The lifts of surfaces in neutral 4-manifolds into the 2-Grassmann bundles

Naoya Ando (Kumamoto University)

Contents

1. Minimal surfaces in Riemannian 4-manifolds

2. Space-like surfaces with zero mean curvature vector in Lorentzian 4-manifolds and Willmore surfaces in 3-dimensional space forms

3. Space-like surfaces with zero mean curvature vector in neutral 4-manifolds

4. Time-like surfaces with zero mean curvature vector in neutral 4-manifolds
1. Minimal surfaces in Riemannian 4-manifolds

$N$: an oriented Riemannian 4-dimensional manifold with its metric $h$.

$\Rightarrow$ For $a \in N$, the eigenvalues of $*: \bigwedge^2 T_aN \to \bigwedge^2 T_aN$ are $\pm 1$, and the corresponding eigenspaces are of dimension 3.

We have a bundle decomposition

$$\bigwedge^2 TN = \bigwedge^2_+ TN \oplus \bigwedge^2_- TN$$

(notice the double covering $SO(4) \to SO(3) \times SO(3)$).

We see that $\bigwedge^2_{\pm} TN$ are locally generated by

$$\frac{1}{\sqrt{2}}(\theta_{12} \pm \theta_{34}), \quad \frac{1}{\sqrt{2}}(\theta_{13} \pm \theta_{42}), \quad \frac{1}{\sqrt{2}}(\theta_{14} \pm \theta_{23}),$$

where $\theta_{ij} := e_i \wedge e_j$ and $(e_1, e_2, e_3, e_4)$ is a local ordered orthonormal frame field of $TN$ giving the orientation of $N$. 
• If $N$ is hyperKähler, then one of $\bigwedge^{2}_{\pm}TN$ is a product bundle.
• If $N = E^{4}$, then both of $\bigwedge^{2}_{\pm}TN$ are product bundles.

The twistor spaces associated with $N$ are the sphere bundles in $\bigwedge^{2}_{\pm}TN$:

$$U\left(\bigwedge^{2}_{+}TN\right) := \left\{ \Theta \in \bigwedge^{2}_{+}TN \mid \hat{h}(\Theta, \Theta) = 1 \right\},$$

$$U\left(\bigwedge^{2}_{-}TN\right) := \left\{ \Theta \in \bigwedge^{2}_{-}TN \mid \hat{h}(\Theta, \Theta) = 1 \right\}.$$
$M$: a Riemann surface,

$F : M \rightarrow N$: a conformal immersion of a Riemann surface $M$ into $N$.

$\Theta_{F,\pm}$: sections of $U\left(\bigwedge_{\pm}^2 F^*TN\right)$ defined by $\Theta_{F,\pm} := \frac{1}{\sqrt{2}}(\xi_1 \wedge \xi_2 \pm \xi_3 \wedge \xi_4)$, where $\xi_1, \xi_2, \xi_3, \xi_4$ form a local orthonormal frame field of $F^*TN$ s.t.

- $(\xi_1, \xi_2, \xi_3, \xi_4)$ gives the orientation of $N$,
- $\xi_1, \xi_2 \in dF(TM)$ so that $(\xi_1, \xi_2)$ gives the orientation of $M$.

$I_{F,\pm}$: the complex structures of $F^*TN$ corresponding to $\Theta_{F,\pm}$.

Then $\Theta_{F,\pm} = \frac{1}{\sqrt{2}}(e \wedge I_{F,\pm}(e) + e^\perp \wedge I_{F,\pm}(e^\perp))$, where $e$ (respectively, $e^\perp$) is a unit tangent (respectively, normal) vector of $F$.

If $N$ is hyperKähler so that $\bigwedge_{\pm}^2 TN$ (respectively, $\bigwedge_{-}^2 TN$) is a product bundle, then we can consider $\Theta_{F,+}$ (respectively, $\Theta_{F,-}$) to be a map from $M$ into $\mathbb{C}P^1$. 
Theorem (A, 2020)

Suppose that $N$ is hyperKähler and that $F : M \rightarrow N$ is minimal. Then one of $\Theta_{F,+}, \Theta_{F,-}$ is a holomorphic map from $M$ into $\mathbb{C}P^1$.

In particular, we have the following corollary, which is a well-known theorem (see pp. 16–22 in D. A. Hoffman and R. Osserman, *The geometry of the generalized Gauss map*, Memoirs of AMS 236, 1980).

Corollary

$F : M \rightarrow E^4 :$ a conformal and minimal immersion of $M$ into $E^4$. Then the Gauss map $G_F : M \rightarrow \mathbb{C}P^1 \times \mathbb{C}P^1$ of $F$ is holomorphic.
Proof of the theorem
Suppose that $\Lambda^2_+ TN$ is a product bundle.
Then we can suppose that

$$\Theta_{+,1} := \frac{1}{\sqrt{2}}(\theta_{12} + \theta_{34}), \quad \Theta_{+,2} := \frac{1}{\sqrt{2}}(\theta_{13} + \theta_{42}), \quad \Theta_{+,3} := \frac{1}{\sqrt{2}}(\theta_{14} + \theta_{23})$$

are horizontal. These sections form an orthonormal frame field of $\Lambda^2_+ TN$.

$g_{F,+}$: a $\mathbb{CP}^1$-valued function satisfying

$$
\Theta_{F,+} = \frac{1 - |g_{F,+}|^2}{1 + |g_{F,+}|^2} \Theta_{+,1} + \frac{2\text{Re} \ g_{F,+}}{1 + |g_{F,+}|^2} \Theta_{+,2} + \frac{2\text{Im} \ g_{F,+}}{1 + |g_{F,+}|^2} \Theta_{+,3}.
$$

$w$: a local complex coordinate of $M$.
If we set $dF\left(\frac{\partial}{\partial w}\right) = \sum_{i=1}^{4} \psi^i e_i$, then we obtain $g_{F,+} = \sqrt{-1} \frac{\psi^1 + \sqrt{-1} \psi^2}{\psi^3 - \sqrt{-1} \psi^4}$. 

Suppose that $F : M \longrightarrow N$ is minimal. Then $\nabla_{\partial / \partial w} dF \left( \frac{\partial}{\partial w} \right) = 0$.

We set $\nabla e_i = \sum_{j=1}^{4} \omega^j_i e_j \ (i = 1, 2, 3, 4)$.

$\implies$ • $\omega^i_j = -\omega^j_i$,

• $\omega^3_2 = -\omega^4_1, \ \omega^4_2 = \omega^1_3, \ \omega^4_3 = -\omega^2_1$,

• $\frac{\partial \psi^i}{\partial w} + \sum_{j \neq i} \psi^j \omega^j_i \left( \frac{\partial}{\partial w} \right) = 0 \ (i = 1, 2, 3, 4)$.

Using these, we can obtain $\frac{\partial g_{F,+}}{\partial w} = 0$. \qed
\( F : M \rightarrow N \): a conformal and minimal immersion of \( M \) into \( N \),
\[ \Psi := dF(\partial/\partial w). \]

\[ \implies \Psi dw \text{ gives a section of } F^*TN \otimes \mathbb{C} \otimes T^*M \text{ on } M. \]

\( \nabla \): the connection of \( F^*TN \otimes \mathbb{C} \otimes T^*M \) given by the Levi-Civita connection \( \nabla \) of \( h \).

\[ \implies \nabla_{\partial/\partial w}(\Psi dw) = \sigma \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right) dw \quad (\sigma: \text{the 2nd fundamental form of } F). \]

We see that

\[ Q := h \left( \sigma \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right), \sigma \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right) \right) dw \otimes dw \otimes dw \otimes dw \]

does not depend on the choice of a local complex coordinate \( w \) and we can define a complex quartic differential \( Q \) on \( M \).

If \( N \) is a 4-dimensional Riemannian space form, then we see by the equations of Codazzi that \( Q \) is holomorphic.
Theorem  The following are mutually equivalent:

(a) at each point of $M$, principal curvatures do not depend on the choice of a unit normal vector of $F$;

(b) $h(\sigma(T_1, T_1), \sigma(T_1, T_1)) = h(\sigma(T_1, T_2), \sigma(T_1, T_2))$, $h(\sigma(T_1, T_1), \sigma(T_1, T_2)) = 0$ for $T_1 := dF(\partial/\partial u), T_2 := dF(\partial/\partial v)$;

(c) $Q \equiv 0$;

(d) one of $\Theta_F,+, \Theta_F,-$ is horizontal w.r.t. the connection $\hat{\nabla}$ of $\Lambda^2 F^*TN$ induced by $\nabla$;

(e) one of $I_F,\pm$ is parallel w.r.t. $\nabla$;

(f) we have one of $I_F,\pm \sigma(T_1, T_1) = \sigma(T_1, T_2)$. 
We say that a minimal immersion $F$ is \emph{isotropic} if one of $(a) \sim (f)$ in the above theorem holds.

We easily see

- $(a)$, $(b)$, $(c)$ and $(f)$ are mutually equivalent,
- $(d)$ and $(e)$ are equivalent.

In addition, $(a)$ and $(d)$ are equivalent (Friedrich).
Suppose $N = S^4$.
Bryant showed that an isotropic minimal surface (superminimal surface) is
given by the composition of

- the twistor map
  \[ \mathbb{CP}^3 \longrightarrow S^4 (= \mathbb{HP}^1), \quad a \mathbb{C} \mapsto a \mathbb{H} \quad (a \in \mathbb{C}^4 \setminus \{0\} = \mathbb{H}^2 \setminus \{0\}) \]
  associated with $S^4$,
- a holomorphic immersion $\hat{F} : M \longrightarrow \mathbb{CP}^3$ which is horizontal in
  the twistor space $\mathbb{CP}^3 (= Sp(2)/U(2) \cong SO(5)/U(2))$. 
Suppose $N = E^4$.
Then a conformal immersion $F : M \to E^4$ is an isotropic minimal immersion if and only if
the composition of $F$ with an isometry of $E^4$ is a holomorphic immersion into $C^2 = E^4$.

Suppose that $N$ is hyperKähler.
Then a conformal immersion $F : M \to N$ is an isotropic minimal immersion compatible with the orientation of $N$
if and only if
$F$ is a complex curve w.r.t. a complex structure given by
the hyperKähler structure of $N$. 
Suppose that $N$ is a Kähler surface.
Then a conformal immersion $F : M \rightarrow N$ is an isotropic minimal immersion which is compatible with the orientation of $N$ and equipped with at least one complex point if and only if $F$ is a complex curve w.r.t. the complex structure given by the Kähler structure of $N$. 
$R$: the curvature tensor of $\nabla$:
\[ R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \]
\[ \hat{R}: \text{the curvature tensor of } \hat{\nabla}. \]

\[ \Rightarrow \hat{R}(X_1, X_2)(Y_1 \wedge Y_2) = (R(X_1, X_2)Y_1) \wedge Y_2 + Y_1 \wedge R(X_1, X_2)Y_2. \]

$(e_1, e_2)$: a local ordered orthonormal frame field of $TM$ giving the orientation of $M$.

- If one of $\Theta_{F, \pm}$ is horizontal, then $\hat{R}(e_1, e_2)\Theta_{F, +} = 0$ or $\hat{R}(e_1, e_2)\Theta_{F, -} = 0$.
- If $\Theta_{F, \pm}$ are horizontal, then $\hat{R}(e_1, e_2)\Theta_{F, \pm} = 0$ and $F$ is totally geodesic.
Theorem (A, 2020)

$F : M \rightarrow N$: a conformal and minimal immersion s.t. $\hat{R}(e_1, e_2)\Theta_{F, \pm} = 0$. Then $Q$ is holomorphic.

In addition, if $\hat{\nabla}\Theta_{F, \pm} \neq 0$, then we can choose $(e_1, e_2, e_3, e_4)$ satisfying

(a) the connection forms $\omega$, $\omega^\perp$ given by $\omega := h(\nabla e_1, e_2)$, $\omega^\perp := h(\nabla e_3, e_4)$ satisfy $d \ast \omega = 0$ and $d \ast \omega^\perp = 0$;

(b) the 2nd fundamental form of $F$ is constructed by a solution of an over-determined system s.t. the compatibility condition is given by $d \ast \omega = 0$ and $d \ast \omega^\perp = 0$.

Remark  If $N$ is a space form, then $\hat{R}(e_1, e_2)\Theta_{F, \pm} = 0$.

Remark  The condition $d \ast \omega = 0$ means that on a neighborhood of each point of $M$, there exists a local complex coordinate $w = u + \sqrt{-1}v$ satisfying $e_1 = e^{-\lambda}dF(\partial/\partial u)$, $e_2 = e^{-\lambda}dF(\partial/\partial v)$ for a function $\lambda$. 
Proof of the theorem

Since $F$ is minimal, we have $\nabla \partial / \partial w \Psi = 0$.

Since $\hat{R}(e_1, e_2) \Theta F, \pm = 0$, we have $\hat{R} \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial \overline{w}} \right) \left( \frac{\partial}{\partial w} \wedge \frac{\partial}{\partial \overline{w}} \right) = 0$.

Therefore we obtain $\nabla \perp \partial / \partial w \sigma \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right) = 0$ and this means that $Q$ is holomorphic.

Suppose $\hat{\nabla} \Theta F, \pm \neq 0$.

Then principal curvatures of $F$ at each point depend on the choice of a unit normal vector.

$e_3$: a locally defined unit normal vector field which gives the maximum of the absolute values of principal curvatures of $F$ at each point.

Then the maximum is positive and therefore we can suppose that $e_1, e_2$ give principal directions of $F$ w.r.t. $e_3$. 
\( e_4 \): a unit normal vector field perpendicular to \( e_3 \).

\[
\sigma_{ij}^k := h(\sigma(e_i, e_j), e_k) \quad (i, j = 1, 2, k = 3, 4).
\]

\[
\implies \sigma_{11}^k + \sigma_{22}^k = 0 \quad (k = 3, 4), \quad \sigma_{12}^3 = 0, \quad \sigma_{11}^4 = 0.
\]

\[
f_{\pm} := \sigma_{11}^3 \pm \sigma_{12}^4 \implies f_{\pm} \neq 0.
\]

\[
p^j := 2\omega(e_j), \quad q^j := (-1)^{3-j}\omega^\perp(e_3-j) \quad (j = 1, 2).
\]

Then \( \hat{R}(e_1, e_2)\Theta_F,\pm = 0 \) mean

\[
e_1(\log |f_\pm|) = -p^2 \pm q^1, \quad e_2(\log |f_\pm|) = p^1 \pm q^2.
\]

Since \( \nabla \) is torsion-free, we obtain \( 2[e_1, e_2] + p^1e_1 + p^2e_2 = 0 \).

Therefore we obtain

- \( e_1(p^1) + e_2(p^2) = 0, \) i.e., \( d \ast \omega = 0 \),
- \( e_2(q^1) - e_1(q^2) = \frac{1}{2}(p^1q^1 + p^2q^2), \) i.e., \( d \ast \omega^\perp = 0 \).
2. Space-like surfaces with zero mean curvature vector in Lorentzian 4-manifolds and Willmore surfaces in 3-dimensional space forms

$N$: an oriented Lorentzian 4-dimensional manifold with its metric $h$,

$F : M \rightarrow N$: a space-like and conformal immersion of $M$ into $N$ with zero mean curvature vector.

$$\nabla_{\partial/\partial w}(\Psi dw) = \sigma \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right) dw \quad \left( \Psi := dF \left( \frac{\partial}{\partial w} \right) \right).$$

We can define a complex quartic differential $Q$ on $M$ by

$$Q := h \left( \sigma \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right), \sigma \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right) \right) dw \otimes dw \otimes dw \otimes dw.$$
We see that $Q \equiv 0$ if and only if the 2nd fundamental form is light-like or zero, that is, the shape operator of a light-like normal vector field vanishes.

If $N$ is a 4-dimensional Lorentzian space form, then we see by the equations of Codazzi that $Q$ is holomorphic, and $Q \equiv 0$ means that a light-like normal vector field is contained in a constant direction.

**Remark** $L_0$: the constant sectional curvature of $N$.

- $L_0 = 0 \implies N = E^4_1$.
- $L_0 > 0 \implies N = S^4_1(L_0) = \left\{ x \in E^5_1 \mid \langle x, x \rangle_{4,1} = \frac{1}{L_0} \right\}$.
- $L_0 < 0 \implies N = H^4_1(L_0) = \left\{ x \in E^5_2 \mid \langle x, x \rangle_{3,2} = \frac{1}{L_0} \right\}$. 
\(\iota : M \rightarrow S^3 = \{x \in E^5_1 \mid \langle x, x \rangle_{4,1} = 0, \ x^5 = 1\}\): a conformal immersion,

\(e_3\): a unit normal vector field of \(\iota\) in \(S^3\),

\(H\): the mean curvature of \(\iota\) w.r.t. \(e_3\).

\[\implies \gamma_\iota := e_3 + H\iota\] is a map from \(M\) into the de Sitter 4-space \(S^4_1 = \{x \in E^5_1 \mid \langle x, x \rangle_{4,1} = 1\}\).

\(\text{Reg}(\iota)\): the set of non-umbilical points of \(\iota\).

\[\implies \gamma_\iota|_{\text{Reg}(\iota)}\] is a space-like immersion s.t. the induced metric \(g\) is given by

\[g = \varepsilon^2 g^M, \text{ where } \varepsilon := \sqrt{H^2 - K^M + 1}, \text{ and } K^M \text{ is the curvature of the induced metric } g^M \text{ by } \iota.\]

We call \(\gamma_\iota : M \rightarrow S^4_1\) the conformal Gauss map of \(\iota\).

We see that \(\iota\) is a light-like normal vector field of \(\gamma_\iota|_{\text{Reg}(\iota)}\) and that the trace of the shape operator of \(\gamma_\iota|_{\text{Reg}(\iota)}\) w.r.t. \(\iota\) vanishes.
\( \nu \): a light-like normal vector field of \( \gamma_\nu|_{\text{Reg}(\nu)} \) s.t. \( \langle \nu, \iota \rangle_{4,1} = -1 \).

\[ \implies \text{The trace of the shape operator of } \gamma_\nu|_{\text{Reg}(\nu)} \text{ w.r.t. } \nu \text{ is given by} \]
\[ - (\Delta H + 2H) \quad (\Delta: \text{the Laplacian on } \text{Reg}(\nu) \text{ w.r.t. } g). \]

Since \( \Delta H + 2H = \frac{1}{\varepsilon^2}(\Delta^M H + 2\varepsilon^2 H) \), we obtain

**Theorem (Bryant)** An immersion \( \iota \) is Willmore if and only if the mean curvature vector of \( \gamma_\iota|_{\text{Reg}(\iota)} \) vanishes.
\( \iota : M \rightarrow S^3 \): a conformal immersion,
\[ \Xi := 2\sigma^M \otimes \text{Hess}^M_H + (H^2 + 1)\sigma^M \otimes \sigma^M - 2dH \otimes \nabla^M \sigma^M, \] where
- \( \sigma^M \): the 2nd fundamental form of \( \iota \),
- \( H \): the mean curvature of \( \iota \),
- \( \text{Hess}^M_H \): the Hessian of \( H \) w.r.t. the Levi-Civita connection \( \nabla^M \) of \( g^M \).

We consider \( \Xi \) to be a complex 4-linear function on the complexification of the tangent space of \( M \) at each point.

**Proposition (Bryant)**

*If \( \iota \) is Willmore, then a complex quartic differential
\[ \tilde{Q} := \Xi \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w'}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right) dw \otimes dw \otimes dw \otimes dw \]

is holomorphic.*
Theorem (A) $M$: a Riemann surface, 
$\iota: M \rightarrow S^3$: a conformal and Willmore immersion. 
Then the holomorphic quartic differential $Q$ for a conformal immersion $F := \gamma|_{\text{Reg}(\iota)}$ coincides with $\tilde{Q}$ on $\text{Reg}(\iota)$ up to a nonzero constant.

Remark We can have analogous discussions for $\iota: M \rightarrow H^3 = \{x \in E^5_1 \mid \langle x, x \rangle_{4,1} = 0, \ x^1 = 1, \ x^5 > 0\}$ or $E^3 = \{x \in E^5_1 \mid \langle x, x \rangle_{4,1} = 0, \ x^5 = x^1 + 1\}$. 
\[ \iota : M \longrightarrow S^3_1 = \{ x \in E^5_2 \mid \langle x, x \rangle_{3,2} = 0, \ x^5 = 1 \} : \]
a space-like and conformal immersion,

\[ e_3 : \] a normal vector field of \( \iota \) in \( S^3_1 \) s.t. \( \langle e_3, e_3 \rangle_{3,2} = -1 \),

\( H \): the mean curvature of \( \iota \) w.r.t. \( e_3 \).

\[ \Rightarrow \ \bullet \ \gamma_\iota := -e_3 + H \iota \] is a map from \( M \) into the anti-de Sitter 4-space \( H^4_1 = \{ x \in E^5_2 \mid \langle x, x \rangle_{3,2} = -1 \} \),

\[ \bullet \ |_{\text{Reg} (\iota)} \gamma_\iota \] is a space-like immersion s.t. \( g = \varepsilon^2 g^M \)

\[ \left( \varepsilon := \sqrt{H^2 + K^M - \delta} \right). \]

We call \( \gamma_\iota : M \longrightarrow H^4_1 \) the conformal Gauss map of \( \iota \).

We can show that an immersion \( \iota \) is Willmore if and only if the mean curvature vector of \( \gamma_\iota |_{\text{Reg} (\iota)} \) vanishes.
$$\Xi := 2\sigma^M \otimes \text{Hess}^M_H - (H^2 - \delta)\sigma^M \otimes \sigma^M - 2dH \otimes \nabla^M \sigma^M.$$  

**Proposition (A)** If $\iota$ is Willmore, then $\tilde{Q}$ is holomorphic.

**Theorem (A)** $M$: a Riemann surface,  
$\iota: M \rightarrow S^3_1$: a conformal and Willmore immersion.  
Then the holomorphic quartic differential $Q$ for a conformal immersion $F := \gamma|_{\text{Reg}(\iota)}$ coincides with $\tilde{Q}$ on $\text{Reg}(\iota)$ up to a nonzero constant.

**Remark** We can have analogous discussions for $\iota: M \rightarrow H^3_1 = \{x \in E^5_2 \mid \langle x, x \rangle_{3,2} = 0, \ x^1 = 1\}$  
or $E^3_1 = \{x \in E^5_2 \mid \langle x, x \rangle_{3,2} = 0, \ x^5 = x^1 + 1\}$.
\[ \iota : M \rightarrow L^+ := \{ x \in E^4_1 \mid \langle x, x \rangle_{3,1} = 0, \ x^4 > 0 \} : \]
a space-like and conformal immersion,
\[ \xi : \text{a light-like normal vector field of } \iota \text{ in } E^4_1 \text{ s.t. } \langle \xi, \iota \rangle_{3,1} = -1, \]
\[ H : \text{the mean curvature of } \iota \text{ w.r.t. a normal vector field } \iota. \]
\[ \implies \quad \gamma_\iota := -\xi + H \iota \text{ is a map from } M \text{ into } E^4_1, \]
\[ \gamma_\iota|_{\text{Reg}(\iota)} \text{ is a space-like immersion s.t. } g = \varepsilon^2 g^M \quad (\varepsilon := \sqrt{H^2 - K}). \]

We call \( \gamma_\iota : M \rightarrow E^4_1 \) the conformal Gauss map of \( \iota \).

**Remark** We see that \( H \) is determined by the induced metric \( g^M \).
**Theorem (A)** An immersion \( \iota \) satisfies \( \Delta^M H - 2\varepsilon^2 = 0 \) if and only if the mean curvature vector of \( \gamma_\iota|_{\text{Reg}(\iota)} \) vanishes.

**Remark** The Euler-Lagrange equation for Willmore surfaces in \( L^+ \) is given by \( \Delta^M H + 2H^2 = 0 \).

\[
\Xi := \sigma^M \otimes \text{Hess}_H^M - H\sigma^M \otimes \sigma^M - dH \otimes \nabla^M \sigma^M,
\]
where \( \sigma^M \) is the 2nd fundamental form of \( \iota \) w.r.t. a normal vector field \( \iota \).

**Proposition (A)** If \( \iota \) satisfies \( \Delta^M H - 2\varepsilon^2 = 0 \), then \( \tilde{Q} \) is holomorphic.

**Theorem (A)** \( M \): a Riemann surface,
\( \iota : M \longrightarrow L^+ \subset E^4_1 \): a conformal immersion s.t. \( \Delta^M H - 2\varepsilon^2 = 0 \).

Then the holomorphic quartic differential \( Q \) for a conformal immersion \( F := \gamma_\iota|_{\text{Reg}(\iota)} \) coincides with \( \tilde{Q} \) on \( \text{Reg}(\iota) \) up to a nonzero constant.
3. Space-like surfaces with zero mean curvature vector in neutral 4-manifolds

\((N, h)\): an oriented neutral 4-dimensional manifold.

\[\Rightarrow\] The metric \(h\) induces an indefinite metric \(\hat{h}\) of \(\bigwedge^2 TN\) defined by

\[
\hat{h}(x_i \wedge x_j, x_k \wedge x_l) = h(x_i, x_k)h(x_j, x_l) - h(x_i, x_l)h(x_j, x_k).
\]

\((e_1, e_2, e_3, e_4)\): a local ordered pseudo-orthonormal frame field of \(TN\) giving the orientation of \(N\).

\[
\Theta_{\pm,1} := \frac{1}{\sqrt{2}}(\theta_{12} \pm \theta_{34}), \quad \Theta_{\pm,2} := \frac{1}{\sqrt{2}}(\theta_{13} \pm \theta_{42}), \quad \Theta_{\pm,3} := \frac{1}{\sqrt{2}}(\theta_{14} \pm \theta_{23}).
\]

\[\Rightarrow\] \(\Theta_{\pm,1}, \Theta_{\pm,2}, \Theta_{\pm,3}\) are mutually orthogonal and satisfy

\[
\hat{h}(\Theta_{\pm,1}, \Theta_{\pm,1}) = 1, \quad \hat{h}(\Theta_{\pm,2}, \Theta_{\pm,2}) = \hat{h}(\Theta_{\pm,3}, \Theta_{\pm,3}) = -1.
\]

Therefore \(\hat{h}\) has signature \((2, 4)\).
$\Lambda^2_+ TN, \Lambda^2_- TN$: $SO(2, 2)$-invariant subbundles of $\Lambda^2 TN$ with rank 3 s.t. all the elements of $\Lambda^2_+ TN$ are $SU(1, 1)$-invariant (notice the double covering $SO_0(2, 2) \longrightarrow SO_0(1, 2) \times SO_0(1, 2)$).

$\implies$ Each fiber of $\Lambda^2_+ TN$ (resp. $\Lambda^2_- TN$) is spanned by $\Theta_{-,1}, \Theta_{+,2}, \Theta_{+,3}$ (resp. $\Theta_{+,1}, \Theta_{-,2}, \Theta_{-,3}$).

In particular, we see

- $\Lambda^2 TN = \Lambda^2_+ TN \oplus \Lambda^2_- TN$,
- $\Lambda^2_+ TN \perp \Lambda^2_- TN$ w.r.t. $\hat{h}$,
- The restriction of $\hat{h}$ on each of $\Lambda^2_+ TN, \Lambda^2_- TN$ has signature $(1, 2)$. 
• If $N$ is neutral hyperKähler, then one of $\wedge^2_{\pm}TN$ is a product bundle.

• If $N = E_2^4$, then both of $\wedge^2_{\pm}TN$ are product bundles.

The space-like twistor spaces associated with $N$ are fiber bundles in $\wedge^2_{\pm}TN$ given by

\[ U_+\left(\wedge^2_{\pm}TN\right) := \left\{ \Theta \in \wedge^2_{\pm}TN \mid \hat{h}(\Theta, \Theta) = 1 \right\}, \]
\[ U_+\left(\wedge^2_{-}TN\right) := \left\{ \Theta \in \wedge^2_{-}TN \mid \hat{h}(\Theta, \Theta) = 1 \right\}. \]
$M$: a Riemann surface,

$F : M \longrightarrow N$: a space-like and conformal immersion of $M$ into $N$.

$\Theta_{F, \pm}$: sections of $U_+ \left( \bigwedge^{2 \pm} F^*TN \right)$ defined by $\Theta_{F, \pm} := \frac{1}{\sqrt{2}}(\xi_1 \wedge \xi_2 \mp \xi_3 \wedge \xi_4)$,

where $\xi_1, \xi_2, \xi_3, \xi_4$ form a local pseudo-orthonormal frame field of $F^*TN$ s.t.

- $(\xi_1, \xi_2, \xi_3, \xi_4)$ gives the orientation of $N$,
- $\xi_1, \xi_2 \in dF(TM)$ so that $(\xi_1, \xi_2)$ gives the orientation of $M$.

$I_{F, \pm}$: the complex structures of $F^*TN$ corresponding to $\Theta_{F, \pm}$.

Then $\Theta_{F, \pm} = \frac{1}{\sqrt{2}}(e \wedge I_{F, \pm}(e) - e^\perp \wedge I_{F, \pm}(e^\perp))$,

where

- $e$ is a unit tangent vector of $F$,
- $e^\perp$ is a normal vector of $F$ with $h(e^\perp, e^\perp) = -1$. 
If $N$ is neutral hyperKähler so that $\bigwedge^2_T N$ (respectively, $\bigwedge^2_-TN$) is a product bundle, then we can consider $\Theta_{F,+}$ (respectively, $\Theta_{F,-}$) to be a map from $M$ into $\mathbb{C}H^1$.

**Theorem (A, 2020) Suppose**

- $N$ is neutral hyperKähler,
- $F : M \rightarrow N$ has zero mean curvature vector.

Then one of $\Theta_{F,+}, \Theta_{F,-}$ is a holomorphic map from $M$ into $\mathbb{C}H^1$.

**Corollary (A, 2020)**

$F : M \rightarrow E^4_2$: a space-like and conformal immersion with zero mean curvature vector.

Then the Gauss map $G_F : M \rightarrow \mathbb{C}H^1 \times \mathbb{C}H^1$ of $F$ is holomorphic.
$M$: a Riemann surface,

$F : M \rightarrow N$: a space-like and conformal immersion of $M$ into $N$
with zero mean curvature vector.

\[ \nabla_{\partial/\partial w}(\Psi dw) = \sigma \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right) dw \quad \left( \Psi = dF \left( \frac{\partial}{\partial w} \right) \right). \]

We can define a complex quartic differential $Q$ on $M$ by

\[ Q := h \left( \sigma \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right), \sigma \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right) \right) dw \otimes dw \otimes dw \otimes dw. \]

If $N$ is a 4-dimensional neutral space form,
then we see by the equations of Codazzi that $Q$ is holomorphic.
Theorem  The following are mutually equivalent:

(a) at each point of $M$, principal curvatures do not depend on the choice of a normal vector $e^\perp$ of $F$ with $h(e^\perp, e^\perp) = -1$;

(b) $h(\sigma(T_1, T_1), \sigma(T_1, T_1)) = h(\sigma(T_1, T_2), \sigma(T_1, T_2))$, $h(\sigma(T_1, T_1), \sigma(T_1, T_2)) = 0$ for $T_1 := dF(\partial/\partial u)$, $T_2 := dF(\partial/\partial v)$;

(c) $Q \equiv 0$;

(d) one of $\Theta_F, +$, $\Theta_F, -$ is horizontal w.r.t. $\hat{\nabla}$;

(e) one of $I_F, \pm$ is parallel w.r.t. $\nabla$;

(f) we have one of $I_F, \pm \sigma(T_1, T_1) = \sigma(T_1, T_2)$.

We say that $F$ is isotropic if one of (a) $\sim$ (f) in the above theorem holds.
Theorem (A, 2020)

\[ F : M \rightarrow N: \text{a space-like and conformal immersion} \]

with zero mean curvature vector and \( \hat{R}(e_1, e_2)\Theta_{F,\pm} = 0 \).

Then \( Q \) is holomorphic.

In addition, if \( \hat{\nabla}\Theta_{F,\pm} \neq 0 \), then we can choose \((e_1, e_2, e_3, e_4)\) satisfying

(a) the connection forms \( \omega, \omega^\perp \) given by \( \omega := h(\nabla e_1, e_2), \omega^\perp := h(\nabla e_3, e_4) \) satisfy \( d*\omega = 0 \) and \( d*\omega^\perp = 0 \);

(b) the 2nd fundamental form of \( F \) is constructed by a solution of an over-determine system s.t. the compatibility condition is given by \( d*\omega = 0 \) and \( d*\omega^\perp = 0 \).

Remark If \( N \) is a 4-dimensional neutral space form, then \( \hat{R}(e_1, e_2)\Theta_{F,\pm} = 0 \).
4. Time-like surfaces with zero mean curvature vector in neutral 4-manifolds

The time-like twistor spaces associated with $N$ are fiber bundles in $\bigwedge_{\pm}^2 TN$ given by

$$U_-(\bigwedge_{\varepsilon}^2 TN) := \left\{ \Theta \in \bigwedge_{\varepsilon}^2 TN \mid \hat{h}(\Theta, \Theta) = -1 \right\} \quad (\varepsilon = +, -).$$

$M$: a Lorentz surface (two-dimensional manifold with a holomorphic system of paracomplex coordinate neighborhoods),

$F: M \rightarrow N$: a time-like and conformal immersion of $M$ into $N$.

$\Theta_{F, \pm}$: sections of $U_-(\bigwedge_{\pm}^2 F^*TN)$ defined by $\Theta_{F, \pm} := \frac{1}{\sqrt{2}}(\xi_1 \wedge \xi_3 \pm \xi_4 \wedge \xi_2)$,

where $\xi_1, \xi_2, \xi_3, \xi_4$ form a local pseudo-orthonormal frame field of $F^*TN$ (we suppose that $\xi_1, \xi_2$ are space-like) s.t.

- $(\xi_1, \xi_2, \xi_3, \xi_4)$ gives the orientation of $N$,
- $\xi_1, \xi_3 \in dF(TM)$ so that $(\xi_1, \xi_3)$ gives the orientation of $M$. 
$J_{F,\pm}$: the paracomplex structures of $F^*TN$ corresponding to $\Theta_{F,\pm}$.

Then $\Theta_{F,\pm} = \frac{1}{\sqrt{2}}(e \wedge J_{F,\pm}(e) - e^\perp \wedge J_{F,\pm}(e^\perp)),$

where

- $e$ is a unit tangent vector of $F$,
- $e^\perp$ is a normal vector of $F$ with $h(e^\perp, e^\perp) = -1$.

If $N$ is neutral hyperKähler so that $\wedge^2_+ TN$ (respectively, $\wedge^2_- TN$) is a product bundle, then we can consider $\Theta_{F,+}$ (respectively, $\Theta_{F,-}$) to be a map from $M$ into $\tilde{\mathcal{C}H}^1$ (a hyperboloid of one sheet as a Lorentz surface).
A hyperboloid of one-sheet is given by $H^2_1 = \{ x \in E^3_2 \mid \langle x, x \rangle_{1,2} = -1 \}$.

Let $R_+, R_-$ be open subsets of $H^2_1$ defined by

$R_+ := \{ x = (x^1, x^2, x^3) \in H^2_1 \mid x^3 \neq 1 \}$,

$R_- := \{ x = (x^1, x^2, x^3) \in H^2_1 \mid x^3 \neq -1 \}$.

$\tilde{\mathbb{C}}$: the paracomplex plane $= \{ \tilde{w} = u + jv \mid u, v \in \mathbb{R} \}$

$(j$: the paraimaginary unit),

$|\tilde{w}|^2 := \overline{\tilde{w}}\tilde{w} = u^2 - v^2$,

$C_\delta := \{ \tilde{w} \in \tilde{\mathbb{C}} \mid |\tilde{w}|^2 = \delta \}$ ($\delta = 0, 1$).

The stereographic projections $\text{pr}_{\pm}$ are bijective maps from $R_\pm$ onto $\tilde{\mathbb{C}} \setminus C_1$ defined by

$$\text{pr}_\pm^{-1}(\tilde{w}) = \left( \frac{\text{Re} \, \tilde{w}}{1 - |\tilde{w}|^2}, \mp \frac{\text{Im} \, \tilde{w}}{1 - |\tilde{w}|^2}, \mp \frac{1 + |\tilde{w}|^2}{1 - |\tilde{w}|^2} \right) \quad (\tilde{w} \in \tilde{\mathbb{C}} \setminus C_1).$$
The geometric definition of $\text{pr}_+$
Since $\text{pr}_\pm(R_+ \cap R_-) = \tilde{C} \setminus (C_1 \cup C_0)$, we see that the composition
$$\text{pr}_- \circ \text{pr}_+^{-1} : \text{pr}_+(R_+ \cap R_-) \longrightarrow \text{pr}_-(R_+ \cap R_-)$$
is holomorphic.
Therefore, noticing $R_+ \cup R_- = H^2_1$, we can consider $H^2_1$ to be a Lorentz surface, which is denoted by $\tilde{\mathcal{C}}H^1$.

**Theorem (A, 2020)** Suppose

- $N$ is neutral hyperKähler,
- $F : M \longrightarrow N$ has zero mean curvature vector.

Then one of $\Theta_{F,+}, \Theta_{F,-}$ is a holomorphic map from $M$ into $\tilde{\mathcal{C}}H^1$.

**Corollary (A, 2020)**

$F : M \longrightarrow E^4_2$: a time-like and conformal immersion with zero mean curvature vector,

Then the Gauss map $\mathcal{G}_F : M \longrightarrow \tilde{\mathcal{C}}H^1 \times \tilde{\mathcal{C}}H^1$ of $F$ is holomorphic.
$M$: a Lorentz surface,
$F : M \rightarrow N$: a time-like and conformal immersion of $M$ into $N$ with zero mean curvature vector,

$w = u + jv$: a local paracomplex coordinate of $M$,

$\Psi := dF \left( \frac{\partial}{\partial w} \right) \left( \frac{\partial}{\partial w} = \frac{1}{2} \left( \frac{\partial}{\partial u} + j \frac{\partial}{\partial v} \right) \right)$.

$\nabla_{\partial/\partial w}(\Psi dw) = \sigma \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right) dw$.

We can define a paracomplex quartic differential $Q$ on $M$ by

$Q := h \left( \sigma \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right), \sigma \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right) \right) dw \otimes dw \otimes dw \otimes dw$.

If $N$ is a 4-dimensional neutral space form, then we see by the equations of Codazzi that $Q$ is holomorphic.
Theorem \textbf{The following are equivalent:}\n(a) $h(\sigma(T_1, T_1), \sigma(T_1, T_1)) = -h(\sigma(T_1, T_2), \sigma(T_1, T_2)),$
\hspace{1cm}$h(\sigma(T_1, T_1), \sigma(T_1, T_2)) = 0$ for $T_1 := dF(\partial/\partial u), T_2 := dF(\partial/\partial v);$\n(b) $Q \equiv 0.$

We say that $F$ is isotropic if one of (a), (b) in the above theorem holds.

\textbf{Theorem (A, 2020) The following are mutually equivalent:}\n(a) one of $\Theta_F,+, \Theta_F, -$ is horizontal \ w.r.t. $\hat{\nabla};$\n(b) one of $J_F,\pm$ is parallel \ w.r.t. $\nabla;$\n(c) we have one of $J_F,\pm \sigma(T_1, T_1) = \sigma(T_1, T_2).$
\textbf{In addition, if $F$ satisfies one of (a), (b), (c), then $F$ is isotropic.}\n
We say that $F$ is strictly isotropic if one of (a), (b), (c) in the above theorem holds for the orientation of $N.$
It is possible that although $F$ is isotropic, none of the covariant derivatives of $\Theta_{F,+}, \Theta_{F,-}$ w.r.t. $\hat{\nabla}$ become zero.

**Proposition (A, 2020)**

*If both $\hat{\nabla}\Theta_{F,+}$ and $\hat{\nabla}\Theta_{F,-}$ are light-like, then one of the following holds:*

(a) the shape operator of a light-like normal vector field vanishes and then $Q$ vanishes;

(b) the shape operator of any normal vector field is zero or light-like, and then $Q$ is zero or null.
Remark
Suppose that $N$ is a 4-dimensional neutral space form.

- Condition (a) implies that a light-like normal vector field of the surface is contained in a constant direction.

  The conformal Gauss map of a time-like surface in a 3-dimensional Lorentzian space form of Willmore type with $Q \equiv 0$ has this property.

- We can characterize surfaces with condition (b), based on the Gauss-Codazzi-Ricci equations.
$M$: an oriented two-dimensional manifold,

$\iota : M \longrightarrow N_1^3 = S_1^3$, $E_1^3$ or $H_1^3$: a time-like immersion

(we consider $S_1^3$, $E_1^3$, $H_1^3$ to be subsets of $E_2^5$),

e_3: a unit normal vector field of $\iota$ in $N_1^3$,

$H$: the mean curvature of $\iota$ w.r.t. $e_3$,

$\gamma_\iota := e_3 + H\iota$,

$\Lambda := H^2 - K^M + \delta$

($\delta = 1, 0$ or $-1$, $K^M$: the curvature of the induced metric $g^M$ by $\iota$),

Reg ($\iota$): the set of nonzero points of $\Lambda$.

$\implies \gamma_\iota|_{\text{Reg}(\iota)}$ is a time-like immersion of Reg ($\iota$) into $S_2^4$ s.t.

the induced metric $g$ by $\gamma_\iota|_{\text{Reg}(\iota)}$ is given by $g = \Lambda g^M$.

We call $\gamma_\iota : M \longrightarrow S_2^4$ the conformal Gauss map of $\iota : M \longrightarrow N_1^3$. 
• $\iota$ is a light-like normal vector field of a time-like immersion $\gamma|_{\text{Reg}(\iota)}$, 
• the trace of the shape operator of $\gamma|_{\text{Reg}(\iota)}$ w.r.t. $\iota$ is zero, 
• if we denote by $\nu$ a light-like normal vector field of $\gamma|_{\text{Reg}(\iota)}$ satisfying $\langle \iota, \nu \rangle_{3,2} = -1$, then the trace of the shape operator of $\gamma|_{\text{Reg}(\iota)}$ w.r.t. $\nu$ is given by $-\frac{1}{\Lambda}(\Delta^M H + 2\Lambda H)$.

Since $\Lambda \equiv 0$ means that $\Delta^M H = 0$, we obtain

**Theorem (A)** An immersion $\iota : M \longrightarrow N_1^3$ satisfies $\Delta^M H + 2\Lambda H = 0$ if and only if the mean curvature vector of $\gamma|_{\text{Reg}(\iota)} : \text{Reg}(\iota) \longrightarrow S^4_2$ vanishes.

We say that $\iota$ is of Willmore type $\iff \Delta^M H + 2\Lambda H = 0$. 
$M$: a Lorentz surface,
$\iota: M \rightarrow N_1^3$: a time-like and conformal immersion,
$\Xi := 2\sigma^M \otimes \text{Hess}^M_H + (H^2 + \delta)\sigma^M \otimes \sigma^M - 2dH \otimes \nabla^M \sigma^M$ 
($\sigma^M$: the 2nd fundamental form of $\iota$).

**Proposition (A)** If $\iota: M \rightarrow N_1^3$ is of Willmore type, then a paracomplex quartic differential

$$\tilde{Q} := \Xi\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right) dw \otimes dw \otimes dw \otimes dw$$

is holomorphic ($w = u + jv$: a local paracomplex coordinate of $M$).
Theorem (A)

\[ \iota : M \rightarrow N^3_1 : \text{a time-like and conformal immersion of Willmore type.} \]

On \( \text{Reg(}\iota) \), the following hold:

(a) the null points of the differential \( Q \) for \( F := \gamma_{\iota}|_{\text{Reg(}\iota)} \) coincide with the null points of \( \tilde{Q} \), and a null point of \( Q \) is just given by a condition that the shape operator of \( F \) w.r.t. \( \nu \) is light-like;

(b) except the null points, \( Q \) coincides with \( \tilde{Q} \) up to a nonzero constant;

(c) \( Q \equiv 0 \) if and only if a light-like normal vector field \( \nu \) of \( F \) is contained in a constant direction.

Remark

Suppose

- \( \iota \) as in the above theorem satisfies \( \tilde{Q} \equiv 0 \);
- \( (\nabla_{T_1} T_1)^\perp \neq \pm (\nabla_{T_1} T_2)^\perp \) \( (T_1 = dF(\partial/\partial u), T_2 = dF(\partial/\partial v)) \).

\[ \implies \text{For } \Theta_{F,\pm} \text{ with } F = \gamma_{\iota}|_{\text{Reg(}\iota)}, \hat{\nabla}\Theta_{F,\pm} \text{ are light-like.} \]
(\(e_1, e_3\)): a local ordered pseudo-orthonormal frame field of \(TM\) giving the orientation of \(M\).

**Theorem (A, 2020)**

\[ F : M \rightarrow N : \text{a time-like and conformal immersion} \]

with zero mean curvature vector and \(\hat{R}(e_1, e_3)\Theta_{F, \pm} = 0\).

Then \(Q\) is holomorphic and

the 2nd fundamental form of \(F\) is constructed by solutions of four families of ordinary differential equations defined along integral curves of light-like vector fields \(e_1 \pm e_3\) and given by the connection forms \(\omega := -h(\nabla e_1, e_3)\), \(\omega^\perp := -h(\nabla e_2, e_4)\).
If $\hat{\nabla} \Theta_{F, \pm}$ are zero or light-like, then $\hat{R}(e_1, e_3) \Theta_{F, \pm}$ are zero or light-like.

**Theorem (A, 2020)**

$F : M \longrightarrow N$: a time-like and conformal immersion with zero mean curvature vector s.t. $\hat{R}(e_1, e_3) \Theta_{F, \pm}$ are zero or light-like.

Then the 2nd fundamental form of $F$ is constructed by solutions of suitable two families of ordinary differential equations of the four families in the previous theorem.
THE FIRST TALK HAS ENDED.