Totally geodesic surfaces in symmetric spaces and applications

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The 23rd International Differential Geometry Workshop on Submanifolds in Homogeneous Spaces and Related Topics
- The 19th RIRCM-OCAMI Joint Differential Geometry Workshop
(online), 03/July/2021

Introduction - (1/4)

Totally geodesic submanifolds (in symmetric spaces) would be most fundamental class of submanifolds.

Def.

A submfd M in $(\overline{M}, g)$ is **totally geodesic** (**TG**)
$:\Leftrightarrow$ the second fundamental form $\equiv 0$
$\Leftrightarrow$ every geodesic in M is also a geodesic in $(\overline{M}, g)$.

Note

- Usually just consider connected and complete ones.
- All totally geodesic submfds in symmetric spaces are (intrinsically) symmetric.

Introduction - (2/4)

A classification problem of TG submfds in symmetric spaces is hard and widely open.

Known Classification

- Klein (2008-10) classified the case
  - $\overline{M}$ irreducible rank ≤ 2, and M maximal TG;
- $\nexists$ general classification for rank ≥ 3;
- $\exists$ classification for some special case;
  - (Satake, Ihara) $\overline{M}$ Hermitian, M cplx TG;
  - (Leung) M reflective;
  - ...

Introduction - (3/4)

We study **TG surfaces** in symmetric spaces — which would be "building blocks" of TG submfds.

Prop.

- Let $\overline{M}$ be a semisimple symmetric space.
- Then every nonflat TG submfd in $\overline{M}$ contains a nonflat TG surface.

Note

- If we understand all TG surfacfes well, then it would helpful to understand all TG submfds.
- In previous studies "maximal" TG submfds are considered; on the other hand, TG surfaces are "smallest possible";

Introduction - (4/4)

We describe TG surfaces in terms of "nilpotent elements", and obtain several application.

Result (one of applications)

- $\overline{M}$ a (irr.) Hermitian symmetric space;
- cplx curve := surface which is a cplx submfd.
- Then we have

$$\#(\{\text{TG cplx curves in } \overline{M}\}/\text{cong}) = \mathrm{rank}(\overline{M}).$$

Note

- This talk is based on several joint works with Kentaro Kimura, Akira Kubo, Katsuya Mashimo, Takayuki Okuda, ...

TG surfaces - (1/3)

Fact

- TG submfds are preserved by "duality";
- TG submfd $\leftrightarrow$ Lie Triple System (**LTS**);
- Flat TG submfds are well-understood.

Ex.

- $\mathrm{Sym}^0(n, \mathbb{R}) := \{X \in M(n, \mathbb{R}) \mid \text{symm, trace } 0\}$.
- A subspace $V \subset \mathrm{Sym}^0(n, \mathbb{R})$ is **LTS** $:\Leftrightarrow [V, [V, V]] \subset V$ (where $[u, v] := uv - vu$).

Note

- LTSs in $\mathrm{Sym}^0(n, \mathbb{R})$ correspond to TG submfds in $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$.
- In general, TG submfds in G/K correspond to LTSs in $\mathfrak{p}$, the subspace in the Cartan decomp.

TG surfaces - (2/3)

Fact

- TG surfaces $\leftrightarrow$ LTS of dim 2;
- nonflat TG surfaces $\leftrightarrow$ nonabelian LTS of dim 2.

Thm (Fujimaru-Kubo-T. 2014)

$\exists$ correspondence between

- 2-dim. nonabelian LTSs in $\mathrm{Sym}^0(n, \mathbb{R})$;
- upper triangular $X \in M(n, \mathbb{R})$ such that
  (i) $[X, {}^tX] = \mathrm{diag}(a_1, \ldots, a_n)$ with $a_1 \geq \cdots \geq a_n$;
  (ii) $\exists c > 0 : [X, [X, {}^tX]] = cX$.

Note

- TRICK: 2-dim. object $\leftrightarrow$ 1-dim. nilpotent element.
- This theorem can be stated for G/K of noncpt type in terms of root systems...

TG surfaces - (3/3)

Cor.

- In $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$, up to isometric congruence,
  - $n = 3 \Rightarrow \exists$ exactly 2 nonflat TG surfaces;
  - $n = 4 \Rightarrow \exists$ exactly 4 nonflat TG surfaces;
  - $n = 5 \Rightarrow \exists$ exactly 6 nonflat TG surfaces.

Note

- Classifying all TG surfaces in general symmetric spaces is more involved, for which we need further techniques (in progress).

Application 1 - (1/3)

Note

- TG cplx curve := TG surface which is cplx (in Hermitian symmetric space)

Thm (Kubo-Okuda-T.)

- For an irr. Hermitian symm. sp. $\overline{M}$, we have
$$\#(\{\text{TG cplx curves in } \overline{M}\}/\text{cong}) = \mathrm{rank}(\overline{M}).$$

Furthermore we can describe them explicitly.

Ex.

- Let $\overline{M} := \mathbb{C}\mathrm{H}^n$.
- Then $\exists! M = \mathbb{C}\mathrm{H}^1$ which is TG.
- Note: $\mathbb{C}\mathrm{H}^1 \subset \mathbb{C}\mathrm{H}^2 \subset \cdots \subset \mathbb{C}\mathrm{H}^{n-1} \subset \mathbb{C}\mathrm{H}^n$.

Application 1 - (2/3)

Ex.

- Let $\overline{M} := G_2^*(\mathbb{R}^n)$ with $n \geq 4$ (real Grassmann);
- $\mathrm{rank}(\overline{M}) = 2$;
- Observe

$$\overline{M} \supset G_2^*(\mathbb{R}^4) = \mathbb{C}\mathrm{H}^1 \times \mathbb{C}\mathrm{H}^1.$$

- Then $\exists$ two TG cplx curves, namely $(z, 0)$ and (z, z).

Ex.

- Let $\overline{M} := G_k^*(\mathbb{C}^n)$ with $2k \leq n$ (cplx Grassmann);
- $\mathrm{rank}(\overline{M}) = k$;
- Observe

$$\overline{M} \supset G_k^*(\mathbb{C}^{2k}) \supset (G_1^*(\mathbb{C}^2))^k = (\mathbb{C}\mathrm{H}^1)^k.$$

- Then $\exists$ k different TG cplx curves.

Fact (polydisc theorem)

- Let $\overline{M}$ be Hermitian with $\mathrm{rank}(\overline{M}) = r$;
- Then there is a TG submfd

$$(\mathbb{C}\mathrm{H}^1)^r \subset \overline{M}.$$

Application 1 - (3/3)

Idea of Proof

- By the polydisc theorem, we have r-TG cplx curves.
- They can be distinguished by the sectional curvatures.
- They exhaust all — this is a consequence of our thm on TG surfaces.
  (Note: this result itself is known by Satake.)

Note

- As mentioned before, counting all TG surfaces is more involved (in progress).

Application 2 - (1/2)

Prop.

- Let $\overline{M}$ be a symmetric space (semisimple), and M be a nonflat TG submfd in $\overline{M}$.
- Then M contains a nonflat TG surface.

Note

- Therefore TG surfaces would be useful for the classification problem of TG submfds.
- Possible Steps:
  (1) Classify all nonflat TG surfaces Σ (in given $\overline{M}$);
  (2) For each Σ, classify TG submfd M satisfying
  $$\Sigma \subset M \subset \overline{M}.$$

Application 2 - (2/2)

Thm (Kimura-Kubo-Okuda-T.)

- The maximal TG submfds in $\mathrm{SL}(4, \mathbb{R})/\mathrm{SO}(4)$ are:
  (1) $[\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)] \times \mathbb{R}^+$;
  (2) $[\mathrm{Sp}(2, \mathbb{R})/\mathrm{U}(2)]$;
  (3) $\mathbb{R}\mathrm{H}^2 \times \mathbb{R}\mathrm{H}^2 \times \mathbb{R}^+$;
  (4) $\mathrm{SO}^0(2, 2)/\mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(2))$;
  (5) $\mathrm{SO}^0(1, 3)/\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(3))$.

Note

- This would be the first complete classification result for a rank 3 ambient space.
- It may possible to apply this strategy for other spaces, but the calculations are quite involved.

Summary

The main point

- TG submfds are fundamental objects, but many questions remain open.
- We propose a new strategy for studying TG submfds in symmetric spaces
  — focus on TG surfaces as "building blocks".

Our Results

- TG surfaces correspond to some "nilpotent" elements;
- Classification of TG surfaces in some spaces;
- Classification of TG cplx curves;
- Classification of (maximal) TG submfds in some spaces.

Thank you!