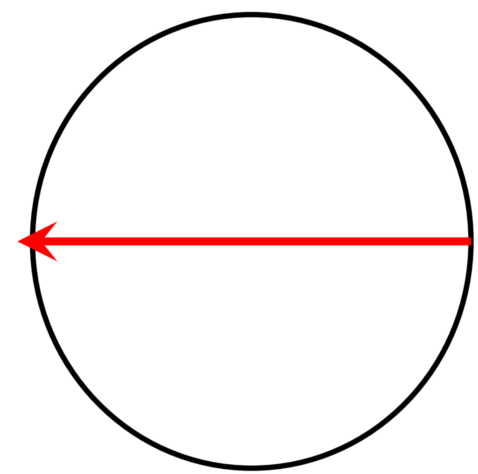
$i(a_0) =$

Example

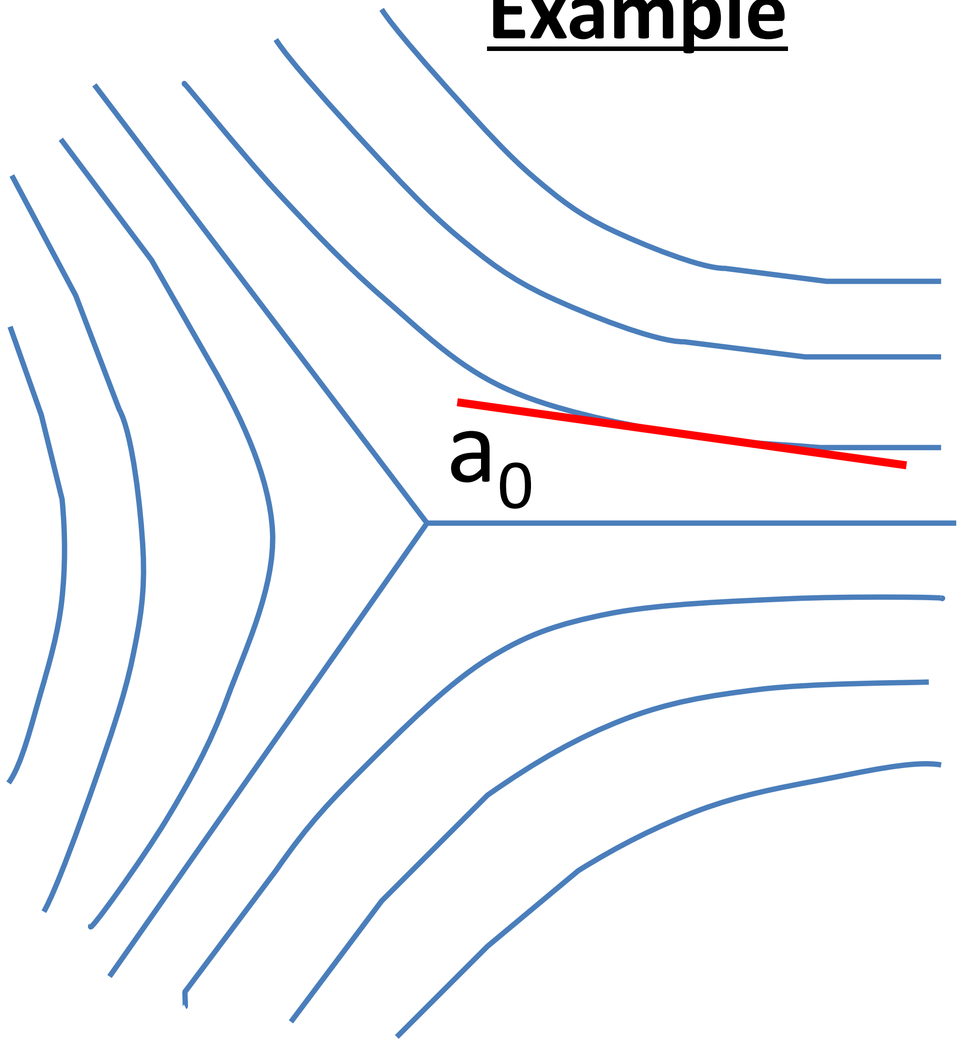


If curvature lines are as in the left figure, then

$i(a_0) =$

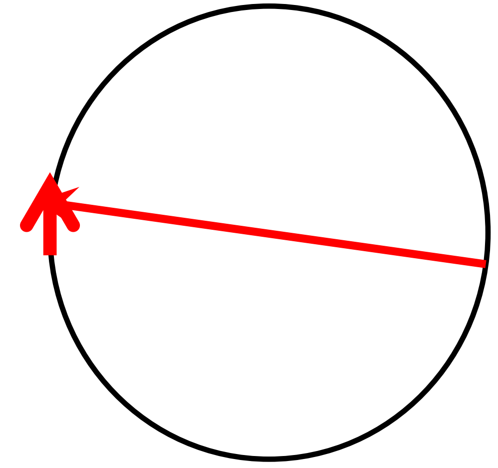


Example

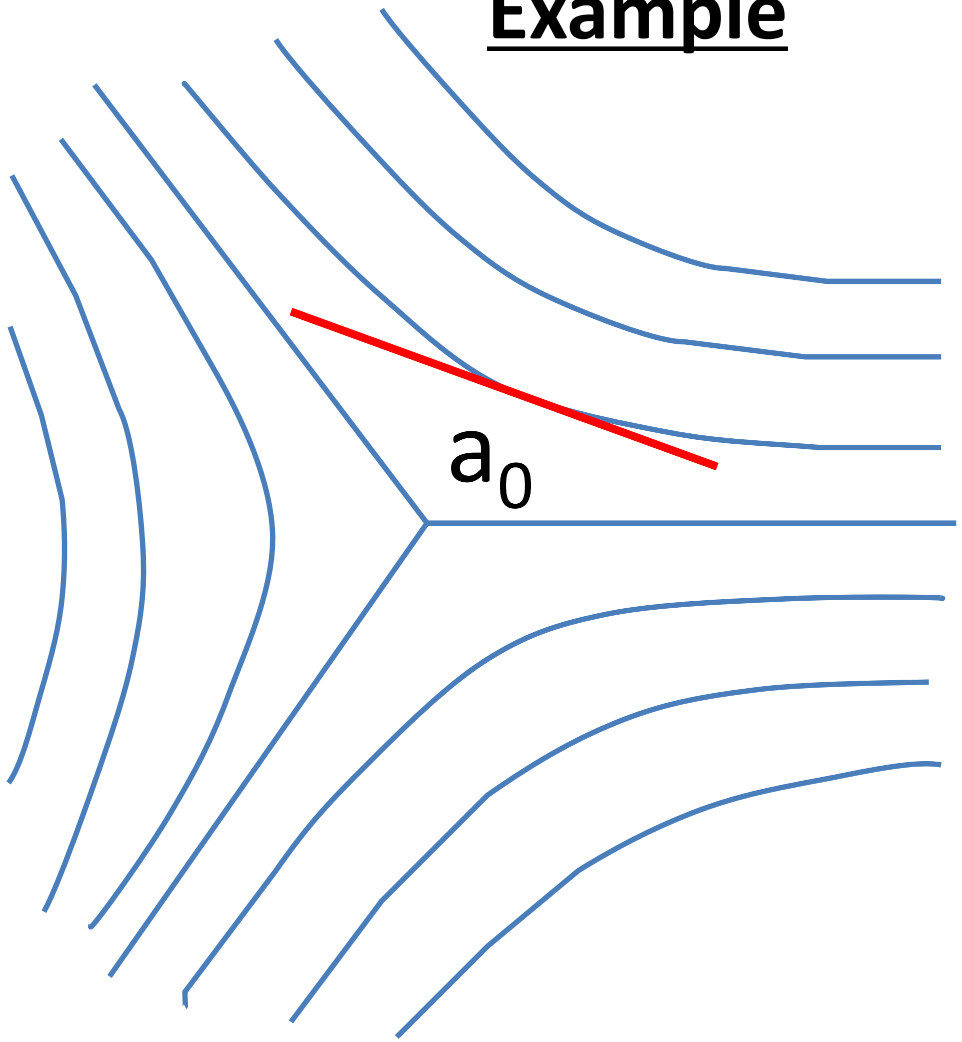


If curvature lines are as in the left figure, then

$i(a_0) =$

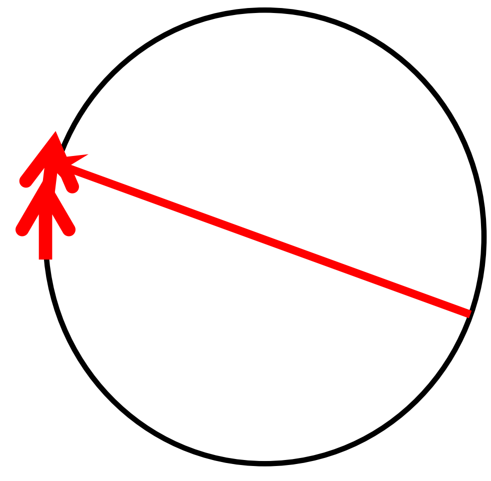


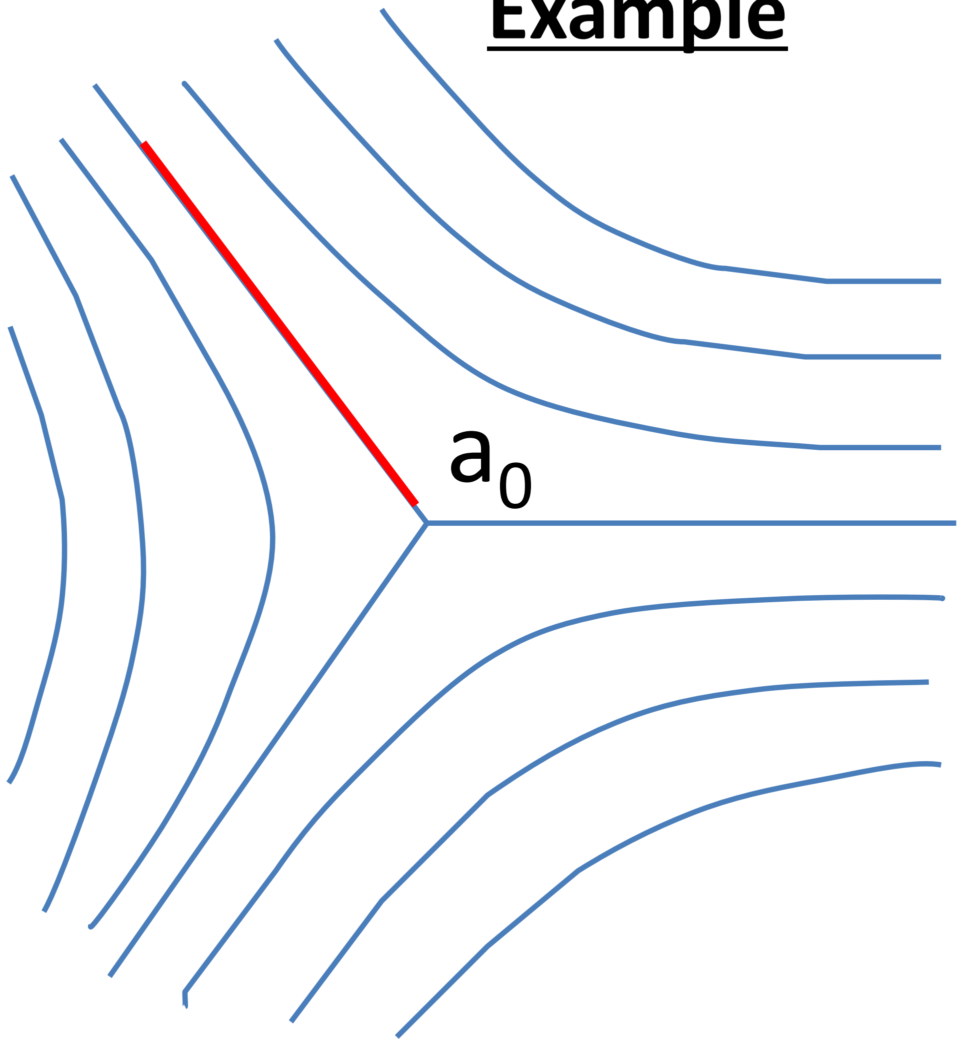
Example



If curvature lines are as in the left figure, then

$i(a_0) =$

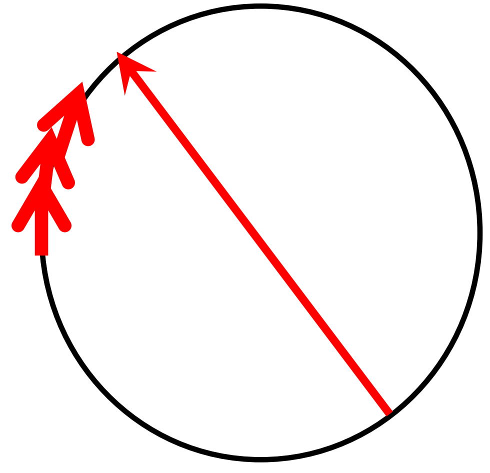




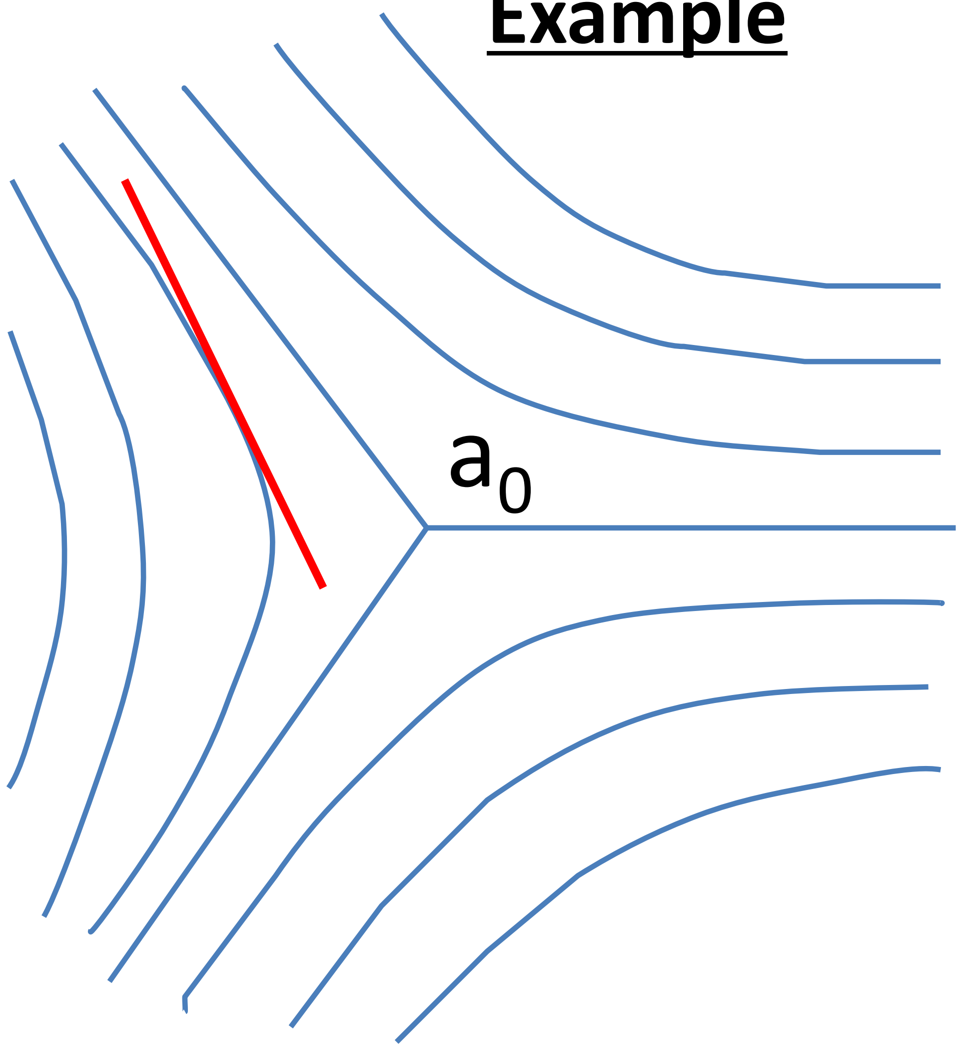
Example

If curvature lines are as in the left figure, then

$i(a_0) =$

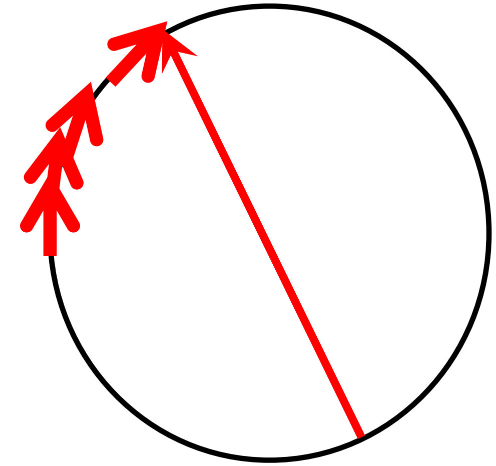


<u>Example</u>

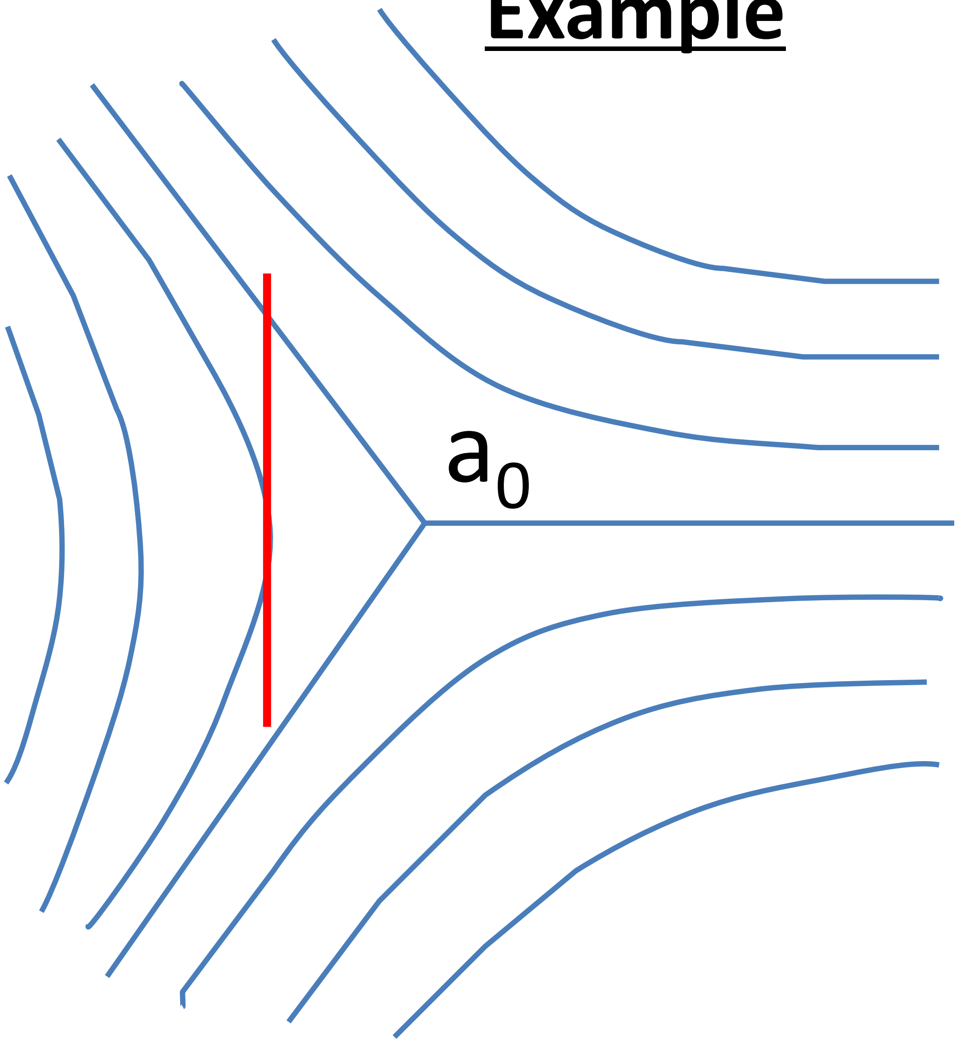


If curvature lines are as in the left figure, then

$i(a_0) =$

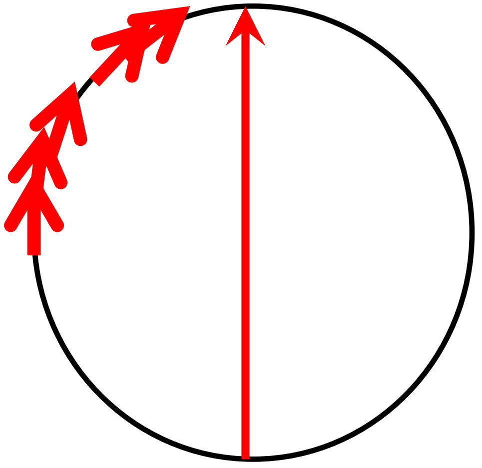


Example

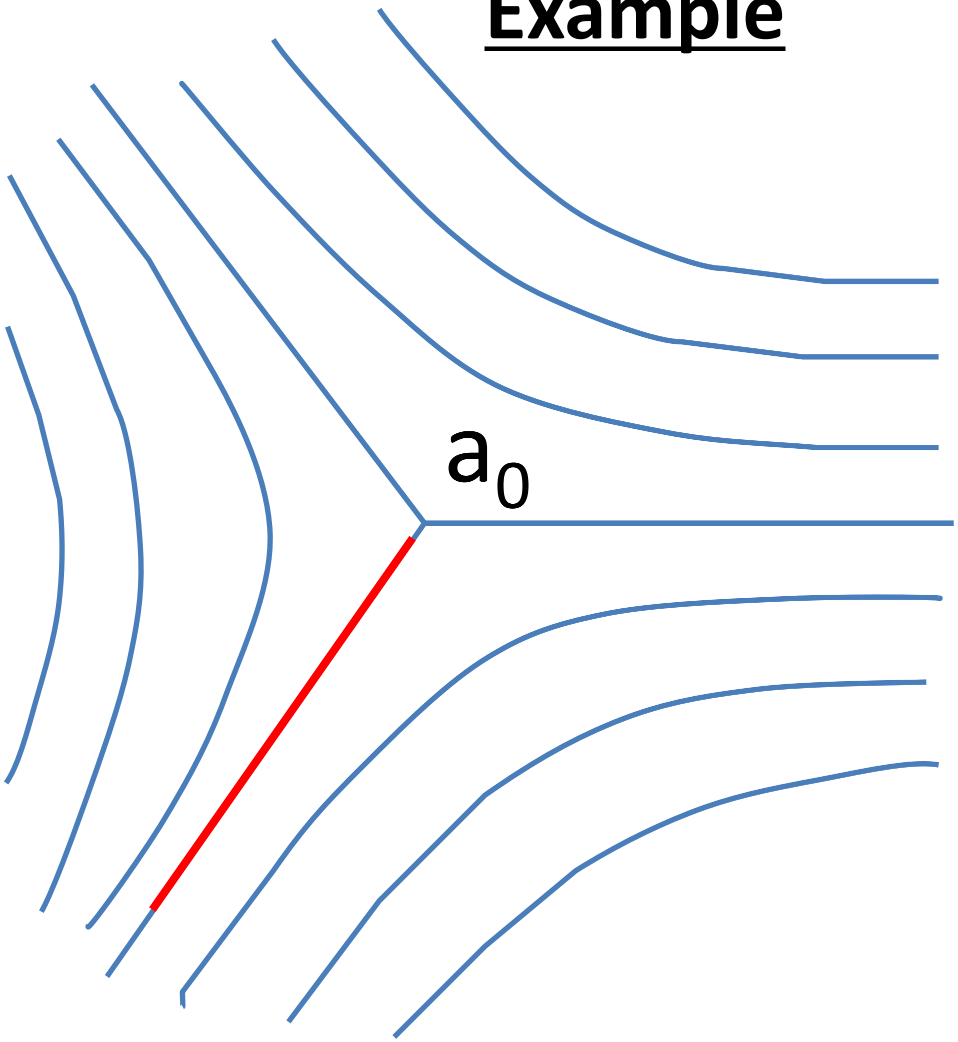


If curvature lines are as in the left figure, then

$i(a_0) =$

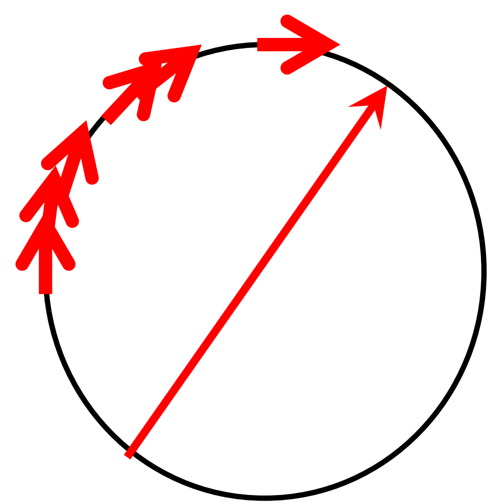


Example

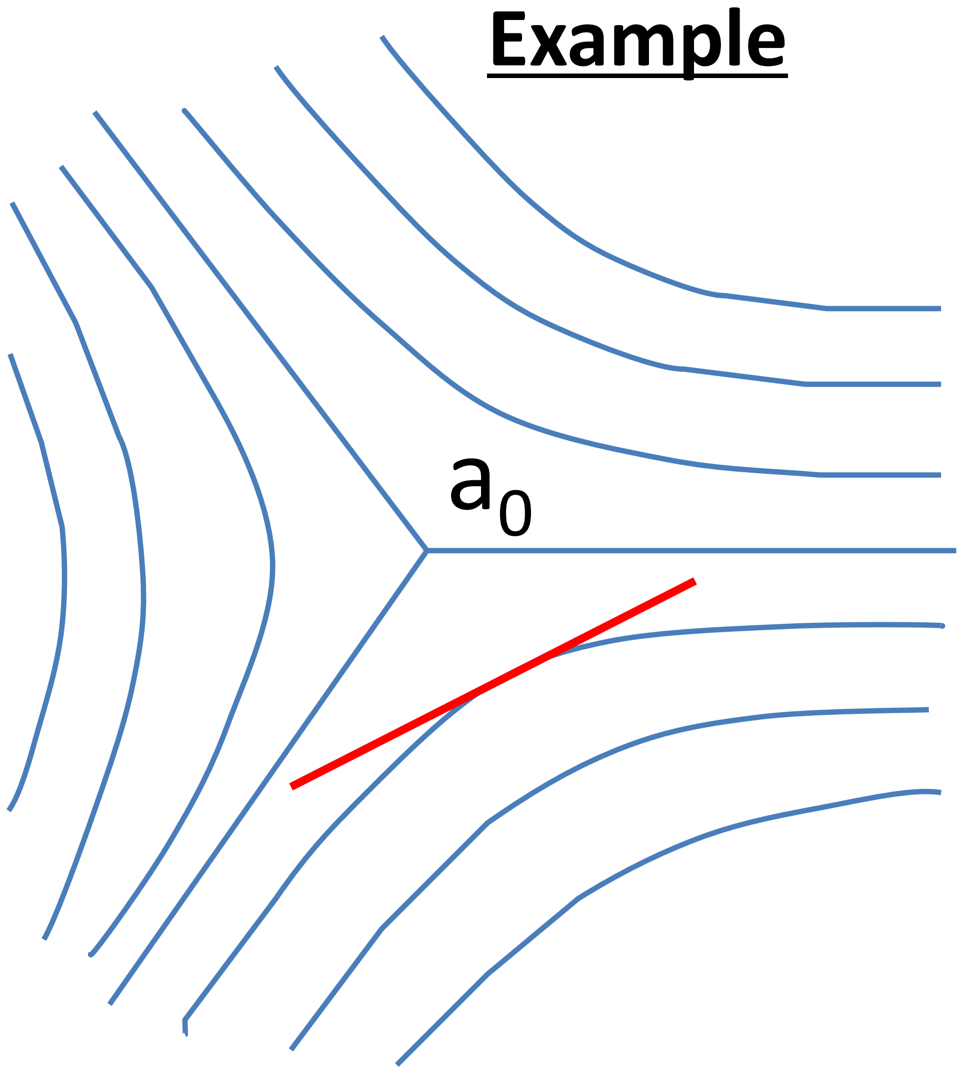


If curvature lines are as in the left figure, then

$i(a_0) =$

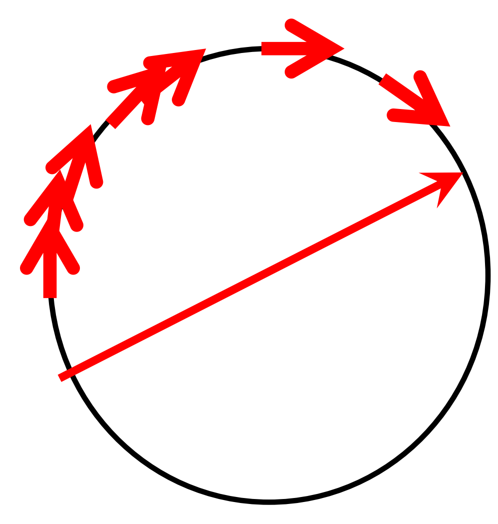


Example

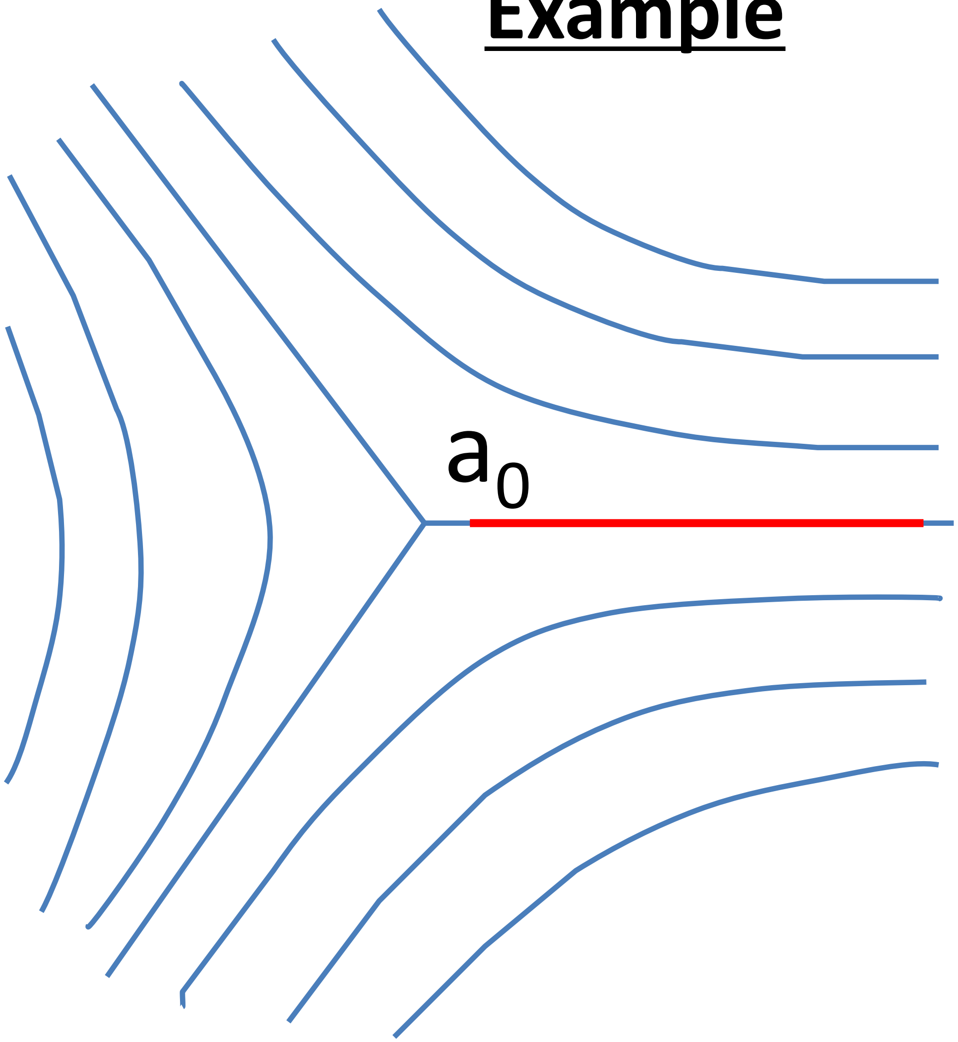


If curvature lines are as in the left figure, then

$i(a_0) =$

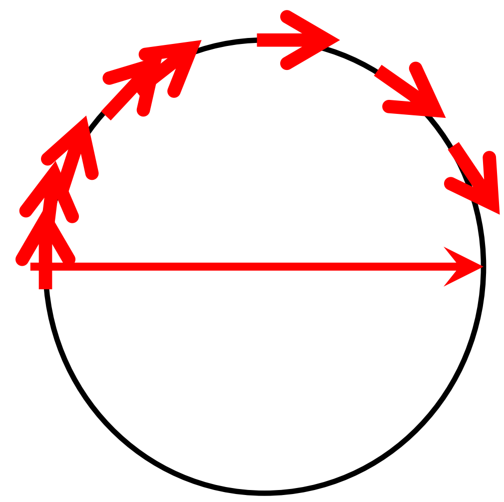


Example

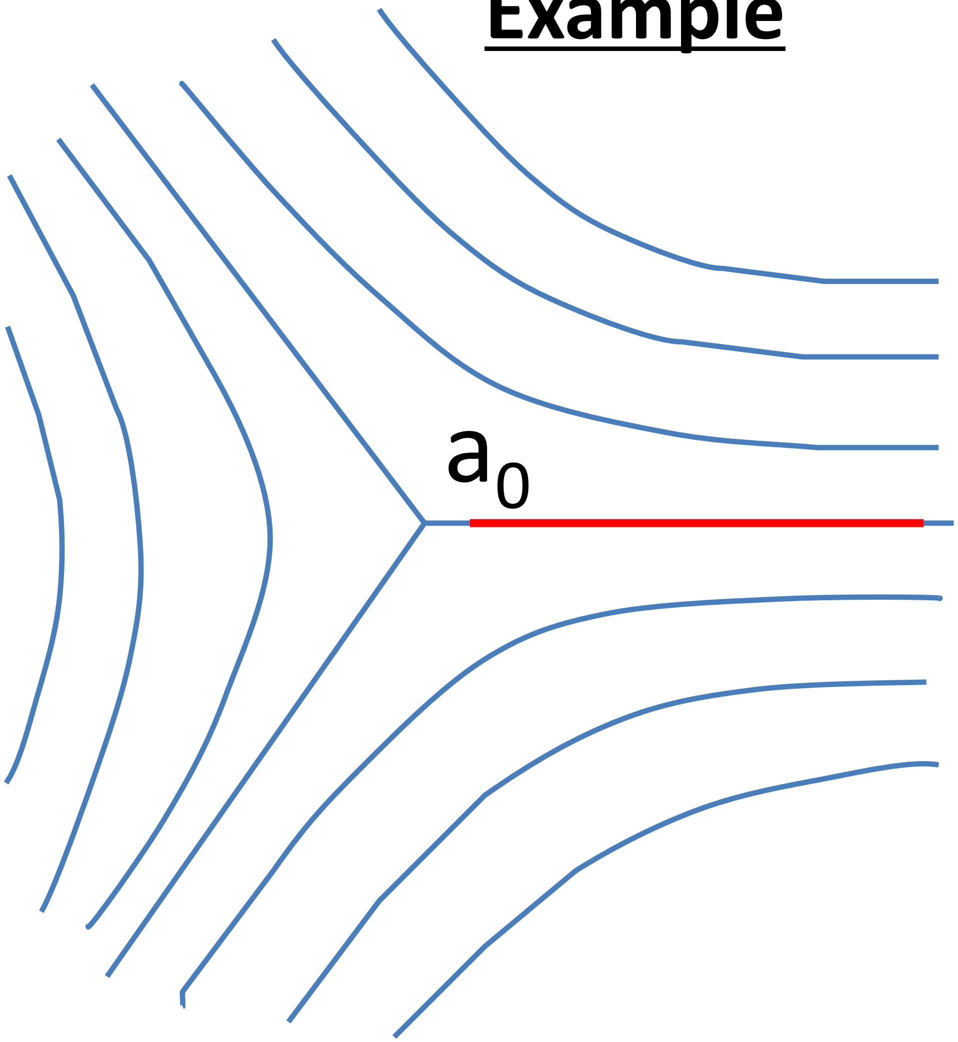


If curvature lines are as in the left figure, then

$i(a_0) =$

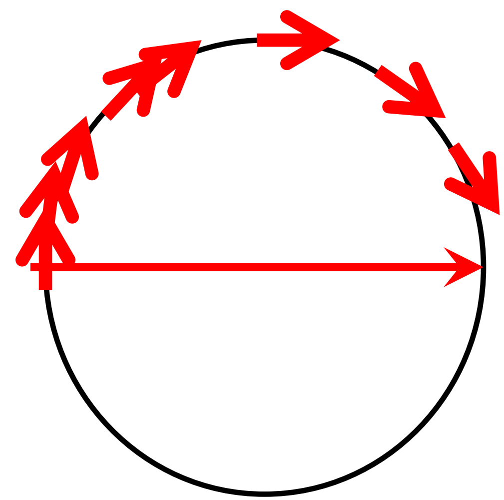


Example



If curvature lines are as in the left figure, then

$$i(a_0) = \boxed{-\frac{1}{2}}.$$



Outline of the proof

Outline of the proof

Q: the Hopf differential on S

Outline of the proof

Q: the Hopf differential on S

⇒ ▪Since the mean curvature is constant, Q is holomorphic;

Outline of the proof

Q: the Hopf differential on S

⇒ ▪ Since the mean curvature is constant, Q is holomorphic;

▪ an umbilical point of S is just a zero point of Q.

Outline of the proof

Q: the Hopf differential on S

$\Rightarrow$ ▪ Since the mean curvature is constant, Q is holomorphic;

▪ an umbilical point of S is just a zero point of Q.

$\therefore$ If S is not totally umbilical, then any umbilical point is isolated.

a_0: an isolated umbilical point of S,

a_0: an isolated umbilical point of S,

(u, v): isothermal coordinates on
a neighborhood U_0 of a_0 s.t.
a_0 corresponds to (0, 0).

a_0: an isolated umbilical point of S,

(u, v): isothermal coordinates on

a neighborhood U_0 of a_0 s.t.

a_0 corresponds to (0, 0),

$w := u + iv$.

If we represent Q as $Q = \Phi dw^2$,

then $\Phi = w^n f(w)$, where $f(0) \neq 0$.

r: a positive number.

r: a positive number.

We set $f(re^{it}) = \rho(t)\exp(i\theta(t))$.

r: a positive number.

We set $f(re^{it}) = \rho(t)\exp(i\theta(t))$.

$\Rightarrow$ We can suppose $\rho \neq 0$ and $\theta(2\pi) = \theta(0)$.

r: a positive number.

We set $f(re^{it}) = \rho(t)\exp(i\theta(t))$.

$\Rightarrow$ We can suppose $\rho \neq 0$ and $\theta(2\pi) = \theta(0)$.

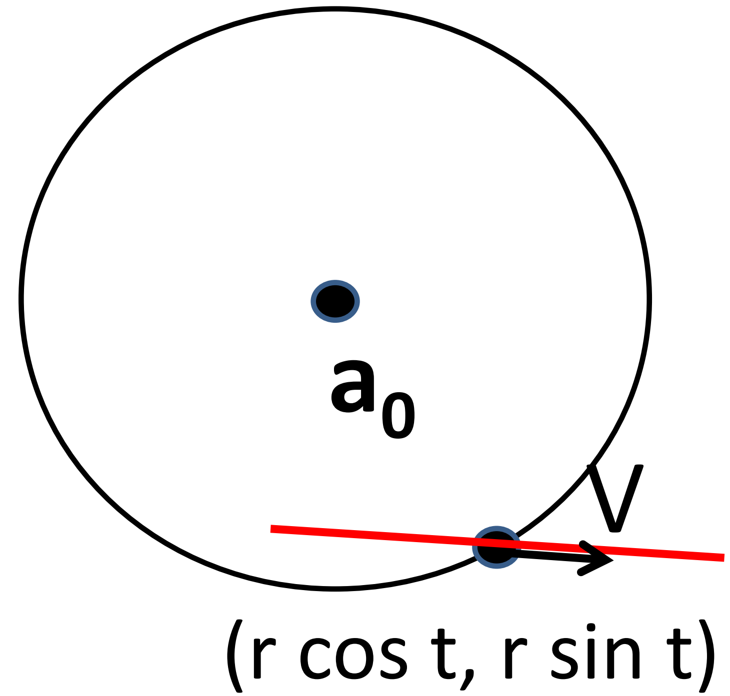
ϕ: a smooth function on **R** s.t.

$$V(t) := \cos\phi(t)\frac{\partial}{\partial u} + \sin\phi(t)\frac{\partial}{\partial v}$$

is contained in a principal direction at $(r\cos t, r\sin t)$ for $\forall t \in \mathbf{R}$.

Since Im(Q(V, V))=0,

Im (Φ exp(2i ϕ)) = 0.

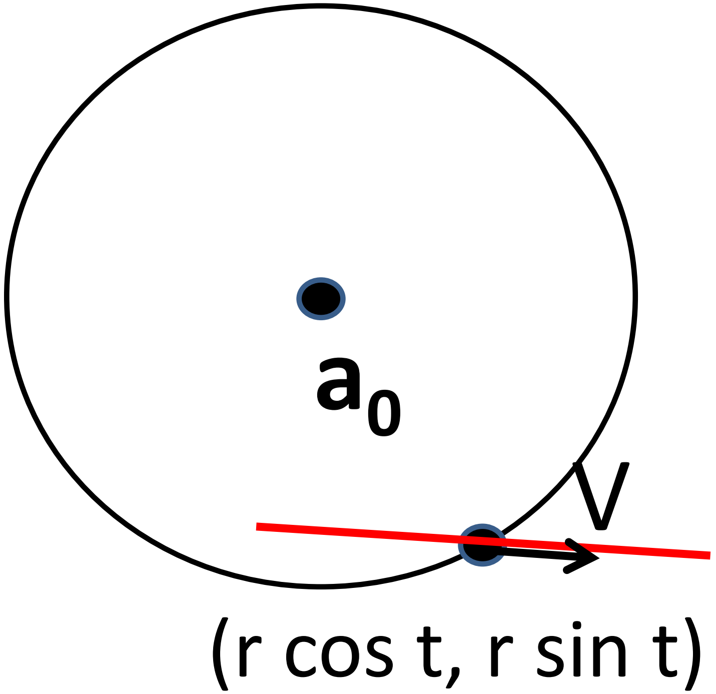


Since $\mathrm{Im}(Q(V, V))=0$,

$\mathrm{Im}\,(\Phi \exp(2i\,\phi)) = 0$.

$\therefore \quad nt + \theta(t) + 2\phi(t)$

$= \exists N\pi \;\; (\forall t \in \mathbf{R})$.



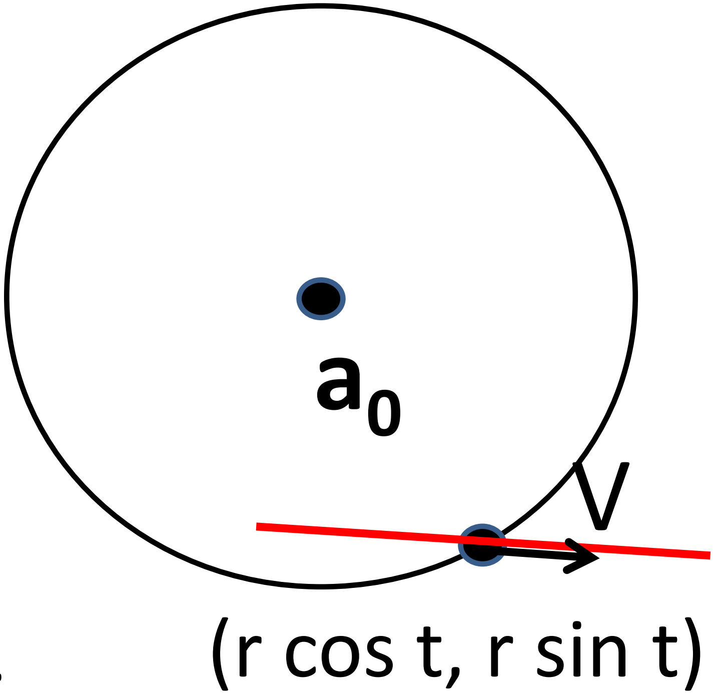
Since Im(Q(V, V))=0,

Im $(\Phi \exp(2i\,\phi)) = 0$.

$\therefore \quad nt + \theta(t) + 2\phi(t)$

$= \exists\, N\pi \;\; (\forall\, t \in \mathbf{R})$.

$\therefore \quad \phi(2\pi) - \phi(0) = -\,n\pi$

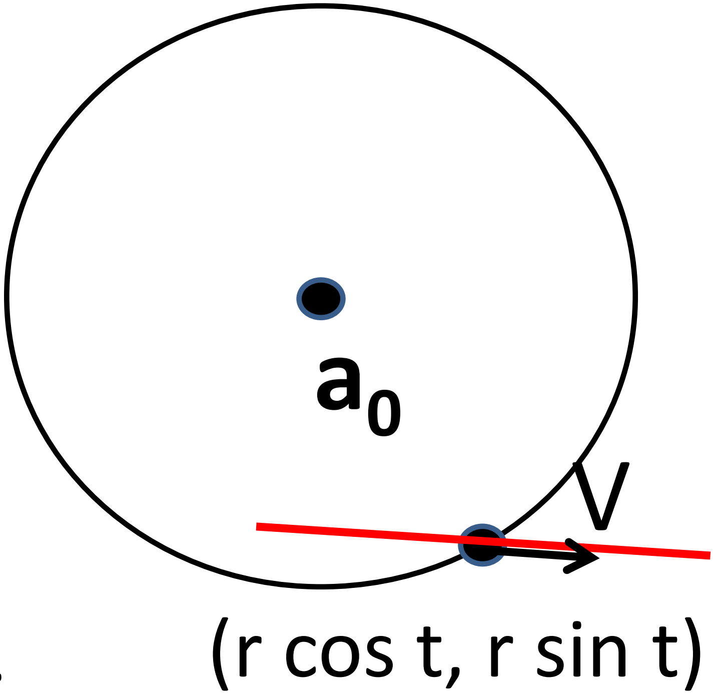


Since $\mathrm{Im}(Q(V, V))=0$,

$\mathrm{Im}\ (\Phi \exp(2i\ \phi)) = 0.$

$\therefore \quad nt + \theta(t) + 2\phi(t)$

$= \exists N \pi \ \ (\forall t \in \mathbf{R}).$



$\therefore \quad \phi(2\pi) - \phi(0) = -n\pi$

$$\therefore \quad i(a_0) = \frac{\phi(2\pi) - \phi(0)}{2\pi} = -\frac{n}{2} < 0$$

Since S is homeomorphic to S^2,
if S is not totally umbilical, then S has at most finite umbilical points.

Since S is homeomorphic to S^2,
if S is not totally umbilical, then S has at most finite umbilical points.
According to Hopf-Poincare's theorem, the sum of all the indices is equal to the Euler number of S (= 2).

Since S is homeomorphic to S^2,
if S is not totally umbilical, then S has at most finite umbilical points.
According to Hopf-Poincare's theorem, the sum of all the indices is equal to the Euler number of S (= 2).

Contradiction!

Generalizations of Hopf’s theorem:

Generalizations of Hopf's theorem:

The same result holds for

<u>Generalizations of Hopf's theorem</u>:

The same result holds for

- special Weingarten surfaces (Hartman-Wintner, Chern);

<u>Generalizations of Hopf's theorem</u>:

The same result holds for

- special Weingarten surfaces (Hartman-Wintner, Chern);
- surfaces with constant anisotropic mean curvature. (Koiso-Palmer, A).

Special Weingarten surfaces

<u>Special Weingarten surfaces</u>

A surface S in E^3 is said to be <u>Weingarten</u>

$\overset{\text{def}}{\Leftrightarrow}$

<u>Special Weingarten surfaces</u>

A surface S in E^3 is said to be <u>Weingarten</u>

$$\overset{\text{def}}{\Leftrightarrow} \ \exists\, W\ (\not\equiv 0)\text{: a smooth function of two variables s.t. } W(k_1, k_2) \equiv 0$$

on S, where k_1 and k_2 are principal curvatures of S.

A Weingarten surface S is said to be

<u>special</u>

$\overset{\text{def}}{\Leftrightarrow}$

A Weingarten surface S is said to be <u>special</u>

$\overset{\text{def}}{\Leftrightarrow}$ We can choose W s.t.

$$\frac{\partial W}{\partial X}(k_1, k_2)\frac{\partial W}{\partial Y}(k_1, k_2) > 0$$

at any umbilical point of S.

A Weingarten surface S is said to be <u>special</u>

$\overset{\text{def}}{\Leftrightarrow}$ We can choose W s.t.

$$\frac{\partial W}{\partial X}(k_1, k_2)\frac{\partial W}{\partial Y}(k_1, k_2) > 0$$

at any umbilical point of S.

<u>**Remark**</u> A surface with constant mean curvature is special Weingarten.

A Weingarten surface S is said to be

<u>special</u> $W(X, Y) := X + Y - 2H_0$

$\overset{\text{def}}{\Leftrightarrow}$ We can choose W s.t.

$$\frac{\partial W}{\partial X}(k_1, k_2)\frac{\partial W}{\partial Y}(k_1, k_2) > 0$$

at any umbilical point of S.

<u>**Remark**</u> A surface with constant mean curvature is special Weingarten.

Hartman-Wintner proved that
if S is special Weingarten and not
totally umbilical,

Hartman-Wintner proved that if S is special Weingarten and not totally umbilical, then any umbilical point a_0 of S is isolated and $i(a_0) < 0$.

f: a function of u, v s.t the graph of f is a neighborhood of a_0 in S.

f: a function of u, v s.t the graph of f is a neighborhood of a_0 in S.

$\Rightarrow$ f is a solution of an elliptic equation of 2^{nd} order:

f: a function of u, v s.t the graph of f is a neighborhood of a_0 in S.

$\Rightarrow$ f is a solution of an elliptic equation of 2^{nd} order:

$$\Psi(u, v, f, p_f, q_f, r_f, s_f, t_f) = 0,$$

where Ψ is a function of eight variables s.t. $\frac{\partial \Psi}{\partial r}\frac{\partial \Psi}{\partial t} - \frac{1}{4}\left(\frac{\partial \Psi}{\partial s}\right)^2 > 0.$

<u>The key point of Hartman-Wintner's result:</u>

The key point of Hartman-Wintner's result:

If f_0 is a solution of the same equation s.t. $f_0(0, 0) = f(0, 0)$ and $f_0 \not\equiv f$,

<u>The key point of Hartman-Wintner's result:</u>

If f_0 is a solution of the same equation s.t. $f_0(0, 0) = f(0, 0)$ and $f_0 \not\equiv f$, then

$$f - f_0 = p_k(u, v) + o((u^2 + v^2)^{k/2}),$$

where p_k is a homogeneous polynomial of degree $k \in \mathbf{N}$.

Chern devised another proof of Hartman-Wintner’s result.

Chern devised another proof of Hartman-Wintner's result.

(u, v): isothermal coordinates on
a neighborhood U_0 of a_0 s.t.
a_0 corresponds to $(0, 0)$,

$w := u + iv$.

If S is special Weingarten and not totally umbilical, If S is special Weingarten and not totally umbilical, then Chern proved

$$\Phi(w, \overline{w}) = cw^n + o(|w|^n),$$

where $c \in \mathbf{C} \setminus \{0\}$ and $n \in \mathbf{N}$.

If S is special Weingarten and not totally umbilical, then Chern proved

$$\Phi(w, \overline{w}) = cw^n + o(|w|^n),$$

where $c \in \mathbf{C} \setminus \{0\}$ and $n \in \mathbf{N}$.

This implies that a_0 is an isolated umbilical point and $i(a_0) = -\frac{n}{2}$.

Surfaces with constant anisotropic mean curvature

Surfaces with constant anisotropic mean curvature

W: a surface in E^3.

Surfaces with constant anisotropic mean curvature

W: a surface in E^3.

Suppose that

- W is homeomorphic to S^2,

Surfaces with constant anisotropic mean curvature

W: a surface in E^3.

Suppose that

- W is homeomorphic to S^2,
- the Gaussian curvature of W is everywhere positive.

S: a surface in E^3,

S: a surface in E^3,

$g: S \longrightarrow W$: a smooth map s.t.

$$T_a(S) \mathbin{/\!/} T_{g(a)}(W) \text{ in } E^3 \ (\forall a \in S).$$

S: a surface in E^3,

g: $S \longrightarrow$ W: a smooth map s.t.

$$T_a(S) \,/\!/\, T_{g(a)}(W) \text{ in } E^3 \ (\forall a \in S).$$

We call g the **anisotropic Gauss map**.

S: a surface in E^3,

g: S $\longrightarrow$ W: a smooth map s.t.

$$T_a(S) \,/\!/\, T_{g(a)}(W) \text{ in } E^3 \ (\forall a \in S).$$

We call g the **anisotropic Gauss map**.

Remark

If W is the unit sphere, then g is

a usual Gauss map.

Since $T_a(S) \,/\!/\, T_{g(a)}(W)$, we can consider the differential dg of g to be a smooth tensor field on S of type (1, 1).

Since $T_a(S) \parallel T_{g(a)}(W)$, we can consider the differential dg of g to be a smooth tensor field on S of type (1, 1).

We call $A := -dg$ the **anisotropic shape operator**

Since $T_a(S) \;/\!/\; T_{g(a)}(W)$, we can consider the differential dg of g to be a smooth tensor field on S of type (1, 1).

We call $A := -dg$ the **anisotropic shape operator** and $\Lambda := \mathrm{tr}\, A$ (the trace of A) the **anisotropic mean curvature**.

Koiso-Palmer proved that

if S is homeomorphic to S^2 and

if Λ is constant, then S is similar to W

in E^3

Koiso-Palmer proved that
if S is homeomorphic to S^2 and
if Λ is constant, then S is similar to W
in E^3:
they showed that if S is not similar to
W, then any anisotropic umbilical point
a_0 of S is isolated and $i(a_0) < 0$.

THANK YOU!