Codimension one Ricci soliton subgroups of solvable Iwasawa groups

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Abstract (1/2)

Theme

- An interaction between
  - homogeneous Ricci solitons,
  - submanifolds in symmetric spaces.

Joint work with

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Abstract (2/2)

Setting

- $M = G/K$: irreducible symmetric space of noncpt type,
- $G = KAN$: Iwasawa decomposition,
- AN : the solvable Iwasawa group.

Note

- $M \cong AN$;
- $AN \supset S$: codim 1 subgroup
  $\Rightarrow S$ $(\cong S.o)$ is a homog. hypersurface.

Main Result

- We classify codim 1 Ricci soliton subgr. in AN.
- In other words, we classify particular Ricci soliton hypersurfaces in M.

Motivation 1 - (1/2)

Basic Question

- Which homogeneous spaces G/K admit G-invariant Einstein or Ricci soliton metrics?

Def.

- Riem. mfd (M, g) is **Ricci soliton** if $\exists c \in \mathbb{R}, \exists X \in \mathfrak{X}(M) : \mathrm{Ric}_g = cg + \mathfrak{L}_X g$.

Note

- A situation of homogeneous Einstein/Ricci solitons depends on the signature of c.

Fact (Naber, Petersen-Wylie)

- (M, g) : homogeneous Ricci soliton $(c > 0)$ $\Rightarrow M \cong$ [homog. Einstein with $\mathrm{sc} > 0$] $\times$ [flat].

Fact (Alekseevskii-Kimel'fel'd)

- (M, g) : homogeneous Ricci soliton $(c = 0)$ $\Rightarrow$ flat.

Motivation 1 - (2/2)

(Generalized) Alekseevskii Conjecture

- (M, g) : homog. Einstein (Ricci soliton), $c < 0$
  $\Rightarrow M$ is a solvmanifold?
  (solvable Lie group with left-invariant metric?)

Fact (Jablonski)

- AC and GAC are equivalent.

Fact (Böhm-Lafuente, arXiv:2107)

- AC is true.

Motivating Question

- Which solvable Lie groups admit left-invariant Einstein or Ricci soliton metrics?

Note

- Classification is known only for dim ≤ 6 (Will).
- Typical examples are (irr.) symmetric spaces of noncpt type $M = G/K \cong AN$.

Motivation 2 - (1/4)

Basic Question

- Find/classify submanifolds with some nice property in symmetric spaces.

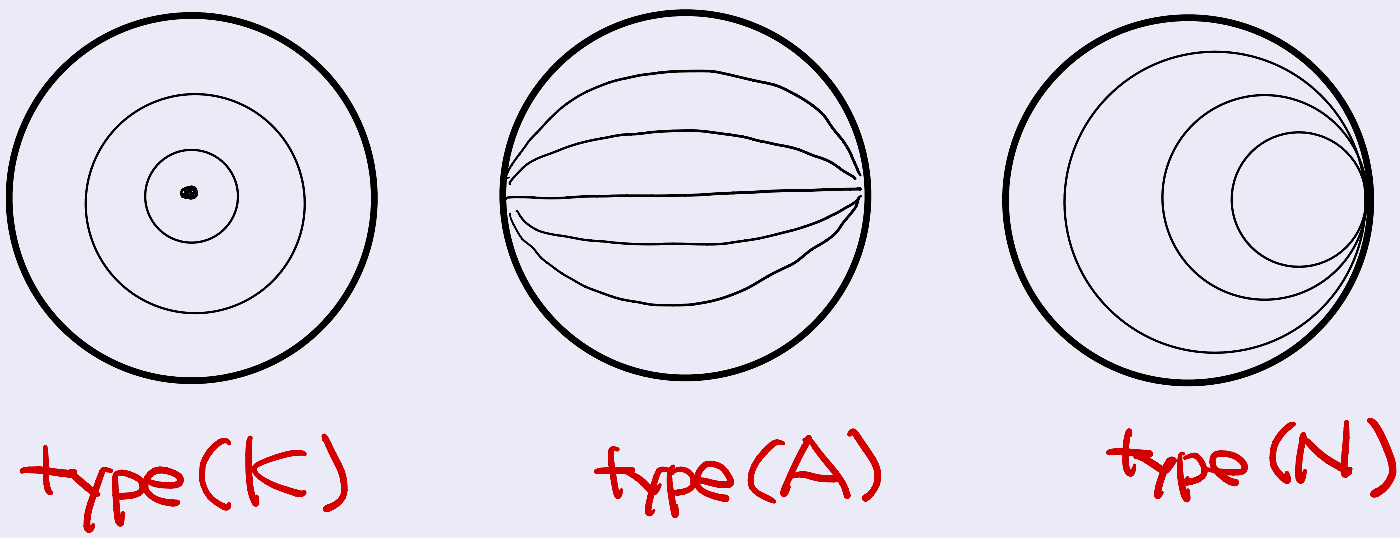
Our Question

- Study/classify homogeneous hypersurfaces in symmetric spaces of noncompact type.
- Equivalently, cohomogeneity one action := isometric action with regular orbit of codim 1.

Fact (Berndt-T.)

- M : irr. symmetric space of noncpt type,
- $H \curvearrowright M$: cohom. one (H connected) satisfies
  (K) $\exists 1$ singular orbit;
  (A) $\nexists$ singular orbit, $\exists 1$ minimal orbit;
  (N) $\nexists$ singular orbit, all orbits are congruent.

Motivation 2 - (2/4)



Ex. (E. Cartan)

Homogeneous hypersurfaces in $\mathbb{R}\mathrm{H}^n$ are

- (K) a tube around totally geodesic $\mathbb{R}\mathrm{H}^k$ with $k \in \{0, \ldots, n-2\}$;
- (A) a totally geodesic $\mathbb{R}\mathrm{H}^{n-1}$ and its equidistant hypersurfaces;
- (N) horospheres.

Motivation 2 - (3/4)

Note (classification of (A), Berndt-T.)

- Actions of (A) are given by some codim 1 subgroups in AN;
- there are almost $\mathrm{rk}(M)$ such actions.

Note (classification of (N), Berndt-T.)

- Actions of (N) are given by some codim 1 subgroups in AN;
- $\exists$ continuously many such actions if $\mathrm{rk} > 1$.

Note (classification of (K))

- $\mathrm{rk} = 1$: completely classified:
  - $\mathbb{R}\mathrm{H}^n$ by Cartan;
  - $\mathbb{C}\mathrm{H}^n$, $\mathbb{O}\mathrm{H}^2$ by Berndt-T. (2007);
  - $\mathbb{H}\mathrm{H}^n$ by DiazRamos et al. (2021);
- $\mathrm{rk} = 2$: classified for many cases:
  - Berndt-T. (2012);
  - Berndt-DominguezVazquez (2015);
  - Solonenko (arXiv);
- $\mathrm{rk} \geq 3$: widely open.

Motivation 2 - (4/4)

- We are interested in Ricci soliton hypersurfaces.

Ex. (for $\mathbb{R}\mathrm{H}^n$)

(A) a totally geodesic $\mathbb{R}\mathrm{H}^{n-1}$ and its equidistant hypersurfaces have const sectional curvatures;

(N) horospheres are flat.

Ex. (for $\mathbb{C}\mathrm{H}^n$)

- $\mathbb{C}\mathrm{H}^n$ admits no Einstein hypersurfaces;
- The horosphere is Ricci soliton (isometric to the Heisenberg group with left-inv. metric);

Motivation 3 - (1/1)

- We study Ricci soliton subgroups in the solvable Iwasawa groups AN.

Fact (T.)

- $\exists$ interesting examples $(\mathrm{codim} > 1)$:
  - For every parabolic subgroup $Q_\Phi \subset G$, the solvable part $A_\Phi N_\Phi$ of the Langlands decomposition $Q_\Phi = M_\Phi A_\Phi N_\Phi$ is Einstein.

Fact (Jablonski)

- Every Ricci soliton solvmfd (S, g) can be isometrically embedded into some AN.
  (In general, codim is very high.)

Result - (1/3)

Setting

- $M = G/K$: irr. symmetric sp. of noncpt type;
- $G = KAN$: Iwasawa decomposition;
- AN : the solvable Iwasawa group.

Note

- $M \cong (AN, \exists\langle , \rangle)$: isometric;
- For $S \subset AN$ we equip the induced metric.

Main Thm.

codim 1 subgroup S ($\subset AN$) is Ricci soliton iff

- M is any, and S contains N (i.e., type (N));
- $M = \mathbb{C}\mathrm{H}^2$, and S is "ruled minimal";
- $M = \mathbb{R}\mathrm{H}^n$, and S is any.

Result - (2/3)

Note

- Case for $\mathbb{R}\mathrm{H}^n$ is well-known (as before).

Note

- Case for type (N) is known:
- Recall that all orbits are congruent;
- All orbits are Ricci soliton (cf. Cho-Hashinaga-Kubo-Taketomi-T.);
- If $\mathrm{rk} > 1$, then $\exists$ Einstein ones.

Note

- Case for $\mathbb{C}\mathrm{H}^n$:
- type (N) = horosphere, which is Ricci soliton;
- type (A) = homogeneous ruled minimal hypersurface W^{2n-1}, and its equidistant ones;
- W^{2n-1} is Ricci soliton iff $n = 2$. (cf. Kubo-Hashinaga-T.)

Result - (3/3)

Idea of Proof

- S ($\subset AN$) is completely solvable;
- completely solvable solvmfd is Ricci soliton iff algebraic Ricci soliton, i.e.,

$$\mathrm{Ric} = c \cdot \mathrm{id} + D \quad (c \in \mathbb{R}, \ D \in \mathrm{Der}(\mathfrak{s}))$$

- so we have only to check this condition...

Note

- By a similar method, we also classify codim. 1 Ricci soliton subgroups of H-type groups and Damek-Ricci spaces, which contain some new examples.

Problems - (1/3)

Problem 1

- Classify Ricci soliton subgroups in AN with ...
  e.g., low codimension;
  e.g., inducing (hyper)polar foliations.

Note

- For $S_\Phi = A_\Phi N_\Phi$ in parabolic subgroups,
  - some of them are of codim 2;
  - the action of S_Φ is polar (inducing a foliation);
  - all S_Φ-orbits are congruent (like (N)-type);
  - all orbit are minimal.

Problems - (2/3)

Problem 2

- Classify Ricci soliton homogeneous hypersurfaces in M (irr. symmetric space of noncpt type).

Note

- Cases (A) and (N) are solved by our work;
- It remains (K) only;
- By (the proof of) Alekseevskii conjecture, seems $\nexists$ Ricci soliton with $c < 0$;
- Do not forget geodesic spheres in $\mathbb{R}\mathrm{H}^n$, which is Einstein (standard sphere)...

Problems - (3/3)

Problem 3

- Classify Ricci soliton subgroups in N with codim 1.

Note

- For $S_\Phi = A_\Phi N_\Phi$ in parabolic subgroups,
  - N_Φ is Ricci soliton in N;
  - some N_Φ is of codim 1;
  - so more examples can be expected.

Summary

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Thank you!